A simple upper bound for Lebesgue constants associated with Leja points on the real line

Vladimir Andrievskii, Fedor Nazarov

Department of Mathematical Sciences, Kent State University

Running head: Lebesgue constants for Leja points on the line

Mailing address:

V. Andrievskii, F. Nazarov
Department of Mathematical Sciences
Kent State University
Kent, OH 44242, USA

E-mail addresses:
andriyev@math.kent.edu,   nazarov@math.kent.edu

Phones: (330) 672 9029, (330) 672 9013

Abstract

Let $K \subset \mathbf{R}$ be a regular compact set and let $g(z) = g_{\overline{\mathbf{C}} \setminus K}(z, \infty)$ be the Green function for $\overline{\mathbf{C}} \setminus K$ with pole at infinity. For $\delta > 0$, define

$$G(\delta) := \max\{g(z) : z \in \mathbf{C}, \operatorname{dist}(z, K) \leq 2\delta\}.$$

Let $\{x_n\}_{n=0}^{\infty}$ be a Leja sequence of points of K. Then the uniform norm $\|T_n\| = \Lambda_n, n = 1, 2, \ldots$ of the associated interpolation operator T_n, i.e., the n-th Lebesgue constant, is bounded from above by

$$\min_{\delta > 0} 2n \left[\frac{\operatorname{diam}(K)}{\delta} e^{nG(\delta)} \right]^{9/8}.$$

In particular, when K is a uniformly perfect subset of $\mathbf{R}$, the Lebesgue constants grow at most polynomially in n.

To the best of our knowledge, the result is new even when K is a finite union of intervals.

Key Words: Leja points, Green's function, interpolation, uniformly perfect sets.

AMS classification: 41A05, 41A10

1. Introduction

Let $K \subset \mathbf{R}$ be any compact set. We say that a sequence of points $\{x_n\}_{n=0}^{\infty}$ in K is a *Leja sequence* if for every $k \geq 1$, x_k is a point of global maximum of the product $\prod_{j:j<k} |x - x_j|$ on K ($x_0 \in K$ can be arbitrary). Let $C(K)$ be the space of continuous on K functions $f : K \to \mathbf{C}$ endowed with the uniform norm $\|\cdot\|$ and let $T_n : C(K) \to C(K)$ be the *Lagrange interpolation operator* associated with the points $x_0, \ldots, x_{n-1}$, i.e.,

$$T_n f(x) = \sum_{k=0}^{n-1} f(x_k) L_{k,n}(x)$$

where

$$L_{k,n}(x) = \prod_{j:0 \leq j \leq n-1, j \neq k} \frac{x - x_j}{x_k - x_j} .$$

A natural question is for what classes of functions we have $T_n f \to f$ in $C(K)$ and what is the rate of convergence. In [5, 6, 2, 4] the reader can find known results, a discussion of the rich history of this question, and further references.

Let $E_m(f) := \inf\{\|f - p\| : p \text{ is a polynomial of degree} \leq m\}$ be the error of the best approximation of f by polynomials of degree at most m on K and let $\Lambda_n = \|T_n\|$ be the n-th *Lebesgue constant*. Then, denoting by p_{n-1}^* the polynomial of best approximation of degree at most $n - 1$ and taking into account that $T_n p_{n-1}^* = p_{n-1}^*$, we can write

$$\|f - T_n f\| = \|(f - p_{n-1}^*) - T_n(f - p_{n-1}^*)\|$$

$$\leq (1 + \Lambda_n)\|f - p_{n-1}^*\| = (1 + \Lambda_n)E_{n-1}(f).$$

Thus, the condition $\Lambda_n E_{n-1}(f) \to 0$ as $n \to \infty$ is sufficient for the convergence. Many natural classes of functions can be described in terms of the rate at which $E_n(f)$ tends to 0. For instance, let $I \supset K$ be a closed interval. Then, for every fixed $k \in \mathbf{N}$, the condition that the function f is a restriction to K of a function having continuous k-th derivative on I implies that $E_n(f) = O(n^{-k})$. So knowing a good upper bound for Λ_n and comparing it to the rate of decay of $E_n(f)$ can help one to decide quickly if the Leja interpolation scheme is guaranteed to converge.

3

Another reason to care about Λ_n is that often even if we do know a priori that f is very good, the actual data we interpolate may be noisy, the noise coming either from the measurement errors, or simply from rounding in the numerical computations. In this case, we technically want to estimate not $\|f - T_n f\|$, but rather $\|f - T_n(f + \Delta f)\|$ where Δf is the noise. Then, even when the former is small, the latter may be huge because $\|T_n \Delta f\|$ is large. Since the noise can be completely arbitrary, it is hard to control the norm $\|T_n \Delta f\|$ by anything better than $\Lambda_n \|\Delta f\|$, so, again, having a reasonably clear idea of what Λ_n is allows one to estimate a priori what level of noise is acceptable.

It is worth noting that in practice Λ_n can be evaluated a posteriori once the points $x_0, \ldots, x_{n-1}$ have been chosen, so the a priori theoretical bounds are not necessarily a must, especially when they fall far short from what is really observed in the computations. Still, we believe that they may hold some value, if not as a prediction, then, at least, as an explanation of the pretty decent efficiency of the Leja interpolation scheme.

2. Main results

Let $K \subset \mathbf{R}$ be a regular compact set, i.e., the Green function $g(z) = g_{\overline{\mathbf{C}} \setminus K}(z, \infty)$ of $\overline{\mathbf{C}} \setminus K$ with pole at infinity is continuous on $\mathbf{C}$ (by definition, $g_{|K} = 0$).

