Tractability properties of the discrepancy in Orlicz norms

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Dedicated to Gerhard Larcher on the occasion of his 60$^{\text{th}}$ birthday[1].

Abstract

We show that the minimal discrepancy of a point set in the d-dimensional unit cube with respect to Orlicz norms can exhibit both polynomial and weak tractability. In particular, we show that the ψ_α-norms of exponential Orlicz spaces are polynomially tractable.

Keywords: Discrepancy, Orlicz norm, tractability, quasi-Monte Carlo
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1 Introduction and main results

The discrepancy of an N-element point set $\mathcal{P} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$ in the unit cube $[0,1]^d$ measures the deviation of the empirical distribution of $\mathcal{P}$ from the uniform measure. This concept has important applications in numerical analysis, where so-called Koksma-Hlawka inequalities establish a deep connection between norms of the discrepancy function and worst case errors of quasi-Monte Carlo integration rules determined by the point set $\mathcal{P}$. For a comprehensive introduction and exposition on this subject we refer the reader to [8, 13, 16] and the references cited therein.

To define the concept of discrepancy, we first introduce the local discrepancy function $\Delta_{\mathcal{P}} : [0,1]^d \to \mathbb{R}$ defined as

$$\Delta_{\mathcal{P}}(\boldsymbol{t}) = \frac{\#\{j \in \{1,2,\ldots,N\} : \boldsymbol{x}_j \in [\boldsymbol{0}, \boldsymbol{t})\}}{N} - \mathrm{Vol}([\boldsymbol{0}, \boldsymbol{t}]),$$

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[1]Heinrich *et al.* mention in their seminal paper [11, p. 280] that the question about the dependence on d for the star-discrepancy was first raised by Gerhard Larcher in 1998.

where $[\mathbf{0}, \boldsymbol{t}) = [0, t_1) \times [0, t_2) \times \ldots \times [0, t_d)$ for $\boldsymbol{t} = (t_1, t_2, \ldots, t_d) \in [0, 1]^d$ and $\mathrm{Vol}(\cdot)$ stands for the d-dimensional Lebesgue measure. We now apply a norm $\|\cdot\|_\bullet$ to the local discrepancy function to obtain the discrepancy $\|\Delta_{\mathcal{P}}\|_\bullet$ of the point set $\mathcal{P}$ with respect to the norm $\|\cdot\|_\bullet$. Of particular interest are the norms on the usual Lebesgue spaces L_p $(1 \le p \le \infty)$ of p-integrable functions on the unit cube $[0, 1]^d$. Those lead to the central notions of L_p-discrepancy for $1 \le p < \infty$, and the L_∞-discrepancy, which is usually called the star-discrepancy, when $p = \infty$.

The N^{th} minimal discrepancy with respect to the norm $\|\cdot\|_\bullet$ in dimension d is the best possible discrepancy over all point sets of size N in the d-dimensional unit cube $[0, 1]^d$, i.e.,

$$\mathrm{disc}_\bullet(N, d) = \inf_{\substack{\mathcal{P} \subseteq [0,1]^d \\ |\mathcal{P}| = N}} \|\Delta_{\mathcal{P}}\|_\bullet.$$

We compare this value with the initial discrepancy given by the discrepancy of the empty point set $\|\Delta_\emptyset\|_\bullet$. Since the initial discrepancy may depend on the dimension, we use it to normalize the N^{th} minimal discrepancy when we study the dependence of $\mathrm{disc}_\bullet(N, d)$ on the dimension d. We therefore define the inverse of the N^{th} minimal discrepancy in dimension d as the number $N_\bullet(\varepsilon, d)$ which is the smallest number N such that a point set with N points exists that reduces the initial discrepancy at least by a factor of $\varepsilon \in (0, 1)$,

$$N_\bullet(\varepsilon, d) = \min\left\{N \in \mathbb{N} \,:\, \mathrm{disc}_\bullet(N, d) \le \varepsilon \|\Delta_\emptyset\|_\bullet\right\}.$$

In this paper we are interested in how $N_\bullet(\varepsilon, d)$ depends simultaneously on ε and the dimension d. In general, the dependence of the inverse of the N^{th} minimal discrepancy can take different forms. For instance, if the dependence on the dimension d or on ε^{-1} is exponential, then we call the discrepancy intractable. If the inverse of the N^{th} minimal discrepancy grows exponentially fast in d, then the discrepancy is said to suffer from the curse of dimensionality. On the other hand, if $N_\bullet(\varepsilon, d)$ increases at most polynomially in d and ε^{-1}, as d increases and ε tends to zero, then the discrepancy is said to be polynomially tractable. This leads us to the following definition.

