ORTHOGONAL PROJECTORS ONTO SPACES OF PERIODIC SPLINES

MARKUS PASSENBRUNNER

Abstract. The main result of this paper is a proof that for any integrable function f on the torus, any sequence of its orthogonal projections $(\widetilde{P}_n f)$ onto periodic spline spaces with arbitrary knots $\widetilde{\Delta}_n$ and arbitrary polynomial degree converges to f almost everywhere with respect to the Lebesgue measure, provided the mesh diameter $|\widetilde{\Delta}_n|$ tends to zero. We also give a proof of the fact that the operators $\widetilde{P}_n$ are bounded on L^∞ independently of the knots $\widetilde{\Delta}_n$.

1. Introduction

1.1. **Splines on an interval.** In this article we prove some results about the periodic spline orthoprojector. In order to achieve this, we rely on existing results for the non-periodic spline orthoprojector on a compact interval, so we first describe some of those results for the latter operator. Let $k \in \mathbb{N}$ and $\Delta = (t_i)_{i=\ell}^{r+k}$ a knot sequence satisfying

$$t_i \le t_{i+1}, \qquad t_i < t_{i+k},$$
$$t_\ell = \cdots = t_{\ell+k-1}, \qquad t_{r+1} = \cdots = t_{r+k}.$$

Associated to this knot sequence, we define $(N_i)_{i=\ell}^r$ as the sequence of L^∞-normalized B-spline functions of order k on Δ that have the properties

$$\operatorname{supp} N_i = [t_i, t_{i+k}], \qquad N_i \ge 0, \qquad \sum_{i=\ell}^r N_i \equiv 1.$$

We write $|\Delta| = \max_{\ell \le j \le r}(t_{j+1} - t_j)$ for the maximal mesh width of the partition Δ. Then, define the space $\mathcal{S}_k(\Delta)$ as the set of polynomial splines of order k (or at most degree $k-1$) with knots Δ, which is the linear span of the B-spline functions $(N_i)_{i=\ell}^r$. Moreover, let P_Δ be the orthogonal projection operator onto the space $\mathcal{S}_k(\Delta)$ with respect to the ordinary (real) inner product $\langle f, g\rangle = \int_{t_\ell}^{t_{r+1}} f(x)g(x)\,\mathrm{d}x$, i.e.,

$$\langle P_\Delta f, s\rangle = \langle f, s\rangle \qquad \text{for all } s \in \mathcal{S}_k(\Delta).$$

The operator P_Δ is also given by the formula

$$P_\Delta f = \sum_{i=\ell}^r \langle f, N_i\rangle N_i^*, \tag{1.1}$$

Date: May 11, 2018.
2010 *Mathematics Subject Classification.* 40A05, 41A15, 46E30.
Key words and phrases. Periodic splines, Almost everywhere convergence.
The author is supported by the FWF, project number P27723.

where $(N_i^*)_{i=\ell}^r$ denotes the dual basis to (N_i) defined by the relations $\langle N_i^*, N_j\rangle = 0$ when $j \neq i$ and $\langle N_i^*, N_i\rangle = 1$ for all $i = \ell, \ldots, r$. A famous theorem by A. Shadrin states that the L^∞ norm of this projection operator is bounded independently of the knot sequence Δ:

Theorem 1.1 ([8])**.** *There exists a constant c_k depending only on the spline order k such that for all knot sequences $\Delta = (t_i)_{i=\ell}^{r+k}$ as above,*

$$\|P_\Delta : L^\infty[t_\ell, t_{r+1}] \to L^\infty[t_\ell, t_{r+1}]\| \leq c_k.$$

We are also interested in the following equivalent formulation of this theorem, which is proved in [1]: for a knot sequence Δ, let (a_{ij}) be the matrix $(\langle N_i^*, N_j^*\rangle)$, which is the inverse of the banded matrix $(\langle N_i, N_j\rangle)$. Then, the assertion of Theorem 1.1 is equivalent to the existence of two constants $K_0 > 0$ and $\gamma_0 \in (0,1)$ only depending on the spline order k such that

$$|a_{ij}| \leq \frac{K_0 \gamma_0^{|i-j|}}{\max\{\kappa_i, \kappa_j\}}, \qquad \ell \leq i, j \leq r, \tag{1.2}$$

where κ_i denotes the length of $\operatorname{supp} N_i$. The proof of this equivalence uses Demko's theorem [4] on the geometric decay of inverses of band matrices and de Boor's stability (see [2] or [5, Chapter 5, Theorem 4.2]) which states that for $0 < p \leq \infty$, the L^p norm of a B-spline series is equivalent to a weighted ℓ^p norm of its coefficients, i.e. there exists a constant D_k only depending on the spline order k such that:

$$D_k k^{-1/p} \Big(\sum_j |c_j|^p \kappa_j\Big)^{1/p} \leq \Big\| \sum_j c_j N_j \Big\|_{L^p} \leq \Big(\sum_j |c_j|^p \kappa_j\Big)^{1/p}.$$