For $\delta > 0$, let

$$G(\delta) := \max\{g(z) : z \in \mathbf{C}, \, \mathrm{dist}(z, K) \leq 2\delta\}.$$

Theorem 1. *Let* $x_0, \ldots, x_{n-1} \in K$ *be a Leja sequence. Then, for every* $\delta > 0$,
$$\Lambda_n \leq 2n \left[\frac{\mathrm{diam}(K)}{\delta} e^{nG(\delta)} \right]^{9/8}.$$

This bound, though, probably, still suboptimal for a general K, is fairly decent.

Indeed, on the one hand, it shows that for any regular compact subset $K \subset \mathbf{R}$, Λ_n is subexponential. To see this, just take any $\varepsilon > 0$ and choose

$\delta > 0$ so that $G(\delta) < (8/9)\varepsilon$ to obtain

$$\limsup_{n \to \infty} \frac{\log \Lambda_n}{n} < \varepsilon.$$

This already vastly extends the class of compact sets for which the subexponential upper bound was known.

On the other hand, consider a *uniformly perfect set* K. Recall that according to Beardon and Pommerenke [1] it means that there exists a constant $0 < \gamma_K < 1$ such that for every $x \in K$ and every $r \in (0, \mathrm{diam}(K))$, we have

$$K \cap \{\zeta \in \mathbf{R} : \gamma_K r \leq |x - \zeta| \leq r\} \neq \varnothing.$$

Any finite union of closed intervals is uniformly perfect as well as the classical Cantor set. If $K \subset \mathbf{R}$ is uniformly perfect, then $G(\delta) = O(\delta^\beta)$ as $\delta \to 0$ for some $\beta > 0$ (see [3, pp. 562-563]) and, choosing $\delta = n^{-1/\beta}$, we conclude that $\Lambda_n = O(n^{1+(9/8)/\beta})$. In particular, when K is a finite union of intervals, we have $\beta = 1/2$ and $\Lambda_n = O(n^{13/4})$.

The disadvantage of our technique is that it is unclear how to generalize it to the complex setting. The main obstacle is that the *Key Lemma* (Lemma 2) fails for complex points and we do not know a good substitute for it.

While Theorem 1 seems quite satisfactory from the purely theoretical point of view, in practice the issue is further complicated by the fact that one can find the maximum of a polynomial only with a certain precision, so to ensure that the Leja interpolation scheme is robust, we must also show that a small error in the maximization problem at each step does not result in high instability of the bound given by Theorem 1. To formalize this small error possibility, we shall make the following

Definition. *Let $0 < \tau \leq 1$. A sequence $x_0, \ldots, x_{n-1} \in K$ is called τ-quasi Leja if for every $k = 1, \ldots, n-1$, we have*

$$\prod_{j:j<k} |x_k - x_j| \geq \tau \max_{x \in K} \prod_{j:j<k} |x - x_j|.$$

In the ideal theoretical case $\tau = 1$, while in practice it can be made very close to 1 but, strictly speaking, the value $\tau = 1$ is unattainable.

Theorem 2. *If $x_0, \ldots, x_{n-1} \in K$ is τ-quasi Leja, then for every $\delta > 0$,*

$$\Lambda_n \le \frac{2}{\tau^2} n \left[\frac{\operatorname{diam}(K)}{\tau \delta} e^{nG(\delta)} \right]^{9/8 + 2\lambda^{-1} \log(1/\tau)},$$

where $\lambda = 0.24565978\ldots$ is the positive root of the equation $e^{e^\lambda}(e^\lambda - 1) = 1$.

Theorem 2 thus shows that the Leja scheme can exhibit at most moderate numerical instability: for τ close to 1, the theoretical bound of Theorem 1 is just raised to some power slightly bigger than 1.

3. The idea of the proof of Theorem 2

It is well known that

$$\Lambda_n = \sup_{x \in K} \sum_{k=0}^{n-1} |L_{k,n}(x)| \le n \sup_{x \in K, 0 \le k \le n-1} |L_{k,n}(x)|.$$

Thus the main issue is to get a good bound for $|L_{k,n}(x)|$ for individual k. The question here is how to use the τ-quasi Leja condition in a simple but reasonably efficient way. Our suggestion is just to notice that for every n' with $k \le n' < n$, we can use the τ-quasi Leja property of $x_{n'}$ to write

$$\prod_{j:j<n} |x - x_j| = \prod_{j:j<n'} |x - x_j| \prod_{j:n' \le j<n} |x - x_j|$$

$$\le \tau^{-1} \prod_{j:j<n'} |x_{n'} - x_j| \prod_{j:n' \le j<n} |x - x_j|.$$

Repeating this trick several times, we see that if $x_0, \ldots, x_{n-1} \in K$ are τ-quasi Leja and $x = x_n \in K$ is arbitrary, then for every sequence $k = n_0 < n_1 < \ldots < n_m = n$, we have

$$\prod_{j:j<n} |x - x_j| \le \tau^{-m} \prod_{j:j<k} |x_k - x_j| \prod_{l=0}^{m-1} \prod_{j:n_l \le j<n_{l+1}} |x_{n_{l+1}} - x_j|.$$

Now,

$$|L_{k,n}(x_n)| = \frac{\prod_{j:0 \le j \le n-1, j \ne k} |x_n - x_j|}{\prod_{j:0 \le j \le n-1, j \ne k} |x_k - x_j|} = \frac{\prod_{j:0 \le j \le n-1} |x_n - x_j|}{\prod_{j:0 \le j \le n, j \ne k} |x_k - x_j|}.$$

6

The above observations imply that we can replace the numerator on the right hand side with any product of differences similar to the one marked by arcs on Figure 1, in which we can change the current "reference point" (initially x_n) to the last subtracted point any time at the cost of an extra τ^{-1} factor.
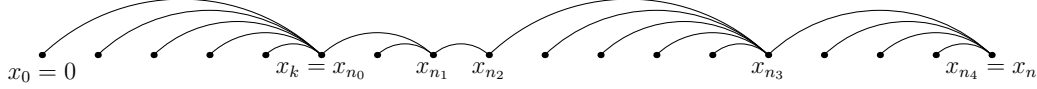

Figure 1

Note that the order of points x_j on this picture has nothing to do with their actual order on $\mathbf{R}$.