Definition 1. *The discrepancy with respect to the norm $\|\cdot\|_\bullet$ is polynomially tractable if there are numbers $C \in (0, \infty)$, $\tau \in (0, \infty)$, and $\sigma \in (0, \infty)$ such that*

$$N_\bullet(\varepsilon, d) \le C\, d^\tau \varepsilon^{-\sigma}, \quad \textit{for all } \varepsilon \in (0, 1) \textit{ and all } d \in \mathbb{N}. \tag{1}$$

The infimum over all exponents $\tau \in (0, \infty)$ such that a bound of the form (1) holds is called the d-exponent of polynomial tractability.

To cover cases between polynomial tractability and intractability, we now introduce the concept of weak tractability, where $N_\bullet(\varepsilon, d)$ is not exponential in ε^{-1} and d. This encodes the absence of intractability.

Definition 2. *The discrepancy with respect to the norm $\|\cdot\|_\bullet$ is weakly tractable, if*

$$\lim_{d + \varepsilon^{-1} \to \infty} \frac{\log N_\bullet(\varepsilon, d)}{d + \varepsilon^{-1}} = 0.$$

(Throughout this paper log *means the natural logarithm.)*

The subject of tractability of multivariate problems is a very popular and active area of research and we refer the reader to the books [19, 20] by Novak and Woźniakowski for an introduction into tractability studies of discrepancy and an exhaustive exposition.

A famous result by Heinrich, Novak, Wasilkowski, and Woźniakowski [11] based on the theory of empirical processes and Talagrand's majorizing measure theorem shows that the star-discrepancy is polynomially tractable. In fact, they show that τ in Definition 1 can be set to one and hence in this case the inverse of the star-discrepancy $N_{L_\infty}(\varepsilon, d)$ depends at most linearly on the dimension d. It was shown in [11] and [12] that $\tau = 1$ is the minimal possible τ in Definition 1 for the star-discrepancy. Determining the optimal exponent σ for ε^{-1} is an open problem. On the other hand, the L_2-discrepancy is known to be intractable, as shown by Woźniakowski [21] (see also [20]). The behavior of the inverse of the L_p-discrepancy in between, where $p \notin \{2, \infty\}$, seems to be unknown.

Note that due to the normalization with the initial discrepancy, we cannot infer a continuous change in the behavior of $N_{L_p}(\varepsilon, d)$ as p goes from 1 to ∞. A natural assumption seems to be that the L_p-discrepancy is intractable for any $p \in [1, \infty)$. If correct, this would mean that there is a sharp change from intractability to polynomial tractability as one goes from $p \in [1, \infty)$ to $p = \infty$. A natural question which hence arises is what happens between those two cases $p \in [1, \infty)$ and $p = \infty$.

To study this question we shall work in the setting of (specific) Orlicz spaces. Let us recall that a function $M : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function if $M(0) = 0$, M is convex, and $M(t) > 0$ for $t > 0$. If $\lim_{x\to 0} M(x)/x = \lim_{x\to\infty} x/M(x) = 0$, then M is called an N-function. The previous limit assumptions simply guarantee that the convex-dual is again an N-function. Now if $D \subseteq \mathbb{R}^d$ is a compact set, we define the Orlicz space L_M to be the space of (equivalence classes of) Lebesgue measurable functions f on D for which

$$\|f\|_M := \inf \left\{ K > 0 : \int_D M\left(\frac{|f(\boldsymbol{x})|}{K} \right) \mathrm{d}\boldsymbol{x} \leq 1 \right\} < \infty.$$

The latter functional is a norm on L_M known as Luxemburg norm, named after W. A. J. Luxemburg [18], which turns L_M into a Banach space. One commonly just speaks of Orlicz functions, Orlicz norms, and Orlicz spaces. An introduction to the theory of Orlicz spaces can be found in [15].