In fact, for a_{ij}, we actually have the following improvement of (1.2) (see [6]): There exist two constants $K > 0$ and $\gamma \in (0,1)$ that depend only on the spline order k such that

$$|a_{ij}| \leq \frac{K \gamma^{|i-j|}}{h_{ij}}, \qquad \ell \leq i, j \leq r, \tag{1.3}$$

where h_{ij} denotes the length of the convex hull of $\operatorname{supp} N_i \cup \operatorname{supp} N_j$. This inequality can be used to obtain almost everywhere convergence for spline projections of L^1 functions:

Theorem 1.2 ([6])**.** *For all $f \in L^1[t_\ell, t_{r+1}]$ there exists a subset $A \subset [t_\ell, t_{r+1}]$ of full Lebesgue measure such that for all sequences (Δ_n) of partitions of $[t_\ell, t_{r+1}]$ such that $|\Delta_n| \to 0$, we have*

$$\lim_{n\to\infty} P_{\Delta_n} f(x) = f(x), \qquad x \in A.$$

Our aim in this article is to prove an analogue of Theorem 1.2 for orthoprojectors on periodic spline spaces. In this case, we do not have a periodic version of (1.3) at our disposal, since the proof of this inequality does not carry over to the periodic setting. However, by comparing orthogonal projections onto periodic spline spaces to suitable non-periodic projections, we are able to obtain a periodic version of Theorem 1.2.

In the course of the proof of the periodic version of Theorem 1.2, we also need a periodic version of Theorem 1.1, which can be proved by first establishing the same assertion for infinite point sequences and then by viewing periodic functions as defined on the whole real line [A. Shadrin, private communication]. The proof of Theorem 1.1 for infinite point sequences is announced in [8] and carried out [3]. In this article we give a different proof of the periodic version of Shadrin's theorem by employing a similar comparison of periodic and non-periodic projection operators as in the proof of the periodic version of Theorem 1.2. This proof directly passes from the interval case to the periodic result without recourse to infinite point sequences.

1.2. **Periodic splines.** Let $n \geq k$ be a natural number and $\widetilde{\Delta} = (s_j)_{j=0}^{n-1}$ be a sequence of distinct points on the torus $\mathtt{T} = \mathtt{R}/\mathtt{Z}$ identified canonically with $[0, 1)$, such that for all j we have

$$s_j \leq s_{j+1}, \qquad s_j < s_{j+k},$$

and we extend $(s_j)_{j=0}^{n-1}$ periodically by

$$s_{rn+j} = r + s_j$$

for $r \in \mathtt{Z} \setminus \{0\}$ and $0 \leq j \leq n-1$.

Now, the main result of this article reads as follows:

Theorem 1.3. *For all functions $f \in L^1(\mathtt{T})$ there exists a set $\widetilde{A}$ of full Lebesgue measure such that for all sequences of partitions $(\widetilde{\Delta}_n)$ on $\mathtt{T}$ as above with $|\widetilde{\Delta}_n| \to 0$, we have*

$$\lim_{n\to\infty} \widetilde{P}_n f(x) = f(x), \qquad x \in \widetilde{A},$$

where $\widetilde{P}_n$ denotes the orthogonal projection operator onto the periodic spline space of order k with knots $\widetilde{\Delta}_n$.

In order to prove this result, we also need a periodic version of Theorem 1.1:

Theorem 1.4. *There exists a constant c_k depending only on the spline order k such that for all knot sequences $\widetilde{\Delta} = (s_j)_{j=0}^{n-1}$ on $\mathtt{T}$, the associated orthogonal projection operator $\widetilde{P}$ satisfies the inequality*

$$\|\widetilde{P} : L^\infty(\mathtt{T}) \to L^\infty(\mathtt{T})\| \leq c_k.$$

The idea of the proofs of Theorems 1.3 and 1.4 is to estimate the difference between the periodic projection operator $\widetilde{P}$ and the non-periodic projection operator P for certain non-periodic point sequences associated to $\widetilde{\Delta} = (s_i)_{i=0}^{n-1}$.

The article is organized as follows. In Section 2, we prove a simple lemma on the growth behaviour of linear combinations of non-periodic B-spline functions which is frequently needed later in the proofs of both Theorem 1.3 and Theorem 1.4. Section 3 is devoted to the proof of Theorem 1.4, which is needed for the proof of Theorem 1.3 in Section 4. Finally, in Section 5, we also apply our method of proof to recover Shadrin's theorem for infinite point sequences (see [3, 8]).

