Estimates of the asymptotic Nikolskii constants for spherical polynomials

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Abstract. Let Π_n^d denote the space of spherical polynomials of degree at most n on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ that is equipped with the surface Lebesgue measure $d\sigma$ normalized by $\int_{\mathbb{S}^d} d\sigma(x) = 1$. This paper establishes a close connection between the asymptotic Nikolskii constant,

$$\mathcal{L}^*(d) := \lim_{n\to\infty} \frac{1}{\dim \Pi_n^d} \sup_{f\in\Pi_n^d} \frac{\|f\|_{L^\infty(\mathbb{S}^d)}}{\|f\|_{L^1(\mathbb{S}^d)}},$$

and the following extremal problem:

$$\mathcal{I}_\alpha := \inf_{a_k} \Big\| j_{\alpha+1}(t) - \sum_{k=1}^{\infty} a_k j_\alpha\big(q_{\alpha+1,k} t/q_{\alpha+1,1}\big) \Big\|_{L^\infty(\mathbb{R}_+)}$$

with the infimum being taken over all sequences $\{a_k\}_{k=1}^\infty \subset \mathbb{R}$ such that the infinite series converges absolutely a.e. on $\mathbb{R}_+$. Here j_α denotes the Bessel function of the first kind normalized so that $j_\alpha(0) = 1$, and $\{q_{\alpha+1,k}\}_{k=1}^\infty$ denotes the strict increasing sequence of all positive zeros of $j_{\alpha+1}$. We prove that for $\alpha \ge -0.272$,

$$\mathcal{I}_\alpha = \frac{\int_0^{q_{\alpha+1,1}} j_{\alpha+1}(t) t^{2\alpha+1}\, dt}{\int_0^{q_{\alpha+1,1}} t^{2\alpha+1}\, dt} = {}_1F_2\Big(\alpha+1; \alpha+2, \alpha+2; -\frac{q_{\alpha+1,1}^2}{4}\Big).$$

As a result, we deduce that the constant $\mathcal{L}^*(d)$ goes to zero exponentially fast as $d \to \infty$:

$$0.5^d \le \mathcal{L}^*(d) \le (0.857\cdots)^{d\,(1+\varepsilon_d)} \qquad \text{with } \varepsilon_d = O(d^{-2/3}).$$

1. Introduction

Let $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} \colon |x| = 1\}$ denote the unit sphere of $\mathbb{R}^{d+1}$, and $d\sigma$ the surface Lebesgue measure on $\mathbb{S}^d$ normalized by $\int_{\mathbb{S}^d} d\sigma(x) = 1$, where $|\cdot|$ denotes the Euclidean

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norm of $\mathbb{R}^{d+1}$. Denote by $\omega_d := \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$ the surface area of $\mathbb{S}^d$. Given $0 < p \le \infty$, let $L^p(\mathbb{S}^d)$ denote the Lebesgue L^p-space defined with respect to the measure $d\sigma$ on $\mathbb{S}^d$, and $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{S}^d)}$ the quasi-norm of $L^p(\mathbb{S}^d)$. We denote by $R_k^{(\alpha,\beta)}$, $\alpha, \beta \in \mathbb{R}$, the usual Jacobi polynomial of degree n normalized by $R_k^{(\alpha,\beta)}(1) = 1$, and by C_k^μ, $\mu > 0$, the Gegenbauer polynomial of degree k.

A spherical polynomial of degree at most n on $\mathbb{S}^d$ is the restriction to $\mathbb{S}^d$ of an algebraic polynomial in $d+1$ variables of total degree at most n. Let Π_n^d denote the space of all spherical polynomials of degree at most n on $\mathbb{S}^d$. As is well known (see, e.g., [**13**, Chap. 1]),

$$\dim \Pi_n^d = \frac{2n+d}{n+d}\binom{n+d}{n} = \frac{2n^d}{\Gamma(d+1)}\,(1 + O(n^{-1})), \quad n \to \infty. \tag{1.1}$$

The classical Nikolskii inequality for spherical polynomials ([**27**]) asserts that there exists a positive constant C_d depending only on the dimension d such that for any $0 < p < q \le \infty$,

$$\|f\|_{L^q(\mathbb{S}^d)} \le C_d\big(\dim \Pi_n^d\big)^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^p(\mathbb{S}^d)}, \quad \forall\, f \in \Pi_n^d.$$

In this paper, we are mainly interested in the best Nikolskii constant defined as follows:

$$\mathcal{N}(\mathbb{S}^d; n)_{p,q} := \sup\Big\{\|f\|_{L^q(\mathbb{S}^d)} :\ f \in \Pi_n^d \quad \text{and} \quad \|f\|_{L^p(\mathbb{S}^d)} = 1\Big\}, \tag{1.2}$$

where $0 < p < q \le \infty$ and $n \in \mathbb{N}$.

Exact values of the constants $\mathcal{N}(\mathbb{S}^d; n)_{p,q}$ are known only in the case of $p = 2$ and $q = \infty$, where one has (see [**14**])

$$\mathcal{N}(\mathbb{S}^d; n)_{2,\infty} = \sqrt{\dim \Pi_n^d}. \tag{1.3}$$

For the general case, the following estimates are known ([**27, 14**]):

$$0 < c_d < \frac{\mathcal{N}(\mathbb{S}^d; n)_{p,q}}{(\dim \Pi_n^d)^{\frac{1}{p}-\frac{1}{q}}} \le C_d < \infty, \quad 0 < p < q \le \infty, \tag{1.4}$$

where $C_d = 1$ in the case of $0 < p \le 2$. However, it is a long-standing open problem to find the exact values of the Nikolskii constants $\mathcal{N}(\mathbb{S}^d; n)_{p,q}$ for $(p, q) \ne (2,\infty)$ and $0 < p < q \le \infty$. In fact, this problem is open even in the case of $d = 1$ ([**4, 22**]).

For $d = 1$, Levin and Lubinsky [**29, 30**] established very close connections between the asymptotic behaviour of the quantity $\frac{\mathcal{N}(\mathbb{S}^1;n)_{p,q}}{(2n+1)^{\frac{1}{p}-\frac{1}{q}}}$ as $n \to \infty$ and the best Nikolskii constant for entire functions of exponential type on $\mathbb{R}$. Their results were recently extended to the higher-dimensional case by the current authors [**14**]. To state these results more precisely, we recall that an entire function f of d-complex variables is said to be of spherical exponential type at most $\sigma > 0$ if for every $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that $|f(z)| \le A_\varepsilon e^{(\sigma+\varepsilon)|z|}$ for all $z = (z_1, \cdots, z_d) \in \mathbb{C}^d$. Given $0 < p \le \infty$, we denote by $\mathcal{E}_p^d$ the class of all entire functions f in d-variables of spherical exponential type at most 1 whose restrictions to $\mathbb{R}^d$ belong to the space $L^p(\mathbb{R}^d)$ ([**36**, Ch. 3]). For $0 < p < q \le \infty$, define

$$\mathcal{N}(\mathbb{R}^d)_{p,q} := \sup\Big\{\|f\|_{L^q(\mathbb{R}^d)} :\ f \in \mathcal{E}_p^d \ \text{ and } \ \|f\|_{L^p(\mathbb{R}^d)} = 1\Big\}.$$

Throughout this paper, we will consider the Nikolskii constants for real-valued functions (i.e., those functions in $\mathcal{E}_p^d$ whose restrictions to $\mathbb{R}^d$ are real-valued). This will not cause any problem as for every $f \in \mathcal{E}_p^d$, $g(z) := \frac{1}{2}\big(f(z) + \overline{f(\bar{z})}\big)$ is a function in $\mathcal{E}_p^d$ whose restriction to $\mathbb{R}^d$ is real-valued (see also [**17**, Theorem 1.1]).

The following result was proved first by Levin and Lubinsky [**29, 30**] for $d = 1$ and later by the current authors [**14**] for $d \geq 2$. [1]

Theorem A ([**29, 30, 14**]). *For $0 < p < \infty$, we have*

$$\lim_{n\to\infty} \frac{\mathcal{N}(\mathbb{S}^d; n)_{p,\infty}}{(\dim \Pi_n^d)^{\frac{1}{p}}} = \Big(\frac{(2\pi)^d}{V_d}\Big)^{\frac{1}{p}} \mathcal{N}(\mathbb{R}^d)_{p,\infty} =: \mathcal{L}^*_{p,\infty}(d), \tag{1.5}$$

where $V_d := \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$ denotes the volume of the unit ball in $\mathbb{R}^d$. Furthermore, if $0 < p < q \leq \infty$, then

$$\liminf_{n\to\infty} \frac{\mathcal{N}(\mathbb{S}^d; n)_{p,q}}{(\dim \Pi_n^d)^{\frac{1}{p}-\frac{1}{q}}} \geq \Big(\frac{(2\pi)^d}{V_d}\Big)^{\frac{1}{p}-\frac{1}{q}} \mathcal{N}(\mathbb{R}^d)_{p,q} =: \mathcal{L}^*_{p,q}(d).$$

Note that (1.5) implies that

$$\mathcal{N}(\mathbb{S}^d; n)_{p,\infty} = \mathcal{L}^*_{p,\infty}(d)\big(\dim \Pi_n^d\big)^{\frac{1}{p}}\big(1 + o(1)\big), \quad 0 < p < \infty, \quad \text{as } n \to \infty. \tag{1.6}$$

Furthermore, by (1.3), (1.1) and (1.6), we obtain

$$\mathcal{N}(\mathbb{R}^d)_{2,\infty} = \frac{\sqrt{V_d}}{(2\pi)^{d/2}} = \Big(\frac{1}{2^d\Gamma(\frac{d}{2}+1)\pi^{d/2}}\Big)^{1/2}.$$

In this paper, we continue the research of [**14**]. We shall establish more explicit duality formulas for the constants $\mathcal{N}(\mathbb{S}^d; n)_{p,\infty}$ and $\mathcal{N}(\mathbb{R}^d)_{p,\infty}$ with $1 \leq p < \infty$. For example, in Section 4, we prove

$$\mathcal{N}(\mathbb{S}^d; n)_{1,\infty} = (\dim \Pi_n^d) \inf_{a_k} \Big\| R_n^{(\frac{d}{2},\frac{d-2}{2})} - \sum_{k=n+1}^{\infty} a_k C_k^{\frac{d-1}{2}} \Big\|_{L^\infty[-1,1]} \tag{1.7}$$

with the infimum being taken over all sequences $\{a_k\}_{k=n+1}^\infty \subset \mathbb{R}$ such that the series $\sum_{k=n+1}^\infty a_k C_k^{\frac{d-1}{2}}(t)$ converges to an essentially bounded function in L^2-norm with respect to the measure $(1-t^2)^{\frac{d-2}{2}} dt$ on $[-1,1]$. One of our main goals is to apply these duality formulas to estimate the constant $\mathcal{L}^*_{p,\infty}(d)$ in the asymptotic expansion (1.6) for $p = 1$. For simplicity, we write

$$\mathcal{L}^*(d) := \mathcal{L}^*_{1,\infty}(d).$$

Note that by (1.3) and (1.4), if $0 < p \leq 2$, then for any d,

$$\mathcal{L}^*_{p,\infty}(d) \leq 1 \tag{1.8}$$

with equality for $p = 2$.