Thus, $|L_{k,n}(x_n)| \leq I_\tau(x_k, \ldots, x_n)$, where for arbitrary $x_0, \ldots, x_q \in \mathbf{R}$,

$$(3.1) \quad I_\tau(x_0, \ldots, x_q) = \min_{0 = n_0 < \ldots < n_m = q} \tau^{-m} \frac{\prod_{l=0}^{m-1} \prod_{j:n_l \leq j < n_{l+1}} |x_{n_{l+1}} - x_j|}{\prod_{j=1}^{q} |x_0 - x_j|}.$$

This inequality makes one tempted to forget completely about the Leja (or τ-quasi Leja) property from this point on and just bound $I_\tau(x_0, \ldots, x_q)$ for an arbitrary sequence $x_0, \ldots, x_q$ of pairwise distinct real numbers.

Unfortunately, this strategy does not quite work as it can be seen by examining the sequence $x_0 = 0, x_j = (-\Lambda)^{j-1}, j \geq 1$ with $\Lambda = 1 + \tau^{-1}$ being the positive root of the equation $\Lambda^2 - 1 = \tau^{-1}(\Lambda + 1)$.

For this sequence, it is beneficial to switch at every step, resulting in

$$\begin{aligned} I_\tau(x_0, \ldots, x_q) &= \tau^{-q} \prod_{j=1}^{q} \frac{|x_j - x_{j-1}|}{|x_j - x_0|} \\ &= \tau^{-q} \left(1 + \frac{1}{\Lambda}\right)^{q-1} = \frac{1}{\tau} \left(\frac{1}{\tau} + \frac{1}{\tau + 1}\right)^{q-1}, \end{aligned}$$

which is exponential in q. However, as we shall see in Section 5, it still does yield a universal bound

$$I_\tau(x_0, \ldots, x_q) \leq \frac{1}{\tau} \left(\frac{2}{\tau}\right)^{q-1}$$

7

and the subsequent bounds

$$|L_{k,n}(x)| \leq \frac{1}{\tau}\left(\frac{2}{\tau}\right)^{n-k-1}, \quad x \in K,$$

and

$$\Lambda_n \leq \frac{1}{\tau}\left(\frac{2}{\tau}\right)^n.$$

To go below an exponential bound, we need some more information about the points $x_0, \ldots, x_{n-1}$. Fortunately, we do not need to know too much in the case $K \subset \mathbf{R}$ and the following simple *Separation Lemma* already suffices.

Lemma 1. *If K is a regular compact set and $x_0, \ldots, x_{n-1} \in K$ is a τ-quasi Leja sequence, then for every $\delta > 0$,*

$$|x_i - x_j| \geq \Delta = \tau \delta e^{-nG(\delta)}, \quad i, j = 0, \ldots, n-1; i \neq j.$$

Combined with the *Key Lemma* below, it immediately yields Theorem 2 and, thereby, Theorem 1 as well.

Lemma 2. *Let $x_0, \ldots, x_q \in \mathbf{R}$. Assume that $|x_0 - x_j| \leq D$ for all $j = 1, \ldots, q$, and that $|x_0 - x_j| \geq \Delta$ for $j = 1, \ldots, q-1$ with some $D \geq \Delta > 0$. Then for every $\tau \in (0, 1]$, we have*

$$I_\tau(x_0, \ldots, x_q) \leq \frac{2}{\tau^2}\left(\frac{D}{\Delta}\right)^{9/8 + 2\lambda^{-1}\log(1/\tau)},$$

where, as before, λ is the positive root of the equation $e^{e^\lambda}(e^\lambda - 1) = 1$.

Note that we require no separation for x_q in the assumptions of Lemma 2!

It remains to prove the two lemmas now.

8

4. Proof of the Separation Lemma

Let k satisfy $1 \le k \le n-1$. Let $p(z) = \prod_{j:j<k}(z-x_j)$, $z \in \mathbf{C}$. Finally, let $M = \max_{x \in K} |p(x)|$, so $|p(x_k)| \ge \tau M$.

Consider the function $u(z) = \log(M^{-1}|p(z)|)$ on $\mathbf{C} \setminus K$. It is harmonic in $\mathbf{C} \setminus K$ and satisfies $u(z) = k \log|z| + O(1) < ng(z)$ as $z \to \infty$ and $\limsup_{z \to K} u(z) \le 0$. Thus, by the classical maximum principle, $u(z) \le ng(z)$ on $\mathbf{C} \setminus K$. This inequality, clearly, holds on K as well, so we get

$$|p(z)| \le Me^{ng(z)} \le Me^{nG(\delta)}$$

for every $z \in \mathbf{C}$ with $|z - x_k| \le 2\delta$.