2. A simple upper estimate for B-spline sums

Let A be a subset of $[t_\ell, t_{r+1}]$. Then, define the set of indices $\mathfrak{i}(A)$ whose B-spline supports intersect with A as

$$\mathfrak{i}(A) := \{i : A \cap \operatorname{supp} N_i \neq \emptyset\}.$$

We also write $\mathfrak{i}(x)$ for $\mathfrak{i}(\{x\})$. If we have two subsets U, V of indices, we write $d(U,V)$ for the distance between U and V induced by the metric $d(i,j) = |i-j|$.

We will use the notation $A(t) \lesssim B(t)$ to indicate the existence of a constant C that depends only on the spline order k such that for all t we have $A(t) \leq CB(t)$, where t denotes all explicit or implicit dependencies that the expressions A and B might have.

The fact that B-spline functions are localized, so a fortiori the set $\mathfrak{i}(x)$ is localized for any $x \in [t_\ell, t_{r+1}]$, can be used to derive the following lemma:

Lemma 2.1. *Let J be a subset of the index set $\{\ell, \ell+1, \ldots, r-1, r\}$, $f = \sum_{j\in J} \langle h, N_j\rangle N_j^*$ and $p \in [1,\infty]$. Then, for all $x \in [t_\ell, t_{r+1}]$, we have the estimate*

$$\begin{aligned} |f(x)| &\lesssim \gamma^{d(\mathfrak{i}(x),J)} \|h\|_p \max_{m\in\mathfrak{i}(x), j\in J} \frac{\kappa_j^{1/p'}}{h_{jm}} \\ &\leq \gamma^{d(\mathfrak{i}(x),J)} \|h\|_p \max_{m\in\mathfrak{i}(x), j\in J} \big(\max\{\kappa_m, \kappa_j\}\big)^{-1/p} \\ &\leq \gamma^{d(\mathfrak{i}(x),J)} \|h\|_p \cdot |I(x)|^{-1/p}, \qquad 1 \leq p \leq \infty, \end{aligned}$$

where $\gamma \in (0,1)$ is the constant appearing in (1.3), $I(x)$ is the interval $I = [t_i, t_{i+1})$ containing the point x and the exponent p' is such that $1/p + 1/p' = 1$.

Proof. Since $N_j^* = \sum_m a_{jm} N_m$,

$$f(x) = \sum_{j\in J} \sum_{m\in\mathfrak{i}(x)} a_{jm} \langle h, N_j\rangle N_m(x).$$

This implies

$$|f(x)| \lesssim \max_{m\in\mathfrak{i}(x)} \Big(\sum_{j\in J} \frac{\gamma^{|j-m|}}{h_{jm}} \|h\|_p \|N_j\|_{p'} \Big),$$

where we used inequality (1.3) for a_{jm}, Hölder's inequality with the conjugate exponent $p' = p/(p-1)$ to p and the fact that the B-spline functions N_m form a partition of unity. Using again the uniform boundedness of N_j, we obtain

$$|f(x)| \lesssim \max_{m\in\mathfrak{i}(x)} \Big(\sum_{j\in J} \frac{\gamma^{|j-m|}}{h_{jm}} \|h\|_p \kappa_j^{1/p'} \Big).$$

Summing the geometric series now yields the first estimate. The second and the third estimate are direct consequences of the first one. □

Remark 2.2. We note that we directly obtain the second estimate in the above lemma if we use the weaker inequality (1.2) instead of (1.3).

3. The periodic spline orthoprojector is uniformly bounded on L^∞

In this section, we give a direct proof of Theorem 1.4 on the boundedness of periodic spline projectors without recourse to infinite knot sequences. Here, we will only use the geometric decay of the matrix (a_{jm}) defined above for splines on an interval.

A vital tool in the proofs of both Theorem 1.1 and Theorem 1.2 are B-spline functions. We will also make extensive use of them and introduce their periodic version, cf. [7]. Associated to the periodic point sequence $(s_j)_{j=0}^{n-1}$ and its periodic extension as in Section 1.2 we define the non-periodic point sequence

$$t_j = s_j, \qquad \text{for } j = -k+1, \dots, n+k-1$$

and denote the corresponding non-periodic B-spline functions by $(N_j)_{j=-k+1}^{n-1}$ with $\operatorname{supp} N_j = [t_j, t_{j+k}]$. Then we define for $x \in [0,1)$

$$\widetilde{N}_j(x) = N_j(x), \qquad j = 0, \dots, n-k,$$

if we canonically identify $\mathtt{T}$ with $[0,1)$. Moreover, for $j = n-k+1, \dots, n-1$,

$$\widetilde{N}_j(x) = \begin{cases} N_{j-n}(x), & \text{if } x \in [0, s_j], \\ N_j(x), & \text{if } x \in (s_j, 1). \end{cases}$$