[1] Note that the definition of the constant $\mathcal{N}(\mathbb{S}^d, n)_{p,q}$ here is slightly different from that of the constant $C(n, d, p, q)$ in [**14**] due to the normalization of the surface Lebesgue measure $d\sigma$. Indeed, we have $\mathcal{N}(\mathbb{S}^d; n)_{p,q} = \omega_d^{\frac{1}{p}-\frac{1}{q}} C(n, d, p, q)$.

The estimates for the Nikolskii constant $\mathcal{L}^*(d)$ are important in many applications. Let us mention only a few of them here. First of all, the constant $\mathcal{L}^*(d)$ appears very naturally in problems on best L^1-approximation (see, e.g., [**8, 18, 19, 31**]). It can be used to obtain certain tight bounds in the Remez-type problem about the concentration of L^1-norm of entire functions of the spherical exponential type ([**8, 33**], see also [**43**]). Some details can be found in Section 7. The next example is widely known. The constant $\mathcal{L}^*(d)$ can be used to obtain some lower tight-bounds for spherical designs (see [**28**]). Moreover, the Nikolskii constants play an important role in approximation of smooth, multivariate functions defined on irregular domains by polynomial frame approximation method [**7**]. More detailed historical comments on the constant $\mathcal{L}^*(d)$ and related background information will be given in Section 2.

While it remains to be very challenging to find the exact values of the constants $\mathcal{L}^*(d)$, in [**14**] we solved this problem for non-negative functions in the class $\mathcal{E}_1^d$:

THEOREM 1.1 ([**14**]). *For $d \in \mathbb{N}$, we have*

$$\mathcal{L}^+(d) := \frac{(2\pi)^d}{V_d} \sup_{\substack{f \in \mathcal{E}_1^d \setminus \{0\}, \\ f|_{\mathbb{R}^d} \geq 0}} \frac{\|f\|_{L^\infty(\mathbb{R}^d)}}{\|f\|_{L^1(\mathbb{R}^d)}} = 2^{-d}.$$

One of the main results in this paper asserts that the estimate (1.8) can be significantly improved for $p = 1$, and the constant $\mathcal{L}^*(d)$ goes to zero exponentially fast as $d \to \infty$:

THEOREM 1.2. *For $d \in \mathbb{N}$, we have*

$$2^{-d} \leq \mathcal{L}^*(d) \leq {}_1F_2\Big(\frac{d}{2}; \frac{d}{2}+1, \frac{d}{2}+1; -\frac{\beta_d^2}{4}\Big) = \frac{\int_0^{\beta_d} j_{d/2}(t) t^{d-1}\, dt}{\int_0^{\beta_d} t^{d-1}\, dt},$$

where ${}_1F_2$ denotes the usual hypergeometric function, $j_{d/2}$ is the normalized Bessel function, and $\beta_d = q_{d/2,1}$ is the smallest positive zero of the Bessel function $J_{d/2}$ of the first kind.

COROLLARY 1.1. *For $d \geq 2$, we have*

$$2^{-d} \leq \mathcal{L}^*(d) \leq (\sqrt{2/e})^{d\,(1+\varepsilon_d)},$$

where $\sqrt{2/e} = 0.857\cdots$, and $\varepsilon_d = O(d^{-2/3})$ as $d \to \infty$.

Using Theorem 1.2, we may obtain the numerical upper estimates of $\mathcal{L}^*(d)$ for $d = 1, 2, \ldots, 10$, listed in the following table:

d	1	2	3	4	5	6	7	8	9	10
upper bounds	0.589	0.382	0.261	0.184	0.133	0.098	0.073	0.055	0.042	0.032

Note that for $d = 1$, we recover the upper bound of $\mathcal{L}^*(1)$ previously obtained in [**25, 2**], while for $d = 2$, our method with more delicate calculations leads to the following estimate:

$$\mathcal{N}(\mathbb{S}^2; n)_{1,\infty} = \mathcal{L}^*(2) n^2 (1 + o(1)) \quad \text{with} \quad \mathcal{L}^*(2) \in (0.2820, 0.3822),$$

which improves the corresponding known estimate in [**1, 24**].

Let us give a few comments on the proof of Theorem 1.2. Clearly, the lower estimate in Theorem 1.2 follows directly from Theorem 1.1. However, the proof of the upper estimate in Theorem 1.2 is much more involved. It relies on a duality argument (see, for instance, (1.7)). The crucial ingredient in the proof is to solve an extremal problem on L^∞-approximation by the Bessel functions of the first kind on $\mathbb{R}_+ = [0,\infty)$, which seems to be of independent interest.

To be more precise, we need to introduce several notations. For $\alpha \in \mathbb{C}$, let J_α denote the Bessel function of the first kind, and j_α the normalized Bessel function given by

$$j_\alpha(z) := 2^\alpha \Gamma(\alpha+1)\,\frac{J_\alpha(z)}{z^\alpha}, \quad z \in \mathbb{C}.$$

Let $\{q_{\alpha,k}\}_{k=1}^\infty$ denote the strictly increasing sequence of all positive zeros of $j_\alpha(z)$. For $\alpha > -\frac{1}{2}$, we denote by X_α the set of all functions $F \in L^\infty[0,\infty)$ that can be represented as an infinite sum of the form

$$F(t) := \sum_{k=1}^\infty a_k j_\alpha\big(r_{\alpha+1,k} t\big), \quad t \geq 0, \quad a_k \in \mathbb{R}, \quad r_{\alpha+1,k} = \frac{q_{\alpha+1,k}}{q_{\alpha+1,1}}, \quad k \in \mathbb{N}.$$

Here we assume that the series converges absolutely to F almost everywhere on $\mathbb{R}_+$. In the proof of the upper estimate in Theorem 1.2 , we are required to solve the following extremal problem for $\alpha = \frac{d}{2} - 1$:

$$\mathcal{I}_\alpha := \inf_{F \in X_\alpha} \|j_{\alpha+1} - F\|_{L^\infty(\mathbb{R}_+)}. \tag{1.9}$$

In this paper, we find the exact value of $\mathcal{I}_\alpha$ for $\alpha \geq -0.272$, from which the upper estimate in Theorem 1.2 will follow:

THEOREM 1.3. *Let $\alpha \geq -0.272$ and let $\mathcal{I}_\alpha$ be defined in* (1.9). *Then*

$$\mathcal{I}_\alpha = {}_1F_2\Big(\alpha+1; \alpha+2, \alpha+2; -\frac{q_{\alpha+1,1}^2}{4}\Big) = \frac{\int_0^{q_{\alpha+1,1}} j_{\alpha+1}(t) t^{2\alpha+1}\,dt}{\int_0^{q_{\alpha+1,1}} t^{2\alpha+1}\,dt}. \tag{1.10}$$

The identity (1.10) extends the following result (see [**2, 22**]):

$$\inf_{a_k \in \mathbb{R}} \Big\| \frac{\sin t}{t} - \sum_{k=1}^\infty a_k \cos kt \Big\|_{L^\infty(\mathbb{R}_+)} = \frac{1}{\pi} \int_0^\pi \frac{\sin x}{x}\,dx. \tag{1.11}$$

We point out that the proof of this last formula in [**22**] relies on the fact that the corresponding extremal function is a periodic function, which does not seem to work in our situation. Our proof of (1.10) in this paper is different from that in [**22**].

This paper is organized as follows. Section 2 contains some background information and historical comments on sharp Nikolskii constants. Some preliminary materials on spherical harmonics and Bessel functions are given in Section 3. In Section 4, we deduce more explicit duality formulas for the Nikolskii constants, and connect our problem with several other extremal problems in approximation theory. We also study the existence, uniqueness and characterizations of the corresponding extremal functions for these extremal problems in Section 4. After that, in Section 5, we prove the main theorem, Theorem 1.3, from which the upper estimates in Theorem 1.2 will follow. The proof of Corollary 1.1 is given in Section 6. Finally, in Section 7, we show how our results on the Nikolskii constants can be used to deduce certain interesting Remez-type results.

2. Historical background

In this section, we give some background information and historical comments on the Nikolskii constants. Nikolskii inequalities have been playing crucial roles in approximation theory and harmonic analysis, particularly in the embedding theory of function spaces (see [**36, 16**]).