By the Cauchy bound, it follows that $|p'(z)| \le M\delta^{-1}e^{nG(\delta)}$ when $|z - x_k| \le \delta$. In particular, when $|z - x_k| < \tau\delta e^{-nG(\delta)} = \Delta$, we get

$$|p(z)| \ge |p(x_k)| - M\delta^{-1}e^{nG(\delta)}|z - x_k| > \tau M - \tau M = 0,$$

so no root $x_j, j = 0, \ldots, k-1$ of p can lie at distance smaller than Δ from x_k. This completes the proof of Lemma 1.

5. Proof of the Key Lemma

Replacing x_j by $x_j - x_0$, we can assume without loss of generality that $x_0 = 0$. Then, replacing x_j by $-x_j$ if needed, we can assume that $x_q > 0$. The ratio of the products participating in the definition of $I_\tau(x_0, \ldots, x_q)$ can be written as

$$\frac{|X_0|}{|X_{q-1}|} \prod_{j=1}^{q-1} \frac{|X_j - x_j|}{|x_j|},$$

where X_j is the reference point at the moment of subtracting x_j, i.e., $X_j = x_{n_{l+1}}$ when $n_l \le j < n_{l+1}$. We start with $X_{q-1} = x_q$ and then, when going from j to $j-1$, can either keep the reference point (i.e., put $X_{j-1} = X_j$), or switch it to the point subtracted at the previous step (i.e., put $X_{j-1} = x_j$). The number m in the prefactor τ^{-m} is just the number of switches plus 1.

Our task is to find a good switching strategy, which, on the one hand, will allow us to control the total number of switches and, on the other hand, will

keep the majority of the ratios $|X_j - x_j|/|x_j|$ small. The naive switching strategy is to switch every time when $|x_j| < |X_j|$. It guarantees that for each $j = 1, \ldots, q - 1$ where the reference point is kept, we have

$$\frac{|X_j - x_j|}{|x_j|} \le 2 = 2\frac{|X_j|}{|X_{j-1}|}$$

and at each switch,

$$\frac{|X_j - x_j|}{|x_j|} = \frac{|X_j - x_j|}{|X_j|}\frac{|X_j|}{|x_j|} = \frac{|X_j - x_j|}{|X_j|}\frac{|X_j|}{|X_{j-1}|} \le 2\frac{|X_j|}{|X_{j-1}|}$$

as well.

Thus, the full product of the ratios is at most

$$\frac{|X_0|}{|X_{q-1}|}\prod_{j=1}^{q-1}\left(2\frac{|X_j|}{|X_{j-1}|}\right) = 2^{q-1}.$$

Since with this strategy we may, in principle, switch at every step, the total number of switches cannot be bounded by anything better than $q - 1$, so we only get the bound $I_\tau(0, x_1, \ldots, x_q) \le \tau^{-q}2^{q-1}$ mentioned earlier.

The switching strategy (or, rather, the family of switching strategies) we will consider instead is the following. Let, as before, λ be the positive root of the equation $e^{e^\lambda}(e^\lambda - 1) = 1$. Let q' be the largest index for which $x_{q'} < 0$ or 0 if the sequence $x_1, \ldots, x_q$ contains only positive numbers. Set $X_{q-1} = x_q$. For j with $q' < j \le q - 1$, put $X_{j-1} = X_j$ if $x_j \ge e^{-\lambda}X_j$ and $X_{j-1} = x_j$ if $x_j < e^{-\lambda}X_j$. That is, when going over the positive tail, we switch the reference point if it becomes more than e^λ times smaller after the switch and keep it otherwise.

If $q' = 0$, or, which is the same, $x_j > 0$ for all $j = 1, \ldots, q$, then we always have

$$\frac{|X_j - x_j|}{|x_j|} \le \frac{|X_j|}{|X_{j-1}|}, \quad 1 \le j \le q - 1.$$

Indeed, if we made no switch, then the right hand side is 1, while the left hand one is either $(x_j - X_j)/x_j < 1$ if $x_j \ge X_j$, or

$$\frac{X_j - x_j}{x_j} \le \frac{(1 - e^{-\lambda})X_j}{e^{-\lambda}X_j} = e^\lambda - 1 < 1$$

10

if $e^{-\lambda}X_j \leq x_j < X_j$.

On the other hand, if we switch, then $|x_j| = |X_{j-1}|$ and, since $0 < x_j < e^{-\lambda}X_j$, we have
$$|X_j - x_j| = X_j - x_j < X_j = |X_j|,$$
so the inequality holds again.

Also, since each switch makes the reference point e^{λ} times smaller, we cannot have more than $\lambda^{-1}\log(D/\Delta)$ switches. This is clear if $x_q \geq \Delta$. But if $x_q < \Delta$, then we cannot have any switches at all, so the estimate is still valid.

Thus, in this case
$$I_\tau(0, x_1, \ldots, x_q) \leq \tau^{-(\lambda^{-1}\log(D/\Delta)+1)} \frac{|X_0|}{|X_{q-1}|} \prod_{j=1}^{q-1} \frac{|X_j|}{|X_{j-1}|} = \frac{1}{\tau}\left(\frac{D}{\Delta}\right)^{\lambda^{-1}\log(1/\tau)}.$$

Otherwise, for $j = q'$, keep both options (switching to $x_{q'}$ and staying with $X_{q'}$) available. Denote the corresponding reference points by $b_1 = X_{q'} > 0$ and $-a_1 = x_{q'} < 0$.

From this step on, we shall always have two options for the current reference point with the switching rules as follows. When going left from $q' - 1$, if $x_j \notin (-e^{-\lambda}a_1, e^{-\lambda}b_1)$, just keep the reference point as it was (either $-a_1$ or b_1, whichever was chosen as $X_{q'-1}$).