We denote by $\widetilde{P}$ the orthogonal projection operator onto the space of periodic splines of order k with knots $(s_j)_{j=0}^{n-1}$, which is the linear span of the B-spline functions $(\widetilde{N}_j)_{j=0}^{n-1}$ and similarly to the non-periodic case we define

$$\mathfrak{i}(A) = \{0 \le j \le n-1 : A \cap \operatorname{supp} \widetilde{N}_j \neq \emptyset\}, \qquad A \subset \mathtt{T}.$$

Lemma 3.1. *Let f_i be a function on $\mathtt{T}$ with $\operatorname{supp} f_i \subset [s_i, s_{i+1}]$ for some index i in the range $0 \le i \le n-1$. Then, for any $x \in \mathtt{T}$,*

$$|\widetilde{P} f_i(x)| \lesssim \gamma^{\widetilde{d}(\mathfrak{i}(x), \mathfrak{i}(\operatorname{supp} f_i))} \|f_i\|_\infty,$$

where $\widetilde{d}$ is the distance function induced by the canonical metric in $\mathtt{Z}/n\mathtt{Z}$ and $\gamma \in (0,1)$ is the constant appearing in inequality (1.3).

Proof. We assume that the index i is chosen such that $s_i < s_{i+1}$, since if $s_i = s_{i+1}$, the function f_i is identically zero in L^∞.

Given a function f on $\mathtt{T}$, we associate a non-periodic function Tf defined on $[s_i, s_{i+n+1}]$ given by

$$Tf(t) = f(\pi(t)), \qquad t \in [s_i, s_{i+n+1}],$$

where $\pi(t)$ is the quotient mapping from $\mathtt{R}$ to $\mathtt{T}$. We observe that T is a linear operator, $\|T : L^2(\mathtt{T}) \to L^2([s_i, s_{i+n+1}])\| = \sqrt{2}$ and $\|T : L^\infty(\mathtt{T}) \to L^\infty([s_i, s_{i+n+1}])\| = 1$. Moreover, for $x \in \mathtt{T}$, let $r(x)$ be the representative of x in the interval $[s_i, s_{i+n})$. We want to estimate $\widetilde{P} f_i(x)$. In order to do this, we first decompose

$$\widetilde{P} f_i(x) = T\widetilde{P} f_i(r(x)) = PTf_i(r(x)) + (T\widetilde{P} f_i - PTf_i)(r(x)), \tag{3.1}$$

where P is the orthogonal projection operator onto the space of splines of order k corresponding to the point sequence $\Delta = (t_j)_{j=-k+1}^{n+k}$ associated to the non-periodic grid points in the interval $[s_i, s_{i+n+1}]$, i.e.,

$$t_j = s_{i+j}, \qquad j = 0, \ldots, n+1,$$
$$t_{-k+1} = \cdots = t_{-1} = s_i, \qquad t_{n+2} = \cdots = t_{n+k} = s_{i+n+1}.$$

Also, let $(N_j)_{i=-k+1}^{n}$ be the L^∞-normalized B-spline basis corresponding to this point sequence.

We estimate the first term $PTf_i(r(x))$ from the decomposition in (3.1) of $\widetilde{P}f_i(x)$. Since P is a projection operator onto splines on an interval, we use representation (1.1) to get

$$PTf_i(r(x)) = \sum_{j=-k+1}^{n} \langle Tf_i, N_j \rangle N_j^*(r(x)),$$

and, since $\operatorname{supp} Tf_i \subset [s_i, s_{i+1}] \cup [s_{i+n}, s_{i+n+1}] = [t_0, t_1] \cup [t_n, t_{n+1}]$ by definition of f_i and T and $\operatorname{supp} N_j \subset [t_j, t_{j+k}]$ for all $j = -k+1, \ldots, n$,

$$PTf_i(r(x)) = \sum_{j \in J_1} \langle Tf_i, N_j \rangle N_j^*(r(x)),$$

with $J_1 = \{-k+1, \ldots, 0\} \cup \{n-k+1, \ldots, n\}$. Employing now Lemma 2.1 with $p = \infty$ to this sum, we obtain

$$|PTf_i(r(x))| \lesssim \gamma^{d(\mathrm{i}(r(x)), J_1)} \|Tf_i\|_\infty \lesssim \gamma^{\widetilde{d}(\mathrm{i}(x), \mathrm{i}(\operatorname{supp} f_i))} \|f_i\|_\infty. \tag{3.2}$$

