

ABSORBING-REFLECTING FACTORIZATIONS FOR BIRTH-DEATH CHAINS ON THE INTEGERS AND THEIR DARBOUX TRANSFORMATIONS

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ABSTRACT. We consider a new way of factorizing the transition probability matrix of a discrete-time birth-death chain on the integers by means of an absorbing and a reflecting birth-death chain to the state 0 and viceversa. First we will consider reflecting-absorbing factorizations of birth-death chains on the integers. We give conditions on the two free parameters such that each of the factors is a stochastic matrix. By inverting the order of the factors (also known as a Darboux transformation) we get new families of “almost” birth-death chains on the integers with the only difference that we have new probabilities going from the state 1 to the state -1 and viceversa. On the other hand an absorbing-reflecting factorization of birth-death chains on the integers is only possible if both factors are splitted into two separated birth-death chains at the state 0. Therefore it makes more sense to consider absorbing-reflecting factorizations of “almost” birth-death chains with extra transitions between the states 1 and -1 and with some conditions. This factorization is now unique and by inverting the order of the factors we get a birth-death chain on the integers. In both cases we identify the spectral matrices associated with the Darboux transformation, the first one being a Geronimus transformation and the second one a Christoffel transformation of the original spectral matrix. We apply our results to examples of chains with constant transition probabilities.

1. INTRODUCTION

This paper is a continuation of our work [11] about stochastic factorizations of the transition probability matrix P of a discrete-time birth-death chain on the integers \mathbb{Z} (doubly infinite tridiagonal matrix) and the relation between the spectral matrices after performing the so-called discrete Darboux transformation (inverting the order of the factors). In [11] we considered UL and LU stochastic factorizations of P . That means that each of the factors represents either a pure-birth or a pure-death chain. We consider here a new approach consisting of having one of the factors as a *reflecting* birth-death chain on \mathbb{Z} from the state 0 and the other factor as an *absorbing* birth-death chain on \mathbb{Z} to the state 0 (see (2.5) and (2.6) below). We will name them RA or AR factorizations, respectively. In the case of birth-death chains on the nonnegative integers $\mathbb{Z}_{\geq 0}$ (see [8]) there is no difference between UL (LU) stochastic factorizations and RA (AR) factorizations, but when the state space is \mathbb{Z} these factorizations represent different chains. The main motivation for these stochastic factorizations is to divide the probabilistic model into two different and simpler experiments, and combine them together to obtain a simpler description of the original probabilistic model (see applications to urn models in [8, 9]).

The importance of having the spectral matrices is that it is easy to analyze the corresponding Markov chains in terms of polynomials which arise as a solution of the eigenvalue equation. For the case of birth-death chains on $\mathbb{Z}_{\geq 0}$ this was first done in a series of papers by S. Karlin and J. McGregor in the 1950s (see [14, 15, 16]). Apart from an explicit expression of the n -step transition probabilities and the invariant measure, it is possible to study some other probabilistic properties using spectral methods such as recurrence, absorbing times, first

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return times or limit theorems. In the last section of [16] one can find the first attempt to perform the spectral analysis of a discrete-time birth-death chain on \mathbb{Z} using orthogonal polynomials. We will recall this approach in Section 2.1. After that, apart from [1], there are not so many references concerning the spectral analysis of doubly infinite tridiagonal (or Jacobi) operators acting on $\ell^2_\pi(\mathbb{Z})$. In [21], W.E. Pruitt studied the case of bilateral birth-and-death processes. An example of this approach can be found in the last section of [13]. For a more theoretical work about the spectral theory of doubly infinite tridiagonal operators see [18, 3].

In Section 2 we analyze the conditions under which we can perform a stochastic RA factorization of the form $P = P_R P_A$. There are two important differences with our previous work in [11]. First, we will have now *two free parameters*, while in the case of UL or LU factorizations there was only one. As in [11] (see also [8]) we will show that these two free parameters have to be bounded from below by certain continued fractions if we want to guarantee that the factors are still stochastic matrices. Second, after relabeling the states, P will be equivalent to a semi-infinite 2×2 block tridiagonal matrix \mathbf{P} (see (2.2) and (2.3) below). The RA factorization of P is not a UL factorization, but after relabeling, the corresponding matrix factorization of \mathbf{P} , denoted by $\mathbf{P} = \mathbf{P}_R \mathbf{P}_A$ (see (2.8) below), *will be a UL block matrix factorization*. However, in [11], the corresponding matrix factorization of \mathbf{P} did not preserve the UL block structure of the original factorization of P . This fact will allow us to use techniques from the area of matrix-valued orthogonal polynomials, something that we could not do in [11]. After that we perform a discrete Darboux transformation. The two-parameter family of new matrices $\tilde{P} = P_A P_R$ are also stochastic but they are not strictly tridiagonal matrices since there will be new probability transitions between the states 1 and -1 . Nevertheless the block Darboux transformation $\tilde{\mathbf{P}} = \mathbf{P}_A \mathbf{P}_R$ will preserve the block tridiagonal structure of \mathbf{P} . We will obtain a relation between the spectral matrices Ψ and $\tilde{\Psi}$ associated with P and \tilde{P} , respectively, even if \tilde{P} is not tridiagonal. This relation is given by

$$\tilde{\Psi}(x) = S_0 \Psi_U(x) S_0^T,$$

for certain constant matrix S_0 and $\Psi_U(x)$ is a *Geronimus transformation* of the original spectral matrix $\Psi(x)$ (see Theorem 2.3 below). We apply our results to the simplest discrete-time birth-death chain on \mathbb{Z} with constant transition probabilities (also known as a random walk).

In Section 3 we perform the same analysis but considering a stochastic AR factorization of the form $P = \tilde{P}_A \tilde{P}_R$. The first thing we realize is that the case where P is a birth-death chain on \mathbb{Z} is not interesting since both factors \tilde{P}_A and \tilde{P}_R will be splitted into two separated birth-death chains at the state 0. Therefore it will be better to start with an “almost” birth-death chain similar to the one mentioned in the previous paragraph, i.e., with extra probability transitions between the states 1 and -1 . These transition probabilities must be related with the nearest transition probabilities between the $-1, 0$ and 1 states (see (3.9) below). Although now P is not tridiagonal, the equivalent block matrix \mathbf{P} after relabeling will be a block tridiagonal matrix and the corresponding matrix factorization of \mathbf{P} , denoted by $\mathbf{P} = \tilde{\mathbf{P}}_A \tilde{\mathbf{P}}_R$, *will be a LU block matrix factorization*. We then analyze under what conditions we get a stochastic AR factorization and show that we also need some bounds related with certain continued fractions. An important difference in this case is that the stochastic factorization, if possible, is *unique* and there will be no extra free parameters. After that we consider the Darboux transformation $\hat{P} = \tilde{P}_R \tilde{P}_A$ which it is now a tridiagonal matrix, i.e., a birth-death chain on \mathbb{Z} . Similarly the Darboux transformation $\hat{\mathbf{P}} = \tilde{\mathbf{P}}_R \tilde{\mathbf{P}}_A$ will preserve the block tridiagonal structure of \mathbf{P} . Again we will obtain a relation between the spectral matrices Ψ and $\hat{\Psi}$ associated with P and \hat{P} , respectively. This relation is given by a *Christoffel transformation* of $\Psi(x)$ of the form

$$\hat{\Psi}(x) = x \tilde{S}_0^{-1} \Psi(x) \tilde{S}_0^{-1},$$

for certain constant matrix \tilde{S}_0 (see Theorem 3.3 below). Finally we apply our results to an “almost” birth-death chain with constant transition probabilities. The main difficulty now is to compute the spectral matrix $\Psi(x)$. However, using the block tridiagonal structure of \mathbf{P} , we will be able to use some results of the theory of matrix-valued orthogonal polynomials to obtain an explicit expression of the spectral matrix.

2. REFLECTING-ABSORBING FACTORIZATION

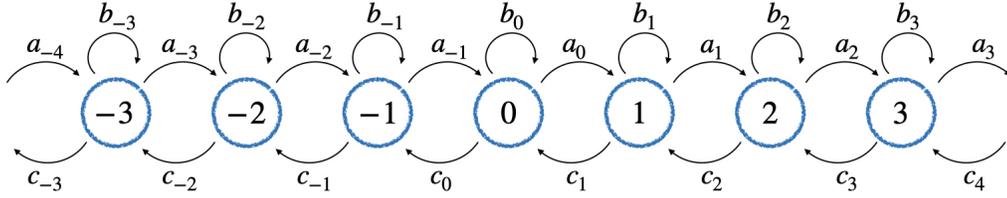
Let $\{X_t : t = 0, 1, \dots\}$ be an irreducible discrete-time birth-death chain on the integers \mathbb{Z} with transition probability matrix given by

$$P = \left(\begin{array}{ccc|ccc} \cdots & \cdots & \cdots & & & \\ & c_{-1} & b_{-1} & a_{-1} & & \\ \hline & & c_0 & b_0 & a_0 & \\ & & & c_1 & b_1 & a_1 \\ & & & & \cdots & \cdots & \cdots \end{array} \right). \quad (2.1)$$

Since the birth-death chain is irreducible, i.e., it is possible to get to any state from any state, then $0 < a_n, c_n < 1, n \in \mathbb{Z}$. Also, since P is stochastic, it has nonnegative entries and

$$c_n + b_n + a_n = 1, \quad n \in \mathbb{Z}.$$

A diagram of the transitions between the states is given by It is possible to relabel the states in such a way



that all the information of P is collected in a semi-infinite block tridiagonal matrix \mathbf{P} with blocks of size 2×2 . Indeed, after the new labeling

$$\{0, 1, 2, \dots\} \rightarrow \{0, 2, 4, \dots\}, \quad \text{and} \quad \{-1, -2, -3, \dots\} \rightarrow \{1, 3, 5, \dots\}, \quad (2.2)$$

we have that P (doubly infinite tridiagonal) is equivalent to a semi-infinite 2×2 block tridiagonal matrix \mathbf{P} of the form

$$\mathbf{P} = \left(\begin{array}{cc|cc|cc|cc|cc} b_0 & c_0 & a_0 & 0 & & & & & & \\ a_{-1} & b_{-1} & 0 & c_{-1} & & & & & & \\ \hline c_1 & 0 & b_1 & 0 & a_1 & 0 & & & & \\ 0 & a_{-2} & 0 & b_{-2} & 0 & c_{-2} & & & & \\ \hline & & c_2 & 0 & b_2 & 0 & a_2 & 0 & & \\ & & 0 & a_{-3} & 0 & b_{-3} & 0 & c_{-3} & & \\ \hline & & & & \ddots & & \ddots & & \ddots & \end{array} \right) = \left(\begin{array}{cccc} B_0 & A_0 & & \\ C_1 & B_1 & A_1 & \\ & C_2 & B_2 & A_2 \\ & & \ddots & \ddots & \ddots \end{array} \right), \quad (2.3)$$

where

$$\begin{aligned} B_0 &= \begin{pmatrix} b_0 & c_0 \\ a_{-1} & b_{-1} \end{pmatrix}, & B_n &= \begin{pmatrix} b_n & 0 \\ 0 & b_{-n-1} \end{pmatrix}, & n &\geq 1, \\ A_n &= \begin{pmatrix} a_n & 0 \\ 0 & c_{-n-1} \end{pmatrix}, & n &\geq 0, & C_n &= \begin{pmatrix} c_n & 0 \\ 0 & a_{-n-1} \end{pmatrix}, & n &\geq 1. \end{aligned} \tag{2.4}$$

The Markov chain generated by \mathbf{P} takes values in the two-dimensional state space $\mathbb{Z}_{\geq 0} \times \{1, 2\}$. These type of processes are called discrete-time *quasi-birth-and-death processes*. In general these processes allow transitions between all two-dimensional adjacent states (see [17, 20] for a general reference). The spectral analysis of these processes has been considered for instance in [4, 5, 6, 7, 12].