However, if $x_j \in (-e^{-\lambda}a_1, e^{-\lambda}b_1)$ and $x_j > 0$, then force the switch from b_1 to $b_2 = x_j$ and allow the switch from $-a_1$ to b_2, so, beyond this step, we shall have the options b_2 and $-a_1$ instead of b_1 and $-a_1$. For notational convenience, we will denote them $-a_2$ and b_2, a_2 being just the same as a_1. Similarly, if $x_j < 0$, then x_j becomes $-a_2$, b_2 stays the same as b_1, and the switch from $-a_1$ to $-a_2$ is forced while the switch from b_1 to $-a_2$ is optional.

Now repeat the same procedure with the interval $(-e^{-\lambda}a_2, e^{-\lambda}b_2)$ instead of $(-e^{-\lambda}a_1, e^{-\lambda}b_1)$, and so on until we reach $j = 0$ with two options $-a_l$ or b_l for X_0 with some $l > 0$. Our first task will be to bound the total number of switches. Consider first the case when $x_q \geq \Delta$. Then as long as we go along the positive tail, each switch decreases X_j at least e^{λ} times, so the number of switches made during this part of the strategy is at most
$$\lambda^{-1}\log\frac{X_{q-1}}{X_{q'}} \leq \lambda^{-1}\log\frac{D}{X_{q'}}.$$

Then there may be one switch to $x_{q'}$. Beyond that, every time we allow a switch, the product $a_s b_s$ decreases at least e^λ times, so the remaining number of switches is at most

$$\lambda^{-1} \log \frac{a_1 b_1}{a_l b_l} \le \lambda^{-1} \log \frac{D X_{q'}}{\Delta^2} \, .$$

Adding everything up, we conclude that in this case we can have at most $2\lambda^{-1} \log(D/\Delta) + 1$ switches, so

(5.1) $$\tau^{-m} \le \tau^{-(2\lambda^{-1} \log(D/\Delta)+2)} = \frac{1}{\tau^2} \left(\frac{D}{\Delta} \right)^{2\lambda^{-1} \log(1/\tau)} .$$

Now suppose that $x_q < \Delta$. Then we have no switches in the positive tail and $b_1 = X_{q'} = x_q$. We still may have one switch to $x_{q'}$. After that we cannot decrease b_1, so every time we switch, we decrease a_s at least e^λ times and the remaining number of switches is at most $\lambda^{-1} \log(a_1/a_l) \le \lambda^{-1} \log(D/\Delta)$.

Thus, in this case we can have at most $\lambda^{-1} \log(D/\Delta) + 1$ switches and

$$\tau^{-m} \le \tau^{-(\lambda^{-1} \log(D/\Delta)+2)} = \frac{1}{\tau^2} \left(\frac{D}{\Delta} \right)^{\lambda^{-1} \log(1/\tau)} ,$$

which is an even stronger bound than that in (5.1).

Now it is time to estimate

$$\frac{|X_0|}{|X_{q-1}|} \prod_{j=1}^{q-1} \frac{|X_j - x_j|}{|x_j|} \, .$$

The same argument as in the positive case shows that

$$\prod_{j:j>q'} \frac{|X_j - x_j|}{|x_j|} \le \frac{X_{q-1}}{X_{q'}} = \frac{X_{q-1}}{b_1} .$$

For $j = q'$, we have $\frac{|X_j - x_j|}{|x_j|} = \frac{a_1 + b_1}{a_1}$. Thus

(5.2) $$\frac{1}{|X_{q-1}|} \prod_{j:j \ge q'} \frac{|X_j - x_j|}{|x_j|} \le \frac{a_1 + b_1}{a_1 b_1} .$$

12

This leaves us with $|X_0| \prod_{j=1}^{q'-1}(|X_j - x_j|/|x_j|)$. The product here is not unique: we have a whole family of admissible strategies, not a single one. So we shall estimate some multiplicative average of this quantity over all of them.

Let $q_2 > \ldots > q_l$ be the indices at which the values of a_s and b_s change, i.e., $x_{q_{s+1}} \in (-e^{-\lambda}a_s, e^{-\lambda}b_s)$ while $x_j \notin (-e^{-\lambda}a_s, e^{-\lambda}b_s)$ for $q_{s+1} < j < q_s, s = 1, \ldots, l$ (we set $q_1 = q', q_{l+1} = 0$ here). Let

$$\alpha_s = \frac{b_s}{a_s + b_s}, \quad \beta_s = \frac{a_s}{a_s + b_s}.$$

Then for every j with $q_{s+1} < j < q_s$, we have

$$\left(\frac{|-a_s - x_j|}{|x_j|}\right)^{\alpha_s} \left(\frac{|b_s - x_j|}{|x_j|}\right)^{\beta_s} \leq 1$$

(see Elementary Inequality 1 in the Appendix).

Thus, if we denote

$$\overline{P}_s = \prod_{j: q_{s+1} < j < q_s} \frac{|-a_s - x_j|}{|x_j|}, \quad \overline{Q}_s = \prod_{j: q_{s+1} < j < q_s} \frac{|b_s - x_j|}{|x_j|},$$

we have $\overline{P}_s^{\alpha_s} \overline{Q}_s^{\beta_s} \leq 1$ as well.

Consider now $j = q_{s+1}$. If $s < l$, then two cases are possible: $x_j = -a_{s+1}$ or $x_j = b_{s+1}$.