Now we turn to the second term on the right hand side of (3.1). Let $g := (T\widetilde{P} - PT)f_i$. Observe that $g \in \mathcal{S}_k(\Delta)$ since the range of both $T\widetilde{P}$ and P is contained in $\mathcal{S}_k(\Delta)$. Moreover,

$$\langle (T\widetilde{P} - T)f_i, N_j \rangle = \langle \widetilde{P}f_i - f_i, \widetilde{N}_{j+i} \rangle, \qquad j = 0, \ldots, n-k+1,$$

where we take the latter subindex $j+i$ to be modulo n. This equation is true in the given range of the parameter j, since in this case, the functions N_j and $\widetilde{N}_{j+i}$ coincide on their supports. The fact that $\widetilde{P}$ is an orthogonal projection onto the span of the functions $(\widetilde{N}_j)_{j=0}^{n-1}$ then implies

$$\langle T\widetilde{P}f_i - Tf_i, N_j \rangle = \langle \widetilde{P}f_i - f_i, \widetilde{N}_{j+i} \rangle = 0, \qquad j = 0, \ldots, n-k+1.$$

Combining this with the fact

$$\langle PTf_i - Tf_i, N_j \rangle = 0, \qquad j = -k+1, \ldots, n,$$

since P is an orthogonal projection onto a spline space as well, we obtain that

$$\langle g, N_j \rangle = 0, \qquad j = 0, \ldots n-k+1.$$

Therefore, we can expand g as a B-spline sum

$$g = \sum_{j \in J_2} \langle g, N_j \rangle N_j^*,$$

with $J_2 = \{-k+1, \dots, -1\} \cup \{n-k+2, \dots, n\}$. Now, we employ Lemma 2.1 on the function g with the parameter $p = 2$ to get for the point $y = r(x)$

$$|g(y)| \lesssim \gamma^{d(\mathfrak{i}(y), J_2)} \|g\|_2 \max_{j \in J_2} |\operatorname{supp} N_j|^{-1/2}.$$

Since $g = (T\widetilde{P} - PT)f_i$ and the operator $T\widetilde{P} - PT$ has norm $\leq 2\sqrt{2}$ on L^2, we get

$$|g(y)| \lesssim \gamma^{d(\mathfrak{i}(y), J_2)} \|f_i\|_2 |\operatorname{supp} f_i|^{-1/2},$$

where we also used the fact that $\operatorname{supp} N_j \supset [s_i, s_{i+1}] = [t_0, t_1]$ or $\operatorname{supp} N_j \supset [s_{i+n}, s_{i+n+1}] = [t_n, t_{n+1}]$ for $j \in J_2$. Since $d(\mathfrak{i}(y), J_2) \geq \widetilde{d}(\mathfrak{i}(x), \mathfrak{i}(\operatorname{supp} f_i))$ and $\|f_i\|_2 \leq \|f_i\|_\infty |\operatorname{supp} f_i|^{1/2}$, we finally get

$$|g(y)| \lesssim \gamma^{\widetilde{d}(\mathfrak{i}(x), \mathfrak{i}(\operatorname{supp} f_i))} \|f_i\|_\infty.$$

Looking at (3.1) and combining the latter estimate with (3.2), the proof is completed. □

This lemma can be used directly to prove Theorem 1.4 on the uniform boundedness of periodic orthogonal spline projection operators on L^∞:

Proof of Theorem 1.4. We just decompose the function f as $f = \sum_{i=0}^{n-1} f \cdot \mathbb{1}_{[s_i, s_{i+1})}$ and apply Lemma 3.1 to each summand and the assertion $\|\widetilde{P}f\|_\infty \lesssim \|f\|_\infty$ follows after summation of a geometric series. □

Remark 3.2. (i) Since $\widetilde{P}$ is a selfadjoint operator, Theorem 1.4 also implies that $\widetilde{P}$ is bounded as an operator from $L^1(\mathbb{T})$ to $L^1(\mathbb{T})$ by the same constant c_k as in the above theorem. Moreover, by interpolation, $\widetilde{P}$ is also bounded by c_k as an operator from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ for any $p \in [1, \infty]$.

(ii) In the proof of Lemma 3.1, we only use the second inequality of Lemma 2.1 which follows from inequality (1.2) on the inverse of the B-spline Gram matrix and does not need its stronger form (1.3). Similarly to the equivalence of Shadrin's theorem and (1.2) in the non-periodic case, we can derive the equivalence of Theorem 1.4 and the estimate

$$|\widetilde{a}_{ij}| \leq \frac{K\gamma^{\widetilde{d}(i,j)}}{\max(\widetilde{\kappa}_i, \widetilde{\kappa}_j)}, \qquad 0 \leq i, j \leq n-1,$$

where $(\widetilde{a}_{ij})$ denotes the inverse of the Gram matrix $(\langle \widetilde{N}_i, \widetilde{N}_j \rangle)$, $K > 0$ and $\gamma \in (0,1)$ are constants only depending on the spline order k, $\widetilde{\kappa}_i$ denotes the length of the support of $\widetilde{N}_i$ and $\widetilde{d}$ is the canonical distance in $\mathbb{Z}/n\mathbb{Z}$. The proof of this equivalence uses the same tools as the proof in the non-periodic case: a periodic version of both Demko's theorem and de Boor's stability.