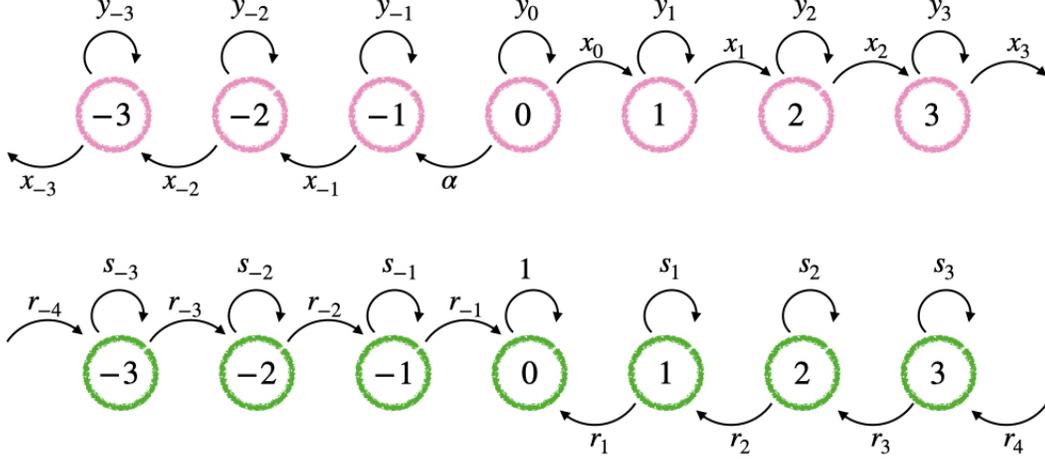
In [11] we considered stochastic UL and LU factorizations of P , which means that each of the factors can be viewed as a pure-birth or a pure-death chain, corresponding to the U or L matrix in the factorization, respectively. We consider here a different approach, given by a stochastic factorization of P where the first factor is a *reflecting* birth-death chain from the state 0 and the second factor is an *absorbing* birth-death chain to the state 0. From now on we will call this factorization an *RA factorization*. Therefore we are looking for a stochastic factorization of the transition probability matrix P in (2.1) of the form $P = P_R P_A$ where

$$P_R = \left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & & & \\ & x_{-2} & y_{-2} & 0 & & \\ & & x_{-1} & y_{-1} & 0 & \\ \hline & & & \alpha & y_0 & x_0 \\ & & & & 0 & y_1 & x_1 \\ & & & & & \ddots & \ddots & \ddots \end{array} \right), \tag{2.5}$$

and

$$P_A = \left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & & & \\ & 0 & s_{-2} & r_{-2} & & \\ & & 0 & s_{-1} & r_{-1} & \\ \hline & & & 0 & 1 & 0 \\ & & & & r_1 & s_1 & 0 \\ & & & & & \ddots & \ddots & \ddots \end{array} \right). \tag{2.6}$$

Observe that we have to add a new probability α in (2.5) in order to connect the reflecting birth-death chain from the state 0 to the state -1 . Also the state 0 is an absorbing state in the chain (2.6). Diagrams of the possible transitions between the states of both birth-death chains are given by



If we relabel the states as in (2.2), then the matrix P_R is equivalent to a semi-infinite 2×2 upper block matrix \mathbf{P}_R of the form

$$\mathbf{P}_R = \left(\begin{array}{cc|cc|cc|} y_0 & \alpha & x_0 & 0 & & & \\ 0 & y_{-1} & 0 & x_{-1} & & & \\ \hline & & y_1 & 0 & x_1 & 0 & \\ & & 0 & y_{-2} & 0 & x_{-2} & \\ \hline & & & & \ddots & & \ddots \end{array} \right),$$

while P_A is equivalent to a semi-infinite 2×2 lower block matrix \mathbf{P}_A of the form

$$\mathbf{P}_A = \left(\begin{array}{cc|cc|cc|} 1 & 0 & & & & & \\ r_{-1} & s_{-1} & & & & & \\ \hline r_1 & 0 & s_1 & 0 & & & \\ 0 & r_{-2} & 0 & s_{-2} & & & \\ \hline & & \ddots & & \ddots & & \end{array} \right).$$

If we call

$$\begin{aligned} Y_0 &= \begin{pmatrix} y_0 & \alpha \\ 0 & y_{-1} \end{pmatrix}, & Y_n &= \begin{pmatrix} y_n & 0 \\ 0 & y_{-n-1} \end{pmatrix}, & n \geq 1, & X_n &= \begin{pmatrix} x_n & 0 \\ 0 & x_{-n-1} \end{pmatrix}, & n \geq 0, \\ S_0 &= \begin{pmatrix} 1 & 0 \\ r_{-1} & s_{-1} \end{pmatrix}, & S_n &= \begin{pmatrix} s_n & 0 \\ 0 & s_{-n-1} \end{pmatrix}, & n \geq 1, & R_n &= \begin{pmatrix} r_n & 0 \\ 0 & r_{-n-1} \end{pmatrix}, & n \geq 0, \end{aligned} \tag{2.7}$$

then we can write P_R and P_A as

$$P_R = \begin{pmatrix} Y_0 & X_0 & & & \\ & Y_1 & X_1 & & \\ & & Y_2 & X_2 & \\ & & & \ddots & \ddots \end{pmatrix}, \quad P_A = \begin{pmatrix} S_0 & & & & \\ R_1 & S_1 & & & \\ & R_2 & S_2 & & \\ & & & \ddots & \ddots \end{pmatrix}. \quad (2.8)$$

The RA factorization $P = P_R P_A$, which is not a UL factorization, is equivalent to the block matrix factorization $P = P_R P_A$, which is a UL block matrix factorization. This is one of the main differences between this approach and the one we used in [11], where we performed a UL factorization of P while, after relabeling, the equivalent factorization for P did not have a UL block structure.

A direct computation from $P = P_R P_A$ shows that

$$\begin{aligned} A_n &= X_n S_{n+1}, & n \geq 0, \\ B_n &= Y_n S_n + X_n R_{n+1}, & n \geq 0, \\ C_n &= Y_n R_n, & n \geq 1, \end{aligned} \quad (2.9)$$

or equivalently, using $P = P_R P_A$, we obtain

$$a_n = x_n s_{n+1}, \quad n \geq 0, \quad a_{-n} = y_{-n} r_{-n}, \quad n \geq 1, \quad (2.10)$$

$$\begin{aligned} b_n &= y_n s_n + x_n r_{n+1}, & n \geq 1, & \quad b_0 = y_0 + x_0 r_1 + \alpha r_{-1}, & \quad b_{-n} = y_{-n} s_{-n} + x_{-n} r_{-n-1}, & \quad n \geq 1, \\ c_n &= y_n r_n, & n \geq 1, & \quad c_0 = \alpha s_{-1}, & \quad c_{-n} = x_{-n} s_{-n-1}, & \quad n \geq 1. \end{aligned} \quad (2.11)$$

Now we will see under what conditions we have that P_R and P_A are also stochastic matrices, i.e., all entries are nonnegative and

$$\alpha + x_0 + y_0 = 1, \quad x_n + y_n = 1, \quad s_n + r_n = 1, \quad n \in \mathbb{Z} \setminus \{0\}. \quad (2.12)$$

If we fix α then we can compute $s_{-1}, r_{-1}, y_{-1}, x_{-1}, s_{-2}, \dots$ recursively using (2.10), (2.12) and (2.11). On the other hand, for positive values of the indices, we need to fix a second parameter, say x_0 , in order to obtain recursively $s_1, r_1, y_1, x_1, s_2, \dots$ using again (2.10), (2.12) and (2.11). Using the same arguments as in [11] we will see under what conditions on the free parameters α and x_0 we have that both matrices P_R and P_A are also stochastic matrices. For that consider H and H' the continued fractions

$$H = \cfrac{a_0}{1 + \cfrac{c_1}{1 + \cfrac{a_1}{1 + \cfrac{c_2}{1 + \dots}}}}, \quad H' = \cfrac{c_0}{1 + \cfrac{a_{-1}}{1 + \cfrac{c_{-1}}{1 + \cfrac{a_{-2}}{1 + \dots}}}} \quad (2.13)$$

For each continued fraction, consider the corresponding sequence of convergents $(h_n)_{n \geq 0}$ and $(h'_{-n})_{n \geq 0}$, given by

$$h_n = \frac{A_n}{B_n}, \quad h'_{-n} = \frac{A'_{-n}}{B'_{-n}}. \quad (2.14)$$

We refer to [7, 11] to find more information about the notation and definitions on continued fractions.

Proposition 2.1. *Let H and H' be the continued fractions defined by (2.13) and the corresponding convergents h_n and h_{-n} defined by (2.14). Assume that*

$$0 < A_n < B_n, \quad \text{and} \quad 0 < A'_{-n} < B'_{-n}, \quad n \geq 1.$$

Then both H and H' are convergent. Moreover, let $P = P_R P_A$ and assume that $H + H' \leq 1$. Then, both P_R and P_A are stochastic matrices if and only if we choose α and x_0 in the following ranges

$$\alpha \geq H', \quad x_0 \geq H.$$

Proof. The proof follows the same lines as the proof of Theorem 2.1 of [11] and will be omitted (see also Theorem 2.1 of [8]). \square

2.1. Stochastic Darboux transformation and the associated spectral matrix. We have shown above under what conditions a doubly stochastic matrix P like in (2.1) can be decomposed as an RA factorization, where both factors are still stochastic matrices. This factorization comes with *two free parameters*. Once we have this factorization it is possible to perform what is called as a *discrete Darboux transformation*, consisting of inverting the order of the factors. The Darboux transformation has a long history but probably the first reference of a discrete Darboux transformation like we study here appeared in [19] in connection with the Toda lattice.