Assume that $x_j = -a_{s+1}$. Then, since X_j is either $-a_s$ or b_s, we have $|X_j - x_j|/|x_j|$ equal to either $(a_s - a_{s+1})/a_{s+1}$ or $(b_s + a_{s+1})/a_{s+1}$. By Elementary Inequality 2,

$$\left[\frac{a_s - a_{s+1}}{a_{s+1}}\right]^{\alpha_s} \left[\frac{b_s + a_{s+1}}{a_{s+1}}\right]^{\beta_s} \leq \frac{a_s}{a_{s+1}} \left[\frac{\min(a_s, b_s)}{\min(a_{s+1}, b_s)}\right]^{1/8}$$

$$= \frac{a_s b_s}{a_{s+1} b_{s+1}} \left[\frac{\min(a_s, b_s)}{\min(a_{s+1}, b_{s+1})}\right]^{1/8}$$

(recall that in this case $b_{s+1} = b_s$). The case $x_j = b_{s+1}$ is symmetric to the considered one and results in the same bound.

13

Thus, putting

$$P_s = \overline{P}_s \frac{a_s - a_{s+1}}{a_{s+1}}, \quad Q_s = \overline{Q}_s \frac{b_s + a_{s+1}}{a_{s+1}},$$

we get

$$P_s^{\alpha_s} Q_s^{\beta_s} \le \frac{a_s b_s}{a_{s+1} b_{s+1}} \left[\frac{\min(a_s, b_s)}{\min(a_{s+1}, b_{s+1})} \right]^{1/8}, \quad s = 1, \ldots, l - 1.$$

For $s = l$, we just need to add $|X_0|$ to the product $\prod_{j=1}^{q_l - 1}(|X_j - x_j|/|x_j|)$, which results in either $P_l = \overline{P}_l a_l$ or $Q_l = \overline{Q}_l b_l$. Thus in this case $P_l^{\alpha_l} Q_l^{\beta_l} \le a_l^{\alpha_l} b_l^{\beta_l}$. Multiplying these estimates out, we get

$$\prod_{s=1}^{l} (P_s^{\alpha_s} Q_s^{\beta_s}) \le \frac{a_1 b_1}{a_l b_l} \left[\frac{\min(a_1, b_1)}{\min(a_l, b_l)} \right]^{1/8} a_l^{\alpha_l} b_l^{\beta_l}$$

(5.3)
$$= \frac{a_1 b_1}{a_l^{\beta_l} b_l^{\alpha_l}} \left[\frac{\min(a_1, b_1)}{\min(a_l, b_l)} \right]^{1/8}.$$

Now observe that for every admissible strategy under our rules, the product

$$|X_0| \prod_{j=1}^{q'-1} \frac{|X_j - x_j|}{|x_j|}$$

equals to $\prod_{s=1}^{l} R_s$ where each R_s is either P_s or Q_s and we can start with both P_1 and Q_1, after which we can always follow P_s by P_{s+1} and Q_s by Q_{s+1}, but we also can follow Q_s by P_{s+1} if $a_{s+1} < a_s, b_{s+1} = b_s$ and P_s by Q_{s+1} if $a_{s+1} = a_s, b_{s+1} < b_s$.

14

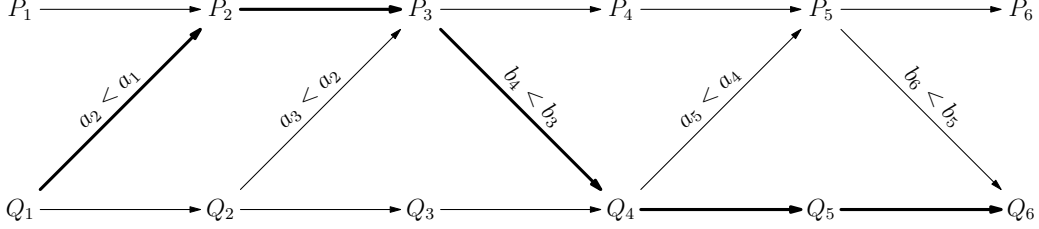In other words, possible products correspond to the paths on a diagram like the one on Figure 2.

Figure 2

For a path π on the diagram, let Γ_π be the corresponding product of P's and Q's. For instance, if π is the path following the thick edges on Figure 2, then $\Gamma_\pi = Q_1 P_2 P_3 Q_4 Q_5 Q_6$.

We want to show that a certain multiplicative average $\prod_\pi \Gamma_\pi^{\omega_\pi}$ with some $\omega_\pi > 0, \sum_\pi \omega_\pi = 1$ is exactly $\prod_{s=1}^l (P_s^{\alpha_s} Q_s^{\beta_s})$ in the sense that if P_s and Q_s on the diagram are viewed as free variables, then the equality

$$\prod_\pi \Gamma_\pi^{\omega_\pi} = \prod_{s=1}^l (P_s^{\alpha_s} Q_s^{\beta_s})$$

becomes an algebraic identity. We prove it by induction on the length l of the diagram.

If $l = 1$, then the only paths are P_1 and Q_1, so we can just put $\omega_{P_1} = \alpha_1, \omega_{Q_1} = \beta_1$.