4. Almost everywhere convergence

In this section we prove Theorem 1.3 on the a.e. convergence of periodic spline projections.

Proof of Theorem 1.3. Without loss of generality, we assume that $\widetilde{\Delta}_n$ has n points. Let $\widetilde{\Delta}_n = (s_j^{(n)})_{j=0}^{n-1}$ and $(\widetilde{N}_j^{(n)})_{j=0}^{n-1}$ be the corresponding periodic B-spline functions. Associated to it, define the non-periodic point sequence $\Delta_n = (t_j^{(n)})_{j=-m}^{n+k-1}$ with the boundary points 0 and 1 as

$$\begin{aligned} t_j^{(n)} &= s_j^{(n)}, \qquad j = 0, \dots, n-1, \\ t_{-m}^{(n)} &= \cdots = t_{-1}^{(n)} = 0, \qquad t_n^{(n)} = \cdots = t_{n+k-1}^{(n)} = 1. \end{aligned}$$

We choose the integer m such that the multiplicity of the point 0 in Δ_n is k and denote by $(N_j^{(n)})_{j=-m}^{n-1}$ the non-periodic B-spline functions corresponding to this point sequence and by P_n the orthogonal projection operator onto the span of $(N_j^{(n)})_{j=-m}^{n-1}$.

We will show that $\widetilde{P}_n f(x) \to f(x)$ for all x in the set A from Theorem 1.2 of full Lebesgue measure such that $\lim P_n Tf(x) = Tf(x)$ for all $x \in A$, where T is just the operator that canonically identifies a function defined on $\mathtt{T}$ with the corresponding function defined on $[0,1)$ and we write x for a point in $\mathtt{T}$ as well as for its representative in the interval $[0,1)$.

So, choose an arbitrary (non-zero) point $x \in A$ and decompose $\widetilde{P}_n f(x)$:

$$\widetilde{P}_n f(x) = T\widetilde{P}_n f(x) = P_n Tf(x) + \big(T\widetilde{P}_n f(x) - P_n Tf(x)\big). \tag{4.1}$$

For the first term of (4.1), $P_n Tf(x)$, we have that $\lim_{n\to\infty} P_n Tf(x) = Tf(x) = f(x)$ since $x \in A$.

It remains to estimate the second term $g_n(x) = T\widetilde{P}_n f(x) - P_n Tf(x) = T\widetilde{P}_n f(x) - Tf(x) + Tf(x) - P_n Tf(x)$ of (4.1). First, note that $g_n \in \mathcal{S}_k(\Delta_n)$. Moreover,

$$\langle T\widetilde{P}_n f - Tf, N_j^{(n)}\rangle = \langle \widetilde{P}_n f - f, \widetilde{N}_j^{(n)}\rangle = 0, \qquad j = 0, \dots, n-k-1,$$

since $\widetilde{P}_n$ is the projection operator onto the span of the B-spline functions $(\widetilde{N}_j^{(n)})$, and

$$\langle Tf - P_n Tf, N_j^{(n)}\rangle = 0, \qquad j = -m, \dots, n-1,$$

since P_n is the projection operator onto the span of the functions $(N_j^{(n)})$. Therefore, $g_n \in \mathcal{S}_k(\Delta_n)$ can be written as

$$g_n = \sum_{j \in J_n} \langle g_n, N_j^{(n)}\rangle N_j^{(n)*},$$

with $J_n = \{-m, \dots, -1\} \cup \{n-k, \dots, n-1\}$ and $(N_j^{(n)*})$ being the dual basis to $(N_j^{(n)})$. We now apply Lemma 2.1 with $p = 1$ to g_n and get

$$|g_n(x)| \lesssim \gamma^{d(\mathfrak{i}_n(x), J_n)} \|g_n\|_1 \max_{\ell \in \mathfrak{i}_n(x), j \in J_n} \frac{1}{h_{\ell j}^{(n)}},$$

where $h_{\ell j}^{(n)}$ denotes the length of the convex hull of $\operatorname{supp} N_\ell^{(n)} \cup \operatorname{supp} N_j^{(n)}$ and $\mathfrak{i}_n(x)$ is the set of indices i such that x is in the support of $N_i^{(n)}$. Since for $\ell \in \mathfrak{i}_n(x)$, the point x is contained in $\operatorname{supp} N_\ell^{(n)}$ and for $j \in J_n$ either the point 0 or the point 1 is contained in $\operatorname{supp} N_j^{(n)}$, we can further estimate

$$|g_n(x)| \lesssim \gamma^{d(\mathfrak{i}_n(x),J_n)} \|g_n\|_1 \frac{1}{\min(x, 1-x)}.$$