In the case of $d = 1$, the problem of finding the exact values of the constants $\mathcal{N}(\mathbb{S}^1; n)_{1,\infty}$ has a very long history, starting with the work of Jackson [**26**] in 1933. A closed form of the constant $\mathcal{N}(\mathbb{S}^1; n)_{1,\infty}$, which is not very useful in applications, was found by Geronimus [**18**]. Stechkin (see [**41, 42**]) proved that there is a constant c such that $\mathcal{N}(\mathbb{S}^1; n)_{1,\infty} = cn + o(n)$ as $n \to \infty$, while Taikov [**41**]) further proved that $\mathcal{N}(\mathbb{S}^1; n)_{1,\infty} = cn + O(1)$ with $c \in (0.539, 0.584)$. In [**22, 23**] it was established that $c = \mathcal{L}^*(1)$ and for any n and $0 < p < \infty$

$$(2n)^{1/p}\mathcal{L}^*_{p,\infty}(1) \le \mathcal{N}(\mathbb{S}^1; n)_{p,\infty} \le (2n + 2\lceil p^{-1}\rceil)^{1/p}\mathcal{L}^*_{p,\infty}(1)$$

(see also [**29, 17**]). In the limiting case of $p = 0$, Arestov [**3**] found the exact values of the Nikolskii constants for the trigonometric polynomials on the unit circle $\mathbb{S}^1$. Finally, in the case of $d \ge 2$ and $0 < p < q = \infty$, Arestov and Deikalova [**4**] proved that the supremum in (1.2) can be achieved by zonal polynomials, and as a result, the Nikolskii constant $\mathcal{N}(\mathbb{S}^2; n)_{1,\infty}$ for spherical polynomials coincides with the Nikolskii constant for algebraic polynomials in $L^1([-1, 1])$ [**1, 24**].

As was mentioned in the introduction, of crucial importance in the proofs of the main results in this paper are the duality formulas for the Nikolskii constants, which will be given in the next section. In the case of $\mathbb{S}^1$ this approach was introduced by Taikov [**41**], who established the classical Bernstein result on the best approximation of $\cos nx$ by functions $\sum_{k=n+1}^{\infty} a_k \cos kx \in L^\infty[0, 2\pi)$. L. Hörmander and B. Bernhardsson [**25**] proved that

$$\mathcal{L}^*(1) = \inf_v \left\| \frac{\sin x}{x} - v(x) \right\|_{L^\infty(\mathbb{R})}, \quad \widehat{v} = 0 \quad \text{in} \quad (-1, 1), \tag{2.1}$$

and described general properties of the extremal function $G \in \mathcal{E}^1_1$ satisfying $\mathcal{L}^*(1) = \frac{\|G\|_{L^\infty(\mathbb{R})}}{\|G\|_{L^1(\mathbb{R})}}$. Furthermore, they also computed the following very precise numerical value: $\mathcal{L}^*(1) \approx 0.54092882$ (cf. with [**22, 23**]).

In the particular case when v has the form $\sum_{k=1}^{\infty} a_k \cos kt$ the problem (2.1) was considered by Andreev, Konyagin, and Popov [**2**] (see (1.11)), who studied a constant that is equivalent to $\mathcal{L}^*(1)$ via the Fourier transform, that is (see also [**22**])

$$\mathcal{L}^*(1) = \sup_{F \ne 0} \frac{|F(0)|}{\|F\|_{L^1(\mathbb{R})}}, \quad \widehat{F} = 0 \quad \text{in} \quad [-1, 1]^c. \tag{2.2}$$

Some interesting applications of the Nikolskii constants in number theory can be found in the paper by Carneiro, Milinovich, and Soundararajan [**10**], who considered a family of problems related to the Nikolskii constant (2.2) and applied the resulting estimates to study the problem on the distribution of prime numbers. The paper [**10**] also considers a version of the Nikolskii problem when $\widehat{F} \le 0$ outside $[-1, 1]$. This problem for $F \ge 0$ corresponds to the extremal Cohn-Elkies problem (also called the Delsarte problem) that is connected with the problem of sphere packing (see, e.g., [**20, 12, 11, 44**]).

We also refer to [**9, 36, 35, 8, 4, 17, 5**] for more background information on classical Nikolskii constants.

3. Preliminaries

In this section, we present some preliminary materials on spherical harmonics and Bessel functions, most of which can be found in [**13**], [**6**, Chap. 7], [**37**], and [**45**].

First, a spherical harmonic of degree n on $\mathbb{S}^d$ is the restriction to $\mathbb{S}^d$ of a homogeneous harmonic polynomial in $d+1$ variables of total degree n. We denote by $\mathcal{H}_n^d$ the space of all spherical harmonics of degree n on $\mathbb{S}^d$. As is well known, the spaces $\mathcal{H}_n^d$, $n = 0, 1, \cdots$, are mutually orthogonal with respect to the inner product of $L^2(\mathbb{S}^d)$, and for each non-negative integer n, the function $\frac{k+\lambda}{\lambda} C_k^\lambda(x \cdot y)$, $x, y \in \mathbb{S}^d$ is the reproducing kernel of the space $\mathcal{H}_n^d$, where $\lambda = \frac{d-1}{2}$ and C_n^λ denotes the usual Gegenbauer polynomial of degree n, as defined in [**39**]. Thus,

$$f(x) = \frac{k+\lambda}{\lambda} \int_{\mathbb{S}^d} f(y) C_k^\lambda(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^d, \quad f \in \mathcal{H}_k^d.$$

As a result, the reproducing kernel of the space Π_n^d of spherical polynomials of degree at most n on $\mathbb{S}^d$ is given by

$$G_n(x \cdot y) := \sum_{k=0}^n \frac{k+\lambda}{\lambda} C_k^\lambda(x \cdot y) = (\dim \Pi_n^d) R_n^{(\frac{d}{2}, \frac{d-2}{2})}(x \cdot y), \tag{3.1}$$

where $R_n^{(\alpha,\beta)}$ denotes the normalized Jacobi polynomial of degree n:

$$R_n^{(\alpha,\beta)}(t) = \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)}.$$

Second, an entire function f of d-complex variables is of spherical exponential type at most σ if for every $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that $|f(z)| \le A_\varepsilon e^{(\sigma+\varepsilon)|z|}$ for all $z = (z_1, \cdots, z_d) \in \mathbb{C}^d$ (see [**36**, Chap. 3]). Given $0 < p \le \infty$, denote by $\mathcal{E}_p^d$ the class of all entire functions of spherical exponential type at most in d-variables whose restrictions to $\mathbb{R}^d$ belong to the space $L^p(\mathbb{R}^d)$. If $0 < p < q \le \infty$, then $\mathcal{E}_p^d \subset \mathcal{E}_q^d$ and there exists a constant $C = C_{d,p,q}$ such that $\|f\|_q \le C\|f\|_p$ for all $f \in \mathcal{E}_p^d$. Moreover, every function $f \in \mathcal{E}_p^d$ is bounded on $\mathbb{R}^d$ and satisfies $|f(z)| \le \|f\|_{L^\infty(\mathbb{R}^d)} e^{\sigma|\mathrm{Im}(z)|}$, $\forall\, z \in \mathbb{C}^d$. According to the Palay-Wiener theorem, each function $f \in \mathcal{E}_p^d$ can be identified with a function in $L^p(\mathbb{R}^d)$ whose distributional Fourier transform is supported in the unit ball $\mathbb{B}^d := \{x \in \mathbb{R}^d \colon |x| \le 1\}$. Here we recall that the Fourier transform of $f \in L^1(\mathbb{R}^d)$ is defined by

$$\mathcal{F}_d f(\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi}\, dx, \quad \xi \in \mathbb{R}^d,$$

while the inverse Fourier transform is given by

$$\mathcal{F}_d^{-1} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi) e^{ix \cdot \xi}\, d\xi, \quad f \in L^1(\mathbb{R}^d), \quad x \in \mathbb{R}^d.$$

Finally, we present some well-known properties of the Bessel functions, most of which can be found in [**6**, Chap. 7], and [**45**]. The Bessel function J_α of the first kind is the solution to the differential equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0 \tag{3.2}$$

such that the limit $\lim_{x\to 0} x^{-\alpha}J_\alpha(x)$ exists and is finite. Denote by j_α the normalized Bessel function given by

$$j_\alpha(z) := 2^\alpha\Gamma(\alpha+1)\,\frac{J_\alpha(z)}{z^\alpha}, \quad z \in \mathbb{C}.$$

As is well known, $j_\alpha(z)$ is an even entire function of exponential type 1 satisfying that $j_\alpha(0) = 1$ and

$$(x^{2\alpha+2}j_{\alpha+1}(x))' = (2\alpha+2)x^{2\alpha+1}j_\alpha(x), \quad j_\alpha'(x) = -\frac{xj_{\alpha+1}(x)}{2\alpha+2}. \tag{3.3}$$

Moreover,

$$|j_\alpha(x)| \le C(1+|x|)^{-\alpha-\frac{1}{2}}, \quad x \in \mathbb{R}. \tag{3.4}$$

Note that (3.3) also implies (see [**6**, 7.2.8 (56)])

$$-\frac{zj_{\alpha+2}(z)}{2(\alpha+2)} = j_{\alpha+1}'(z) = \frac{2(\alpha+1)}{z}\left(j_\alpha(z) - j_{\alpha+1}(z)\right). \tag{3.5}$$

If $\alpha = \frac{d}{2}-1$, then the function $j_\alpha(|\cdot|)$ is the Fourier transform of the normalized surface Lebesgue measure on the sphere $\mathbb{S}^{d-1}$, while if $\alpha = \frac{d}{2}$, then the function $\frac{V_d}{(2\pi)^d}\,j_\alpha(|\cdot|)$ is the Fourier transform of the characteristic function $\chi_{\mathbb{B}^d}$ of the unit ball $\mathbb{B}^d$. That is,

$$j_{\frac{d}{2}-1}(|\xi|) = \int_{\mathbb{S}^{d-1}} e^{-ix\cdot\xi}\,d\sigma(x) = \widehat{\sigma_{d-1}}(\xi), \quad \xi \in \mathbb{R}^d,$$

and

$$\frac{V_d}{(2\pi)^d}\,\mathcal{F}_d\Big(j_{\frac{d}{2}}(|\cdot|)\Big)(\xi) = \chi_{\mathbb{B}^d}(\xi), \quad \xi \in \mathbb{R}^d, \tag{3.6}$$

where the Fourier transform $\mathcal{F}_d$ is understood in a distributional sense or in the space of $L^2(\mathbb{R}^d)$.

