SIGN INTERMIXING FOR RIESZ BASES AND FRAMES MEASURED IN THE KANTOROVICH–RUBINSTEIN NORM

NIKOLAI NIKOLSKI AND ALEXANDER VOLBERG

Abstract. We measure a sign interlacing phenomenon for Bessel sequences (u_k) in L^2 spaces in terms of the Kantorovich–Rubinstein mass moving norm $\|u_k\|_{KR}$. Our main observation shows that, quantitatively, the rate of the decreasing $\|u_k\|_{KR} \longrightarrow 0$ heavily depends on S. Bernstein n-widths of a compact of Lipschitz functions. In particular, it depends on the dimension of the measure space.

1. What this note is about.

Let (Ω, ρ) be a metric space, and m a finite continuous (with no point masses) Borel measure on Ω. It is known [NV2019] that for every *frame* $(u_k)_{k\geq 1}$ *in* $L^2_{\mathbb{R}}(\Omega, m)$, the "$l^2$-masses" of positive and negative values $u_k^{\pm}(x)$ are infinite:

$$\sum_k u_k^+(x)^2 = \sum_k u_k^-(x)^2 = \infty \text{ a.e. on } \Omega$$

(and moreover, $\forall f \in L^2_{\mathbb{R}}(\Omega)$, $f \neq 0 \Rightarrow \sum_k (f, u_k^{\pm})^2_{L^2} = \infty$), where as usual $u_k^{\pm}(x) = max(0, \pm u_k(x))$, $x \in (0,1)$. So, at almost every point $x \in \Omega$, there are many positive and many negative values $u_k(x)$. Here, we show that for a fixed k, positive and negative values are heavily intermixed.

Precisely, we show that the measures $u_k^{\pm} dm$ should be closely interlaced, in the sense that the Kantorovich-Rubinstein (KR) distances $\|u_k\|_{KR} = \|u_k^+ - u_k^-\|_{KR}$ (see below) must be small enough. It is easy to see that if the supports $\mathrm{supp}(u_k^{\pm})$ are distance separated from each other than $\|u_k\|_{KR} \approx \|u_k\|_{L^1(m)}$, whereas in reality, as we will see, these norms are much smaller, and so, the sets $\{x : u^+(x) > 0\}$ and $\{x : u^+(x) < 0\}$ should be increasingly mixed. In this connection, it is interesting to recall one of the first (and classical) results in this direction, that of O. Kellogg [Ke1916], showing that on the unit interval $\Omega = I =: (0,1)$, the consecutive supports $\mathrm{supp}(u_k^{\pm})$ are interlacing under quite general hypothesis on

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an orthonormal sequence (u_k). (Later on, the sign interlacing phenomena were intensively studied for (orthogonal) polynomials (starting from P. Chebyshev, and earlier, see any book on orthogonal polynomials), so that, quite a recent survey [Fi2008] counts about 780 pages and hundreds references; many new quantitative results are also presented).

Our results are most complete for the classical case $\Omega = I^d$ $(d \geq 1)$ in $\mathbb{R}^d$, $I = (0,1)$, and $m = m_d$ the Lebesgue measure and ρ the Euclidean distance on the cube. They also suggest that in general, the magnitudes of $\left\|u_k\right\|_{KR}$ are defined by certain (unknown) interrelations between m and ρ, and by a kind of the dimension of Ω. In fact, all depends on and is expressed in terms of a compact subset Lip_1 of Lipschitz functions in $L^2(\Omega, m)$.

Plan of the rest:

2. Definitions and comments
3. Statements on the generic behaviour of $\left\|u_k\right\|_{KR}$
4. Proofs
5. Further examples and comments; numerical examples to Theorem 3.2; direct comparisons $\left\|u_k\right\|_{KR}$ with Bernstein widths $b_k(Lip_1)$; an explicit expression for $\left\|u\right\|_{KR}$.
6. The fastest rates of decreasing $\left\|u_k\right\|_{KR} \searrow 0$ for frames/bases on $L^2(I^d)$.

Main results are Theorem 3.1, Theorem 3.2, and Theorem

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2. Definitions and comments

In order to simplify the statements, we *always assume that our sequences* $(u_k)_{k\geq 1}$ *(frames, bases, etc) lay in an one codimensional subspace*

$$L_0^2(\Omega, m) = \{f \in L_{\mathbb{R}}^2(\Omega, m) : \textstyle\int_\Omega f dm = 0\}.$$

The most of results below are still true for all *Bessel sequences* $u = (u_k)_{k\geq 1}$ in L_0^2, i.e. the sequences with

$$\sum_k \left|(f, u_k)\right|^2 \leq B(u)^2 \|f\|_2^2, \forall f \in L_0^2$$

where $B(u) > 0$ stands for the best possible constant in such inequality. Recall also that *a frame* (in L^2_0) is a sequence having

$$b\|f\|_2^2 \le \sum_k \Big|(f, u_k)\Big|^2 \le B\|f\|_2^2, \quad \forall f \in L^2_0,$$

with some constants $0 < b, B < \infty$, and *a Riesz basis* is (by definition) an isomorphic image of an orthonormal basis.

We always assume that *the space* (Ω, ρ) *is compact* (unless the contrary explicitly follows from the context) and *the measure* m *is finite and continuous (has no point masses).*

Below, $\|u\|_{KR}$ *stands for the Kantorovich–Rubinstein* (also called *Wasserstein*) *norm (KR)* of a zero mean ($\int udm = 0$) signed measure udm; that norm evaluates the work needed to transport the positive mass u^+dm into the negative one u^-dm. In fact, the KR distance $d(u_k^+dx, u_k^-dx)$ between measures $u_k^{\pm}dx$ (first invented by L. Kantorovich as early as in 1942, see [K1942]) is a partial case of a more general setting. Namely, given nonnegative measures μ, ν on Ω of an equal total mass, $\mu(\Omega) = \nu(\Omega)$, the KR-distance $d(\mu, \nu)$ is defined as the optimal "transfer plan" of the mass distribution μ to the mass distribution ν:

$$d(\mu, \nu) = \inf \Big\{ \int_{\Omega \times \Omega} \rho(x, y) d\psi(x, y) : \psi \in \Psi(\mu, \nu) \Big\},$$

where the family $\Psi(\mu, \nu)$ consists of all "admissible transfer plans" ψ, i.e. nonnegative measures on $\Omega \times \Omega$ satisfying the balance (marginal) conditions $\psi(\Omega \times \sigma) - \psi(\sigma \times \Omega) = (\mu - \nu)(\sigma)$ for every $\sigma \subset \Omega$ (the value $\psi(\sigma \times \sigma')$ has the meaning of how many mass is supposed to transfer from σ to σ'). The KR-norm of a real (signed) measure $\mu = \mu_+ - \mu_-$, $\mu(\Omega) = 0$, is defined as

$$\|\mu\|_{KR} = d(\mu_+, \mu_-).$$

It is shown in Kantorovich–Rubinstein theory (see, for example [KR1957] or [KA1977], Ch.VIII, §4) that the KR-norm of a real (signed) measure $\mu, \mu(\Omega) = 0$, is the dual norm of the Lipschitz space

$$\mathrm{Lip} := \mathrm{Lip}(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} : |f(x) - f(y)| \le c\rho(x, y)\}$$

modulo the constants, where the least possible constant c defines the norm $\mathrm{Lip}(f)$. Namely,

$$\|\mu\|_{KR} = d(\mu_+, \mu_-) = \sup \Big\{ \int_I f d\mu : \mathrm{Lip}(f) \le 1 \Big\},$$

where, in fact, it suffices to test only functions $f \in \mathrm{lip}$, $\mathrm{lip} := \{f \in \mathrm{Lip} : |f(x) - f(y)| = o(\rho(x, y))$ as $\rho(x, y) \longrightarrow 0\}$. Of course, one can extend the above definition to an arbitrary real valued measure μ setting $\|\mu\| = \|\mu - \mu(\Omega)\|_{KR} + |\mu(\Omega)|$. It makes possible to apply our results to $L^2_{\mathbb{R}}$ spaces instead of $L^2_{\mathbb{R},0}$ (using that in the case of Bessel sequences, the sequence $\int_\Omega u_k dm = (1, u_k)$ is in l^2). The KR-norm and its variations (with various cost function $h(x,y)$ instead of the distance $\rho(x,y)$) are largely used in the Monge/Kantorovich transportation problems, in ergodic theory, etc. We refer to [KA1977] for a basic exposition and references, and to [BK2012], [BKP2017] for extensive and very useful surveys of the actual state of the fields.