Suppose now that the statement holds for $l - 1$. Note that if we remove P_1 and Q_1 from a diagram of length l (together with three edges coming out of them), we shall get a diagram of the same kind but of length $l - 1$ and starting with P_2 and Q_2. Thus, by the induction assumption, we can find weights $\omega_{\pi'}, \omega_{\pi''} > 0$ with $\sum_{\pi'} \omega_{\pi'} + \sum_{\pi''} \omega_{\pi''} = 1$, where π' and π'' run over all paths starting with P_2 and Q_2 respectively, such that

$$\prod_{\pi'} \Gamma_{\pi'}^{\omega_{\pi'}} \prod_{\pi''} \Gamma_{\pi''}^{\omega_{\pi''}} = \prod_{s=2}^l (P_s^{\alpha_s} Q_s^{\beta_s}).$$

Comparing the powers at P_2 and Q_2, we see that we must have $\sum_{\pi'} \omega_{\pi'} = \alpha_2$ and $\sum_{\pi''} \omega_{\pi''} = \beta_2$. Now if $a_2 < a_1$, say (i.e., if the switch from Q_1 to P_2 is

15

allowed), we have $\alpha_2 > \alpha_1$ and the admissible paths π in the full diagram are $P_1\pi', Q_1\pi'$ and $Q_1\pi''$. Put

$$\omega_{P_1\pi'} = \frac{\alpha_1}{\alpha_2}\omega_{\pi'}, \quad \omega_{Q_1\pi'} = \left(1 - \frac{\alpha_1}{\alpha_2}\right)\omega_{\pi'}, \quad \omega_{Q_1\pi''} = \omega_{\pi''}.$$

Then in the product $\prod_\pi \Gamma_\pi^{\omega_\pi}$ for the full diagram, the power of P_1 is

$$\frac{\alpha_1}{\alpha_2}\sum_{\pi'}\omega_{\pi'} = \frac{\alpha_1}{\alpha_2}\alpha_2 = \alpha_1,$$

the power of Q_1 is

$$\left(1 - \frac{\alpha_1}{\alpha_2}\right)\sum_{\pi'}\omega_{\pi'} + \sum_{\pi''}\omega_{\pi''} = \left(1 - \frac{\alpha_1}{\alpha_2}\right)\alpha_2 + \beta_2 = \alpha_2 + \beta_2 - \alpha_1 = 1 - \alpha_1 = \beta_1,$$

and the powers of P_s and Q_s with $s \geq 2$ are exactly the same as in the product $\prod_{\pi'}\Gamma_{\pi'}^{\omega_{\pi'}}\prod_{\pi''}\Gamma_{\pi''}^{\omega_{\pi''}}$, i.e., α_s and β_s respectively.

The case when $b_2 < b_1$ (so $\beta_2 > \beta_1$ and the switch from P_1 to Q_2 is possible) is similar just with the roles of P's and Q's as well as α's and β's swapped.

The upshot is that combining (5.2) and (5.3), we conclude that there exists a switching strategy in our family such that

(5.4) $$\frac{|X_0|}{|X_{q-1}|}\prod_{j=1}^{q-1}\frac{|X_j - x_j|}{|x_j|} \leq \frac{a_1 + b_1}{a_l^{\beta_l}b_l^{\alpha_l}}\left[\frac{\min(a_1, b_1)}{\min(a_l, b_l)}\right]^{1/8}.$$

Now consider two cases.

Case 1: $x_q \geq \Delta$. Then $a_l, b_l \geq \Delta$ and the right hand side of (5.4) is at most

$$\frac{2D}{\Delta}\left(\frac{D}{\Delta}\right)^{1/8} = 2\left(\frac{D}{\Delta}\right)^{9/8}.$$

Case 2: $x_q < \Delta$. Then $b_1 = b_l = x_q < \Delta$, so the second factor on the right hand side equals 1, while the first one can be estimated as

$$\frac{a_1 + b_1}{a_l^{\beta_l}b_l^{\alpha_l}} = \frac{a_l + b_l}{a_l^{\beta_l}b_l^{\alpha_l}}\frac{a_1 + b_1}{a_l + b_l} \leq 2\frac{a_1 + b_1}{a_l + b_l} \leq 2\frac{a_1}{a_l} \leq 2\frac{D}{\Delta}$$

(see Elementary Inequality 3).

Thus, in both cases, the right hand side of (5.4) does not exceed $2(D/\Delta)^{9/8}$. Combining this estimate with the previously obtained bound (5.1), we get the statement of the lemma.

6. Appendix: Elementary inequalities

Elementary Inequality 1. Let $a, b > 0, x \notin (-e^{-\lambda}a, e^{-\lambda}b)$. Then

$$\left[\frac{|x+a|}{|x|}\right]^{b/(a+b)} \left[\frac{|x-b|}{|x|}\right]^{a/(a+b)} \leq 1.$$

Since the statement does not change if we multiply x, a and b by any positive number, we can assume that $a + b = 1$. Our inequality then becomes

$$\left[\frac{|x+a|}{|x|}\right]^{b} \left[\frac{|x-b|}{|x|}\right]^{a} \leq 1.$$

If $x \notin [-a, b]$, we can just use the concavity of the function $t \mapsto \log t$ for $t > 0$ to write

$$\left[\frac{|x+a|}{|x|}\right]^{b} \left[\frac{|x-b|}{|x|}\right]^{a} \leq b\frac{|x+a|}{|x|} + a\frac{|x-b|}{|x|} = 1.$$

Suppose now that $x \in [-a, b] \setminus (-e^{-\lambda}a, e^{-\lambda}b) = [-a, -e^{-\lambda}a] \cup [e^{-\lambda}b, b]$.

Since the inequality does not change if we replace the triple (a, b, x) by $(b, a, -x)$, we can assume without loss of generality that $x \in [-a, -e^{-\lambda}a]$, i.e., $x = -ae^{-\Lambda}$ with $0 \leq \Lambda \leq \lambda$.