The zeros of $j_\alpha(z)$ are all simple and real. Let $\{q_{\alpha,k}\}_{k=1}^\infty$ denote the sequence of all positive zeros of $j_\alpha(z)$ arranged so that $0 < q_{\alpha,1} < q_{\alpha,2} < \dots$. For convenience, we also set $q_{\alpha,0} = 0$. Then $q_{\alpha,k} \sim \pi k$ as $k \to \infty$, and for $\alpha > 0$, the smallest positive zero of $j_\alpha(z)$ satisfies

$$\sqrt{\alpha(\alpha+2)} < q_{\alpha,1} < \sqrt{\alpha+1}\,(\sqrt{\alpha+2}+1). \tag{3.7}$$

Moreover,

$$j_\alpha(z) = \prod_{k=1}^\infty\Big(1 - \frac{z^2}{q_{\alpha,k}^2}\Big), \quad z \in \mathbb{C}. \tag{3.8}$$

The positive zeros of $j_\alpha(z)$ and $j_{\alpha+1}(z)$ are interplaced:

$$0 < q_{\alpha,1} < q_{\alpha+1,1} < q_{\alpha,2} < q_{\alpha+1,2} < \dots. \tag{3.9}$$

The following result on the zeros of the Bessel functions will be used repeatedly in later sections:

LEMMA 3.1. *For $\alpha > -1/2$ and $k = 1, 2, \cdots$, we have*

$$\max_{z \ge q_{\alpha+1,k}} |j_\alpha(z)| = |j_\alpha(q_{\alpha+1,k})| = (-1)^k j_\alpha(q_{\alpha+1,k}) > 0. \tag{3.10}$$

PROOF. For the sake of completeness, we include a short proof of this lemma here. Since

$$\left(j_\alpha(z)^2\right)' = 2 j_\alpha(z) j_\alpha'(z) = -\frac{z j_{\alpha+1}(z) j_\alpha(z)}{\alpha+1},$$

the function $(j_\alpha(z))^2$ achieves its local maxima on $(0,\infty)$ at the positive zeros of $j_{\alpha+1}(z)$, on which we also have $(j_\alpha(z))^2 = (j_\alpha(z))^2 + (j_\alpha'(z))^2$. However, it is easily seen from (3.2) that

$$\Big((j_\alpha(z))^2 + (j_\alpha'(z))^2\Big)' = -\frac{2(2\alpha+1)}{z}\,(j_\alpha'(z))^2,$$

which implies that the function $(j_\alpha(z))^2 + (j_\alpha'(z))^2$ is strictly decreasing on $(0,\infty)$ if $\alpha > -1/2$. Thus, the sequence

$$\left\{\Big(j_\alpha(q_{\alpha+1,k})\Big)^2\right\}_{k=1}^\infty = \left\{\Big(j_\alpha(q_{\alpha+1,k})\Big)^2 + \left\{\Big(j_\alpha'(q_{\alpha+1,k})\Big)^2\right\}_{k=1}^\infty\right.$$

is strictly increasing. It then follows that

$$\max_{z \ge q_{\alpha+1,k}} |j_\alpha(z)| = |j_\alpha(q_{\alpha+1,k})|.$$

Finally, the second equality in (3.10) is a direct consequence of (3.9). $\square$

For $\alpha > -\frac{1}{2}$, the Fourier–Bessel expansion of a function $f \in L^1([0,1], t^{2\alpha+1}\,dt)$ with respect to the orthogonal basis $\{j_\alpha(q_{\alpha+1,k}x)\}_{k=0}^\infty$ is given by

$$f(t) = \sum_{k=0}^\infty h_k^{-1} c_k(f) j_\alpha(q_k t), \quad t \in [0,1], \tag{3.11}$$

where

$$h_0 = \int_0^1 t^{2\alpha+1}\,dt = \frac{1}{2\alpha+2}, \quad h_k = \int_0^1 j_\alpha^2(q_k t) t^{2\alpha+1}\,dt = \frac{j_\alpha^2(q_k)}{2}, \quad k = 1, 2, \cdots,$$

$$c_k(f) = \int_0^1 f(t) j_\alpha(q_k t) t^{2\alpha+1}\,dt, \quad k = 0, 1, \cdots.$$

If $f \in C^1([0,1])$, then the series (3.11) converges absolutely outside of a neighbourhood of the origin.

For later applications, we also record here the following two useful formulas on Bessel functions (see [**32**, Sect. 6.2.10]):

$$\int_0^1 j_\alpha(at) j_\alpha(bt) t^{2\alpha+1}\,dt = \frac{a^2 j_{\alpha+1}(a) j_\alpha(b) - b^2 j_\alpha(a) j_{\alpha+1}(b)}{2(\alpha+1)(a^2-b^2)}, \quad a > b > 0, \tag{3.12}$$

$$\int_0^z t^{2\alpha+1} j_{\alpha+1}(t)\,dt = \frac{z^{2\alpha+2}}{2\alpha+2}\, {}_1F_2\Big(\alpha+1; \alpha+2, \alpha+2; -\frac{z^2}{4}\Big), \quad z > 0. \tag{3.13}$$

4. Duality formulas and characterizations of certain extremal functions

The main goals in this section are to prove some duality formulas for the Nikolskii constants $\mathcal{N}(\mathbb{S}^d; n)_{p,\infty}$ and $\mathcal{N}(\mathbb{R}^d)_{p,\infty}$, and to characterize the corresponding extremal functions in the dual spaces. These results will play an important role in the proofs of our main theorems in the next section. For simplicity, we shall write $\mathcal{N}(\mathbb{S}^d; n)_p = \mathcal{N}(\mathbb{S}^d; n)_{p,\infty}$ and $\mathcal{N}(\mathbb{R}^d)_p = \mathcal{N}(\mathbb{R}^d)_{p,\infty}$.

We start with some necessary notations. Let $w_d(t) = c_d(1-t^2)^{d/2-1}$, where $c_d > 0$ is a normalization constant such that $\int_{-1}^1 w_d(t)\,dt = 1$. For $1 \le p \le \infty$, we denote by $L^p([-1,1]; w_d) \equiv L^p(w_d)$ the usual Lebesgue L^p-space defined with respect to the measure $w_d(t)\,dt$ on $[-1,1]$, and $\|\cdot\|_{L^p(w_d)}$ the Lebesgue L^p-norm of the space $L^p(w_d)$. Denote by $\mathcal{P}_n$ the space of all univariate algebraic polynomials of degree at most n. Define

$$\mathcal{P}_n^\perp = \Big\{F \in L^1(w_d)\colon \quad \int_{-1}^1 F(t)t^j w_d(t)\,dt = 0, \quad j = 0, 1, \cdots, n\Big\},$$

and $\mathcal{P}_{n,p}^\perp = \mathcal{P}_n^\perp \cap L^p(w_d)$ for $1 \le p \le \infty$. Finally, given a normed linear space $(X, \|\cdot\|)$, the distance of a vector $x \in X$ from a set $E \subset X$ is defined by

$$\operatorname{dist}(x, E)_X := \inf_{y\in E} \|x - y\|.$$

As is well known, if E is a linear subspace of X, then one has (see, for instance, [**15**, p. 61, Theorem 1.3]),

$$\operatorname{dist}(x, E)_X = \max_{\substack{\ell\in E^\perp \\ \|\ell\|_{X^*}\le 1}} |\langle \ell, x\rangle|, \tag{4.1}$$

where

$$E^\perp := \Big\{\ell \in X^*, \quad \langle \ell, y\rangle = 0, \quad \forall\, y \in E\Big\}$$

and X^* denotes the dual of X.

Next, recall that $j_\alpha(z) = \Gamma(\alpha+1)(t/2)^{-\alpha} J_\alpha(t)$ denotes the normalized Bessel function of the first kind. Let $K(|x|) := \frac{V_d}{(2\pi)^d}\, j_{d/2}(|x|)$. For convenience, we will use a slight abuse of the notation that $f(x) = f(|x|)$ for a radial function on $\mathbb{R}^d$. By (3.6), $\mathcal{F}_d K(\xi) = \widehat{K}(\xi) = \chi_{\mathbb{B}^d}(\xi)$ for every $\xi \in \mathbb{R}^d$, and by (3.4), $K(|\cdot|) \in L^q(\mathbb{R}^d)$ for $q > \frac{2d}{d+1}$. It then follows that for each $1 \le p < \frac{2d}{d-1}$,

$$f(x) = \int_{\mathbb{R}^d} f(y) K(|x-y|)\,dy, \quad x \in \mathbb{R}^d, \quad f \in \mathcal{E}_p^d. \tag{4.2}$$

In particular, for a radial function $f(|\cdot|) \in \mathcal{E}_p^d$ with $1 \le p < \frac{2d}{d-1}$,

$$f(0) = \int_0^\infty K(t) f(t)\, v_d(t)\,dt,$$

where $v_d(t) := \omega_{d-1} t^{d-1}$. Let $L^p(v_d)$ denote the Lebesgue L^p-space defined with respect to the measure $v_d(t)dt$ on $[0,\infty)$. Clearly, for each $f \in L^p(v_d)$, $\|f(|\cdot|)\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(v_d)}$.