For our purpose, we introduce for $\alpha \in [1, \infty)$ the exponential Orlicz norms $\|\cdot\|_{\psi_\alpha}$, which for a measurable function f defined on $[0, 1]^d$ are given by

$$\|f\|_{\psi_\alpha} = \inf \left\{ K > 0 : \int_{[0,1]^d} \psi_\alpha \left(\frac{|f(\boldsymbol{x})|}{K} \right) \mathrm{d}\boldsymbol{x} \leq 1 \right\},$$

where $\psi_\alpha(x) = \exp(x^\alpha) - 1$. The assumption $\alpha \geq 1$ guarantees the convexity of ψ_α. These norms play an important role in the study of the concentration of mass in high-dimensional convex bodies [3, 6, 17] and have recently found applications in the tractability study of multivariate numerical integration [14]. They have appeared earlier in discrepancy theory and the related multivariate integration problems in fixed dimension [4, 5, 7]. As we shall see later, the discrepancy with respect to ψ_α-norms turns out to be polynomially tractable as well.

In our context it is interesting to also study variations of these norms exhibiting different types of behavior of $N_\bullet(\varepsilon, d)$ as a function of the dimension d. In fact, we may write ψ_α as the series

$$\psi_\alpha(x) = \frac{x^\alpha}{1!} + \frac{x^{2\alpha}}{2!} + \frac{x^{3\alpha}}{3!} + \cdots$$

and consider the more general case where ψ_α is replaced by a function

$$\psi_{\alpha,\varphi}(x) = \frac{x^\alpha}{(\varphi(\alpha))^\alpha} + \frac{x^{2\alpha}}{(\varphi(2\alpha))^{2\alpha}} + \frac{x^{3\alpha}}{(\varphi(3\alpha))^{3\alpha}} + \cdots \tag{2}$$

for a non-decreasing function $\varphi : [0,\infty) \to (0,\infty)$ with $\lim_{x\to\infty} \varphi(x) = \infty$. Note that the growth condition on φ guarantees, according to the ratio test, the absolute convergence of the series (2) for all $x \in [0,\infty)$. Choosing $\varphi(p\alpha) = (p!)^{1/(p\alpha)}$ takes us back to the ψ_α-norm, which is therefore a special case of the more general setting.

Below we will characterize functions φ for which the discrepancy with respect to $\|\cdot\|_{\psi_{\alpha,\varphi}}$, given by

$$\|f\|_{\psi_{\alpha,\varphi}} = \inf\left\{K > 0 : \int_{[0,1]^d} \psi_{\alpha,\varphi}\left(\frac{|f(\boldsymbol{x})|}{K}\right) \mathrm{d}\boldsymbol{x} \le 1\right\},$$

is polynomially tractable and weakly tractable.

The aim of this paper is to show the following result.

Theorem 1. *Let $\alpha \in [1,\infty)$. Then the following hold:*

1. *The discrepancy with respect to the ψ_α-norm $\|\cdot\|_{\psi_\alpha}$ is polynomially tractable.*

2. *For any non-decreasing $\varphi : [0,\infty) \to (0,\infty)$ with $\lim_{x\to\infty} \varphi(x) = \infty$ for which there exists an $r \ge 0$ and a constant $C \in (0,\infty)$ such that for all $p \ge 1$*

$$\varphi(p) \le C\, p^r, \tag{3}$$

   the discrepancy with respect to $\|\cdot\|_{\psi_{\alpha,\varphi}}$ is polynomially tractable. The d-exponent of polynomial tractability is at most $3 + 2r$.

3. *For any non-decreasing $\varphi : [0,\infty) \to (0,\infty)$ with $\lim_{x\to\infty} \varphi(x) = \infty$ which satisfies*

$$\lim_{p\to\infty} \frac{\log \varphi(p)}{p} = 0, \tag{4}$$

   the discrepancy with respect to $\|\cdot\|_{\psi_{\alpha,\varphi}}$ is weakly tractable.

Remark 1. Note that by choosing $\psi_{\alpha,\varphi}(p) = p^\alpha$ we obtain the classical L_α-norm. In this case $\varphi(\alpha) = 1$ and $\varphi(x) = \infty$ for all $x > \alpha$. This choice of φ does not satisfy any of the conditions in Theorem 1.

An example of a function φ that satisfies condition (4) for weak tractability is $\varphi(p) = \exp(p^\tau)$ with some $\tau \in (0,1)$. This function does not satisfy condition (3).

We can in fact provide a more accurate estimate for the exponential Orlicz norms and the d-exponent of polynomial tractability.