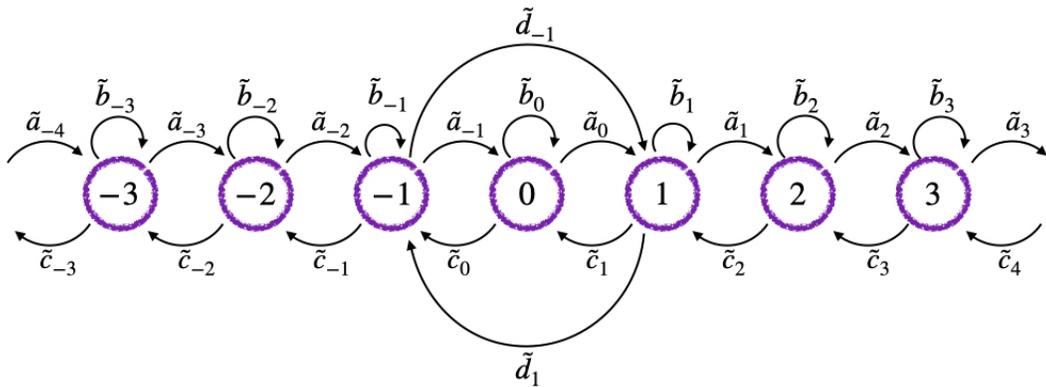
If $P = P_R P_A$ as in (2.5) and (2.6), then, by inverting the order of the factors, we obtain another stochastic matrix of the form $\tilde{P} = P_A P_R$, since the multiplication of two stochastic matrices is again a stochastic matrix. This matrix \tilde{P} is *not tridiagonal* but *pentadiagonal*, so we do not obtain again a birth-death chain on \mathbb{Z} , as in the case of UL (or LU) factorization in [11]. However we obtain a two-parameter family of new Markov chains on \mathbb{Z} , which we denote by $\{\tilde{X}_t : t = 0, 1, \dots\}$, which is “almost” a birth-death chain. The only difference is that we have new transition probabilities between the state 1 to the state -1 and viceversa. The tridiagonal coefficients of \tilde{P} are given by

$$\begin{aligned} \tilde{a}_n &= s_n x_n, & \tilde{a}_{-n-1} &= y_{-n} r_{-n-1}, & n &\geq 0, \\ \tilde{b}_0 &= y_0, & \tilde{b}_{-1} &= r_{-1} \alpha + s_{-1} y_{-1}, & \tilde{b}_n &= r_n x_{n-1} + s_n y_n, & \tilde{b}_{-n} &= r_{-n} x_{-n+1} + s_{-n} y_{-n}, & n &\geq 1, \\ \tilde{c}_0 &= \alpha, & \tilde{c}_n &= r_n y_{n-1}, & \tilde{c}_{-n} &= s_{-n} x_{-n}, & n &\geq 1, \end{aligned} \quad (2.15)$$

while the probability transitions between the state 1 and -1 are given by

$$\tilde{d}_1 = \mathbb{P}(\tilde{X}_1 = -1 | \tilde{X}_0 = 1) = r_1 \alpha, \quad \tilde{d}_{-1} = \mathbb{P}(\tilde{X}_1 = 1 | \tilde{X}_0 = -1) = r_{-1} x_0. \quad (2.16)$$

A diagram of the transitions between the states of this new family of Markov chains is given by



Another way to obtain these coefficients is by considering the block matrix factorization $P = P_R P_A$, where P is given by (2.3) (see also (2.4)) and P_R and P_A are given by (2.8) (see also (2.7)). The Darboux transformation

will be given by $\tilde{\mathbf{P}} = \mathbf{P}_A \mathbf{P}_R$, i.e.,

$$\tilde{\mathbf{P}} = \begin{pmatrix} \tilde{B}_0 & \tilde{A}_0 & & & \\ \tilde{C}_1 & \tilde{B}_1 & \tilde{A}_1 & & \\ & \tilde{C}_2 & \tilde{B}_2 & \tilde{A}_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} S_0 & & & & \\ R_1 & S_1 & & & \\ & R_2 & S_2 & & \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} Y_0 & X_0 & & & \\ & Y_1 & X_1 & & \\ & & Y_2 & X_2 & \\ & & & \ddots & \ddots \end{pmatrix}. \quad (2.17)$$

A direct computation shows

$$\begin{aligned} \tilde{A}_n &= S_n X_n, & n \geq 0, \\ \tilde{B}_0 &= S_0 Y_0, & \tilde{B}_n = R_n X_{n-1} + S_n Y_n, & n \geq 1, \\ \tilde{C}_n &= R_n Y_{n-1}, & n \geq 0. \end{aligned} \quad (2.18)$$

Using the notation in (2.4) we can obtain again the probabilities (2.15) and (2.16). Since $\mathbf{P} = \mathbf{P}_R \mathbf{P}_A$ is a block UL matrix factorization then $\tilde{\mathbf{P}} = \mathbf{P}_A \mathbf{P}_R$ will be also a block tridiagonal matrix, and will be again a family of discrete-time quasi-birth-and-death processes.

Now let us focus on the following question: given the spectrum of the doubly infinite matrix P , how can we compute the spectrum of the Darboux transformation \tilde{P} of the RA factorization? We will see that this will be related with what is called a *Geronimus transformation*. Before that let us introduce some notation to study the spectral measures associated with the original birth-death chain P on \mathbb{Z} .

We will follow the last section of [16] (see also [11]). For P like in (2.1) consider the eigenvalue equation $xq^\alpha(x) = Pq^\alpha(x)$ where $q^\alpha(x) = (\cdots, Q_{-1}^\alpha(x), Q_0^\alpha(x), Q_1^\alpha(x), \cdots)^T$, $\alpha = 1, 2$. For each x real or complex there exist two polynomial families of linearly independent solutions $Q_n^\alpha(x)$, $\alpha = 1, 2, n \in \mathbb{Z}$, depending on the initial values at $n = 0$ and $n = -1$. These polynomials are given by

$$\begin{aligned} Q_0^1(x) &= 1, & Q_0^2(x) &= 0, \\ Q_{-1}^1(x) &= 0, & Q_{-1}^2(x) &= 1, \\ xQ_n^\alpha(x) &= a_n Q_{n+1}^\alpha(x) + b_n Q_n^\alpha(x) + c_n Q_{n-1}^\alpha(x), & n \in \mathbb{Z}, & \alpha = 1, 2. \end{aligned} \quad (2.19)$$

Observe that

$$\begin{aligned} \deg(Q_n^1) &= n, & n \geq 0, & & \deg(Q_n^2) &= n - 1, & n \geq 1, \\ \deg(Q_{-n-1}^1) &= n - 1, & n \geq 1, & & \deg(Q_{-n-1}^2) &= n, & n \geq 0. \end{aligned}$$

Let us define the *potential coefficients* as

$$\pi_0 = 1, \quad \pi_n = \frac{a_0 a_1 \cdots a_{n-1}}{c_1 c_2 \cdots c_n}, \quad \pi_{-n} = \frac{c_0 c_{-1} \cdots c_{-n+1}}{a_{-1} a_{-2} \cdots a_{-n}}, \quad n \geq 1. \quad (2.20)$$

These coefficients are defined as the solutions of the *symmetry equations* $P_{ij}\pi_i = P_{ji}\pi_j$ normalized by the condition $\pi_0 = 1$. In particular we have that $\pi P = \pi$, i.e., $\pi = (\pi_n)_{n \in \mathbb{Z}}$ is an invariant vector of P . As a consequence of these symmetry equations, the matrix P gives rise to a self-adjoint operator of norm ≤ 1 in the Hilbert space $\ell_\pi^2(\mathbb{Z})$, which we will denote by P , abusing the notation. Applying the spectral theorem we obtain three unique measures $\psi_{\alpha\beta}$, $\alpha, \beta = 1, 2$ ($\psi_{12} = \psi_{21}$), which are supported on the interval $[-1, 1]$ such that we have the following orthogonality relation

$$\sum_{\alpha, \beta=1}^2 \int_{-1}^1 Q_i^\alpha(x) Q_j^\beta(x) d\psi_{\alpha\beta}(x) = \frac{\delta_{i,j}}{\pi_j}, \quad i, j \in \mathbb{Z}. \quad (2.21)$$

For more details see [11, 16, 18]. The measures ψ_{11} and ψ_{22} are positive (in fact ψ_{11} is a probability measure but ψ_{22} is not, since $\int_{-1}^1 d\psi_{22}(x) = 1/\pi_{-1}$). The measure ψ_{12} is a signed measure satisfying $0 = \int_{-1}^1 d\psi_{12}(x)$. For simplicity let us assume that the three measures are continuously differentiable with respect to the Lebesgue measure, i.e., $d\psi_{\alpha\beta}(x) = \psi_{\alpha\beta}(x)dx$, $\alpha, \beta = 1, 2$, abusing the notation. These 3 measures can be written in matrix form as the 2×2 matrix

$$\Psi(x) = \begin{pmatrix} \psi_{11}(x) & \psi_{12}(x) \\ \psi_{12}(x) & \psi_{22}(x) \end{pmatrix}, \quad (2.22)$$

so that the orthogonality relation (2.21) can be written in matrix form as

$$\int_{-1}^1 (Q_i^1(x), Q_i^2(x)) \Psi(x) \begin{pmatrix} Q_j^1(x) \\ Q_j^2(x) \end{pmatrix} dx = \frac{\delta_{i,j}}{\pi_j}, \quad i, j \in \mathbb{Z}.$$

The matrix $\Psi(x)$ in (2.22) is called the *spectral matrix* associated with P . The orthogonality conditions are valid for any indexes $i, j \in \mathbb{Z}$. With this information we can compute the n -step transition probabilities of the random walk $\{X_t : t = 0, 1, \dots\}$, given by the so-called *Karlin-McGregor integral representation formula* (see [16])

$$P_{ij}^{(n)} \doteq \mathbb{P}(X_n = j | X_0 = i) = \pi_j \int_{-1}^1 x^n (Q_i^1(x), Q_i^2(x)) \Psi(x) \begin{pmatrix} Q_j^1(x) \\ Q_j^2(x) \end{pmatrix} dx, \quad i, j \in \mathbb{Z}.$$