Our duality results for the Nikolskii constants on the sphere can be stated as follows:

THEOREM 4.1. *If $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then for every positive integer n,*

$$\mathcal{N}(\mathbb{S}^d; n)_p = \operatorname{dist}(G_n, \mathcal{P}_{n,p'}^{\perp})_{L^{p'}(w_d)}, \tag{4.3}$$

where G_n is the reproducing kernel of the space Π_n^d given in (3.1)*. Moreover, there exists a minimizer $F_* \in \mathcal{P}_{n,p'}^{\perp}$ of the form*

$$F_* = G_n - \frac{P^*(1)|P^*|^{p-1} \operatorname{sign} P^*}{\|P^*\|_{L^p(w_d)}^p}, \tag{4.4}$$

such that $\|G_n - F_*\|_{L^{p'}(w_d)} = \operatorname{dist}(G_n, \mathcal{P}_{n,p'}^{\perp})_{L^{p'}(w_d)}$, where P^* denotes the unique algebraic polynomial of degree n such that

$$\|P^*\|_{L^p(w_*)} = \operatorname{dist}(x^n, \mathcal{P}_{n-1})_{L^p(w_*)} \quad \text{with} \quad w_d^*(t) = w_d(t)(1-t).$$

Before stating the similar duality results on $\mathbb{R}^d$, we first note that

$$\mathcal{N}(\mathbb{R}^d)_p = \sup\Big\{|f(0)|: \quad f \in \mathcal{E}_p^d, \quad \|f\|_p = 1\Big\}, \quad 1 \le p < \infty. \tag{4.5}$$

This holds because each $f \in \mathcal{E}_p^d$ achieves its maximum on $\mathbb{R}^d$ (due to the fact that $f(x) \to 0$ as $|x| \to \infty$ [**36**, 3.2.5]) and the space $\mathcal{E}_p^d$ is invariant under the usual translations on $\mathbb{R}^d$.

Duality formulas for functions in $\mathcal{E}_p^d$ can now be stated as follows:

THEOREM 4.2. *The following statements hold:*

(i) *For $1 \le p < \infty$, there exists an unique radial extremizer $f_* \in \mathcal{E}_p^d$ for the supremum in* (4.5) *such that $\|f_*\|_{L^p(\mathbb{R}^d)} = 1$ and $f_*(0) = \mathcal{N}(\mathbb{R}^d)_p$. Furthermore, such an extremizer can be characterized via the following identity:*

$$g(0) = f_*(0) \int_{\mathbb{R}^d} g(x) |f_*(|x|)|^{p-1} \operatorname{sign} f_*(|x|)\, dx, \quad \forall\, g \in \mathcal{E}_p^d; \tag{4.6}$$

that is, a radial function $f_*(|\cdot|) \in \mathcal{E}_p^d$ with $\|f_*\|_{L^p(v_d)} = 1$ is an extremizer for (4.5) *if and only if the condition* (4.6) *is satisfied.*

(ii) *If $1 \le p < \frac{2d}{d-1}$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then*

$$\mathcal{N}(\mathbb{R}^d)_p = \frac{V_d}{(2\pi)^d} \operatorname{dist}(j_{d/2}, \mathcal{E}_{p'}^{\perp})_{L^{p'}(v_d)},$$

where $\mathcal{E}_{p'}^{\perp}$ denotes the space of all functions $f \in L^{p'}(v_d)$ such that

$$\int_0^\infty f(t) g(t) v_d(t)\, dt = 0 \quad \text{whenever } g(|\cdot|) \in \mathcal{E}_p^d.$$

(iii) *For each $1 \le p < \frac{2d}{d-1}$, there exists an unique extremizer $F_* \in \mathcal{E}_{p'}^{\perp}$, which takes the form*

$$F_*(t) = \frac{V_d}{(2\pi)^d} j_{d/2}(t) - f_*(0)|f_*(t)|^{p-1} \operatorname{sign} f_*(t), \quad t \ge 0, \tag{4.7}$$

such that

$$\Big\| F_* - \frac{V_d}{(2\pi)^d}\, j_{d/2} \Big\|_{L^{p'}(v_d)} = \frac{V_d}{(2\pi)^d} \operatorname{dist}(j_{d/2}(|\cdot|), \mathcal{E}_p^{\perp})_{L^{p'}(v_d)}.$$

Here f_* denotes the extremal function in (i).

As pointed out in the introduction, the main goal in this paper is to estimate the following normalized Nikolskii constant for $p=1$:

$$\mathcal{L}^*(d) = (2\pi)^d V_d^{-1} \mathcal{N}(\mathbb{R}^d)_1.$$

By the Paley–Wiener–Schwarz theorem [**35**], if $f \in \mathcal{E}_1^d$, then $\operatorname{supp} \widehat{f} \subset \mathbb{B}^d$, and hence for any $r \geq 1$,

$$\begin{aligned} 0 &= \int_{\mathbb{S}^{d-1}} \widehat{f}(r\xi)\, d\sigma(\xi) = (\widehat{D_r f}) * (d\sigma)(0) \\ &= \int_{\mathbb{R}^d} (D_r f)(-x) j_{\frac{d}{2}-1}(|x|)\, dx = \int_{\mathbb{R}^d} f(x) j_{\frac{d}{2}-1}(r|x|)\, dx, \end{aligned}$$

where $D_r f(x) = r^{-d} f(x/r)$, and in the third step we used the fact that the distributional Fourier transform of $j_{d/2-1}(|\cdot|)$ is the normalized Lebesgue measure $d\sigma$ on $\mathbb{S}^{d-1}$. This shows that $j_{\frac{d}{2}-1}(r|\cdot|) \in \mathcal{E}_\infty^\perp$ for any $r \geq 1$, as desired.

Since $j_{\frac{d}{2}-1}(r\cdot) \in \mathcal{E}_\infty^\perp$ for any $r \geq 1$, by Theorem 4.2 we obtain

Corollary 4.1. *For $d \geq 2$, we have*

$$\mathcal{L}^*(d) \leq \inf_{a_k \in \mathbb{R}, r_k \geq 1} \Big\| j_{\frac{d}{2}}(\cdot) - \sum_{k=1}^\infty a_k j_{\frac{d}{2}-1}(r_k \cdot) \Big\|_{L^\infty(\mathbb{R})}$$

with the infimum being taken over all sequences $\{a_k\}_{k=1}^\infty \subset \mathbb{R}$ and $\{r_k\}_{k=1}^\infty \subset [1,\infty)$ such that $\sum_{k=1}^\infty a_k j_{\frac{d}{2}-1}(r_k\cdot)$ converges absolutely to an essentially bounded function on $[0,\infty)$.

4.1. Proof of Theorem 4.1. For simplicity, we write $d\mu_d(t) = w_d(t)\, dt$ and $d\mu_d^*(t) = w_d^*(t)\, dt$ in the proof below.

We start with the proof of (4.3). Using orthogonality of spherical harmonics, we have that for any $f \in \Pi_n^d$ and $x \in \mathbb{S}^d$,

$$f(x) = \int_{\mathbb{S}^d} (G_n(x\cdot y) - F(x\cdot y)) f(y)\, d\sigma(y), \quad \forall\, F \in \mathcal{P}_n^\perp \cap L^{p'}(w_d).$$

It follows by Hölder's inequality that

$$\begin{aligned} \mathcal{N}(\mathbb{S}^d, n)_p = \sup_{0\neq f \in \Pi_n^d} \frac{\|f\|_\infty}{\|f\|_p} &\leq \inf\big\{ \|G_n - F\|_{L^{p'}(w_d)} \colon F \in \mathcal{P}_n^\perp \cap L^{p'}(w_d) \big\} \\ &= \operatorname{dist}(G_n, \mathcal{P}_{n,p'}^\perp)_{L^{p'}(w_d)}. \end{aligned}$$

To show the lower estimate

$$\mathcal{N}(\mathbb{S}^d, n)_p \geq \operatorname{dist}(G_n, \mathcal{P}_{n,p'}^\perp)_{L^{p'}(w_d)},$$

we use the duality formula (4.1) with $E := \mathcal{P}_{n,p'}^\perp \subset L^{p'}(w_d)$. Here, if $p=1$, then we use $C[-1,1]$ in place of L^∞, and recall that $(C[-1,1])^*$ is the space of Radon measures on $[-1,1]$ with the norm given by the total variation of a measure. Since $G_j \subset E$ for any

$j > n$, it follows that $E^{\perp} = \mathcal{P}_n$. Thus, using (4.1) , we obtain

$$\begin{aligned}\operatorname{dist}(G_n, \mathcal{P}_{n,p'}^{\perp})_{L^{p'}(w_d)} &= \sup_{\substack{\ell \in (\mathcal{P}_{n,p'}^{\perp})^{\perp} \\ \|\ell\| \le 1}} |\langle \ell, G_n \rangle| = \sup_{\substack{\|P\|_{L^p(w_d)} \le 1 \\ P \in \mathcal{P}_n}} \Big| \int_{-1}^{1} P(t) G_n(t) d\mu_d(t) \Big| \\ &= \sup_{\substack{\|P\|_{L^p(w_d)} \le 1 \\ P \in \mathcal{P}_n}} |P(1)| \le \mathcal{N}(\mathbb{S}^d; n)_p. \end{aligned} \tag{4.8}$$

This proves (4.3).

Next, we show the existence of the extremal function F_* and the formula (4.4). The proof relies on the following characterization of best approximants in L^p-spaces.