Then we want to prove that

$$\left[\frac{a(1 - e^{-\Lambda})}{ae^{-\Lambda}}\right]^{b} \left[\frac{b + ae^{-\Lambda}}{ae^{-\Lambda}}\right]^{a} = [e^{\Lambda} - 1]^{b} \left[1 + \frac{b}{a}e^{\Lambda}\right]^{a} \leq 1.$$

However, since $1 + t \leq e^{t}$ for every $t \geq 0$, this follows from

$$[e^{\Lambda} - 1]^{b} e^{(b/a)e^{\Lambda}a} = [(e^{\Lambda} - 1)e^{e^{\Lambda}}]^{b} \leq [(e^{\lambda} - 1)e^{e^{\lambda}}]^{b} = 1.$$

Elementary Inequality 2. Let $A, B > 0, 0 < a < A$. Then

$$\left[\frac{A - a}{a}\right]^{B/(A+B)} \left[\frac{B + a}{a}\right]^{A/(A+B)} \leq \frac{A}{a} \left[\frac{\min(A, B)}{\min(a, B)}\right]^{1/8}.$$

Since the statement does not change if we multiply A, B, a by any positive number, we can assume without loss of generality that $A = 1$. Multiplying both sides by a, we see that we need to prove that

$$(1 - a)^{B/(1+B)}(B + a)^{1/(1+B)} \leq \left[\frac{\min(1, B)}{\min(a, B)}\right]^{1/8}.$$

17

Since

$$(1-a)^{B/(1+B)}(1+aB)^{1/(1+B)} \leq \frac{B}{1+B}(1-a) + \frac{1}{1+B}(1+aB) = 1$$

by the concavity of logarithm, it is enough to show that

$$\left[\frac{B+a}{1+aB}\right]^{1/(1+B)} \leq \left[\frac{\min(1,B)}{\min(a,B)}\right]^{1/8}.$$

If $B \leq 1$, then $1 + aB - (B+a) = (1-B)(1-a) \geq 0$, so the left hand side is at most 1 and the right hand side is at least 1.

Otherwise $\min(1,B) = 1, \min(a,B) = a$ and we arrive at

$$\left[\frac{B+a}{1+aB}\right]^{1/(1+B)} \leq \left[\frac{1}{a}\right]^{1/8}, \quad 0 < a < 1, B > 1.$$

Put $a = 1 - t, 0 < t < 1$, and rewrite the inequality as

$$\frac{1}{1+B}\left[\log\left(1 - \frac{t}{B+1}\right) - \log\left(1 - \frac{Bt}{B+1}\right)\right] \leq \frac{1}{8}\log\frac{1}{1-t},$$

or

$$\frac{1}{1+B}\left[\varphi\left(\frac{Bt}{B+1}\right) - \varphi\left(\frac{t}{B+1}\right)\right] \leq \frac{1}{8}[\varphi(t) - \varphi(0)],$$

where $\varphi(t) = -\log(1-t)$. The left hand side is then $\frac{B-1}{(1+B)^2}t$ times the average of φ' over the interval $\left[\frac{t}{B+1}, \frac{Bt}{B+1}\right]$, while the right hand side is t times the average of φ' over the interval $[0, t]$. Since the intervals are concentric and $\varphi'(t) = 1/(1-t)$ is convex, the latter average is greater, so it suffices to show that

$$\frac{B-1}{(B+1)^2} \leq \frac{1}{8}.$$

However $(B+1)^2 = (B-1+2)^2 \geq (2\sqrt{2(B-1)})^2 = 8(B-1)$ and we are done.

Elementary Inequality 3. Let $a, b > 0$. Then

$$\frac{a+b}{a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}} \leq 2.$$

Again, the inequality is invariant under the multiplication of a, b by any positive number, so we can assume that $a + b = 1$. Then $b = 1 - a$ and we are to prove that

$$\frac{1}{a^a(1-a)^{1-a}} \leq 2.$$

However $a \mapsto a \log a + (1-a) \log(1-a)$ is a convex function on $(0, 1)$ symmetric around $a = 1/2$, so its minimum is attained at $1/2$ and equals $\log(1/2)$.

Elementary Inequality 4. The positive root λ of the equation $e^{e^\lambda}(e^\lambda - 1) = 1$ satisfies $\lambda > 1/5$.

Indeed, since the function is increasing, it suffices to prove that $e^{e^{1/5}}(e^{1/5} - 1) < 1$. Note that $e^{-1/5} > 1 - 1/5 = 4/5$, so $e^{1/5} < 5/4$. Thus, it is enough to show that $e^{5/4} < 4$. Since $e < 3$, we may check that $3^{5/4} < 4$ instead. However $3^5 = 243 < 256 = 4^4$ and the result follows.

References

[1] A. F. Beardon and Ch. Pommerenke, The Poincaré metric of plane domains, J. London Math. Soc., 18 (1978) 475-483.

[2] J.-P. Calvi and P. Van Manh, On the Lebesgue constant of Leja sequences for the unit disk and its applications to multivariate interpolation, JAT, 163 (2011) 608-622.

[3] L. Carleson and V. Totik, Holder continuity of Green's functions, Acta Sci. Math. (Szeged), 70 (2004) 557-608.

[4] I. Pritsker, Equidistribution of points via energy, Ark. Mat., 49 (2011) 149-173.

[5] L. Reichel, Newton interpolation at Leja points, BIT, 30 (1990) 332-346.

[6] R. Taylor and V. Totik, Lebesgue constants for Leja points, IMA Journal of Numerical Analysis, 30 (2010) 462-486.