If we define the matrix-valued polynomials

$$\mathbf{Q}_n(x) = \begin{pmatrix} Q_n^1(x) & Q_n^2(x) \\ Q_{-n-1}^1(x) & Q_{-n-1}^2(x) \end{pmatrix}, \quad n \geq 0, \quad (2.23)$$

then we have

$$\begin{aligned} x\mathbf{Q}_0(x) &= A_0\mathbf{Q}_1(x) + B_0\mathbf{Q}_0(x), & \mathbf{Q}_0(x) &= I_{2 \times 2}, \\ x\mathbf{Q}_n(x) &= A_n\mathbf{Q}_{n+1}(x) + B_n\mathbf{Q}_n(x) + C_n\mathbf{Q}_{n-1}(x), & n &\geq 1, \end{aligned} \quad (2.24)$$

where $I_{2 \times 2}$ denotes the 2×2 identity matrix and $(A_n)_{n \geq 0}$, $(B_n)_{n \geq 0}$ and $(C_n)_{n \geq 1}$ are given by (2.4). If we denote $\mathbf{Q} = (Q_0^T, Q_1^T, \dots)^T$ then we have $x\mathbf{Q} = \mathbf{P}\mathbf{Q}$, where \mathbf{P} is given by (2.3). The matrix orthogonality is defined in terms of the (matrix-valued) inner product

$$\int_{-1}^1 \mathbf{Q}_n(x) \Psi(x) \mathbf{Q}_m^T(x) dx = \Pi_n^{-1} \delta_{nm},$$

where A^T is the transpose of a matrix A and

$$\Pi_n = \begin{pmatrix} \pi_n & 0 \\ 0 & \pi_{-n-1} \end{pmatrix}, \quad n \geq 0, \quad (2.25)$$

where $\pi = (\pi_n)_{n \in \mathbb{Z}}$ are given by (2.20). An alternative way of writing Π_n , which solves the symmetry equations for \mathbf{P} , is given by (see [10])

$$\Pi_n = (C_1^T \cdots C_n^T)^{-1} \Pi_0 A_0 \cdots A_{n-1}, \quad n \geq 1.$$

Therefore we have (see [4, 5]) the Karlin-McGregor integral representation formula where the 2×2 block entry (i, j) is given by

$$P_{ij}^{(n)} = \left(\int_{-1}^1 x^n \mathbf{Q}_i(x) \Psi(x) \mathbf{Q}_j^T(x) dx \right) \Pi_j, \quad i, j \in \mathbb{Z}_{\geq 0}.$$

Remark 2.2. In several references (see for instance [3, 18, 21]) the two families of polynomials $(Q_n^\alpha)_{n \in \mathbb{Z}}$, $\alpha = 1, 2$, generated by the three-term recurrence relation (2.19) are denoted in a different way and have different initial conditions. If we denote by $(P_n^\beta)_{n \in \mathbb{Z}}$ the families defined in those papers, then the initial conditions are given by $P_n^\beta = \delta_{n\beta}$ for $n, \beta = 0, 1$. We prefer to use our notation (and follow [16] or [1]) since in this case, and after relabeling, the sequence $(Q_n)_{n \geq 0}$ in (2.23) is a proper family of matrix-valued polynomials, i.e., $\deg Q_n = n$ and have nonsingular leading coefficient, something that will not occur if we perform the same labeling to the family $(P_n^\beta)_{n \in \mathbb{Z}}$. In this way we make a direct connection between the theory of orthogonal polynomials on the integers and the theory of 2×2 matrix-valued orthogonal polynomials.

Consider now the matrix-valued polynomials

$$\begin{aligned} \mathbf{U}_0(x) &= S_0 \mathbf{Q}_0(x) = S_0, \\ \mathbf{U}_n(x) &= R_n \mathbf{Q}_{n-1}(x) + S_n \mathbf{Q}_n(x), \quad n \geq 1, \end{aligned} \tag{2.26}$$

where $(S_n)_{n \geq 0}$ and $(R_n)_{n \geq 1}$ are defined by (2.7). If we denote $\mathbf{U} = (\mathbf{U}_0^T, \mathbf{U}_1^T, \dots)^T$, then we have that $\mathbf{U} = \mathbf{P}_A \mathbf{Q}$, where \mathbf{P}_A is given by (2.8). Therefore, from the RA factorization of \mathbf{P} , we get $\mathbf{P}_R \mathbf{U} = \mathbf{P}_R \mathbf{P}_A \mathbf{Q} = \mathbf{P} \mathbf{Q} = x \mathbf{Q}$, or in other words

$$x \mathbf{Q}_n(x) = Y_n \mathbf{U}_n(x) + X_n \mathbf{U}_{n+1}(x), \quad n \geq 0. \tag{2.27}$$

Finally, using the Darboux transformation (2.17), we have

$$\tilde{\mathbf{P}} \mathbf{U} = \mathbf{P}_A \mathbf{P}_R \mathbf{P}_A \mathbf{Q} = x \mathbf{P}_A \mathbf{Q} = x \mathbf{U}.$$

Evaluating at $x = 0$ in (2.27) we can easily get

$$\mathbf{U}_n(0) = (-1)^n X_{n-1}^{-1} Y_{n-1} \cdots X_0^{-1} Y_0 S_0. \tag{2.28}$$

As a consequence we also have

$$\mathbf{U}_{n+1}(0) \mathbf{U}_{n-k}^{-1}(0) = (-1)^{k+1} X_n^{-1} Y_n \cdots X_{n-k}^{-1} Y_{n-k}, \quad k = 0, 1, \dots, n.$$

We can also solve (2.27) recursively in which case we have, using the previous notation,

$$\mathbf{U}_n(x) = \mathbf{U}_n(0) \left[I + x \sum_{k=0}^{n-1} \mathbf{U}_{k+1}^{-1}(0) X_k^{-1} \mathbf{Q}_k(x) \right]. \tag{2.29}$$

From (2.26) we have that $\deg(\mathbf{U}_n(x)) = n$, $n \geq 0$, but $\mathbf{U}_0(x) = S_0 \neq I_{2 \times 2}$. We will be interested in a new family of matrix-valued polynomials $(\tilde{\mathbf{Q}}_n)_{n \geq 0}$ where $\tilde{\mathbf{Q}}_0 = I_{2 \times 2}$. Since S_0 is a constant matrix and has an inverse, this new family can be defined as

$$\tilde{\mathbf{Q}}_n(x) = \mathbf{U}_n(x) S_0^{-1}. \tag{2.30}$$

Finally, if we define $(\tilde{\Pi}_n)_{n \geq 0}$ as the solution of the symmetry equations for $\tilde{\mathbf{P}}$ in (2.17), given by

$$\tilde{\Pi}_n = (\tilde{C}_1^T \cdots \tilde{C}_n^T)^{-1} \tilde{\Pi}_0 \tilde{A}_0 \cdots \tilde{A}_{n-1}, \quad n \geq 1,$$

then, using (2.9), (2.18) and the previous equation, we get

$$\tilde{\Pi}_n = Y_n^T \Pi_n S_n^{-1}, \quad n \geq 0. \tag{2.31}$$

Computing $\tilde{\Pi}_0$ using (2.10) and (2.11), we get

$$\tilde{\Pi}_0 = \begin{pmatrix} y_0 & 0 \\ 0 & \alpha/r_{-1} \end{pmatrix}.$$

Therefore $(\tilde{\Pi}_n)_{n \geq 0}$ are always *diagonal matrices*. We are ready to prove the main result of this section.

Theorem 2.3. Let $\{X_t : t = 0, 1, \dots\}$ be the birth-death chain on \mathbb{Z} with transition probability matrix P given by (2.1) and $\{\tilde{X}_t : t = 0, 1, \dots\}$ be the Markov chain generated by the Darboux transformation of $P = P_R P_A$ with transition probabilities given by (2.15) and (2.16). Assume that $\mathbf{M}_{-1} = \int_{-1}^1 x^{-1} \Psi(x) dx$ is well-defined (entry by entry), where $\Psi(x)$ is the original spectral matrix (2.22). Then the matrix-valued polynomials $(\tilde{\mathbf{Q}}_n)_{n \geq 0}$ defined by (2.30) are orthogonal with respect to the following spectral matrix

$$\tilde{\Psi}(x) = S_0 \Psi_U(x) S_0^T, \quad (2.32)$$

where the constant matrix S_0 is defined by (2.7) and

$$\Psi_U(x) = \frac{\Psi(x)}{x} + \left[\frac{1}{y_0} \begin{pmatrix} 1 & -r_{-1}/s_{-1} \\ -r_{-1}/s_{-1} & (r_{-1}/s_{-1})^2 + y_0 r_{-1}/\alpha s_{-1}^2 \end{pmatrix} - \mathbf{M}_{-1} \right] \delta_0(x). \quad (2.33)$$

Moreover, we have

$$\int_{-1}^1 \tilde{\mathbf{Q}}_n(x) \tilde{\Psi}(x) \tilde{\mathbf{Q}}_m^T(x) dx = \tilde{\Pi}_n^{-1} \delta_{n,m},$$

where $(\tilde{\Pi}_n)_{n \geq 0}$ are defined by (2.31).