Lemma 4.1 ([**40**, 4.2.1, 4.2.2]). *Let Y be a closed real subspace of $L^p(Q, d\mu)$ for some measure space (Q, μ) and $1 \le p < \infty$. Let $f \in L^p(d\mu)$. If $p = 1$, we assume in addition that $f(x) \neq 0$ for μ-a.e. $x \in Q$. Then a function $g \in Y$ is the best approximant to f from the space Y in L^p-metric (i.e., $\|f - g\|_p = \operatorname{dist}(f, Y)_p$) if and only if*

$$\int_Q \Big(|f-g|^{p-1} \operatorname{sign}(f-g) \Big) h \, d\mu = \int_Q \frac{|f-g|^p}{f-g} h \, d\mu = 0, \quad \forall\, h \in Y.$$

Now we continue the proof of Theorem 4.1. Note first that by (4.3) and (4.8),

$$\mathcal{N}(\mathbb{S}^d; n)_p = \max \Big\{ P(1) : \quad P \in \mathcal{P}_n, \quad \|P\|_{L^p(w_d)} \le 1 \Big\}.$$

Let $P_* \in \mathcal{P}_n$ denote the maximizer for the maximum in this last equation. Then $\|P_*\|_{L^p(w_d)} = 1$ and

$$P(1) \le P_*(1) \|P\|_{L^p(w_d)}, \quad \forall\, P \in \mathcal{P}_n.$$

In particular, this implies that for any

$$\mathcal{P}_{n,0} := \{ P \in \mathcal{P}_n : \quad P(1) = 0 \},$$

we have

$$\begin{aligned} P_*(1) &\le P_*(1) \inf_{P \in \mathcal{P}_{n,0}} \|P_* - P\|_{L^p(w_d)} = P_*(1) \operatorname{dist}(P_*, \mathcal{P}_{n,0})_{L^p(w_d)} \\ &\le P_*(1) \|P_*\|_{L^p(w_d)} = P_*(1). \end{aligned}$$

Thus,

$$1 = \|P_*\|_{L^p(w_d)} = \operatorname{dist}(P_*, \mathcal{P}_{n,0})_{L^p(w_d)}.$$

It then follows from Lemma 4.1 that

$$\int_{-1}^{1} (|P_*|^{p-1} \operatorname{sign}(P_*)) P \, d\mu_d = P(1) \int_{-1}^{1} |P_*|^{p-1} \operatorname{sign}(P_*) \, d\mu_d, \quad \forall\, P \in \mathcal{P}_n. \tag{4.9}$$

Setting $P = P_*$ in (4.9), we obtain

$$1 = \|P_*\|_{L^p(w_d)}^p = P_*(1) \int_{-1}^{1} |P_*|^{p-1} \operatorname{sign}(P_*) \, d\mu_d. \tag{4.10}$$

Multiplying both sides of (4.9) by $P_*(1)$ and using (4.10), we have

$$P_*(1) \int_{-1}^{1} (|P_*|^{p-1} \operatorname{sign}(P_*)) P d\mu_d = P(1) = \int_{-1}^{1} P(t) G_n(t) d\mu_d(t), \quad \forall\, P \in \mathcal{P}_n.$$

This implies that

$$F_*(t) := G_n(t) - P_*(1)|P_*(t)|^{p-1}\operatorname{sign}(P_*(t)) \in \mathcal{P}_{n,p'}^{\perp}.$$

Note also that

$$\|G_n - F_*\|_{L^{p'}(w_d)} = P_*(1)\|P_*\|_{L^p(w_d)}^{p-1} = P_*(1) = \operatorname{dist}(G_n, \mathcal{P}_{n,p'})_{L^{p'}(w_d)}.$$

This shows (4.4) and that F_* is the desired extremal function.

Finally, we point out that the connection of P_* with the extremal polynomial P^* was proved in [**4**]. For completeness, we include a proof of the identity $P_* = P^*/\|P^*\|_{L^p(w_d)}$ here. By (4.9), we have

$$\int_{-1}^{1} |P_*|^{p-1}\operatorname{sign}(P_*)\frac{P(t)-P(1)}{1-t} w_d^*(t)\,dt = 0, \quad \forall\, P \in \mathcal{P}_n,$$

or equivalently,

$$\int_{-1}^{1} |P_*|^{p-1}\operatorname{sign}(P_*)P(t) w_d^*(t)\,dt = 0, \quad \forall\, P \in \mathcal{P}_{n-1}. \tag{4.11}$$

By Lemma 4.1, this implies that

$$\|P_*\|_{L^p(w_*)} = \operatorname{dist}(P_*, \mathcal{P}_{n-1})_{L^p(w^*)} = |L_{n,P_*}|\operatorname{dist}(x^n, \mathcal{P}_{n-1})_{L^p(w_d^*)},$$

where L_{n,P_*} denotes the leading coefficient of the n-th degree polynomial P_*. Note that by (4.11), we have $\deg(P_*) = n$ and all the zeros of P_* are simple and inside the interval $(-1,1)$. Since $P_*(1) > 0$, we must have $L_{n,P_*} > 0$. It then follows that

$$P_* = L_{n,P_*}P^* = \frac{P^*}{\|P^*\|_{L^p(w_d)}}.$$

This completes the proof of Theorem 4.1.

4.2. Proof of Theorem 4.2. We start with the proof of (i), which relies on the following compactness result on entire functions of exponential type.

LEMMA 4.2 ([**36**, 3.3.6]). *Let $1 \le p < \infty$ and let $\mathcal{B}_p^d := \{f \in \mathcal{E}_p^d\colon\ \|f\|_p \le 1\}$. Then every sequence of functions from the class $\mathcal{B}_p^d$ contains a subsequence which converges uniformly to a function $f \in \mathcal{B}_p^d$ on every compact subset of $\mathbb{R}^d$.*

By (4.5), there exists a sequence of functions $\{f_n\} \subset \mathcal{B}_p^d$ such that $\lim_{n\to\infty} f_n(0) = \mathcal{N}(\mathbb{R}^d)_p$. By Lemma 4.2, without loss of generality, we may assume that $\{f_n\}_{n=0}^{\infty}$ converges uniformly to a function $f \in \mathcal{B}_p^d$ on every compact subset of $\mathbb{R}^d$ (since otherwise we consider a subsequence of $\{f_n\}$). Then

$$f(0) = \lim_{n\to\infty} f_n(0) = \mathcal{N}(\mathbb{R}^d)_p.$$

Now consider the following radial part of the function f:

$$f_*(|x|) := \int_{\mathbb{S}^{d-1}} f(|x|\xi)\,d\sigma(\xi), \quad x \in \mathbb{R}^d.$$

For convenience, we will identify f_* with the radial function $f_*(|\cdot|)$ on $\mathbb{R}^d$. It is easily seen that $f_* \in \mathcal{E}_p^d$ and $f_*(0) = f(0) = \mathcal{N}(\mathbb{R}^d)_p$. Thus, f_* is the extremal function in (4.5); that is,

$$|f(0)| \le f_*(0)\|f\|_p, \quad \text{for every } f \in \mathcal{E}_p^d. \tag{4.12}$$

The proof of the characterization (4.6) of f_* follows exactly as that for spherical polynomials on the sphere. Indeed, applying (4.12) to $f = f_* - h$ with $g \in \mathcal{E}_p^d$ and $g(0) = 0$, we obtain

$$\|f_*\|_p = \operatorname{dist}(f_*, \mathcal{H})_p,$$

where $\mathcal{H} := \{f \in \mathcal{E}_p^d\colon\ f(0) = 0\}$. By Lemma 4.1, this implies that $|f_*|^{p-1}\operatorname{sign}(f_*)\perp\mathcal{H}$, which is equivalent to (4.6). That (4.6) implies that $f_* \in \mathcal{E}_p^d$ is the desired extremal function follows directly from Hölder's inequality.

Next, we show the uniqueness of f_*. For $1 < p < \infty$, the uniqueness follows directly from the strict convexity of the space L^p. It remains to deal with the case $p = 1$. We consider f_* as a function in one variable so that f_* is an even entire function of exponential type on $\mathbb{C}$. By the classical Hadamard theorem (see, e.g., [**9**, Ch. 2]), it follows that

$$f_*(z) = f_*(0)e^{bz} \prod_{n\in\mathbb{Z}\setminus\{0\}} \Big(1 - \frac{z}{z_n}\Big) \exp\Big(\frac{z}{z_n}\Big),$$

where $b \in \mathbb{C}$ and $\{z_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is the sequence of all nonzero zeros of f_*. Since f_* is an even function, we may assume that $z_n = z_{-n}$ for all positive integers n. Thus, we must have that $b = 0$, and

$$f_*(z) = f_*(0) \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{z_n^2}\Big), \quad z \in \mathbb{C}. \tag{4.13}$$

We further claim that all the zeros of f_* must be simple and real (i.e., the numbers z_n are distinct and real). To see this, we first recall that $f_*\big|_{\mathbb{R}}$ is real valued, which implies $\overline{f_*(z)} = f_*(\bar z)$ for every $z \in \mathbb{C}$. Thus, if the claim were not true, then for a nonzero complex zero w of f, we have the decomposition

$$f(z) = (z^2 - w^2)(z^2 - \bar w^2)g(z) = a(z)g(z),$$

where $g \in \mathcal{E}_1^1$, and $a(z) = z^4 - (w^2 + \bar w^2)z^2 + |w|^4$. It is easily seen that $a(x) \ge 0$ for all $x \in \mathbb{R}$ and the equality holds only in the case when $x = w \in \mathbb{R}$. Thus, $\operatorname{sign}(f_*)(x) = \operatorname{sign}(g)(x)$ for all $x \in \mathbb{R} \setminus \{w\}$. Now consider the functions

$$f_t(z) = a(tz)g(z), \quad z \in \mathbb{C}, \quad t \in \mathbb{R}.$$

Clearly, $f_t(|\cdot|) \in \mathcal{E}_1^d$ and $f_*(0) = f_t(0) > 0$ for all $t \in \mathbb{R}$. Thus, (4.6) implies

$$1 = \frac{f_t(0)}{f_*(0)} = \int_{\mathbb{R}^d} \operatorname{sign}(f_*)(|x|) f_t(|x|)\, dx = \int_{\mathbb{R}^d} a(t|x|)|g(|x|)|\, dx, \quad \forall\, t \in \mathbb{R}.$$

This last term in this last equation is a polynomial in $t \in \mathbb{R}$ of degree 4, which can not be constant. We obtain a contradiction and hence prove the claim.

Now assume that f_{**} is another radial extremizer for the supremum in (4.5). Then (4.6) implies

$$\|f_{**}\|_1 = 1 = \frac{f_{**}(0)}{f_*(0)} = \int_{\mathbb{R}^d} f_{**}(x) \operatorname{sign} f_*(x)\, dx.$$

It follows that $|f_{**}(x)| = f_{**}(x)\operatorname{sign} f_*(x)$ for a.e. $x \in \mathbb{R}$. Since all the zeros of f_* are simple, it follows that f_* changes signs at each of its zeros. By continuity, this further implies that $|f_{**}(x)| = f_{**}(x)\operatorname{sign} f_*(x)$ for every $x \in \mathbb{R}$. By symmetry, we also have $|f_*| \equiv f_* \operatorname{sign} f_{**}$. This means that f_{**} and f_* have common zeros. By (4.13) and the above claim, we conclude that $f_{**} \equiv f_*$, proving the uniqueness.