It is clear from the above definitions that, for measuring the sign intermixing of $u_k dm$ for a Bessel sequence $(u_k) \subset L^2_0$, one can employ certain *size characteristics* of the following compact subset of $L^2(\Omega, m)$,

$$\mathrm{Lip}_1 = \Big\{ f : \Omega \longrightarrow \mathbb{R} : \Big| f(x) - f(y) \Big| \leq \rho(x,y), f(x_0) = 0 \Big\},$$

where $x_0 \in \Omega$ stands for a fixed point of Ω (it will be easily seen that the choice of x_0 does not matter). Below, we do that making use of the known Bernstein width numbers $b_n(\mathrm{Lip}_1)$, or - in the case when there exists a linear Hilbert space operator T for which Lip_1 is the range of the unit ball - simply the singular numbers $s_n(T)$.

Namely, S.Bernstein n-widths $b_n(A, X)$ of a (compact) subset $A \subset X$ (convex, closed and centrally symmetric) of a Banach space X are defined as follows (see [Pi1985]):

$$b_n(A, X) = \sup_{X_{n+1}} \sup \Big\{ \lambda : \lambda B(X_{n+1}) \subset A, \lambda \geq 0 \Big\},$$

where X_{n+1} runs over all linear subspaces in X of $dim X_{n+1} = n+1$, and $B(X_{n+1})$ stands for the closed unit ball of X_{n+1}. A subspace $X_{n+1}(A)$ where $\sup_{X_{n+1}}$ is attained, is called optimal; it does not need to be unique (in general). In the case of a Hilbert space H (as everywhere below), if A is the image of the unit ball with respect to a linear (compact) operator T, $A = TB(H)$, we have $b_n(A, H) = s_n(T)$, where $s_k(T) \searrow 0$ $(k = 0, 1, ...)$ stands for the k-th singular number of T; optimal subspaces $H_{n+1}(T)$ are simply the linear hulls of $y_0, ..., y_n$ from the Schmidt decomposition of T,

$$T = \sum_{k \geq 0} s_k(T)(\cdot, x_k) y_k,$$

(x_k) and (y_k) being orthonormal sequences in H.

3. Statements

Recall that (Ω, ρ) stands for a compact metric space (unless the other is claimed explicitly), and m is a finite Borel measure on Ω *having no point masses* (for convenience normalized to 1).

Lemma 1 below shows what kind of the intermixing of signs we have for free, for every Bessel sequence (u_k). Lemma 2 shows that in no cases, one can have an intermixing better than l^2 smallness of $\|u_k\|_{KR}$. All intermediate cases can occur, following the widths properties of the compact $\mathrm{Lip}_1 \subset L^2(\Omega, m)$, see Theorems 3.1,3.2 and the comments below.

Lemma 1. *For every Bessel sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(\Omega, m)$, we have*

$$\lim_k \|u_k\|_{KR} = 0.$$

Lemma 2. *For every compact measure triple (Ω, ρ, m) (with the above conditions) and every sequence $(\epsilon_k)_{k\geq 1}$, $\epsilon_k \geq 0$, such that $\sum_k \epsilon_k^2 < \infty$, there exists an orthonormal sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(\Omega, m)$ satisfying*

$$\|u_k\|_{KR} \geq c\epsilon_k, k = 1, 2, ... \ (c > 0).$$

In particular, there exists an orthonormal sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(\Omega, m)$ such that

$$\sum_k \|u_k\|_{KR}^{2-\epsilon} = \infty, \forall \epsilon > 0.$$

Lemma 3. *For every sequence $(\epsilon_k)_{k\geq 1}$, $\epsilon_k > 0$, with $\lim_k \epsilon_k = 0$, there exists a compact measure triple (Ω, ρ, m) (with the above conditions) and an orthonormal sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(\Omega, m)$ such that*

$$\|u_k\|_{KR} = c\epsilon_k, k = 1, 2, ... \ (\tfrac{1}{2\sqrt{2}} \leq c \leq \tfrac{2\sqrt{2}}{\pi}).$$

Theorem 3.1. *(1) Given a Bessel sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(I, dx)$, $I = (0, 1)$, we have*

$$\sum_k \|u_k\|_{KR}^2 < \infty.$$

(2) Given a Bessel sequence $(u_k)_{k\geq 1}$ in $L^2_{\mathbb{R}}(I^d, dx)$, $d = 2, 3, ...$, we have

$$\sum_k \|u_k\|_{KR}^{d+\epsilon} < \infty, \ \forall \epsilon > 0.$$

(3) For the Sin orthonormal sequence $(u_n)_{n\in 2\mathbb{N}^d}$ in $L^2_{\mathbb{R}}(I^d, dx)$,

$$u_n(x) = 2^{d/2} Sin(\pi n_1 x_1) Sin(\pi n_2 x_2) ... Sin(\pi n_d x_d) \ (n = (n_1, ..., n_d) \in (2\mathbb{N})^d),$$

we have

$$\sum_n \|u_n\|_{KR}^d = \infty.$$

Remark. For a generic Bessel sequence (or, an orthonormal sequence), the l^2-convergence property (1) is a best possible result (see Lemma 2). However, for certain specific sequences, *(1) can be much sharpen.* For example, let $u \in L^2_{\mathbb{R},0}(\mathbb{T})$ and

$$u_n(\zeta) = u(\zeta^n),\ n = 1, 2, ...$$

Then, as it easy to see,

$$\|u_n\|_{KR} \leq \tfrac{1}{n}\|u\|_{KR}$$

(in fact, there is an equality), and so $\sum\limits_n \|u_n\|_{KR}^{1+\epsilon} < \infty$ $(\forall \epsilon > 0)$. Such a dilated sequence $(u_n)_n$ is Bessel if, and only if, the *Bohr transform* of u, $Bu(\zeta) = \sum_n \hat{u}(n)\zeta^{\alpha(n)}$, $\zeta^\alpha = \zeta_1^{\alpha_1}\zeta^{\alpha_2}...$ $(n = 2^{\alpha_1}3^{\alpha_2}...$ stands for for Euclid prime representation of $n \in \mathbb{N}$) is bounded on the multitorus $\zeta = (\zeta_1, \zeta_2, ...) \in \mathbb{T}^\infty$, see for instance [Ni2017].

In fact, Theorem 3.1, is an immediate corollary of the next Theorem 3.2. We extend the property $(\|u_k\|_{KR}) \in l^2$ to any "one dimensional smooth manifolds", see Proposition 5.1 for the exact statement. Lemma 2 shows that this condition describe the fastest decrease of the KR-norms for a generic Bessel sequence. On the spaces Ω, ρ of "higher dimensions" the property fails.

In Theorem 3.2, we develop the approach mentioned at the end of Section 2: we compare the compact set Lip_1 with the T-range $T(B(L^2))$of the unit ball for an appropriate compact operator T. For a direct comparison $\|u_n\|_{KR}$ with Bernstein numbers $b_n(\mathrm{Lip}_1)$ see Section 5 below.

Theorem 3.2. *Let T be compact linear operator*

$$T : L^2_{\mathbb{R}}(\Omega, m) \longrightarrow L^2_{\mathbb{R}}(\Omega, m),$$

and $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be a continuous increasing function on $[0, \infty)$ whose inverse φ^{-1} satisfies

$$\varphi^{-1}(x) = x^{1/2} r(1/x^{-1/2}) \quad \forall x > 0$$

with a concave (or, pseudo-concave) function $x \longmapsto r(x)$ on $(0, \infty)$.