Theorem 2. *For any $\alpha \in [1,\infty)$, we have*

$$N_{\psi_\alpha}(\varepsilon, d) \le \left\lceil C_\alpha\, d^{\max\{1,2/\alpha\}} \, (\log(d+1))^{2/\alpha}\varepsilon^{-2}\right\rceil,$$

where

$$C_\alpha = 2601 \cdot \alpha^{2/\alpha} \cdot \left(\frac{\sqrt{2\pi}}{\mathrm{e}^{11/12}}\right)^{2/\alpha}.$$

In particular, the d-exponent of polynomial tractability is at most $\max\{1, 2/\alpha\}$.

This upper bound on $N_{\psi_\alpha}(\varepsilon, d)$ shows that for $\alpha \to \infty$ the inverse of the star-discrepancy depends linearly on the dimension, thereby matching the result of Heinrich, Novak, Wasilkowski, and Woźniakowski [11].

In the following Section 2 we present the proofs of our main results, where we start by establishing an equivalence between the norms $\|\cdot\|_{\psi_{\alpha,\varphi}}$ and an expression involving a supremum of classical L_p-norms. Subsection 2.1 is then devoted to the proof of Theorem 1. The proof of Theorem 2 will be presented in Subsection 2.2.

2 The proofs

For the proofs of Theorems 1 and 2 we define another norm which we show to be equivalent to the Orlicz norm $\|\cdot\|_{\psi_{\alpha,\varphi}}$, namely

$$\|f\|_\varphi := \sup_{p\ge 1} \frac{\|f\|_{L_p}}{\varphi(p)} \tag{5}$$

with $\varphi : [0,\infty) \to (0,\infty)$. In the special case of exponential Orlicz norms $\|\cdot\|_{\psi_\alpha}$ such an equivalence is a classical result in asymptotic geometric analysis and may be found, without explicit constants, in the monographs [3, Lemma 3.5.5] and [6, Lemma 2.4.2]. In the context of this paper it is important that these constants do not depend on the dimension d.

Lemma 1. *Let $d \in \mathbb{N}$ and $\alpha \in [1,\infty)$. For any measurable function $f : [0,1]^d \to \mathbb{R}$, we have the estimates*

$$\inf_{p\ge 1} \frac{\varphi(p)}{\max\{\varphi(\alpha), \varphi(p)\}}\ \|f\|_\varphi \le \|f\|_{\psi_{\alpha,\varphi}} \le 2^{1/\alpha}\|f\|_\varphi. \tag{6}$$

In particular, for any $\alpha \in [1,\infty)$, we have

$$\left(\frac{\mathrm{e}^{11/12}}{\sqrt{2\pi}}\right)^{1/\alpha} \|f\|_\alpha \le \|f\|_{\psi_\alpha} \le (2\,\mathrm{e}\,\alpha)^{1/\alpha}\,\|f\|_\alpha, \tag{7}$$

where $\|f\|_\alpha := \sup_{p\ge 1} p^{-1/\alpha}\|f\|_{L_p}$.

Proof. Using the series expansion of $\psi_{\alpha,\varphi}$, we obtain

$$\int_{[0,1]^d} \psi_{\alpha,\varphi}\left(\frac{|f(\boldsymbol{x})|}{K}\right) \mathrm{d}\boldsymbol{x} = \sum_{p=1}^{\infty} \left(\frac{\|f\|_{L_{\alpha p}}}{K\varphi(\alpha p)}\right)^{\alpha p}.$$

By choosing

$$K = 2^{1/\alpha} \sup_{p \ge \alpha} \frac{\|f\|_{L_p}}{\varphi(p)},$$

we obtain

$$\int_{[0,1]^d} \psi_{\alpha,\varphi}\left(\frac{|f(\boldsymbol{x})|}{K}\right) \mathrm{d}\boldsymbol{x} \le \sum_{p=1}^{\infty} 2^{-p} = 1.$$

Therefore, we have

$$\|f\|_{\psi_{\alpha,\varphi}} \le K = 2^{1/\alpha} \sup_{p \ge \alpha} \frac{\|f\|_{L_p}}{\varphi(p)}.$$

This implies the upper bound in (6) for all $\alpha \ge 1$.