Proof. For $n \geq 1$ and $j = 1, \dots, n-1$, we have

$$\begin{aligned} \int_{-1}^1 \tilde{\mathbf{Q}}_n(x) \tilde{\Psi}(x) x^j dx &= \int_{-1}^1 \mathbf{U}_n(x) \Psi_U(x) x^j S_0^T dx = \int_{-1}^1 [R_n \mathbf{Q}_{n-1}(x) + S_n \mathbf{Q}_n(x)] \Psi(x) x^{j-1} S_0^T dx \\ &= \int_{-1}^1 R_n \mathbf{Q}_{n-1}(x) \Psi(x) x^{j-1} S_0^T dx + \int_{-1}^1 S_n \mathbf{Q}_n(x) \Psi(x) x^{j-1} S_0^T dx = \mathbf{0}_{2 \times 2}, \end{aligned}$$

where for the first equality we have used (2.30) and (2.32), for the second equality we have used (2.26) and (2.33), and finally we have used the orthogonality of the family $(\mathbf{Q}_n)_{n \geq 0}$. Now, for $n \geq 1$ we have, using (2.29), that

$$\begin{aligned} \int_{-1}^1 \tilde{\mathbf{Q}}_n(x) \tilde{\Psi}(x) dx &= \int_{-1}^1 \mathbf{U}_n(x) \Psi_U(x) S_0^T dx = \int_{-1}^1 \mathbf{U}_n(0) \left[I + x \sum_{k=0}^{n-1} \mathbf{U}_{k+1}^{-1}(0) X_k^{-1} \mathbf{Q}_k(x) \right] \Psi_U(x) S_0^T dx \\ &= \mathbf{U}_n(0) \left[\int_{-1}^1 \Psi_U(x) dx + \sum_{k=0}^{n-1} \mathbf{U}_{k+1}^{-1}(0) X_k^{-1} \int_{-1}^1 \mathbf{Q}_k(x) \Psi(x) dx \right] S_0^T. \end{aligned}$$

The second part of the previous sum vanishes for $k = 1, \dots, n-1$. Therefore the only nonzero term is for $k = 0$, i.e., $\mathbf{U}_1^{-1}(0) X_0^{-1} \int_{-1}^1 \Psi(x) dx = \mathbf{U}_1^{-1}(0) X_0^{-1} \Pi_0^{-1}$. A direct computation using (2.28), (2.7), (2.25) and (2.20) shows that

$$\mathbf{U}_1^{-1}(0) X_0^{-1} \Pi_0^{-1} = -S_0^{-1} Y_0^{-1} \Pi_0^{-1} = -\frac{1}{y_0} \begin{pmatrix} 1 & -r_{-1}/s_{-1} \\ -r_{-1}/s_{-1} & (r_{-1}/s_{-1})^2 + y_0 r_{-1}/\alpha s_{-1}^2 \end{pmatrix}.$$

From the definition of $\Psi_U(x)$ in (2.33) we obtain that $\int_{-1}^1 \tilde{\mathbf{Q}}_n(x) \tilde{\Psi}(x) dx = \mathbf{0}_{2 \times 2}$.

Finally, for $n \geq 0$, and using (2.30), (2.32), (2.27), (2.26) and the orthogonality properties, we have

$$\begin{aligned}
\int_{-1}^1 \tilde{\mathbf{Q}}_n(x) \tilde{\Psi}(x) \tilde{\mathbf{Q}}_n^T(x) dx &= \int_{-1}^1 \mathbf{U}_n(x) S_0^{-1} S_0 \Psi_{\mathbf{U}}(x) S_0^T S_0^{-T} \mathbf{U}_n^T(x) dx \\
&= \int_{-1}^1 Y_n^{-1} [x \mathbf{Q}_n(x) - X_n \mathbf{U}_{n+1}(x)] \Psi_{\mathbf{U}}(x) \mathbf{U}_n^T(x) dx \\
&= Y_n^{-1} \int_{-1}^1 x \mathbf{Q}_n(x) \Psi_{\mathbf{U}}(x) \mathbf{U}_n^T(x) dx \\
&= Y_n^{-1} \int_{-1}^1 \mathbf{Q}_n(x) \Psi(x) [R_n \mathbf{Q}_{n-1}(x) + S_n \mathbf{Q}_n(x)]^T dx \\
&= Y_n^{-1} \left[\int_{-1}^1 \mathbf{Q}_n(x) \Psi(x) \mathbf{Q}_n^T(x) dx \right] S_n^T \\
&= Y_n^{-1} \Pi_n^{-1} S_n^T = (S_n^{-T} \Pi_n Y_n)^{-1} = \tilde{\Pi}_n^{-T} = \tilde{\Pi}_n^{-1},
\end{aligned}$$

where in the last steps we have used the formula (2.31) and the fact that $(\tilde{\Pi}_n)_{n \geq 0}$ are diagonal matrices. \square

Remark 2.4. Observe that the derivation of the spectral matrix $\tilde{\Psi}(x)$ for the matrix-valued polynomials $(\tilde{\mathbf{Q}}_n)_{n \geq 0}$ is not restricted to the case where we start with a birth-death chain on \mathbb{Z} with transition probability matrix P . For *any* tridiagonal block matrix \mathbf{P} as in (2.3) with $N \times N$ blocks, if we are able to find a factorization of the form $\mathbf{P} = \mathbf{P}_R \mathbf{P}_A$ where \mathbf{P}_R and \mathbf{P}_A are given by (2.8), then we can follow the same steps to compute the spectral matrix associated with the Darboux transformation $\tilde{\mathbf{P}} = \mathbf{P}_A \mathbf{P}_R$. Assuming that we have computed the spectral matrix $\Psi(x)$ associated with \mathbf{P} (which is not always possible, see Theorem 2.1 of [4]) and that $\mathbf{M}_{-1} = \int_{-1}^1 x^{-1} \Psi(x) dx$ is well-defined (entry by entry), the spectral matrix $\tilde{\Psi}(x)$ associated with $\tilde{\mathbf{P}}$ will be given by

$$\tilde{\Psi}(x) = S_0 \left(\frac{\Psi(x)}{x} + [(\Pi_0 Y_0 S_0)^{-1} - \mathbf{M}_{-1}] \delta_0(x) \right) S_0^T.$$

This is what is called a *Geronimus transformation* of the original spectral matrix $\Psi(x)$. Observe that $\tilde{\Psi}(x)$ is not necessarily a proper weight matrix even if $\Psi(x)$ is. For that we need the matrix $(\Pi_0 Y_0 S_0)^{-1} - \mathbf{M}_{-1}$ to be a positive semi-definite matrix. In the case we are treating in this paper we have that $\Pi_0 Y_0 S_0$ is a positive definite matrix (see (2.33)).

2.2. Example: random walk on \mathbb{Z} . Let us consider an irreducible birth-death chain on \mathbb{Z} with transition probability matrix P as in (2.1) with constant transition probabilities

$$a_n = a, \quad b_n = b, \quad c_n = c, \quad n \in \mathbb{Z}, \quad a + b + c = 1, \quad a, c > 0, \quad b \geq 0.$$

This discrete-time birth-death chain is usually called a *random walk*. If we consider an RA factorization we have that the continued fractions (2.13) can be explicitly computed and in this case they are given by

$$H = \frac{1}{2} \left(1 + a - c - \sqrt{(1 + c - a)^2 - 4c} \right), \quad H' = \frac{1}{2} \left(1 + c - a - \sqrt{(1 + c - a)^2 - 4c} \right),$$

with $a \leq (1 - \sqrt{c})^2$ (to ensure convergence). Following Proposition 2.1 we have that the RA stochastic factorization is possible if and only if we take $\alpha \geq H'$ and $x_0 \geq H$ bearing in mind that $\alpha + x_0 \leq 1$. Observe that if we choose $\alpha = H'$ and $x_0 = H$, then we get

$$\begin{aligned}
s_{-n} &= H, & r_{-n} &= H, & y_{-n} &= 1 - H', & x_{-n} &= H', & n &\geq 1, \\
s_n &= 1 - H', & r_n &= H', & y_n &= 1 - H, & x_n &= H, & n &\geq 1,
\end{aligned}$$

and $y_0 = 1 - H - H'$, $s_0 = 1$ and $r_0 = 0$. In this case the coefficients of the Darboux transformation remain “almost” invariant, i.e., $\tilde{a}_n = a$, $\tilde{b}_n = b$ and $\tilde{c}_n = c$ for all n except for \tilde{a}_{-1} , \tilde{a}_0 , \tilde{b}_0 , \tilde{c}_0 , \tilde{c}_1 .

The spectral matrix associated with this example is given by only an absolutely continuous part, i.e.,

$$\Psi(x) = \frac{1}{\pi\sqrt{(x-\sigma_-)(\sigma_+-x)}} \begin{pmatrix} 1 & \frac{x-b}{2c} \\ \frac{x-b}{2c} & a/c \end{pmatrix}, \quad x \in [\sigma_-, \sigma_+], \quad \sigma_{\pm} = 1 - (\sqrt{a} \mp \sqrt{c})^2. \quad (2.34)$$

For details on how to compute this spectral matrix see Section 4.1 of [11]. Also a straightforward computation shows that the moment \mathbf{M}_{-1} of Ψ is given by

$$\mathbf{M}_{-1} = \begin{pmatrix} \frac{1}{\sqrt{\sigma_-\sigma_+}} & \frac{1}{2c} \left(1 - \frac{b}{\sqrt{\sigma_-\sigma_+}}\right) \\ \frac{1}{2c} \left(1 - \frac{b}{\sqrt{\sigma_-\sigma_+}}\right) & \frac{a}{c\sqrt{\sigma_-\sigma_+}} \end{pmatrix}.$$

In order for \mathbf{M}_{-1} to be well-defined we need to assume that $\sigma_- > 0$, i.e., $\sqrt{a} + \sqrt{c} < 1$, or, in other words $a < (1 - \sqrt{c})^2$, which is the condition for convergence of the continued fractions H and H' . Once we have this information it is possible to compute the spectral matrix associated with the Darboux transformation $\tilde{P} = P_A P_R$ at the beginning of Section 2.1, which we recall it is an “almost” birth-death chain except for the states 1 and -1 (see (2.15) and (2.16)) and two free parameters, α and x_0 . In this case we have

$$S_0 = \begin{pmatrix} 1 & 0 \\ 1 - c/\alpha & c/\alpha \end{pmatrix}.$$

Following Theorem 2.3 we have that the spectral matrix associated with \tilde{P} is given by (after some computations)

$$\tilde{\Psi}(x) = \frac{1}{\pi x \sqrt{(x-\sigma_-)(\sigma_+-x)}} [\tilde{\mathbf{A}} + \tilde{\mathbf{B}}x] + \tilde{\mathbf{M}}_{-1} \delta_0,$$

where

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & \frac{2\alpha + H - H' - 1}{2\alpha} \\ \frac{2\alpha + H - H' - 1}{2\alpha} & \frac{(\alpha - H')(H + \alpha - 1)}{\alpha^2} \end{pmatrix}, \quad \tilde{\mathbf{B}} = \frac{1}{2\alpha} \begin{pmatrix} 0 & 1 \\ 1 & \frac{2(\alpha - c)}{\alpha} \end{pmatrix},$$

$$\tilde{\mathbf{M}}_{-1} = \begin{pmatrix} \frac{x_0 - H + \alpha - H'}{y_0(1 - H - H')} & \frac{H' - \alpha}{\alpha(1 - H - H')} \\ \frac{H' - \alpha}{\alpha(1 - H - H')} & -\frac{(\alpha - H')(1 - H - \alpha)}{\alpha^2(1 - H - H')} \end{pmatrix}.$$

Observe that in the case that $\alpha = H'$ and $x_0 = H$ we have that $\tilde{\mathbf{M}}_{-1} = \mathbf{0}_{2 \times 2}$.