(1) If $\mathrm{Lip}_1 \subset T(B(L^2_{\mathbb{R}}(\Omega, m)))$ and $\sum_k \varphi(s_k(T)) < \infty$, then, for every Bessel sequence $(u_k) \subset L^2_{\mathbb{R}}(\Omega, m)$,

$$\sum_{k\geq 1} \varphi(a\|u_k\|_{KR}) < \infty \text{ (for a suitable } a > 0).$$

(2) If $\mathrm{Lip}_1 \supset T(B(L^2_{\mathbb{R}}(\Omega, m)))$*, then there exists an orthonormal sequence* $(u_k)_{k\geq 0} \subset L^2_{\mathbb{R}}(\Omega, m)$*, such that*

$$\|u_k\|_{KR} \geq s_k(T),\ k = 0, 1, \ldots$$

In particular (in order to compare with (1)), $\sum_k h(\|u_k\|_{KR}) = \infty$ *for every* h *for which* $\sum_k h(s_k(T)) = \infty$.

Remark. See Section 5.III below for a version of Theorem 3.2, point (2), employing the Bernstein widths $b_n(\mathrm{Lip}_1)$ instead of $s_n(T)$ (T does not need to exist for the compact set Lip_1).

Corollary. *Let* $\mathrm{Lip}_1 = T(B(L^2_{\mathbb{R}}(\Omega, m)))$ *and* $p(T) := \inf\{\alpha : \sum_k s_k(T)^\alpha < \infty\}$.

(1) If $p(T) < 2$*, then* $\sum_k \|u_k\|^2 < \infty$*, for every Bessel sequence* $(u_k) \subset L^2_{\mathbb{R}}(\Omega, m)$*. On the other hand, there exists* T *with* $p(T) = 1$ *and an orthonormal sequence such that* $\sum_k \|u_k\|_{KR}^{2-\epsilon} = \infty$ $(\forall \epsilon > 0)$ *(see Lemma 2 above)*

(2) If $\sum_k s_k(T)^p < \infty$*,* $p \geq 2$*, then* $\sum_k \|u_k\|^p_{KR} < \infty$ *for every Bessel sequence* $(u_k) \subset L^2_{\mathbb{R}}(\Omega, m)$.

Remark. As we will see, Theorem 3.1, in fact, is a consequence of the last Corollary. Some concrete examples to Theorem 3.2 are presented below, in Section 5.

4. Proofs

I. Proof of Lemma 1. Since $(u_k)_{k\geq 1}$ is a Bessel sequence, it tends weakly to zero: $(u_k, f) \longrightarrow 0$ as $k \longrightarrow \infty$, for every $f \in L^2_{\mathbb{R}}(\Omega, m)$. On a (pre)compact set $f \in \mathrm{Lip}_1$, the limit is uniform:

$$\lim_k \|u_k\|_{KR} = \lim_k \sup \left\{ \int_\Omega u_k f d\mu : f \in \mathrm{Lip}_1 \right\} = 0\,.$$

II. Proof of Lemma 2. The Borel measure m being continuous satisfies the Menger property: the values mE, $E \subset \Omega$ fill in an interval $[0, m(\Omega)]$; if m is normalized - the interval $[0, 1]$ (see [Ha1950], §41 (with many retrospective references, the oldest one is to K.Menger, 1928), and for a complete and short proof [DN2011], Prop. A1, p.645). Below, we use that property many times.

Let $E_i \subset \Omega$ be disjoint Borel sets, $E_1 \bigcap E_2 = \emptyset$, $mE_i = 1/2$, and further, $K_i \subset E_i$ be compacts such that $mK_i = 1/3$ $(i = 1, 2)$. Denote $\delta = dist(K_1, K_2) > 0$, and set

$$f(x) = (1 - \tfrac{2}{\delta} dist(x, K_1))^+ - (1 - \tfrac{2}{\delta} dist(x, K_2))^+,\ x \in \Omega.$$

Then, $f \in \mathrm{Lip}(\Omega, \rho)$, $\mathrm{Lip}(f) \leq 2/\delta$ and $f(x) = 1$ for $x \in K_1$, $f(x) = -1$ for $x \in K_2$.

Now, using the Menger property, one can find two sequences (Δ_k^1), (Δ_k^2), $k = 1, 2, ...$, of pairwise disjoint sets such that $\Delta_k^i \subset K_i$, $\Delta_k^i \bigcap \Delta_j^i = \emptyset$ $(i = 1, 2,\ k \neq j)$, and $m\Delta_k^1 = m\Delta_k^2 = a^2\epsilon_k^2$, where $a > 0$ is chosen in such a way that $a^2 \sum_{k\geq 1} \epsilon_k^2 \leq 1/3$. Setting

$$u_k = c_k(\chi_{\Delta_k^1} - \chi_{\Delta_k^2}), \quad k = 1, 2, ...,$$

with $\|u_k\|_2^2 = 2c_k^2 m\Delta_k^1 = 1$, we obtain an orthonormal sequence $(u_k) \subset L^2(\Omega, m)$ such that

$$\|u_k\|_{KR} \geq \int_\Omega u_k(\frac{\delta}{2}f)dm = \frac{\delta}{2}2c_k m\Delta_k^1 = \frac{\delta}{\sqrt{2}}\sqrt{m\Delta_k^1} = \frac{\delta a}{\sqrt{2}}\epsilon_k.$$

III. Proof of Lemma 3. Let $\Omega = \mathbb{T}^\infty$, the infinite topological product of compact abelian groups $\mathbb{T} \times \mathbb{T} \times ...$, endowed with its normalized Haar measure $m_\infty = m \times m \times ....$ The product topology on Ω is metrizable by a variety of metrics, we choose $\rho = \rho_\epsilon$, $\epsilon = (\epsilon_k)_{k\geq 1}$ defined by

$$\rho_\epsilon(\zeta, \zeta') = max_{k\geq 1}\epsilon_k|\zeta_k - \zeta_k'|,\ \zeta', \zeta = (\zeta_k)_{k\geq 1} \in \mathbb{T}^\infty.$$

Setting

$$u_k(\zeta) = \sqrt{2}Re(\zeta_k), \quad \zeta \in \mathbb{T}^\infty,$$

we define an orthonormal sequence in $L^2(\mathbb{T}^\infty, m_\infty)$ with $|u_k(\zeta) - u_k(\zeta')| \leq \frac{\sqrt{2}}{\epsilon_k}\rho(\zeta, \zeta')$, and so $\mathrm{Lip}(u_k) \leq \sqrt{2}/\epsilon_k$.

Further, we need the following notation: let $f \in \mathrm{Lip}_1(\mathbb{T}^\infty)$, $f(\zeta) = f(\zeta_k, \overline{\zeta})$ where $\zeta = (\zeta_k, \overline{\zeta}) \in \mathbb{T}^\infty = \mathbb{T} \times \mathbb{T}^\infty$, $\overline{\zeta}$ consists of variables different from ζ_k, and

$$\overline{u}_k(\zeta_k) = \sqrt{2}Re(\zeta_k), \quad \zeta_k \in \mathbb{T},$$

(in fact, this is one and the same function $e^{i\theta} \longmapsto \sqrt{2}Cos(\theta)$ for every k). Finally, we set $\overline{f}(\zeta_k) := \int_{\mathbb{T}^\infty} f(\zeta_k, \overline{\zeta})dm_\infty(\overline{\zeta})$ and observe that $\mathrm{Lip}(\overline{f}) \leq \epsilon_k$:

$$\left|\overline{f}(\zeta_k) - \overline{f}(\zeta_k')\right| \leq \int_{\mathbb{T}^\infty} \left|f(\zeta_k, \overline{\zeta}) - f(\zeta_k', \overline{\zeta})\right| dm_\infty(\overline{\zeta}) \leq$$
$$\leq \int_{\mathbb{T}^\infty} \epsilon_k\left|\zeta_k - \zeta_k'\right| dm_\infty(\overline{\zeta}) = \epsilon_k\left|\zeta_k - \zeta_k'\right|.$$

Now,

$$\int_{\mathbb{T}^\infty} u_k(\zeta)f(\zeta_k, \overline{\zeta})dm_\infty(\zeta) = \int_{\mathbb{T}} \overline{u}_k(\zeta_k)\int_{\mathbb{T}^\infty} f(\zeta_k, \overline{\zeta})dm_\infty(\overline{\zeta})dm(\zeta_k) =$$
$$= \int_{\mathbb{T}} \overline{u}_k(\zeta_k)\overline{f}(\zeta_k)dm(\zeta_k) \leq \epsilon_k\|\overline{u}_k\|_{KR(\mathbb{T})},$$

and hence $\|u_k\|_{KR(\mathbb{T}^\infty)} \leq \epsilon_k\|\overline{u}_k\|_{KR(\mathbb{T})}$.