Now, $\|g_n\|_1 = \|(T\widetilde{P}_n - P_n T)f\|_1 \lesssim \|f\|_1$, since the operator T has norm one on L^1 and $\widetilde{P}_n$ and P_n are both bounded on L^1 uniformly in n by Theorem 1.4 (cf. Remark 3.2) and Theorem 1.1, respectively. Since $|\widetilde{\Delta}_n|$ tends to zero, and a fortiori the same is true for $|\Delta_n|$, we have that $d(\mathfrak{i}_n(x), J_n)$ tends to infinity as $n \to \infty$. This implies $\lim_{n\to\infty} g_n(x) = 0$, and therefore, by the choice of the point x and decomposition (4.1), $\lim \widetilde{P}_n f(x) = f(x)$. Since $x \in A$ was arbitrary and A is a set of full Lebesgue measure, we obtain

$$\lim_{n\to\infty} \widetilde{P}_n f(y) = 0, \qquad \text{for a.e. } y \in \mathbb{T},$$

and the proof is completed. □

5. The case of infinite point sequences

In this last section, we use the methods introduced in the previous sections to recover Shadrin's theorem for infinite point sequences (see [8, 3]).

Let $(s_i)_{i\in\mathbb{Z}}$ be a biinfinite point sequence in $\mathbb{R}$ satisfying

$$s_i \le s_{i+1}, \qquad s_i < s_{i+k},$$

with the corresponding B-spline functions $(\widetilde{N}_i)_{i\in\mathbb{Z}}$ satisfying $\operatorname{supp} \widetilde{N}_i = [s_i, s_{i+k}]$. Furthermore, we denote by $\widetilde{P}$ the orthogonal projection operator onto the closed linear span of the functions $(\widetilde{N}_i)_{i\in\mathbb{Z}}$.

Lemma 5.1. *Let f be a function on $(\inf s_i, \sup s_i)$ with compact support. Then, for any $x \in (\inf s_i, \sup s_i)$,*

$$|\widetilde{P} f(x)| \lesssim \gamma^{d(\mathfrak{i}(x), \mathfrak{i}(\operatorname{supp} f))} \|f\|_\infty,$$

where $\gamma \in (0,1)$ is the constant appearing in inequality (1.3).

Proof. For notational simplicity, we assume in this proof that the sequence (s_i) is strictly increasing. Let $x \in (\inf s_i, \sup s_i)$ and let $I(x)$ be the interval $I = [s_i, s_{i+1})$ containing x. Since f has compact support and the sequence (s_i) is biinfinite, we can choose the indices ℓ and r such that $\{x\} \cup \operatorname{supp} f \subset [s_\ell, s_{r+1})$ and with $J = \{\ell - k + 1, \dots, \ell - 1\} \cup \{r - k + 2, \dots, r\}$, the inequality

$$\gamma^{d(\mathfrak{i}(x),J)} |\operatorname{supp} f|^{1/2} |I(x)|^{-1/2} \le \gamma^{d(\mathfrak{i}(x), i(\operatorname{supp} f))}$$

is true.

Next, define the point sequence $\Delta = (t_i)_{i=\ell-k+1}^{r+k}$ by

$$t_i = s_i, \qquad i = \ell, \dots, r+1,$$
$$a = t_{\ell-k+1} = \cdots = t_\ell = s_\ell, \qquad b = t_{r+k} = \cdots = t_{r+1} = s_{r+1},$$

and let the collection $(N_i)_{i=\ell-k+1}^{r}$ be the corresponding B-spline functions and P the associated orthogonal projector. Let T be the operator that restricts a function defined on $(\inf s_i, \sup s_i)$ to the interval $[a, b]$. In order to estimate $\widetilde{P}f(x)$, we decompose

$$\widetilde{P}f(x) = T\widetilde{P}f(x) = PTf(x) + \big(T\widetilde{P}f(x) - PTf(x)\big). \tag{5.1}$$