We point out that the proofs of the duality formulas (4.3) and (4.4) are very similar to those of (4.3) and (4.4) for spherical polynomials on $\mathbb{S}^d$. We skip the details.

Finally, we show the uniquenss of the extremal function F_* defined by (4.7). For $p > 1$, the uniqueness follows directly of strict convexity of the L^p-norm. It remains to consider the case of $p = 1$.

Recall that $K(|\cdot|) = \frac{V_d}{(2\pi)^d} j_{d/2}(|\cdot|)$, $f_*(0) = \mathcal{N}(\mathbb{R}^d)_1$, and $\|f_*\|_{L^1(\mathbb{R}^d)} = 1$. If F is extremal then $\|K - F\|_{L^\infty(\mathbb{R})} = f_*(0)$. By (4.2) and (4.6), we have

$$\int_{\mathbb{R}^d} (K(|x|) - F(|x|)) f_*(|x|)\,dx = f_*(0) - \int_{\mathbb{R}^d} F(|x|) f_*(|x|)\,dx = f_*(0)$$

On the other hand,

$$\begin{aligned} f_*(0) &= \int_{\mathbb{R}^d} (K(|x|) - F(|x|)) f_*(|x|)\,dx \le \int_{\mathbb{R}^d} |K(|x|) - F(|x|)|\,|f_*(|x|)|\,dx \\ &\le \|K - F\|_{L^\infty(\mathbb{R})} \|f_*\|_{L^1(\mathbb{R}^d)} = f_*(0). \end{aligned}$$

Thus, we have the sharp Hölder inequality for $(p, p') = (\infty, 1)$ in the third step. It follows that

$$K(t) - F(t) = f_*(0)\operatorname{sign} f_*(t) \quad \text{a.e. for } t \ge 0.$$

Therefore, $F_* = K - f_*(0)\operatorname{sign} f_*$ is unique.

5. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3.

Note first that the indentify,

$${}_1F_2\Big(\alpha + 1; \alpha + 2, \alpha + 2; -\frac{q_{\alpha+1,1}^2}{4}\Big) = \frac{\int_0^{q_{\alpha+1,1}} j_{\alpha+1}(t) t^{2\alpha+1}\,dt}{\int_0^{q_{\alpha+1,1}} t^{2\alpha+1}\,dt}, \tag{5.1}$$

follows directly from (3.13). It remains to show that

$$\mathcal{I}_\alpha = a_0^* := \frac{\int_0^{q_{\alpha+1,1}} j_{\alpha+1}(t) t^{2\alpha+1}\,dt}{\int_0^{q_{\alpha+1,1}} t^{2\alpha+1}\,dt}, \tag{5.2}$$

where $\mathcal{I}_\alpha$ is defined in (1.9).

For simplicity, we write $r_0 = q_0 = 0$, $q_k = q_{\alpha+1,k}$, and $r_k = r_{\alpha+1,k} = \frac{q_k}{q_1}$ for $k = 1, 2, \cdots$. Recall that $\{j_\alpha(q_k t)\}_{k=0}^\infty$ forms an orthogonal basis of the space $L^2([0,1], t^{2\alpha+1}\,dt)$. In particular, we have

$$0 = q_1^{2\alpha+2} \int_0^1 j_\alpha(q_k t) t^{2\alpha+1}\,dt = \int_0^{q_1} j_\alpha(r_k t) t^{2\alpha+1}\,dt, \quad k = 1, 2, \cdots. \tag{5.3}$$

The crucial step in our proof is to construct an extremal function $F^* \in X_\alpha$ with the following properties:

$$\sup_{t>0} |j_{\alpha+1}(t) - F^*(t)| \leq a_0^* := \frac{\int_0^{q_1} j_{\alpha+1}(t) t^{2\alpha+1}\, dt}{\int_0^{q_1} t^{2\alpha+1}\, dt}, \tag{5.4}$$

and

$$j_{\alpha+1}(t) - F^*(t) \equiv a_0^*, \quad \text{for a.e. } t \in [0, q_1]. \tag{5.5}$$

For the moment, we assume that there exists an extremal function $F^* \in X_\alpha$ satisfying (5.4) and (5.5), and proceed with the proof of (5.2). Indeed, by (5.1) and (5.4), we have

$$\mathcal{I}_\alpha \leq a_0^* = {}_1F_2\Big(\alpha+1; \alpha+2, \alpha+2; -\frac{q_{\alpha+1,1}^2}{4}\Big).$$

Thus, if $\mathcal{I}_\alpha \neq a_0^*$, then $\mathcal{I}_\alpha < a_0^*$, and there exists a function $F \in X_\alpha$ such that

$$\|j_{\alpha+1} - F\|_\infty < a_0^*.$$

However, by (5.5), this implies that

$$j_{\alpha+1}(t) - F(t) < j_{\alpha+1}(t) - F^*(t) \quad \text{for a.e. } t \in [0, q_1],$$

or equivalently,

$$F^*(t) - F(t) < 0 \quad \text{for a.e. } t \in [0, q_1].$$

Integrating this last inequality with respect to the measure $t^{2\alpha+1}dt$ on $[0, q_1]$, we obtain

$$\int_0^{q_1} (F^*(t) - F(t)) t^{2\alpha+1}\, dt < 0,$$

which is impossible since by (5.3), we have $\int_0^{q_1} F(t) t^{2\alpha+1}\, dt = 0$ for every $F \in X_\alpha$. Thus, one must have $\mathcal{I}_\alpha = a_0^*$.

It remains to prove the existence of an extremal function $F^* \in X_\alpha$ satisfying (5.4) and (5.5). Firstly, the condition (5.5) suggests us to consider the Fourier-Bessel series of the function $j_{\alpha+1}(t)$ with respect to the orthogonal basis $\{j_\alpha(q_k t/q_1)\}_{k=0}^\infty$ of the space $L^2([0, q_1], t^{2\alpha+1}dt)$. Indeed, by (3.11), we have

$$j_{\alpha+1}(t) = a_0^* + \sum_{k=1}^\infty a_k^* j_\alpha(r_k t), \quad t \in [0, q_1], \tag{5.6}$$

where

$$a_0^* = \frac{\int_0^{q_1} j_{\alpha+1}(t) t^{2\alpha+1}\, dt}{\int_0^{q_1} t^{2\alpha+1}\, dt}, \quad a_k^* = \frac{2}{(j_\alpha(q_k))^2} \int_0^1 j_{\alpha+1}(q_1 t) j_\alpha(q_k t) t^{2\alpha+1}\, dt, \quad k \geq 1.$$

Using (3.3) , we have

$$\begin{aligned} a_k^* &= \frac{4(\alpha+1)}{(j_\alpha(q_k))^2} q_1^{-2\alpha-2} \int_0^1 j_\alpha(q_k t) t^{-1} \int_0^{q_1 t} x^{2\alpha+1} j_\alpha(x)\, dx dt \\ &= \frac{4(\alpha+1)}{(j_\alpha(q_k))^2} \int_0^1 x^{2\alpha+1} \int_0^1 j_\alpha(q_k t) j_\alpha(q_1 x t) t^{2\alpha+1} dt\, dx, \quad k \geq 1. \end{aligned}$$

It then follows by (3.12) that

$$a_k^* = -\frac{2}{j_\alpha(q_k)} \int_0^1 \frac{j_{\alpha+1}(q_1 x) x^{2\alpha+3}}{r_k^2 - x^2}\, dx, \quad k = 1, 2, \cdots. \tag{5.7}$$

Note that $j_{\alpha+1}(t) > 0$ for $t \in [0, q_1)$ and $r_k \geq 1$ for $k \geq 1$. Thus, (5.7) together with (3.10) implies that

$$j_\alpha(q_k) a_k^* < 0, \quad (-1)^{k+1} a_k^* > 0, \quad k = 1, 2, \cdots. \tag{5.8}$$

Secondly, a straightforward calculation shows that for any $t > 0$,

$$|a_k^* j_\alpha(r_k t)| \leq \frac{C}{k^2} \Bigl(\frac{k}{1+kt}\Bigr)^{\alpha+\frac{1}{2}}, \quad \text{as } k \to \infty. \tag{5.9}$$

This in particular implies that (5.6) holds pointwisely for every $t \in [0, q_1]$. Thus, by (5.8), we have

$$0 = j_{\alpha+1}(q_1) = a_0^* + \sum_{k=1}^\infty a_k^* j_\alpha(q_k) = a_0^* - \sum_{k=1}^\infty |a_k^* j_\alpha(q_k)|. \tag{5.10}$$

Note also that (5.9) implies that the series in (5.6) converges uniformly on every compact subset of $(0, \infty)$ to a function $F^* \in L^\infty[0, \infty)$. Thus, we may use the infinite series on the right hand side of (5.6) to define a function F^* on $\mathbb{R}_+$ as follows:

$$F^*(t) := \begin{cases} \sum_{k=1}^\infty a_k^* j_\alpha(r_k t), & \text{if } t > 0; \\ 1 - a_0^*, & \text{if } t = 0. \end{cases}$$

Note that by (5.6),

$$F^*(t) + a_0^* = j_{\alpha+1}(t), \quad \forall\, t \in [0, q_1].$$

This together with the uniform convergence of the series on compact subsets of $(0, \infty)$ implies that F^* is a uniformly bounded continuous function on $[0, \infty)$.