Conversely, if $h \in \mathrm{Lip}_1(\mathbb{T})$ and $\underline{h}(\zeta) := h(\zeta_k)$ for $\zeta \in \mathbb{T}^\infty$, then $\left|\underline{h}(\zeta_k) - \underline{h}(\zeta'_k)\right| \leq \frac{1}{\epsilon_k}\rho(\zeta,\zeta')$, and so

$$\int_{\mathbb{T}} \overline{u}_k h dm(\zeta_k) = \int_{\mathbb{T}^\infty} dm_\infty(\overline{\zeta}) \int_{\mathbb{T}} \overline{u}_k(\zeta_k) h(\zeta_k) dm(\zeta_k) = \int_{\mathbb{T}^\infty} u_k(\zeta)\underline{h}(\zeta) dm_\infty(\zeta) \leq$$
$$\leq \frac{1}{\epsilon_k}\|u_k\|_{KR(\mathbb{T}^\infty)},$$

which entails $\|\overline{u}_k\|_{KR(\mathbb{T})} \leq \frac{1}{\epsilon_k}\|u_k\|_{KR(\mathbb{T}^\infty)}$. Finally, $\|u_k\|_{KR(\mathbb{T}^\infty)} = \epsilon_k \|\overline{u}_k\|_{KR(\mathbb{T})}$. Moreover, since $\mathrm{Lip}(\overline{u}_k) \leq \sqrt{2}$,

$$\frac{1}{2\sqrt{2}} = \int_{\mathbb{T}} \overline{u}_k(\overline{u}_k/\sqrt{2}) dm(\zeta_k) \leq \|\overline{u}_k\|_{KR(\mathbb{T})} \leq \|\overline{u}_k\|_{L^1(\mathbb{T})} = \frac{2\sqrt{2}}{\pi}.$$

Remark. For the same space $L^2(\mathbb{T}^\infty, \mathrm{m}_\infty)$, but with a *non-compact* (bounded) metric $\rho(\zeta,\zeta') = \sup_{k\geq 1}|\zeta_k - \zeta'_k|$, we have $\|\mathrm{u_k}\|_{\mathrm{KR}} \geq 1$ for $\mathrm{u_k}(\zeta) = \mathrm{Sin}\pi \mathrm{x_k}$, $\zeta = (\mathrm{e}^{\mathrm{ix_1}}, \mathrm{e}^{\mathrm{ix_2}}, ..., \mathrm{e}^{\mathrm{ix_k}}, ...) \in \mathbb{T}^\infty$, so that $(\|\mathrm{u_k}\|_{\mathrm{KR}})_{k\geq 1}$ does not tend to zero.

IV. Proof of Theorem 3.1. *(1)* Since $u_k \in L^2_{\mathbb{R},0}(I, dx)$, $\int_I u_k dx = 0$. Taking a smooth function f with $\mathrm{Lip}(f) \leq 1$ (which are dense in the unit ball of *lip*) and $v_k(x) = Ju_k(x) := \int_0^x u_k dx$, we get $v_k(0) = v_k(1) = 0$, and hence

$$\int_I f u_k dx = (f v_k)_0^1 - \int_I v_k f' dx = -\int_I v_k f' dx.$$

Making sup over all f with $|f'| \leq 1$, we obtain $\|u_k\|_{KR} = \|v_k\|_{L^1}$. But the mapping

$$J : L^2(I) \longrightarrow L^2(I)$$

is a Hilbert-Schmidt operator, and hence $\sum_k \|Ju_k\|^2_{L^2} < \infty$, and so $\sum_k \|u_k\|^2_{KR} = \sum_k \|Ju_k\|^2_{L^1} < \infty$.

The penultimate inequality is obvious if (u_k) is an orthonormal (or only Riesz) sequence, but is still true for every Bessel sequence $(u_k)_{k\geq 1}$. Indeed, taking an auxiliary orthonormal basis $(e_j)_{j\geq 1}$ in $L^2_{\mathbb{R}}(I, dx)$, we can write

$$\sum_k \|Ju_k\|^2_{L^2} = \sum_k \sum_j \left|(Ju_k, e_j)\right|^2 = \sum_j \sum_k \left|(u_k, J^* e_j)\right|^2 \leq$$
$$\leq \sum_j const \cdot \|J^* e_j\|^2 < \infty,$$

since the adjoint J^* is a Hilbert-Schmidt operator.

(2) This is a d-dimensional version of the previous reasoning. Anew, we use the dual formula for the KR norm,

$$\|u_k\|_{KR} = \sup\{\int_{I^d} f u_k dx : f \in C^\infty, \mathrm{Lip}(f) \le 1, \int f dx = 0\},$$

the last requirement does not matter since $\mathrm{Lip}(f) = \mathrm{Lip}(f + const)$. Notice that for $f \in C^\infty(I^d)$, $\mathrm{Lip}(f) \le 1 \Leftrightarrow |\nabla f(x)| \le 1$ $(x \in I^d)$, where ∇f stands for the gradient vector $\nabla f = (\frac{\partial f}{\partial x_j})_{1\le j\le d}$. Now, define a linear mapping on the set $\mathcal{P}_0$ of vector valued trigonometric polynomials of the form $\sum_{n\in\mathbb{Z}^d} c_n \nabla e^{i(n,\cdot)} \in L^2(I^d, \mathbb{C}^d)$ with the zero mean $(c_0 = 0)$ by the formula

$$A(\nabla e^{i(n,x)}) = \Big|n\Big| e^{i(n,x)},\ n \in \mathbb{Z}^d \backslash \{0\}.$$

It is clear that A extends to a unitary operator

$$A : clos_{L^2(I^d,\mathbb{C}^d)}(\nabla \mathcal{P}_0) \longrightarrow L_0^2(I^d).$$

Further, let $M : L_0^2(I^d) \longrightarrow L_0^2(I^d)$ be a (bounded) multiplier,

$$M(e^{i(n,x)}) = \frac{1}{|n|} e^{i(n,x)},\ n \in \mathbb{Z}^d \backslash \{0\},$$

and finally, $T(\nabla f) = f$, $f \in C_0^\infty(I^d)$. Then,

$$\int_{I^d} f u_k dx = \int_{I^d} (T(\nabla f)) u_k dx = \int_{I^d} \nabla f \cdot (T^* u_k) dx,$$

T^* being the adjoint between L^2 Hilbert spaces. It follows

$$\|u_k\|_{KR} \le \sup\Big\{ \int_{I^d} \nabla f (T^* u_k) dx : \Big|\nabla f\Big| \le 1 \Big\} \le \|T^* u_k\|_{L^1(I^d,\mathbb{C}^d)} \le$$

$$\le \|T^* u_k\|_{L^2(I^d,\mathbb{C}^d)}.$$

Moreover, $T = MA$, where A is unitary (between the corresponding spaces) and M in a Schatten-von Neumann class $\mathcal{S}_p$ for every p, $p > d$ (since M is diagonal and $\displaystyle\sum_{n\in\mathbb{Z}^d\backslash\{0\}} \frac{1}{|n|^p} < \infty \Leftrightarrow p > d$). Using the dual definition of the Bessel sequence as $\|\sum a_k u_k\|^2 \le c\,(\sum a_k^2)$ for every real finite sequence (a_k), we can write (u_k) as the image $u_k = Be_k$ of an orthonormal sequence (e_k) under a linear bounded map B. This gives

$$\|u_k\|_{KR} \le \|T^* B e_k\|_{L^2}.$$

For every $p > d$, this implies $\displaystyle\sum_k \|u_k\|_{KR}^p \le \sum_k \|T^* B e_k\|_{L^2}^p < \infty$ since $T^* B \in \mathcal{S}_p$ and $d \ge 2$ (see Remark below).