3. ABSORBING-REFLECTING FACTORIZATION

In this section we will use the same notation as in Section 2, but replacing all parameters, matrices, etc. in the RA factorization by a tilde superscript. For instance $y_n, x_n, s_n, r_n, \alpha$ in (2.5) and (2.6) will be replaced by $\tilde{y}_n, \tilde{x}_n, \tilde{s}_n, \tilde{r}_n, \tilde{\alpha}$, respectively, $\mathbf{P}_R, \mathbf{P}_A, Y_n, X_n, S_n, R_n$ in (2.8) will be replaced by $\tilde{\mathbf{P}}_R, \tilde{\mathbf{P}}_A, \tilde{Y}_n, \tilde{X}_n, \tilde{S}_n, \tilde{R}_n$, respectively, and so on.

As we saw at the beginning of Section 2.1 the multiplication of matrices of the form $\tilde{P}_A \tilde{P}_R$, where \tilde{P}_A and \tilde{P}_R are given by (2.6) and (2.5), gives rise to Markov chain which is “almost” a birth-death chain, except for

the states 1 and -1 where it is possible to go from 1 to -1 and viceversa (see (2.16)). Hence it will not be interesting to start from a birth-death chain P and consider an absorbing-reflecting (or AR) factorization of the form $P = \tilde{P}_A \tilde{P}_R$, since this will imply that both \tilde{P}_A and \tilde{P}_R will be splitted into two separated birth-death chains at the state 0. Therefore it will make more sense to start with an irreducible Markov chain $\{X_t : t = 0, 1, \dots\}$ on \mathbb{Z} with transition probability matrix

$$P = \left(\begin{array}{ccc|cc} \ddots & \ddots & \ddots & & \\ & c_{-2} & b_{-2} & a_{-2} & \\ & & c_{-1} & b_{-1} & a_{-1} & d_{-1} \\ \hline & & & c_0 & b_0 & a_0 \\ & & & d_1 & c_1 & b_1 & a_1 \\ & & & & & c_2 & b_2 & a_2 \\ & & & & & & \ddots & \ddots & \ddots \end{array} \right). \quad (3.1)$$

A diagram of the transitions of this Markov chain is similar to the one given in Section 2.1. If we perform the same labeling as in (2.2) then P is equivalent to a semi-infinite 2×2 block tridiagonal matrix \mathbf{P} of the form

$$\mathbf{P} = \left(\begin{array}{cc|cc|cc|cc|} b_0 & c_0 & a_0 & 0 & & & & & \\ a_{-1} & b_{-1} & d_{-1} & c_{-1} & & & & & \\ \hline c_1 & d_1 & b_1 & 0 & a_1 & 0 & & & \\ 0 & a_{-2} & 0 & b_{-2} & 0 & c_{-2} & & & \\ \hline & & c_2 & 0 & b_2 & 0 & a_2 & 0 & \\ & & 0 & a_{-3} & 0 & b_{-3} & 0 & c_{-3} & \\ \hline & & & & \ddots & & \ddots & & \ddots \end{array} \right), \quad (3.2)$$

where the only difference with the coefficients in (2.4) is the triangular shape of the matrices A_0 and C_1 , given by

$$A_0 = \begin{pmatrix} a_0 & 0 \\ d_{-1} & c_{-1} \end{pmatrix}, \quad C_1 = \begin{pmatrix} c_1 & d_1 \\ 0 & a_{-2} \end{pmatrix}.$$

Now, using the same notation as in (2.8) (see also (2.7)), let us consider the block matrix factorization $\mathbf{P} = \tilde{\mathbf{P}}_A \tilde{\mathbf{P}}_R$, which in this case is a LU block matrix factorization. A direct computation shows

$$\begin{aligned} A_n &= \tilde{S}_n \tilde{X}_n, \quad n \geq 0, \\ B_n &= \tilde{S}_n \tilde{Y}_n + \tilde{R}_n \tilde{X}_{n-1}, \quad n \geq 1, \quad B_0 = \tilde{S}_0 \tilde{Y}_0, \\ C_n &= \tilde{R}_n \tilde{Y}_{n-1}, \quad n \geq 1, \end{aligned} \quad (3.3)$$

or equivalently, using $P = \tilde{P}_A \tilde{P}_R$, we obtain

$$a_n = \tilde{x}_n \tilde{s}_n, \quad n \geq 1, \quad a_0 = \tilde{x}_0, \quad a_{-n} = \tilde{y}_{-n+1} \tilde{r}_{-n}, \quad n \geq 1, \quad (3.4)$$

$$b_n = \tilde{y}_n \tilde{s}_n + \tilde{x}_{n-1} \tilde{r}_n, \quad n \geq 1, \quad b_0 = \tilde{y}_0, \quad b_{-n} = \tilde{y}_{-n} \tilde{s}_{-n} + \tilde{x}_{-n+1} \tilde{r}_{-n}, \quad n \geq 1, \quad (3.5)$$

$$c_n = \tilde{y}_{n-1} \tilde{r}_n, \quad n \geq 1, \quad c_0 = \tilde{\alpha}, \quad c_{-n} = \tilde{x}_{-n} \tilde{s}_{-n}, \quad n \geq 1, \quad (3.6)$$

$$d_{-1} = \tilde{r}_{-1} \tilde{x}_0, \quad d_1 = \tilde{r}_1 \tilde{\alpha}, \quad (3.7)$$

with \tilde{P}_A and \tilde{P}_R stochastic matrices, i.e., all entries are nonnegative and

$$\tilde{\alpha} + \tilde{x}_0 + \tilde{y}_0 = 1, \quad \tilde{x}_n + \tilde{y}_n = 1, \quad \tilde{s}_n + \tilde{r}_n = 1, \quad n \in \mathbb{Z} \setminus \{0\}. \quad (3.8)$$

Observe that, from (3.7) and (3.4), we have $\tilde{r}_{-1} = d_{-1}/\tilde{x}_0 = a_{-1}/\tilde{y}_0$ and $\tilde{r}_1 = d_1/\tilde{\alpha} = c_1/\tilde{y}_0$. Therefore, from (3.4) and (3.5), we get that the factorization is possible if and only if

$$d_{-1} = \frac{a_{-1}a_0}{b_0}, \quad d_1 = \frac{c_0c_1}{b_0}. \quad (3.9)$$

This means that, in order to have a stochastic AR factorization, the Markov chain (3.1) is not a general one but restricted to the conditions (3.9). Additionally we need to have that $0 < d_{-1}, d_1 < 1$, so we have to assume that $b_0 > \max\{a_{-1}a_0, c_0c_1\}$. Once we have all the previous considerations it is possible to get the coefficients $\tilde{y}_0, \tilde{r}_1, \tilde{s}_1, \tilde{x}_1, \tilde{y}_1, \dots$ recursively using (3.5), (3.6) and (3.8) in that order. On the other hand it is possible to get $\tilde{y}_0, \tilde{r}_{-1}, \tilde{s}_{-1}, \tilde{x}_{-1}, \tilde{y}_{-1}, \tilde{r}_{-2}, \dots$ recursively using (3.5), (3.4) and (3.8) in that order. In this case we also have that $\tilde{x}_0 = a_0$ and $\tilde{\alpha} = c_0$. Therefore *there is no free parameter* and the factorization is unique. Let \tilde{H} and \tilde{H}' be the following continued fractions

$$\tilde{H} = \cfrac{c_1}{1} \cfrac{a_1}{1} \cfrac{c_2}{1} \cfrac{a_2}{1} \dots, \quad \tilde{H}' = \cfrac{a_{-1}}{1} \cfrac{c_{-1}}{1} \cfrac{a_{-2}}{1} \cfrac{c_{-2}}{1} \dots \quad (3.10)$$

For each continued fraction, consider the corresponding sequence of convergents $(\tilde{h}_n)_{n \geq 0}$ and $(\tilde{h}'_{-n})_{n \geq 0}$, given by

$$\tilde{h}_n = \frac{\tilde{A}_n}{\tilde{B}_n}, \quad \tilde{h}'_{-n} = \frac{\tilde{A}'_{-n}}{\tilde{B}'_{-n}}. \quad (3.11)$$

Again, we refer to [7, 11] to find more information about the notation and definitions on continued fractions.

Proposition 3.1. *Let \tilde{H} and \tilde{H}' be the continued fractions given by (3.10) and the corresponding convergents \tilde{h}_n and \tilde{h}'_{-n} defined by (3.11). Assume that*

$$0 < A_n < B_n, \quad \text{and} \quad 0 < A'_{-n} < B'_{-n}, \quad n \geq 1.$$

Then both \tilde{H} and \tilde{H}' are convergent. Moreover, let $P = \tilde{P}_A \tilde{P}_R$. Then, both \tilde{P}_A and \tilde{P}_R are stochastic matrices if and only if

$$b_0 > \max\{\tilde{H}, \tilde{H}'\}.$$

Proof. The proof follows the same lines as the proof of Theorem 2.1 of [11] and will be omitted (see also Theorem 2.1 of [8]). \square

Remark 3.2. Observe that the condition of the previous proposition implies that $b_0 \geq \tilde{H}$ and $b_0 \geq \tilde{H}'$. Also, if $b_0 \geq \tilde{H}$, then in particular $b_0 > c_1$ and since we know that $0 < c_0 < 1$, then we have $b_0 > c_1 > c_0c_1$. Similarly, if $b_0 \geq \tilde{H}'$, in particular we have $b_0 > a_{-1}$ and since $0 < a_0 < 1$, then we get $b_0 > a_{-1} > a_0a_{-1}$. This allows us to conclude that if $b_0 \geq \max\{\tilde{H}, \tilde{H}'\}$, then $b_0 \geq \max\{a_{-1}a_0, c_0c_1\}$ and therefore $0 < d_1, d_{-1} < 1$. Observe that this does not mean that the factorization is always possible since we need to have (3.9).

Since $\mathbf{T}_0(x) = \tilde{S}_0^{-1} \neq I_{2 \times 2}$, let us define a new family $(\hat{\mathbf{Q}}_n)_{n \geq 0}$ such that $\hat{\mathbf{Q}}_0 = I_{2 \times 2}$, i.e.,

$$\hat{\mathbf{Q}}_n(x) = \mathbf{T}_n(x) \tilde{S}_0. \quad (3.16)$$

Finally, if we define $(\hat{\Pi}_n)_{n \geq 0}$ as the solution of the symmetry equations for $\hat{\mathbf{P}}$ in (3.12), given by

$$\hat{\Pi}_n = (\hat{C}_1^T \cdots \hat{C}_n^T)^{-1} \hat{\Pi}_0 \hat{A}_0 \cdots \hat{A}_{n-1}, \quad n \geq 1.$$

Then, using (3.3), (3.13) and the previous equation, we get

$$\hat{\Pi}_n = \tilde{Y}_n^{-T} \Pi_n \tilde{S}_n, \quad n \geq 0. \quad (3.17)$$

Computing $\hat{\Pi}_0$ using (3.4) and (3.6), we have

$$\hat{\Pi}_0 = \frac{1}{\tilde{y}_0} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\tilde{\alpha} \tilde{s}_{-1}}{\tilde{y}_{-1} \tilde{r}_{-1}} \end{pmatrix}.$$

Therefore, as in the RA factorization, $(\hat{\Pi}_n)_{n \geq 0}$ are always *diagonal matrices*. We are ready to prove the main result of this section.