For the lower bound, we argue as follows. For any $K \in (0, \infty)$ such that $K \ge \|f\|_{\psi_{\alpha,\varphi}}$, we have

$$1 \ge \int_{[0,1]^d} \psi_{\alpha,\varphi}\left(\frac{|f(\boldsymbol{x})|}{K}\right) \mathrm{d}\boldsymbol{x} = \sum_{p=1}^{\infty} \left(\frac{\|f\|_{L_{\alpha p}}}{K\varphi(\alpha p)}\right)^{\alpha p} \ge \left(\sup_{p \ge 1} \frac{\|f\|_{L_{\alpha p}}}{K\varphi(\alpha p)}\right)^{\alpha p}.$$

This implies that

$$K \ge \sup_{p \ge \alpha} \frac{\|f\|_{L_p}}{\varphi(p)}$$

and since this holds for any such K, we obtain

$$\|f\|_{\psi_{\alpha,\varphi}} \ge \sup_{p \ge \alpha} \frac{\|f\|_{L_p}}{\varphi(p)}.$$

If $\alpha \in [1, \infty)$, and $q \in [1, \alpha]$, then

$$\sup_{p \ge \alpha} \frac{\|f\|_{L_p}}{\varphi(p)} \ge \frac{\|f\|_{L_\alpha}}{\varphi(\alpha)} \ge \frac{\|f\|_{L_q}}{\varphi(q)} \inf_{q \in [1,\alpha]} \frac{\varphi(q)}{\varphi(\alpha)}.$$

Hence,

$$\sup_{p \ge \alpha} \frac{\|f\|_{L_p}}{\varphi(p)} \ge \min\left\{1, \inf_{q \in [1,\alpha]} \frac{\varphi(q)}{\varphi(\alpha)}\right\} \sup_{p \ge 1} \frac{\|f\|_{L_p}}{\varphi(p)}.$$

In any case, for all $\alpha \in [1, \infty)$, we have that

$$\sup_{p \ge \alpha} \frac{\|f\|_{L_p}}{\varphi(p)} \ge \min\left\{1, \inf_{q \ge 1} \frac{\varphi(q)}{\varphi(\alpha)}\right\} \|f\|_{\varphi} = \inf_{q \ge 1} \frac{\varphi(q)}{\max\{\varphi(\alpha), \varphi(q)\}} \|f\|_{\varphi},$$

which implies the result since $\inf_{q \ge 1} \frac{\varphi(q)}{\max\{\varphi(\alpha),\varphi(q)\}} \le 1$.

The bound (7) for the ψ_α-norms can be shown using similar arguments together with Stirling's formula

$$\sqrt{2\pi p}(p/\mathrm{e})^p \le p! \le \sqrt{2\pi p}(p/\mathrm{e})^p \mathrm{e}^{1/(12p)}. \tag{8}$$

We use the Taylor series expansion of the exponential function and obtain

$$\begin{aligned} \int_{[0,1]^d} \psi_\alpha\left(\frac{|f(\boldsymbol{x})|}{K}\right) \mathrm{d}\boldsymbol{x} &= \int_{[0,1]^d} \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left(\frac{|f(\boldsymbol{x})|}{K}\right)^{\alpha\ell} \mathrm{d}\boldsymbol{x} \\ &= \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left(\frac{\|f\|_{L_{\alpha\ell}}}{K}\right)^{\alpha\ell}. \end{aligned} \tag{9}$$

Using Stirling's formula (8) we get

$$\int_{[0,1]^d} \psi_\alpha \left(\frac{|f(\boldsymbol{x})|}{K} \right) \mathrm{d}\boldsymbol{x} \leq \sum_{\ell=1}^{\infty} \frac{\mathrm{e}^\ell}{\ell^\ell} \left(\frac{\|f\|_{L_{\alpha\ell}}}{K} \right)^{\alpha\ell} = \sum_{\ell=1}^{\infty} \left(\frac{\|f\|_{L_{\alpha\ell}} (\mathrm{e}\alpha)^{1/\alpha}}{K(\ell\alpha)^{1/\alpha}} \right)^{\alpha\ell}.$$

If we choose

$$K = (2\mathrm{e}\alpha)^{1/\alpha} \sup_{\ell \geq 1} \frac{\|f\|_{L_{\alpha\ell}}}{(\ell\alpha)^{1/\alpha}},$$

then we obtain

$$\int_{[0,1]^d} \psi_\alpha \left(\frac{|f(\boldsymbol{x})|}{K} \right) \mathrm{d}\boldsymbol{x} \leq \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} = 1.$$