Remark. For the last property, see for example [GoKr1965]. Here is a simple explanation: given a linear bounded operator $S : H \longrightarrow K$ between two Hilbert spaces and an orthonormal sequence (e_k) in H, define a mapping $j : S \longrightarrow (Se_k)$; then, j is bounded as a map $\mathcal{S}_2 \longmapsto l^2(K)$ and as a map $\mathcal{S}_\infty \longmapsto c_0(K)$ (compact operators); by operator interpolation, $j : \mathcal{S}_p \longmapsto l^p(K)$ is also bounded for $2 < p < \infty$.

For $1 \leq p \leq 2$, the things go differently: the best summation property $\sum_k \|Se_k\|^\alpha < \infty$, which one can generally have for $S \in \mathcal{S}_p$, is only for $\alpha = 2$ (look at rank one operators $S = (\cdot, x)y$). This claim explains the strange behavior in exponent from $2 + \epsilon$ for dimension 2 to exactly 2 for dimension 1 (and not $1 + \epsilon$ as one would expect).

(3) We use anew the duality formula

$$\|u_n\|_{KR} = \sup\Big\{ \int_{I^d} f u_n d\mu : \operatorname{Lip}(f) \leq 1 \Big\}.$$

Taking $f = u_n / \operatorname{Lip}(u_n)$ we get $\|u_n\|_{KR} \geq 1/\operatorname{Lip}(u_n)$ where $\operatorname{Lip}(u_n) \leq max|\nabla u_n(x)| \leq 2^{d/2}|n|$, and so

$$\sum_n \|u_n\|_{KR}^d \geq 2^{-d^2/2} \sum_{n \in (2\mathbb{N})^d} \Big|n\Big|^{-d} = \infty.$$

V. Proof of Theorem 3.2. Let $T = \sum_{k\geq 0} s_k(T)(\cdot, x_k)y_k$ be the Schmidt decomposition of a compact operator T acting on a Hilbert space H, $s_k(T) \searrow 0$ being the singular numbers. Let further, $A : H \longrightarrow H$ be a bounded operator, and $(e_k)_{k\geq 0}$ an arbitrary (fixed) orthonormal basis. Given a sequence $\alpha = (\alpha_j)_{j\geq 0}$ of real numbers, $\alpha \in l^\infty$, define a bounded operator

$$T_\alpha = \sum_{k\geq 0} \alpha_k(\cdot, x_k)y_k,$$

and then a mapping

$$j : \alpha \longmapsto (T_\alpha^* A e_k)_{k\geq 0},$$

a H-vector valued sequence in $l^\infty(H)$.

We are using a (partial case of a) J. Gustavsson–J. Peetre interpolation theorem [GuP1977]for Orlicz spaces. Recall that, in the case of sequence spaces, an Orlicz space l^φ, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ = (0, \infty)$ is increasing, continuous, and meets the so-called Δ_2-condition $\varphi(2x) \leq C\varphi(x)$, $x \in \mathbb{R}_+$, is the vector space of real sequences $c = (c_k)$ satisfying $\sum_k \varphi(a|c_k|) < \infty$ for a suitable $a > 0$. Similarly, a vector valued Orlicz space consists of sequences $c = (c_k)$, $c_k \in H$ having $\sum_k \varphi(a\|c_k\|) < \infty$ for a suitable $a > 0$. We need the Hilbert space valued spaces only. The Gustavsson–Peetre interpolation theorem (theorem 9.1 in [GuP1977]) implies that if mappings $j : l^\infty \longrightarrow l^\infty(H)$ and $j : l^2 \longrightarrow l^2(H)$ are bounded, then

$$j : l^\varphi \longrightarrow l^\varphi(H)$$

is bounded whenever the measuring function φ satisfies the conditions given in Theorem 3.2.

(1) Now, in the notation and the assumptions of statement (1), the Bessel sequence (u_k) is of the form $u_k = Ae_k$, where A is a bounded operator and (e_k) an orthonormal sequence. It follows

$$\|u_k\|_{KR} = \sup_{f\in \mathrm{Lip}_1}\Big|(Ae_k, f)\Big| \le \sup_{f\in T(B(L^2))}\Big|(Ae_k, f)_{L^2}\Big| = \|T^*Ae_k\|_{L^2}.$$

For every $\alpha \in l^2$, $T_\alpha \in \mathcal{S}_2$ (Hilbert-Schmidt), and then $T^*_\alpha A \in \mathcal{S}_2$, and hence $j(\alpha) \in l^2(H)$. By Gustavsson–Peetre, $\alpha \in l^\varphi \Rightarrow j(\alpha) \in l^\varphi(H)$. Applying this with $\alpha = (s_k(T))$, we get $\sum_k \varphi(a\|u_k\|_{KR}) \le \sum_k \varphi(a\|T^*Ae_k\|) < \infty$ for a suitable $a > 0$.

(2) In the assumptions of (2), and with the Schmidt decomposition

$$T = \sum_{k\ge 0} s_k(T)(\cdot, x_k)y_k\,,$$

set $u_k = y_k$ $k \ge 0$. Then

$$\|u_k\|_{KR} = \sup_{f\in \mathrm{Lip}_1}\Big|(y_k, f)\Big| \ge \sup_{f\in T(B(L^2))}\Big|(y_k, f)\Big| = \|T^*y_k\|_2 = s_k(T).$$

5. Further examples and comments

I. Fastest and slowest rates of decreasing $\|u_k\|_{KR} \searrow 0$. Lemma 2 shows that, the KR-norms of a generic Bessel sequence don't have to be smaller than required by the condition $\sum_k \|u_k\|^2_{KR} < \infty$.

On the other hand, point (1) of Theorem 3.1 gives an example of (Ω, ρ, dx), where every Bessel sequence meets that property.

Now, we extend this result to measure spaces over (almost) arbitrary 1-dimensional "smooth manifold" of finite length, as follows.

As to the fastest possible decreasing of $\Big\|u_k\Big\|_{KR}$ for frames/bases, we treat the question in Section 6 below for the classical spaces $L^2(I^d)$.

Proposition 5.1. *Let $\varphi : I \longrightarrow X$ be a continuous injection of $I = [0, 1]$ in a normed space X differentiable a.e. (with respect to Lebesgue measure dx), and the distance on I* be *defined by*

$$\rho(x, y) = \|\varphi(x) - \varphi(y)\|_X\,,\ x, y \in I.$$

Let further, μ be a continuous (without point masses) probability measure on I, satisfying

$$\int_I d\mu(y)\int_y^1 \|\varphi'(x)\|_X dx =: C^2(\mu, \varphi) < \infty.$$

Then, every Bessel sequence $u = (u_k)$ in $L^2(\mu) =: L^2_0((I,\mu)$ fulfills

$$\sum_k \|u_k\|^2_{KR} \leq B^2 C(\mu,\varphi) < \infty,$$

where $B(u) > 0$ comes from the Bessel condition.

Proof. Following the proof of Theorem 3.1*(1)* and using that for $f \in C^\infty$,

$$\mathrm{Lip}(f) \leq 1 \Leftrightarrow |f(x) - f(y)| \leq \|\varphi(x) - \varphi(y)\| \Leftrightarrow |f'(x)| \leq \|\varphi'(x)\|_X \ (x \in I),$$

we obtain, for every $h \in L^2_0(\mu)$ and $J_\mu(h)(x) := \int_0^x h d\mu$,

$$\|h\|_{KR} = \sup\Big\{\int_I f\, hd\mu : f \in C^\infty, \mathrm{Lip}(f) \leq 1\Big\} =$$

$$= \sup\Big\{\int_I f' J_\mu(h)dx : \Big|f'(x)\Big| \leq \|\varphi'(x)\|_X\Big\} = \int_I \Big|J_\mu(h)\Big| \cdot \|\varphi'(x)\|_X dx \leq$$

$$\leq \|J_\mu(h)\|_{L^2(I,vdx)},$$

where $v(x) = \|\varphi'(x)\|_X$. A mapping $Th := J_\mu(h)$, $Th(x) := \int_I k(x,y)h(y)d\mu$ acting as $T : L^2(\mu) \longrightarrow L^2(I,vdx)$ is in the Hilbert-Schmidt class $\mathcal{S}_2$ if and only if

$$\|T\|_2^2 = \int\int_{I\times I} \Big|k(x,y)\Big|^2 d\mu(y)v(x)dx = \int_0^1 d\mu(y)\int_y^1 v(x)dx =: C^2(\mu,\varphi) < \infty.$$

If $u = (u_k)$ is Bessel (with $\sum_k |(h,u_k)|^2 \leq B(u)^2\|h\|^2$, $\forall h \in L^2_w$), and the last condition is fulfilled, then $u_k = Ae_k$ where (e_k) is orthonormal and $\|A\| \leq B(u)$, and hence

$$\sum_k \|u_k\|^2_{KR} \leq \sum_k \|(TA)e_k\|_2^2 \leq$$

$$\|TA\|_2^2 \leq \|T\|_2^2\|A\|^2 \leq B^2(u)C^2(\mu,\varphi).$$

Remark. In particular, the following (known?) formula appeared in the proof:

$$\|h\|_{KR} = \int_I \Big|J_\mu(h)\Big| \cdot \|\varphi'(x)\|_X dx;$$

see also comments below.