Theorem 3.3. *Let $\{X_t : t = 0, 1, \dots\}$ be the Markov chain on \mathbb{Z} with transition probability matrix P given by (3.1) and $\{\tilde{X}_t : t = 0, 1, \dots\}$ the birth-death chain generated by the Darboux transformation of $P = \tilde{P}_A \tilde{P}_R$. Then the matrix-valued polynomials $(\hat{\mathbf{Q}}_n)_{n \geq 0}$ defined by (3.16) are orthogonal with respect to the following spectral matrix*

$$\hat{\Psi}(x) = x \tilde{S}_0^{-1} \Psi(x) \tilde{S}_0^{-T}, \quad (3.18)$$

where the constant matrix \tilde{S}_0 is defined by (2.7) and $\Psi(x)$ is the original spectral matrix associated with P . Moreover, we have

$$\int_{-1}^1 \hat{\mathbf{Q}}_n(x) \hat{\Psi}(x) \hat{\mathbf{Q}}_m^T(x) dx = \hat{\Pi}_n^{-1} \delta_{n,m},$$

where $(\hat{\Pi}_n)_{n \geq 0}$ are defined by (3.17).

Proof. For $n \geq 1$ and $j = 0, \dots, n-1$, we have

$$\begin{aligned} \int_{-1}^1 \hat{\mathbf{Q}}_n(x) \hat{\Psi}(x) x^j dx &= \int_{-1}^1 x \mathbf{T}_n(x) \Psi(x) x^j \tilde{S}_0^{-T} dx = \int_{-1}^1 [\tilde{Y}_n \mathbf{Q}_n(x) + \tilde{X}_n \mathbf{Q}_{n+1}(x)] \Psi(x) x^j \tilde{S}_0^{-T} dx \\ &= \tilde{Y}_n \int_{-1}^1 \mathbf{Q}_n(x) \Psi(x) x^j \tilde{S}_0^{-T} dx + \tilde{X}_n \int_{-1}^1 \mathbf{Q}_{n+1}(x) \Psi(x) x^j \tilde{S}_0^{-T} dx = \mathbf{0}_{2 \times 2}, \end{aligned}$$

where for the first equality we have used (3.16) and (3.18), for the second equality we have used (3.14), and finally we have used the orthogonality of the family $(\mathbf{Q}_n)_{n \geq 0}$. Finally, for $n \geq 0$, and using (3.14), (3.15), (3.17) and the orthogonality properties, we have

$$\begin{aligned} \int_{-1}^1 \hat{\mathbf{Q}}_n(x) \hat{\Psi}(x) \hat{\mathbf{Q}}_n^T(x) dx &= \int_{-1}^1 \mathbf{T}_n(x) x \Psi(x) \mathbf{T}_n^T(x) dx = \int_{-1}^1 [\tilde{Y}_n \mathbf{Q}_n(x) + \tilde{X}_n \mathbf{Q}_{n+1}(x)] \Psi(x) \mathbf{T}_n^T(x) dx \\ &= \tilde{Y}_n \int_{-1}^1 \mathbf{Q}_n(x) \Psi(x) \mathbf{T}_n^T(x) dx = \tilde{Y}_n \int_{-1}^1 \mathbf{Q}_n(x) \Psi(x) [\tilde{S}_n^{-1} \mathbf{Q}_n(x) - \tilde{S}_n^{-1} \tilde{R}_n \mathbf{T}_{n-1}(x)]^T dx \\ &= \tilde{Y}_n \int_{-1}^1 \mathbf{Q}_n(x) \Psi(x) \mathbf{Q}_n^T(x) \tilde{S}_n^{-T} dx = \tilde{Y}_n \Pi_n^{-1} \tilde{S}_n^{-T} = (\tilde{S}_n^T \Pi_n \tilde{Y}_n^{-1})^{-1} = \hat{\Pi}_n^{-T} = \hat{\Pi}_n^{-1}, \end{aligned}$$

where in the final step we have used the fact that $(\hat{\Pi}_n)_{n \geq 0}$ are diagonal matrices. \square

Remark 3.4. The weight matrix $\widehat{\Psi}(x)$ in (3.18) is called a *Christoffel transformation* of the original spectral matrix $\Psi(x)$.

3.2. Example: random walk with transitions between the states 1 and -1 . Let us consider an irreducible Markov chain on \mathbb{Z} with transition probability matrix P as in (3.1) where

$$a_n = a, \quad n \in \mathbb{Z} \setminus \{-1\}, \quad b_n = b, \quad n \in \mathbb{Z}, \quad c_n = c, \quad n \in \mathbb{Z} \setminus \{1\},$$

and as usual $a + b + c = 1, a, c > 0, b \geq 0$. From (3.9) the values of a_{-1}, d_{-1}, c_1, d_1 must be given in terms of a, b, c . Indeed,

$$a_{-1} = \frac{ab}{1-c}, \quad d_{-1} = \frac{a^2}{1-c}, \quad c_1 = \frac{bc}{1-a}, \quad d_1 = \frac{c^2}{1-a}.$$

Since $1 - c > 0$ and $1 - a > 0$, this implies that $a_{-1}, d_{-1}, c_1, d_1 > 0$. Now, observe that $a + c \leq 1$ and $ab < a$ since $0 \leq b < 1$. Then we have $ab + c < a + c \leq 1$, which implies that $ab < 1 - c$ and therefore $a_{-1} < 1$. In the same way but using $a^2 < a, bc < c$ and $c^2 < c$ we get $d_{-1} < 1, c_1 < 1$ and $d_1 < 1$, respectively. Therefore, independently of the choice of a, b and c , P is always a stochastic matrix.

If we consider the AR factorization we have that the continued fractions in (3.10) can be explicitly computed and in this case they are given by

$$\tilde{H} = \frac{bc}{J(1-a)}, \quad \tilde{H}' = \frac{ab}{J'(1-c)},$$

where

$$J = \frac{1}{2} \left(1 + c - a + \sqrt{(1+c-a)^2 - 4c} \right), \quad J' = \frac{1}{2} \left(1 + a - c + \sqrt{(1+c-a)^2 - 4c} \right). \quad (3.19)$$

Therefore we get

$$\tilde{H} = \frac{b}{2(1-a)} \left(1 + c - a + \sqrt{(1+c-a)^2 - 4c} \right), \quad \tilde{H}' = \frac{b}{2(1-c)} \left(1 + a - c + \sqrt{(1+c-a)^2 - 4c} \right),$$

with $a \leq (1 - \sqrt{c})^2$ (to ensure convergence). With this condition we immediately have that $b > \tilde{H}, b > \tilde{H}'$ and then $b > \max\{\tilde{H}, \tilde{H}'\}$. Therefore, according to Proposition 3.1, we have that the AR stochastic factorization of P is always possible.

Let us now compute the spectral matrix $\Psi(x)$ associated with this example. Since P is not tridiagonal we do not have a birth-death chain on the integers \mathbb{Z} , so we can not apply the same methodology as we did in the previous section. However we can consider the block tridiagonal structure of P given by \mathbf{P} in (3.2) and use the theory of matrix-valued orthogonal polynomials to compute the spectral matrix $\Psi(x)$. \mathbf{P} is given in this case by

$$\mathbf{P} = \left(\begin{array}{cc|cc|cc|cc|} b & c & a & 0 & & & & & \\ \frac{ab}{1-c} & b & \frac{a^2}{1-c} & c & & & & & \\ \hline \frac{bc}{1-a} & \frac{c^2}{1-a} & b & 0 & a & 0 & & & \\ 0 & a & 0 & b & 0 & c & & & \\ \hline & & c & 0 & b & 0 & a & 0 & \\ & & 0 & a & 0 & b & 0 & c & \\ \hline & & & & \ddots & & \ddots & & \ddots \end{array} \right).$$

Let us now use Theorem 2.1 of [2] to obtain a relation between the Stieltjes transform of the spectral matrix Ψ , given by definition by $B(\Psi; z) = \int_{-1}^1 (z - x)^{-1} d\Psi(x)$, and the Stieltjes transform of the spectral matrix Ψ_0 of the 0 -th associated process \mathbf{P}_0 , which is constructed by removing the first block row and column of \mathbf{P} . Observe

that \mathbf{P}_0 has constant diagonal block entries. Therefore, using Theorem 2.1 of [2], it is easy to compute the corresponding Stieltjes transform $B(\Psi_0; z)$, given by

$$B(\Psi_0; z) = \begin{pmatrix} \frac{z - b \pm \sqrt{(z - \sigma_+)(z - \sigma_-)}}{2ac} & 0 \\ 0 & \frac{z - b \pm \sqrt{(z - \sigma_+)(z - \sigma_-)}}{2c^2} \end{pmatrix}, \quad z \in \mathbb{C} \setminus [\sigma_-, \sigma_+],$$

where σ_{\pm} are given by (2.34). Now using again Theorem 2.1 of [2] we have that the Stieltjes transform $B(\Psi; z)$ of \mathbf{P} satisfies the algebraic equation

$$B(\Psi; z)\Pi_{\Psi} [zI_{2 \times 2} - B_0 - A_0B(\Psi_0; z)\Pi_{\Psi_0}C_1] = I_{2 \times 2}, \quad (3.20)$$

where

$$A_0 = \begin{pmatrix} a & 0 \\ \frac{a^2}{1-c} & c \end{pmatrix}, \quad B_0 = \begin{pmatrix} b & c \\ \frac{ab}{1-c} & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} \frac{bc}{1-a} & \frac{c^2}{1-a} \\ 0 & a \end{pmatrix},$$

and Π_{Ψ} and Π_{Ψ_0} are the inverses of the 0-th norms of each spectral matrix, given in this case by

$$\Pi_{\Psi} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{c(1-c)}{ab} \end{pmatrix}, \quad \Pi_{\Psi_0} = \begin{pmatrix} 1 & 0 \\ 0 & c/a \end{pmatrix}.$$