Hence

$$\|f\|_{\psi_\alpha} \leq K = (2\mathrm{e}\alpha)^{1/\alpha} \sup_{\ell \geq 1} \frac{\|f\|_{L_{\alpha\ell}}}{(\alpha\ell)^{1/\alpha}} = (2\mathrm{e}\alpha)^{1/\alpha} \sup_{p \geq \alpha} \frac{\|f\|_{L_p}}{p^{1/\alpha}} \leq (2\mathrm{e}\alpha)^{1/\alpha} \|f\|_\alpha.$$

On the other hand, from (9) and the upper bound in Stirling's formula (8) we obtain

$$\begin{aligned} \int_{[0,1]^d} \psi_\alpha \left(\frac{|f(\boldsymbol{x})|}{K} \right) \mathrm{d}\boldsymbol{x} &\geq \sum_{\ell=1}^{\infty} \frac{1}{\sqrt{2\pi\ell}\, \mathrm{e}^{1/(12\ell)}} \left(\frac{\mathrm{e}}{\ell} \right)^\ell \left(\frac{\|f\|_{L_{\alpha\ell}}}{K} \right)^{\alpha\ell} \\ &\geq \sup_{\ell \geq 1} \frac{(\mathrm{e}\alpha)^\ell}{\sqrt{2\pi\ell}\, \mathrm{e}^{1/(12\ell)}} \frac{1}{K^{\alpha\ell}} \left(\frac{\|f\|_{L_{\alpha\ell}}}{(\alpha\ell)^{1/\alpha}} \right)^{\alpha\ell}. \end{aligned}$$

Now, in order to have $\int_{[0,1]^d} \psi_\alpha \left(\frac{|f(\boldsymbol{x})|}{K} \right) \mathrm{d}\boldsymbol{x} \leq 1$ we find that K has to satisfy

$$K^{\alpha\ell} \geq \frac{(\mathrm{e}\alpha)^\ell}{\sqrt{2\pi\ell}\, \mathrm{e}^{1/(12\ell)}} \left(\frac{\|f\|_{L_{\alpha\ell}}}{(\alpha\ell)^{1/\alpha}} \right)^{\alpha\ell}$$

for all $\ell \geq 1$. Hence

$$K \geq \frac{(\mathrm{e}\alpha)^{1/\alpha}}{(\sqrt{2\pi\ell}\, \mathrm{e}^{1/(12\ell)})^{1/(\alpha\ell)}} \frac{\|f\|_{L_{\alpha\ell}}}{(\alpha\ell)^{1/\alpha}} \geq \left(\frac{\mathrm{e}\alpha}{\sqrt{2\pi}\, \mathrm{e}^{1/12}} \right)^{1/\alpha} \frac{\|f\|_{L_{\alpha\ell}}}{(\alpha\ell)^{1/\alpha}}$$

for all $\ell \geq 1$. Hence,

$$K \geq \left(\frac{\mathrm{e}\alpha}{\sqrt{2\pi}\, \mathrm{e}^{1/12}} \right)^{1/\alpha} \sup_{\ell \geq 1} \frac{\|f\|_{L_{\alpha\ell}}}{(\alpha\ell)^{1/\alpha}} = \left(\frac{\mathrm{e}\alpha}{\sqrt{2\pi}\, \mathrm{e}^{1/12}} \right)^{1/\alpha} \sup_{p \geq \alpha} \frac{\|f\|_{L_p}}{p^{1/\alpha}}.$$

For any $q \in [1, \alpha]$ we have

$$\sup_{p \geq \alpha} \frac{\|f\|_{L_p}}{p^{1/\alpha}} \geq \frac{\|f\|_{L_\alpha}}{\alpha^{1/\alpha}} \geq \frac{\|f\|_{L_q}}{q^{1/\alpha}} \frac{q^{1/\alpha}}{\alpha^{1/\alpha}} \geq \frac{1}{\alpha^{1/\alpha}} \frac{\|f\|_{L_q}}{q^{1/\alpha}}.$$

Hence

$$\sup_{p \geq \alpha} \frac{\|f\|_{L_p}}{p^{1/\alpha}} \geq \frac{1}{\alpha^{1/\alpha}} \sup_{p \geq 1} \frac{\|f\|_{L_p}}{p^{1/\alpha}}.$$

This implies

$$\|f\|_{\psi_\alpha} \geq \left(\frac{\mathrm{e}^{11/12}}{\sqrt{2\pi}} \right)^{1/\alpha} \|f\|_\alpha$$

as desired. This closes the proof. □

We are now prepared to present the proofs of our main results.