II. Examples of interpolation spaces appearing conspicuously in Theorem 3.2. Lemma 3 above suggests that all decreasing rates of $\|u_k\|_{KR}$ can really occur, and so all cases of convergence/divergence of $\sum_k \varphi(\|u_k\|_{KR})$ are different and non empty. The following partial cases are of interest.

(1) The most known interpolation space between l^2 and l^∞ is l^p, $2 < p < \infty$, which is included in Theorem 3.2 with

$$r(t) = t^{1-\frac{2}{p}};$$

it serves for the case of power-like decreasing of $b_n(\mathrm{Lip}_1)$, or $s_n(T)$ (if $\mathrm{Lip}_1 = T(B(L^2))$), and consequently of $\|u_n\|_{KR}$:

$$\log \tfrac{1}{s_n} \approx \log(n),\ n \longrightarrow \infty.$$

In particular, point (2) of Theorem 3.1 (where $\Omega = I^d$, $d \geq 2$) can be seen now as a partial case of Theorem 3.2 since, in the hypotheses of 3.1(2), $\mathrm{Lip}_1 = TB(L^\infty) \supset TB(L^2)$ and $T \in \bigcap_{p>d} \mathcal{S}_p(L^2 \longrightarrow L^2)$ (which was already observed in the proof of Theorem 3.1).

(2) The following spaces l^φ of slowly decreasing sequences (s_n) are conjectured to appear as s-numbers (or Bernstein n-widths) of Lip_1 for partial cases of the triples $\Omega = \mathbb{T}^\infty$, $\rho = \rho_\epsilon$, m_∞ described in the proof of Lemma 3 above:

- $\displaystyle\sum_n s_n^{C \log\log \frac{1}{s_n}} < \infty$ (corresponding to $\log \frac{1}{s_n} \approx \frac{\log(n)}{\log\log(n)}$; the case is included in Theorem 3.2 with

$$r(t) = t \cdot \exp\Big\{ - \frac{1}{C} \cdot \frac{\log(t^2)}{\log\log(t^2)}(1 + o(1)\Big\}, \text{ as } t \longrightarrow \infty$$

(follows from the known $b^{-1}(y) = \frac{y}{\log(y)}(1 + o(1))$ for $b(x) = x \cdot \log(x)$), which is eventually concave (since $t \longmapsto r(t) = o(t)$ for $t \longrightarrow \infty$ and lies in the Hardy fields, see [Bou1976], L'Appendice du Ch.V);

- $\displaystyle\sum_n s_n^{C(\log \frac{1}{s_n})^\alpha} < \infty$, $\alpha > 1$ (corresponding to $\log \frac{1}{s_n} \approx (\log(n))^{1/\alpha}$; the case is included in Theorem 3.2 with

$$r(t) = t \cdot exp\Big\{ - (\frac{1}{C} \cdot \log(t^2))^{1/\alpha}\Big\},$$

which is eventually concave as $t \longrightarrow \infty$ (by the same argument as above);

- $\displaystyle\sum_n e^{-\frac{C}{s_n^\beta}} < \infty$, $\beta > 0$ (corresponding to $\log \frac{1}{s_n} \approx (c + \frac{1}{\beta} \log\log(n))$; the case is included in Theorem 3.2 with

$$r(t) = Ct/(\log(t^2))^{1/\beta},$$

which is eventually concave as $t \longrightarrow \infty$ (by the same argument as above).

III. In terms of the Bernstein n-widths. It is quite easy to see that a part of Theorem 3.2, namely point (2), is still true with a (slightly?) relaxed hypothesis: we replace the assumption that Lip_1 is of the form $\mathrm{Lip}_1 \supset T(B(L^2))$ for a compact T with a hypothesis that the optimal subspaces for Bernstein widths $b_n(\mathrm{Lip}_1)$ are ordered by inclusion (see Section 2 above for the definitions): $H_n(\mathrm{Lip}_1) \subset H_{n+1}(\mathrm{Lip}_1)$, $n = 1, 2, ...$ Namely, the following property holds.

Proposition 5.2. *Let Ω, ρ, m be a compact probability triple for which there exist Bernstein optimal subspaces $H_n(\mathrm{Lip}_1) \subset L^2(\Omega, m)$ such that*

$$H_n(\mathrm{Lip}_1) \subset H_{n+1}(\mathrm{Lip}_1),\ n = 1, 2, ...$$

Then there exists an orthonormal sequence $(u_k)_{k\geq 0} \subset \mathrm{Lip}(\Omega) \subset L^2_{\mathbb{R}}(\Omega, m)$, such that

$$\|u_n\|_{KR} \geq b_n(\mathrm{Lip}_1),\ n = 1, 2, ...$$

Proof. Let $e_1 \in H_1$, $\|e_1\|_2 = b_1$, and assume that e_k, $k \leq n$ are chosen in a way that $e_k \in H_n$, $e_k \perp e_j$ $(k \neq j)$ and $\|e_k\|_2 = b_k$. Since $b_{n+1}B(H_{n+1}) \subset \mathrm{Lip}_1$, there exists a vector $e_{n+1} \in H_{n+1} \ominus H_n \subset \mathrm{Lip}(\Omega)$ with $\|e_{n+1}\|_2 = b_{n+1}$ (and hence, $e_{n+1} \in \mathrm{Lip}_1$). For the constructed sequence (e_n), we set

$$u_n = e_n/b_n$$

and obtain an orthonormal sequence $(u_n) \subset \mathrm{Lip}(\Omega)$ such that $\mathrm{Lip}(u_n) \leq 1/b_n$, and hence $\|u_n\|_{KR} \geq \int_\Omega u_n e_n dm = b_n(\mathrm{Lip}_1)$.

IV. Remark: an "uncertainty inequality" for $\|u\|_{KR}$. As it is already used several times (in particular in the proof of 5.2 above), for a smooth function $u \in \mathrm{Lip}(\Omega)$ the following inequality holds

$$\|u\|_{KR} \, \mathrm{Lip}(u) \geq \|u\|_2^2.$$

Indeed, $\|u\|_{KR} \geq \int_\Omega u(u/\mathrm{Lip}(u))dm$.

As a consequence, one can observe that for every normalized Bessel sequence (u_k), its Lip norms must be sufficiently large, so that $\sum_k \varphi(\frac{1}{\mathrm{Lip}(u_k)}) < \infty$ for any monotone increasing function $\varphi \geq 0$ for which $\sum_k \varphi(\|u_k\|) < \infty$ (compare with the statements of Section 3).

V. Remark: an explicit formula for $\|u\|_{KR}$. There are some cases where the norm $\|\cdot\|_{KR}$ can be explicitly expressed in term of the triple Ω, ρ, m. In particular, if $\mathrm{Lip}_1 = T(B(L^\infty(\Omega, m))$ then

$$\|u\|_{KR} = \|T^*u\|_{L^1(\Omega,m)},\ \forall u \in L^1(\Omega, m).$$

Indeed,

$$\|u\|_{KR} = \sup\Big\{ \int_\Omega u f dm : f \in \mathrm{Lip}_1 \Big\} = \|T^*u\|_{L^1(\Omega,m)}.$$

In particular, such a formula holds for $(\Omega, m) = (I^d, m_d)$, as it is mentioned in the proof of Theorem 3.1 (the corresponding $T(\sum_{k\neq 0} c_k e^{i(k,x)}) = \sum_{k\neq 0} |k| c_k e^{i(k,x)}$ is a multiplier on L^p_0); for $d = 1$, the formula is mentioned in [Ver2004].