Solving (3.20) and after some tedious but straightforward computations we have that

$$B(\Psi; z) = \begin{pmatrix} B(\psi_{11}; z) & B(\psi_{12}; z) \\ B(\psi_{12}; z) & B(\psi_{22}; z) \end{pmatrix},$$

where

$$B(\psi_{ij}; z) = \frac{p_{ij}(z) + q_{ij}(z)\sqrt{(z - \sigma_+)(z - \sigma_-)}}{r_{ij}(z)}, \quad z \in \mathbb{C} \setminus [\sigma_-, \sigma_+],$$

and $p_{ij}(z), q_{ij}(z), r_{ij}(z)$ are polynomials given by

$$\begin{aligned} p_{11}(z) &= 2(1-a)(1-c)z^3 - 4b(1-a)(1-c)z^2 + \gamma_{11}z - b^2((a-c)^2 - a - c), \\ q_{11}(z) &= b[-2(1-a)(1-c)z - a(1-a) - c(1-c)], \\ r_{11}(z) &= 2(1-a)(1-c)z^4 - 4b(1-a)(1-c)z^3 + (\gamma_{11} - b^2(2ac - a - c))z^2 + 4ab^2cz - 2ab^2c, \\ p_{12}(z) &= b[-(1-a)(1-c)z^3 + b(1-a)(2-3c)z^2 + \gamma_{12}z - bc((1-c)^2 - a(1+c))] \\ q_{12}(z) &= b[-(1-a)(1-c)z^2 + (1-a)(1+2c^2 - a - 3c)z + bc(1-c)] \\ r_{12}(z) &= cr_{11}(z), \\ p_{22}(z) &= b^2[-(1-a)z^3 + (1-a)(b+2(1-c))z^2 + \gamma_{22}z - b(ac - c^2 + a + 2c - 1)] \\ q_{22}(z) &= b[-(1-a)(1-c+a)z^2 + 2b(1-a)(1-c)z - b^2(1-c)] \\ r_{22}(z) &= cr_{11}(z), \end{aligned} \quad (3.21)$$

where

$$\begin{aligned} \gamma_{11} &= 2(1-a)^3 + 2(1-c)^3 - 2 + 2ac(2+a+c-4ac) + b^2(2ac - a - c), \\ \gamma_{12} &= a^3 + a^2(2c^2 + 2c - 3) + a(1-c)(2c^2 - 4c + 3) - (1-c)^2(1-3c), \\ \gamma_{22} &= -2a^2(1+c) + a(2c^2 - 5c + 5) - 3(1-c)^2. \end{aligned}$$

Therefore the Stieltjes transform $B(\Psi; z)$ can be written as

$$B(\Psi; z) = \frac{\sqrt{(z - \sigma_+)(z - \sigma_-)}}{cr_{11}(z)} \begin{pmatrix} cq_{11}(z) & q_{12}(z) \\ q_{12}(z) & q_{22}(z) \end{pmatrix} + \frac{1}{cr_{11}(z)} \begin{pmatrix} cp_{11}(z) & p_{12}(z) \\ p_{12}(z) & p_{22}(z) \end{pmatrix}.$$

Observe that $r_{11}(z)$ is a polynomial of degree 4, so the Stieltjes transform may have at most 4 real poles. We have not been able to compute an explicit expression of these zeros, but if we assume that $c = a$ then it is possible to have an explicit expression of them.

So let us assume from now on that $c = a$. Then $b = 1 - 2a$ and the polynomial $r_{11}(z)$ has now a simpler expression:

$$r_{11}(z) = 2(1 - z)(z(1 - a) + a)[(a - 1)z^2 + b^2z - ab^2].$$

The zeros of $r_{11}(z)$ are given by

$$1, \quad -\frac{a}{1 - a}, \quad \frac{b(b \pm \sqrt{2b^2 - 1})}{1 + b}. \quad (3.22)$$

If $\sqrt{2}/2 < b < 1$, i.e., $0 < a < (2 - \sqrt{2})/4$, there can be at most 4 different real zeros. This means that the spectral matrix will consist of a continuous density plus possibly some Dirac delta masses located at these zeros with certain weights. Let us write the spectral matrix as $\Psi(x) = \Psi_c(x) + \Psi_d(x)$. Using the Stieltjes-Perron inversion formula the continuous part of the spectral matrix is given by

$$\Psi_c(x) = \frac{\sqrt{(\sigma_+ - x)(x - \sigma_-)}}{c\pi r_{11}(x)} \begin{pmatrix} cq_{11}(x) & q_{12}(x) \\ q_{12}(x) & q_{22}(x) \end{pmatrix}, \quad x \in [\sigma_-, \sigma_+] = [1 - 4a, 1], \quad (3.23)$$

where

$$\begin{aligned} q_{11}(x) &= -2(1 - 2a)(1 - a)(x(1 - a) + a), \\ q_{12}(x) &= -(1 - 2a)(1 - a)(x(1 - a) + a)(x - 1 + 2a), \\ q_{22}(x) &= -(1 - 2a)(1 - a)(x^2 - 2(1 - a)(1 - 2a)x + (1 - 2a)^2). \end{aligned}$$

The discrete masses come from the residues at the simple poles of $B(\Psi; z)$, given by (3.22). It is possible to see that these zeros (if real) are in $[-1, 1] \setminus (\sigma_-, \sigma_+)$. A straightforward computation shows that all these discrete masses are identically $\mathbf{0}_{2 \times 2}$. Therefore $\Psi_d(x) = \mathbf{0}_{2 \times 2}$ and $\Psi(x) = \Psi_c(x)$. As a consequence the spectral matrix only have a continuous part given by (3.23). In the general case (if $c \neq a$) we have extensive computational evidences that the spectral matrix has again only a continuous part, given by (3.23).

Now let us consider the stochastic Darboux transformation of the AR factorization. As we said in Section 3 (just before (3.10)), there is no free parameter and the factorization is unique. The Darboux transformation is then given by \widehat{P} in (3.12) (in block matrix form), but if we consider it as a Markov chain on \mathbb{Z} , the matrix \widehat{P} is now a discrete-time birth-death chain on \mathbb{Z} . The transition probabilities of this birth-death chain are highly nontrivial and they can be computed from (3.13). Let us call J the continued fraction

$$J = \cfrac{c}{1} \bigg| \cfrac{a}{1} \bigg| \cfrac{c}{1} \bigg| \cfrac{a}{1} \bigg| \dots$$

and $j_n = \alpha_n/\beta_n$ the corresponding convergents. Observe that J is convergent as long as $a \leq (1 - \sqrt{c})^2$ and the limit is given by (3.19). The sequences α_n and β_n of the convergents j_n can be computed recursively using the following relations:

$$\begin{aligned} \alpha_{2n} &= \alpha_{2n-1} - a\alpha_{2n-2}, \quad n \geq 1, & \alpha_{2n+1} &= \alpha_{2n} - c\alpha_{2n-1}, \quad n \geq 0, & \alpha_{-1} &= 1, & \alpha_0 &= 1, \\ \beta_{2n} &= \beta_{2n-1} - a\beta_{2n-2}, \quad n \geq 1, & \beta_{2n+1} &= \beta_{2n} - c\beta_{2n-1}, \quad n \geq 0, & \beta_{-1} &= 0, & \beta_0 &= 1. \end{aligned}$$

The first few convergents are given by

$$j_0 = 0, \quad j_1 = c, \quad j_2 = \frac{c}{1-a}, \quad j_3 = \frac{c(1-c)}{b}, \quad j_4 = \frac{cb}{b-a(1-a)}, \quad j_5 = \frac{c(b-c(1-c))}{b(1-c)-a(1-a)}, \quad \dots$$

A straightforward computation using (3.4) and (3.6) shows that the coefficients $\tilde{x}_n, \tilde{y}_n, \tilde{r}_n, \tilde{s}_n, n \in \mathbb{Z}$, can be written in terms of the convergents j_n . Indeed,

$$\begin{aligned} \tilde{x}_n &= \frac{a}{1-j_{2n}}, \quad n \geq 0, & \tilde{y}_n &= 1 - \tilde{x}_n, \quad n \geq 1, & \tilde{y}_0 &= b, & \tilde{\alpha} &= c, & \tilde{x}_{-n} &= j_{2n+1}, & \tilde{y}_{-n} &= 1 - j_{2n+1}, \quad n \geq 1, \\ \tilde{r}_n &= j_{2n}, & \tilde{s}_n &= 1 - j_{2n}, \quad n \geq 0, & \tilde{r}_{-n} &= \frac{a}{1-j_{2n-1}}, & \tilde{s}_{-n} &= 1 - \tilde{r}_{-n}, \quad n \geq 1. \end{aligned}$$

Therefore, the coefficients $\hat{a}_n, \hat{b}_n, \hat{c}_n, n \in \mathbb{Z}$, of the birth-death chain \hat{P} are given by

$$\begin{aligned} \hat{a}_n &= \frac{a(1-j_{2n+2})}{1-j_{2n}}, \quad n \geq 0, & \hat{a}_{-n} &= \frac{a(1-j_{2n+1})}{1-j_{2n-1}}, \quad n \geq 1, \\ \hat{c}_n &= \frac{j_{2n}(1-a-j_{2n})}{1-j_{2n}}, \quad n \geq 0, & \hat{c}_{-n} &= \frac{j_{2n+1}(1-a-j_{2n+1})}{1-j_{2n+3}}, \quad n \geq 1, \\ \hat{b}_n &= 1 - \hat{a}_n - \hat{c}_n, \quad n \in \mathbb{Z}. \end{aligned}$$

Although these coefficients are highly nontrivial it is possible to compute the spectral matrix $\hat{\Psi}$ of the birth-death chain \hat{P} using Theorem 2.3. Indeed, the spectral matrix is given by

$$\hat{\Psi}(x) = x \tilde{S}_0^{-1} \Psi(x) \tilde{S}_0^{-T},$$

where $\Psi(x)$ is given by (3.23) and

$$\tilde{S}_0^{-1} = \begin{pmatrix} 1 & 0 \\ -a/b & (1-c)/b \end{pmatrix}.$$

In other words,

$$\hat{\Psi}(x) = \frac{x \sqrt{(\sigma_+ - x)(x - \sigma_-)}}{c\pi r_{11}(x)} \begin{pmatrix} cq_{11}(x) & -\frac{ac}{b}q_{11}(x) + \frac{1-c}{b}q_{12}(x) \\ -\frac{ac}{b}q_{11}(x) + \frac{1-c}{b}q_{12}(x) & \frac{a^2c}{b^2}q_{11}(x) - \frac{2a(1-c)}{b^2}q_{12}(x) + \frac{(1-c)^2}{b^2}q_{22}(x) \end{pmatrix},$$

where $q_{ij}(x)$ are given by (3.21).

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