2.1 The proof of Theorem 1

An important consequence of Lemma 1 is that the constants do not depend on the dimension, and hence the Orlicz norm discrepancy satisfies the same tractability properties as the discrepancy with respect to the norm $\|\cdot\|_\varphi$. Therefore in the following proof we will only use the latter norm.

It is well known and easily checked (see, e.g., [20, p. 54]) that for every $p \in [1,\infty)$, the initial L_p-discrepancy in dimension d satisfies

$$\|\Delta_\emptyset\|_{L_p} = \frac{1}{(p+1)^{d/p}}.$$

If $p=\infty$, then the initial discrepancy is 1 for every dimension $d \in \mathbb{N}$. This implies that

$$\|\Delta_\emptyset\|_\varphi = \sup_{p\geq 1} \frac{1}{\varphi(p)} \frac{1}{(p+1)^{d/p}} \geq \frac{1}{(d+1)\varphi(d)},$$

where we used the choice $p=d$ to obtain the last inequality.

From [11] we know that

$$\mathrm{disc}_{L_\infty}(N,d) \leq C_{\mathrm{PT}} \sqrt{\frac{d}{N}}, \tag{10}$$

for some absolute constant $C_{\mathrm{PT}} \in (0,\infty)$. Aistleitner [1] showed that one can choose $C_{\mathrm{PT}} = 10$, but according to [10] the constant C_{PT} may be reduced to $C_{\mathrm{PT}} = 2.5287$.

Hence, we have

$$\mathrm{disc}_\varphi(N,d) \leq \mathrm{disc}_{L_\infty}(N,d) \cdot \sup_{p\geq 1} \frac{1}{\varphi(p)} \leq C_{\mathrm{PT}} \sup_{p\geq 1} \frac{1}{\varphi(p)} \sqrt{\frac{d}{N}}\,,$$

where $\mathrm{disc}_\varphi(N,d)$ stands for the discrepancy with respect to the norm $\|\cdot\|_\varphi$ introduced in (5). This implies that

$$\begin{aligned} N_\varphi(\varepsilon,d) &\leq \min\left\{ N\in\mathbb{N} \,:\, C_{\mathrm{PT}} \sup_{p\geq 1} \frac{1}{\varphi(p)} \sqrt{\frac{d}{N}} \leq \frac{\varepsilon}{(d+1)\varphi(d)} \right\} \\ &\leq \left\lceil C_{\mathrm{PT}}^2 \frac{d(d+1)^2\varphi^2(d)}{\varepsilon^2} \sup_{p\geq 1} \frac{1}{\varphi^2(p)} \right\rceil, \end{aligned} \tag{11}$$

where for $x \in \mathbb{R}$, $\lceil x \rceil := \min\{n\in\mathbb{Z} \,:\, n \geq x\}$. This concludes the proof of the second statement in Theorem 1. As mentioned above, if we choose $\varphi(\alpha p) = (p!)^{1/(\alpha p)}$, then we obtain the ψ_α-norm. Using Stirling's formula (8) together with the previous result, we can deduce the first part of Theorem 1.

In order to prove the third part of Theorem 1, we apply the logarithm to $N_\varphi(\varepsilon,d)$. From (11) we obtain that

$$\log N_\varphi(\varepsilon,d) \leq C' + 2\log\varepsilon^{-1} + 3\log(d+1) + 2\log\varphi(d)$$

for some $C' \in (0,\infty)$ only depending on φ. Hence,

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\log N_\varphi(\varepsilon,d)}{d+\varepsilon^{-1}} \leq 2 \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\log\varphi(d)}{d+\varepsilon^{-1}} = 0.$$

This implies weak tractability of the discrepancy with respect to $\|\cdot\|_{\psi_{\alpha,\varphi}}$. $\square$