VI. Yet another characteristic of a compact set. The following compactness measure seems to be closely related to the estimates of $\|u_n\|_{KR}$:

$$t(n) = \sup\Big\{ r > 0 : \exists x_j \in \mathrm{Lip}_1, x_i \perp x_k (i \neq k), \|x_j\| \geq r, 1 \leq j \leq n \Big\},\ n \geq 1.$$

It is easy to see that $\sqrt{n} b_n(\mathrm{Lip}_1) \geq t(n) \geq b_n(\mathrm{Lip}_1)$, and in principle, we can use $t(n)$ instead of b_n in the proof of Proposition 5.2. We can also derive the existence of finite orthonormal sequences $(e_j)_{j=1}^n \subset \mathrm{Lip}(\Omega)$ such that $\sum_{j=1}^n \varphi(\|e_j\|_{KR}) \geq n\varphi(b_n(\mathrm{Lip}_1))$, $n = 1, 2, ...$

6. A summary, and the best KR-norms behavior for frames/bases in $L^2(I^d)$.

(A) A summary of the worst (generic) behavior of the KR-norms (all these claims are already proved above). For every Bessel sequence (u_k) in $L^2(I^d)$, we have for $d = 1$: $\sum_k \Big\|u_k\Big\|^2_{KR} < \infty$, and for $d > 1$: $\sum_k \Big\|u_k\Big\|^{d+\epsilon}_{KR} < \infty$, $\forall \epsilon > 0$.

These claims are sharp: for every compact triple (Ω,ρ,m) and for every sequence $(\epsilon_\mathrm{k})_{\mathrm{k}\geq 1}$, $\epsilon_\mathrm{k} \geq 0$, such that $\sum_\mathrm{k} \epsilon_\mathrm{k}^2 < \infty$, there exists an orthonormal sequence $(u_\mathrm{k})_{\mathrm{k}\geq 1}$ in $\mathrm{L}^2_{\mathbb{R}}(\Omega, \mathrm{m})$ such that $\Big\|u_k\Big\|_{KR} \geq c\epsilon_k$, $k = 1, 2, ...$ $(c > 0)$, and in $L^2(I^d)$ there exists an orthonormal sequence (u_k) such that $\sum_k \Big\|u_k\Big\|^d_{KR} = \infty$.

For a generic compact triple Ω,ρ,m, we can only claim $\lim_k \Big\|u_k\Big\|_{KR} = 0$ for every Bessel sequence in $\mathrm{L}^2_{\mathbb{R}}(\Omega, \mathrm{m})$. The property is sharp in the following sense: for every sequence $(\epsilon_\mathrm{k})_{\mathrm{k}\geq 1}$, $\epsilon_\mathrm{k} > 0$, with $\lim_\mathrm{k} \epsilon_\mathrm{k} = 0$, *there exists a compact triple* (Ω,ρ,m) (with usual properties) and an orthonormal sequence $(u_\mathrm{k})_{\mathrm{k}\geq 1}$ in $\mathrm{L}^2_{\mathbb{R}}(\Omega, \mathrm{m})$ such that $\Big\|u_k\Big\|_{KR} = c\epsilon_k$, $k = 1, 2, ...$ $(\frac{1}{2\sqrt{2}} \leq c \leq \frac{2\sqrt{2}}{\pi})$.

(B) Bases/frames with the least possible KR-norms. For the best possible behavior of $\Big\|u_k\Big\|_{KR}$ we replace the words "for every Bessel sequence" by the words "there exists Bessel sequence", meaning that we look for the fastest rate of decrease of $\{\Big\|u_k\Big\|_{KR}\}$. Then for bases/frames/Bessel sequences on $L^2(I^d)$, we have different summation properties, and for $d = 1$ the threshold is 2/3 (and not 2 as above), as follows.

Theorem 6.1. *Let $d = 1, 2, ...$ and $\alpha = \frac{2d}{d+2}$ ($\alpha < 2$). Then, (1) there exists an orthonormal basis (u_k) in $L^2(I^d)$ such that $\sum_k \Big\|u_k\Big\|_{KR}^{\alpha+\epsilon} < \infty$, $\forall\epsilon > 0$, but (2) $\sum_k \Big\|u_k\Big\|_{KR}^{\alpha} = \infty$, for every frame (u_k) in $L^2(I^d)$ (in particular, for every Riesz basis).*

Let (u_n) be the Haar basis in $L_0^2(I^d)$ enumerated with the following notation:

$$h = \chi_{(0,1/2)} - \chi_{(1/2,1)}$$

stands for the *Haar basic wavelet* on $I \subset \mathbb{R}$; taking a subset $\sigma \subset D := \{1, 2, ..., d\}$, $\sigma \neq \emptyset$, and a multiindex $k = (k_1, k_2, ..., k_d) \in \mathbb{Z}_+^d$, where $0 \leq k_s < 2^j$ for every s and $j \in \mathbb{Z}_+$, define *the Haar functions* $(u_n) := (h_{j,k,\sigma})$ as

$$h_{j,k,\sigma}(x) = 2^{dj/2} \prod_{s\in\sigma} h(2^j x_s - k_s) \prod_{s\in D\setminus\sigma} \chi_{(0,1)}(2^j x_s - k_s),$$

where $x = (x_1, x_2, .., x_d) \in I^d$. Then (see for example, [Me1992], Section 3.9), (u_n) forms an orthonormal basis in $L_0^2(I^d)$ (j and k run over all mentioned above values, σ runs a finite set of $2^d - 1$ elements). Obviously,

$$\text{supp}(h_{j,k,\sigma}) = Q_{j,k} := \{x \in \mathbb{R}^d : 2^j x - k \in I^d\} = \textstyle\prod_{s=1}^d [k_s 2^{-j}, (k_s + 1)2^{-j}].$$

Lemma. Let $u \in L^\infty(I^d)$, $\text{supp}(u) \subset Q_{j,k}$ and $\int_{I^d} u dx = 0$. Then,

$$\Big\|u\Big\|_{KR} \leq \tfrac{d}{2}\|u\|_\infty 2^{-(d+1)j}.$$

Proof Since $\int_{I^d} u dx = 0$, we can restrict ourselves in the formula

$$\Big\|u\Big\|_{KR} = \sup\Big\{\int_I uf dx : Lip(f) \leq 1\Big\}$$

to the functions f with $f(l) = 0$, $Lip(f) \leq 1$ where $l = (k_s 2^{-j})_{s=1}^d$, and so $|f(x)| \leq |l - x|$, $x \in Q_{j,k}$. Changing variables, we have

$$\Big\|u\Big\|_{KR} \leq \int_{Q_{j,0}} \Big\|u\Big\|_\infty \Big|x\Big| dx \leq \int_{Q_{j,0}} \Big\|u\Big\|_\infty \sum_{s=1}^d x_s dx =$$
$$= \Big\|u\Big\|_\infty \frac{d}{2} 2^{-2j} 2^{-j(d-1)} = \Big\|u\Big\|_\infty \frac{d}{2} 2^{-j(d+1)}.$$

Proof of Theorem 6.1

(1) Applying Lemma to $u = h_{j,k,\sigma}$,

$$\Big\|h_{j,k,\sigma}\Big\|_{KR} \leq 2^{jd/2} \frac{d}{2} 2^{-j(d+1)}.$$

Summing up (with a $\gamma > \alpha$, $\alpha = \frac{2d}{d+2}$), we get

$$\sum_n \left\|u_n\right\|_{KR}^{\gamma} \leq \sum_\sigma \sum_{j\geq 0} \sum_k \left\|h_{j,k,\sigma}\right\|_{KR}^{\gamma} \leq \sum_\sigma \sum_{j\geq 0} 2^{jd}\Big(2^{jd/2}\frac{d}{2}2^{-j(d+1)}\Big)^{\gamma} < \infty.$$

(2) Recall that the space $L_0^1(I^d)$ endowed with the KR-norm is isometrically embedded into the dual space $(Lip_0)^*$ (with respect to the standard duality $(u,f) = \int_{I^d} ufdm$).