Observe that $PTf = \sum_{n\in F}\langle f, N_n\rangle N_n^*$, where $F = \mathfrak{i}(\operatorname{supp} f) = \{i : \operatorname{supp} f \cap \operatorname{supp} N_i \neq \emptyset\}$. Applying Lemma 2.1 with the exponent $p = \infty$, we obtain

$$|PTf(x)| \lesssim \gamma^{d(\mathrm{i}(x),F)}\|f\|_\infty.$$

We now consider the second part of the decomposition (5.1), the function $g = (T\widetilde{P} - PT)f = (T\widetilde{P} - T + T - PT)f$. We observe that $g \in \mathcal{S}_k(\Delta)$ and, moreover,

$$\langle T\widetilde{P}f - Tf, N_j\rangle = \langle \widetilde{P}f - f, \widetilde{N}_j\rangle = 0, \qquad j = \ell, \ldots, r - k + 1,$$

by definition of the projection operator $\widetilde{P}$, and,

$$\langle Tf - PTf, N_j\rangle = 0, \qquad j = \ell - k + 1, \ldots, r,$$

by definition of the projection operator P. Therefore, we can write the function g as

$$g = \sum_{j\in J}\langle g, N_j\rangle N_j^*$$

with $J = \{\ell - k + 1, \ldots, \ell - 1\} \cup \{r - k + 2, \ldots, r\}$ as defined above. Now, by Lemma 2.1 with the exponent $p = 2$, we get

$$\begin{aligned}|g(x)| &\lesssim \gamma^{d(\mathrm{i}(x),J)}\|g\|_2 \cdot |I(x)|^{-1/2} \lesssim \gamma^{d(\mathrm{i}(x),J)}\|f\|_2 \cdot |I(x)|^{-1/2} \\ &\leq \gamma^{d(\mathrm{i}(x),J)}|\operatorname{supp} f|^{1/2}|I(x)|^{-1/2}\|f\|_\infty.\end{aligned}$$

Finally, due to the choice of ℓ and r,

$$\gamma^{d(\mathrm{i}(x),J)}|\operatorname{supp} f|^{1/2}|I(x)|^{-1/2} \leq \gamma^{d(\mathrm{i}(x),\mathrm{i}(\operatorname{supp} f))},$$

which proves the lemma. □

We can now use this lemma to define $\widetilde{P}f$ for functions $f \in L^\infty(\inf s_i, \sup s_i)$ that are not necessarily in $L^2(\inf s_i, \sup s_i)$ if $\inf s_i = -\infty$ or $\sup s_i = +\infty$. If we let $f_i := f\mathbb{1}_{[s_i, s_{i+1})}$, then f_i has compact support and the above lemma implies that the series

$$\widetilde{P}f(x) := \sum_{i\in\mathbb{Z}}\widetilde{P}f_i(x), \qquad x \in (\inf s_i, \sup s_i),$$

is absolutely convergent and, moreover, there exists a constant C only depending on the spline order k such that

$$\|\widetilde{P}f\|_\infty \leq C\|f\|_\infty.$$

This operator enjoys the characteristic property of an orthogonal projection:

$$\langle \widetilde{P}f - f, \widetilde{N}_i\rangle = 0, \qquad i \in \mathbb{Z}.$$

Remark 5.2. One can combine the proofs of Lemma 5.1 and Lemma 3.1 to also obtain the uniform boundedness of the spline orthoprojector on L^∞ for one-sided infinite point sequences.

References

[1] Z. Ciesielski. Orthogonal projections onto spline spaces with arbitrary knots. In *Function spaces (Poznań, 1998)*, volume 213 of *Lecture Notes in Pure and Appl. Math.*, pages 133–140. Dekker, New York, 2000.

[2] C. de Boor. The quasi-interpolant as a tool in elementary polynomial spline theory. In *Approximation theory (Proc. Internat. Sympos., Univ. Texas, Austin, Tex., 1973)*, pages 269–276. Academic Press, New York, 1973.

[3] C. de Boor. On the (bi)infinite case of Shadrin's theorem concerning the L_∞-boundedness of the L_2-spline projector. *Proc. Steklov Inst. Math.*, 277:s73–s78, 2012.

[4] S. Demko. Inverses of band matrices and local convergence of spline projections. *SIAM J. Numer. Anal.*, 14(4):616–619, 1977.

[5] R. A. DeVore and G. G. Lorentz. *Constructive approximation*, volume 303 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1993.

[6] M. Passenbrunner and A. Shadrin. On almost everywhere convergence of orthogonal spline projections with arbitrary knots. *J. Approx. Theory*, 180:77–89, 2014.

[7] L. L. Schumaker. *Spline functions: basic theory*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2007.

[8] A. Shadrin. The L_∞-norm of the L_2-spline projector is bounded independently of the knot sequence: a proof of de Boor's conjecture. *Acta Math.*, 187(1):59–137, 2001.

Institute of Analysis, Johannes Kepler University Linz, Austria, 4040 Linz, Altenberger Strasse 69

E-mail address: `markus.passenbrunner@jku.at`