2.2 The proof of Theorem 2

First we show the corresponding result for $N_\alpha(\varepsilon, d)$ which is based on the norm $\|\cdot\|_\alpha$. Recall that for a measurable function $f : [0,1]^d \to \mathbb{R}$, we defined $\|f\|_\alpha = \sup_{p\geq 1} p^{-1/\alpha}\|f\|_{L_p}$. Let us start with a lower bound for the initial discrepancy. We have

$$\begin{aligned}
\|\Delta_\emptyset\|_\alpha &= \sup_{p\geq 1} \frac{1}{p^{1/\alpha}} \frac{1}{(p+1)^{d/p}} \\
&\geq \frac{1}{(d\log(d+1))^{1/\alpha}} \frac{1}{(1+d\log(d+1))^{1/\log(d+1)}} \\
&\geq \frac{1}{4(d\log(d+1))^{1/\alpha}}, \qquad (12)
\end{aligned}$$

where we have chosen $p = d\log(d+1)$. The final estimate follows from the fact that

$$d \mapsto \frac{1}{(1+d\log(d+1))^{1/\log(d+1)}}$$

attains its minimum in $d = 20$ with minimal value $0.257944\ldots$.

Now let $d \in \mathbb{N}$. Then from Gnewuch [9, Theorem 3] we obtain that

$$\mathbb{E}\,\|\Delta_{\mathcal{P}}\|_{L_d} \leq 2^{5/4}3^{-3/4}\,N^{-1/2}$$

and from Aistleitner and Hofer [2, Corollary 1] that for any $q \in (0,1)$

$$\mathbb{P}\left[\|\Delta_{\mathcal{P}}\|_{L_\infty} \leq 5.7\sqrt{4.9+\log((1-q)^{-1})}d^{1/2}N^{-1/2}\right] \geq q,$$

where the expectation and probability are with respect to the point set $\mathcal{P}$ consisting of independent and uniformly distributed points. Now Markov's inequality implies that there exists an N-element point set $\mathcal{P}$ in $[0,1)^d$ such that

$$\|\Delta_{\mathcal{P}}\|_{L_d} \leq a\,N^{-1/2} \qquad \text{and} \qquad \|\Delta_{\mathcal{P}}\|_{L_\infty} \leq a\,d^{1/2}N^{-1/2}$$

provided that

$$1 > \frac{2^{5/4}}{3^{3/4}a} + \exp\left(4.9 - \left(\frac{a}{5.7}\right)^2\right).$$

For this point set $\mathcal{P}$, we obtain

$$\begin{aligned}
\|\Delta_{\mathcal{P}}\|_\alpha &= \sup_{p\geq 1} p^{-1/\alpha}\|\Delta_{\mathcal{P}}\|_{L_p} \\
&= \max\left\{\sup_{p\leq d} p^{-1/\alpha}\|\Delta_{\mathcal{P}}\|_{L_p}, \sup_{p\geq d} p^{-1/\alpha}\|\Delta_{\mathcal{P}}\|_{L_p}\right\} \\
&\leq a\,N^{-1/2}\,\max\left\{\sup_{p\leq d} p^{-1/\alpha}, \sup_{p\geq d} p^{-1/\alpha}d^{1/2}\right\} \\
&= a\,N^{-1/2}\,\max\left\{1, d^{1/2-1/\alpha}\right\}. \qquad (13)
\end{aligned}$$

Combining (12) and (13), we obtain the upper bound

$$\begin{aligned}
N_\alpha(\varepsilon, d) &\leq \min\left\{N \in \mathbb{N} : \frac{a}{N^{1/2}} \max\left\{1, d^{1/2-1/\alpha}\right\} \leq \varepsilon\,\frac{1}{4(d\log(d+1))^{1/\alpha}}\right\} \\
&= \left\lceil 16\cdot a^2\,d^{2/\alpha}\,\max\left\{1, d^{1-2/\alpha}\right\}\,\big(\log(d+1)\big)^{2/\alpha}\varepsilon^{-2}\right\rceil \\
&= \left\lceil 16\cdot a^2\,d^{\max\{1,2/\alpha\}}\,\big(\log(d+1)\big)^{2/\alpha}\varepsilon^{-2}\right\rceil
\end{aligned}$$

for all $\alpha \in [1, \infty)$. Note that we may choose $a = 12.75$ leading to $16a^2 = 2601$.

Using the second part of Lemma 1, we obtain

$$N_{\psi_\alpha}(\varepsilon, d) \leq N_\alpha(\varepsilon', d)$$

with

$$\varepsilon' = \varepsilon \left(\frac{\mathrm{e}^{11/12}}{\sqrt{2\pi}} \right)^{1/\alpha} \alpha^{-1/\alpha} .$$

From this we finally obtain the upper bound for $N_{\psi_\alpha}(\varepsilon, d)$. $\square$

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