The plan of the proof (suggested by E. Gluskin) is the following: consider some metric properties of the embedding

$$E^* : L_0^2(I^d) \longrightarrow (Lip_0)^*$$

and its predual embedding

$$E : Lip_0 \longrightarrow L_0^2(I^d)$$

from two different points of view. Namely, assuming that there exists a frame (u_k) in $L_0^2(I^d)$ such that $\sum_k \left\|u_k\right\|_{KR}^{\alpha} < \infty$, we show that

(I) embeddings E, E^* are 2-nuclear operators (see below) and the 2-nuclear approximation numbers $a_N^{(2)}(E^*)$ decrease as $o(1/N^{1/d})$ when $N \longrightarrow \infty$;

(II) on the other hand, one can see that - at least for $N = 2^{jd}$, $j = 1, 2, ...$ - the numbers $a_N^{(2)}(E)$ (which coincide with $a_N^{(2)}(E^*)$) cannot be less than $cN^{-1/d}$.

The above contradiction shows property (2) of Theorem 6.1.

Proof of point (I). A linear operator $T : X \longrightarrow Y$ between Banach spaces X and Y is said *p-nuclear* if $Tx = \sum_k T_k x$, $x \in X$ (weak convergence), $rank(T_k) \leq 1$ and $\sum_k \|T_k\|^p < \infty$; $\inf\Big\{\Big(\sum_k \left\|T_k\right\|^p\Big)^{1/p} : \textit{over all such representations}\Big\} =: \left\|T\right\|_{N(p)}$ is called its p-norm. *N-th p-nuclear approximation number of T* ($N = 1, 2, ...$) is

$$a_N^{(p)}(T) := \inf\Big\{\left\|T - A\right\|_{N(p)} : A : X \longrightarrow Y, rank(A) < N\Big\}.$$

Assume now that there exists a frame (u_k) in $L_0^2(I^d)$ such that $\sum_k \left\|u_k\right\|_{KR}^{\alpha} < \infty$ where $\alpha = \frac{2d}{d+2}$. Let $Sf = \sum_k (f, u_k)u_k$ be the frame operator on $L_0^2(I^d)$; S is an isomorphism $S : L_0^2(I^d) \longrightarrow L_0^2(I^d)$, and $E^*S : L_0^2(I^d) \longrightarrow (Lip_0)^*$ is a 2-nuclear operator,

$$E^*Sf = \textstyle\sum_{k\geq 1}(f, u_k)E^*u_k,$$

since $\|E^*u_k\|_{(Lip_0)^*} = \|u_k\|_{KR}$ and $\alpha < 2$. Moreover, letting (u_k) in the decreasing order of $\left\|u_k\right\|_{KR}$, we get $\left\|u_k\right\|_{KR}^{\alpha} = o(1/k)$ (as $k \longrightarrow \infty$), and hence

$$a_N^{(2)}(E^*S)^2 \leq \sum_{k\geq N} \left\|u_k\right\|_{KR}^2 \leq \left\|u_N\right\|_{KR}^{2-\alpha} \sum_{k\geq N} \left\|u_k\right\|_{KR}^{\alpha} = o(\frac{1}{N^{2/\alpha-1}}),$$

and $a_N^{(2)}(E^*S) = o(\frac{1}{N^{1/\alpha-1/2}}) = o(\frac{1}{N^{1/d}})$, as $N \longrightarrow \infty$ and $1/\alpha = 1/2 + 1/d$. Since S is invertible, and $\left\|UTV\right\|_{N(p)} \leq \|U\| \cdot \|T\|_{N(p)} \cdot \|V\|$ for every T, U, V, we have

$$a_N^{(2)}(E^*) = o(\frac{1}{N^{1/d}}), \text{ as } N \longrightarrow \infty.$$

Proof of point (II). (The proof was suggested by E. Gluskin). We need to show that there exists a constant $c > 0$ such that for every operator $A_N : Lip_0 \longrightarrow L_0^2(I^d)$, $rank(A_N) < N = 2^{jd}$ $(j = 1, 2, ...)$, one has $\|E - A_N\|_{N(2)} \geq cN^{-1/d}$. To this end, we construct two linear mappings $V = V_N : \mathbb{R}^N \longrightarrow Lip_0$ and $U = U_N : L_0^2(I^d) \longrightarrow \mathbb{R}^N$ such that

$$UEV = id_{\mathbb{R}^N},\ \|V : \mathbb{R}^N \longrightarrow Lip_0\| \leq CN^{\frac{1}{2}+\frac{1}{d}},\ \|U : L_0^2(I^d) \longrightarrow \mathbb{R}^N\| = 1,$$

where $C > 0$ does not depend on N.

Having these mappings at hand, we get $U_{2N}(E - A_N)V_{2N} = id_{\mathbb{R}^{2N}} - B_N$, where $rank(B_N) < N$ and so

$$\|U_{2N}(E - A_N)V_{2N}\|_{N(2)} = \|id_{\mathbb{R}^{2N}} - B_N\|_{N(2)} \geq N^{1/2},$$

and on the other hand,

$\|U_{2N}(E - A_N)V_{2N}\|_{N(2)} \leq \|U_{2N}\| \cdot \|E - A_N\|_{N(2)}\|V_{2N}\| \leq C(2N)^{\frac{1}{2}+\frac{1}{d}}\|E - A_N\|_{N(2)}$, which gives $\|E - A_N\|_{N(2)} \geq cN^{-1/d}$.

Construction of the mappings $V = V_N : \mathbb{R}^N \longrightarrow Lip_0$ and $U = U_N : L_0^2(I^d) \longrightarrow \mathbb{R}^N$, $N = 2^{jd}$, $j = 1, 2, ...$ We use the similar scaling procedure as in the above proof of part (1) of Theorem 6.1: let ψ be a smooth function on $\mathbb{R}^d$ such that $\mathrm{supp}(\psi) \subset Q_0 = I^d$, $\|\psi\|_{L^2(I^d)} = 1$, $\int_{I^d} \psi dm = 0$, and, for every $j \in \mathbb{Z}_+$,

$$\psi_k = \psi_{j,k}(x) := 2^{jd/2}\psi(2^j x - k),\ k \in K_j,$$

where $K_j = \{k = (k_1, ..., k_d) \in \mathbb{Z}_+^d\colon\ 0 \leq k_s < 2^j\ (1 \leq s \leq d)\}$. Then, ψ_k $(k \in K_j)$ have pairwise disjoint supports and form an orthonormal family in $L_0^2(I^d)$, $card(K_j) = 2^{jd} := N$. Now, setting

$$Va = \sum_{k\in K_j} a_k\psi_k,\ a \in \mathbb{R}^N,$$

we obtain

$$\Big\|Va\Big\|_{Lip} \leq c \cdot \sup_{x\in I^d} \Big|\nabla(Va)(x)\Big| = c \cdot \max_{k\in K_j} \sup_{x\in I^d} \Big|a_k \nabla\psi_k(x)\Big| \leq C2^{jd/2}2^j \Big\|a\Big\|_{\mathbb{R}^N},$$

where $c > 0$, $C > 0$ depend only on d (and the choice of ψ), which gives the needed $\|V : \mathbb{R}^N \longrightarrow Lip_0\| \leq CN^{\frac{1}{2}+\frac{1}{d}}$.

For $U = U_N : L^2_0(I^d) \longrightarrow \mathbb{R}^N$, we let $Uf = ((f, \psi_k))_{k\in K_j}$, and obviously get $UEV = id_{\mathbb{R}^N}$ and $\|U : L^2_0(I^d) \longrightarrow \mathbb{R}^N\| = 1$.

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Department of Mathematics,Universitè de Bordeaux, France
Email address: Nikolai.Nikolski@math.u-bordeaux.fr (N. Nikolski)

Department of Mathematics, Michigan Sate University, East Lansing, MI. 48823 USA and Hausdorff Center for Mathematics, Bonn 53115, Germany
Email address: volberg@math.msu.edu (A. Volberg)