Finally, to complete the proof, it remains to verify that

$$|j_{\alpha+1}(t) - F^*(t)| \leq a_0^*, \quad \forall\, t \geq q_1. \tag{5.11}$$

Using (3.10) and (5.10), we obtain that for $t \geq q_1$,

$$\begin{aligned} |j_{\alpha+1}(t) - F(t)| &\leq |j_{\alpha+1}(t) - a_1^* j_\alpha(t)| + \sum_{k=2}^\infty |a_k^* j_\alpha(q_k)| \\ &= a_0^* - a_1^* |j_\alpha(q_k)| + |j_{\alpha+1}(t) - a_1^* j_\alpha(t)|. \end{aligned}$$

Thus, for the proof of (5.11), it suffices to show that

$$\max_{t \geq q_1} \Bigl| j_\alpha(t) - \frac{j_{\alpha+1}(t)}{a_1^*} \Bigr| \leq |j_\alpha(q_1)|. \tag{5.12}$$

The proof of (5.12) relies on the following technical lemma, which can be seen as an extension of the property $\max_{t \geq q_{\alpha+1,1}} |j_\alpha(t)| = |j_\alpha(q_1)|$:

Lemma 5.1. *For $\alpha \geq -0.272$, we have*

$$\sup_{t \geq q_{\alpha+1,1}} \Bigl| j_\alpha(t) - u\, j_{\alpha+1}(t) \Bigr| \equiv |j_\alpha(q_{\alpha+1,1})|, \quad 0 \leq u \leq \frac{\alpha+2}{\alpha+1}. \tag{5.13}$$

The proof of Lemma 5.1 is very technical, and will be given in the next subsection. For the moment, we take this lemma for granted and proceed with the proof of (5.12).

By Lemma 5.1, it is enough to show

$$\frac{1}{a_1^*} \le \frac{\alpha+2}{\alpha+1}. \tag{5.14}$$

To this end, we define $g(x) := \frac{j_{\alpha+1}(q_1 x)}{1-x^2}$ for $x \in [0,1)$. Then by (3.8),

$$g(x) = \prod_{k=2}^{\infty}\Big(1 - \frac{x^2}{r_k^2}\Big), \quad x \in [0,1).$$

Since $r_k \ge 1$ for all $k \ge 1$, it is easily seen that $g(x)$ is a strictly decreasing function on $[0,1)$. Thus, for any $x \in [0,1)$,

$$\frac{j_{\alpha+1}(q_1 x)}{1-x^2} = g(x) > \lim_{x\to 1-} g(x) = \frac{q_1 j'_{\alpha+1}(q_1)}{-2} = -(\alpha+1) j_\alpha(q_1),$$

where the last step uses (3.5). It then follows from (5.7) that

$$a_1^* = -\frac{2}{j_\alpha(q_1)} \int_0^1 \frac{j_{\alpha+1}(q_1 x) x^{2\alpha+3}}{1-x^2}\, dx > 2(\alpha+1)\int_0^1 x^{2\alpha+3}\, dx = \frac{\alpha+1}{\alpha+2},$$

which proves (5.14).

5.1. Proof of Lemma 5.1. Write as usual $q_1 := q_{\alpha+1,1}$. We first claim that for the proof of Lemma 5.1, it is enough to show (5.13) for $u = \frac{\alpha+2}{\alpha+1}$, or equivalently,

$$\sup_{t\ge q_1}\Big| j_\alpha(t) - \frac{\alpha+2}{\alpha+1}\, j_{\alpha+1}(t)\Big| = |j_\alpha(q_1)|. \tag{5.15}$$

To see this, consider the function

$$F(t,u) := j_\alpha(t) - u j_{\alpha+1}(t), \quad t \ge q_1, \quad u \ge 0.$$

Note that for $t > q_1$ and $u > 0$,

$$\nabla F(t,u) = 0 \iff \begin{cases} j'_\alpha(t) - u j'_{\alpha+1}(t) = 0 \\ j_{\alpha+1}(t) = 0 \end{cases} \iff j_{\alpha+1}(t) = j_{\alpha+2}(t) = 0,$$

which is impossible since $j_{\alpha+1}$ and $j_{\alpha+2}$ do not have common positive zeros. This means that F does not have any critical points in the domain $\{(t,u) :\ \ t > q_1,\ \ u > 0\}$. On the other hand, however, for any $u > 0$,

$$\lim_{t\to\infty} \max_{0\le v\le u} |F(t,v)| = 0.$$

Thus, for any $u > 0$, $|F|$ has a maximum on the domain $D_u := \{(t,v) :\ \ t \ge q_1,\ \ 0 \le v \le u\}$ which is achieved on its boundary ∂D_u. Since by (3.10),

$$\sup_{t\ge q_1} |F(t,0)| = \sup_{0\le v\le u} |F(q_1,v)| = |j_\alpha(q_1)|, \quad u > 0,$$

it follows that

$$\max_{(t,v)\in D_u} |F(t,v)| = \max_{(t,v)\in \partial D_u} |F(t,v)| = \max_{t\ge q_1} |F(t,u)| =: M(u), \quad u > 0.$$

Since $D_{u_1} \subset D_{u_2}$ for $0 \le u_1 < u_2$, this implies that the function $M(u) := \max_{t \ge q_1} |F(t,u)|$ is increasing on $[0,\infty)$. The claim then follows as $M(u) \ge |F(q_1,u)| = |j_\alpha(q_1)|$ for any $u \ge 0$.

It remains to show (5.15). Define

$$f(t) := j_\alpha(t) - \frac{\alpha+2}{\alpha+1}\, j_{\alpha+1}(t), \quad t \ge 0.$$

We need to prove that

$$\max_{t \ge q_1} |f(t)| = |f(q_1)| = |j_\alpha(q_1)|.$$

Using (3.5) and (3.3), we obtain

$$f'(t) = -\frac{t j_{\alpha+1}(t)}{2(\alpha+1)} + \frac{t j_{\alpha+2}(t)}{2(\alpha+1)} = \frac{t^3 j_{\alpha+3}(t)}{2^3(\alpha+1)(\alpha+2)(\alpha+3)}. \tag{5.16}$$

This in particular implies that the local extrema of f on $(0,\infty)$ can only be attained at positive zeros of $j_{\alpha+3}$ (i.e., at the points $q_{\alpha+3,1}$, $q_{\alpha+3,2}$, $\cdots$). We claim that

$$|f(q_{\alpha+3,k})| \ge |f(q_{\alpha+3,k+1})|, \quad k = 1, 2, \cdots, \tag{5.17}$$

which will imply

$$\max_{t \ge q_1} |f(t)| = \max\Big\{ |f(q_1)|, |f(q_{\alpha+3,1})| \Big\}.$$

To show (5.17), we need a differential equation for the function f. Indeed, using (3.3), (5.16), and the formula

$$j_\alpha''(t) = -\frac{2\alpha+1}{t}\, j_\alpha'(t) - j_\alpha(t),$$

we obtain

$$f''(t) = \frac{2}{t}\, j_\alpha'(t) - \frac{2\alpha+3}{t}\, f'(t) - f(t). \tag{5.18}$$

Furthermore, using (5.16) and (3.5), we can write f' in the form

$$f'(t) = \Big(\frac{2(\alpha+2)}{t} - \frac{t}{2(\alpha+1)} \Big) j_{\alpha+1}(t) - \frac{2(\alpha+2)}{t}\, j_\alpha(t). \tag{5.19}$$

Now combining (5.18) with (5.19), we deduce via a straightforward calculation that

$$A_2 f'' + A_2 f' + A_0 f = 0, \tag{5.20}$$

where

$$A_0 = t^3, \quad A_1 = (2\alpha+1)t^2 + 4(\alpha+2)(2\alpha+3), \quad A_2 = t(t^2 + 4(\alpha+2)).$$

Now let us consider the function $\varphi := f^2 + \frac{A_2}{A_0} f'^2$. Using (5.20), and by a straightforward calculation, we obtain that for $t > 0$,

$$\begin{aligned} \varphi' &= 2f'\Big(f + \frac{A_2}{A_0} f'' \Big) + \Big(\frac{A_2}{A_0} \Big)' f'^2 = \Big(\Big(\frac{A_2}{A_0} \Big)' - 2\,\frac{A_1}{A_0} \Big) f'^2 \\ &= -\frac{2((2\alpha+1)t^2 + 8(\alpha+2)^2)}{t^3}\, f'^2 < 0. \end{aligned}$$

Thus, φ is a decreasing function on $[0,\infty)$. The claim (5.17) then follows since

$$\varphi(q_{\alpha+1,k}) = f^2(q_{\alpha+1,k}), \quad k = 1, 2, \cdots.$$

Thus, to complete the proof of the lemma, it suffices to show that for each $\alpha \geq -0.272$,

$$|f(q_1)| \geq |f(q_{\alpha+3,1})|. \tag{5.21}$$

For simplicity, we write $q_1' = q_{\alpha+3,1}$. Using (3.5), and by straightforward calculations, we obtain

$$|f(q_1)| = \frac{q_1^2 j_{\alpha+2}(q_1)}{4(\alpha+1)(\alpha+2)}, \quad |f(q_1')| = -\frac{(q_1'^2 + 4(\alpha+2))j_{\alpha+2}(q_1')}{4(\alpha+1)(\alpha+2)}.$$

Thus,

$$\rho(\alpha) := \frac{|f(q_1')|}{|f(q_1)|} = -\frac{(q_1'^2 + 4(\alpha+2))j_{\alpha+2}(q_1')}{q_1^2 j_{\alpha+2}(q_1)}. \tag{5.22}$$

To complete the proof of (5.21), it remains to verify that $\rho(\alpha) \leq 1$ for $\alpha \geq -0.272$. We consider the following two cases: (i) $-0.272 \leq \alpha \leq 0.575$, (ii) $\alpha > 0.575$.

For the first case, we use the fact that $\rho(\alpha)$ as given in (5.22) is an analytic function of $\alpha \geq -\frac{1}{2}$. Thus, for $\alpha \leq 0.575$ we can use the very precise approximation of J_α and $q_{\alpha,1}$ realized in Maple to compute $\rho(\alpha)$. Indeed, easy numerical calculations shows that $\alpha_0 = -0.2729\cdots$ is a solution of the equation $\rho(\alpha) = 1$, and the function $\rho(\alpha)$ is decreasing on $(-\frac{1}{2}, 0.575]$, and $\rho(\alpha) \leq 1$ whenever $\alpha \in [\alpha_0, 0.575]$.

To estimate $\rho(\alpha)$ for the second case, we set $y(t) := t^{\frac{1}{2}} J_{\alpha+2}(t)$, and express $\rho(\alpha)$ as

$$\rho(\alpha) = -\Big(1 + \frac{4(\alpha+2)}{q_1'^2}\Big)\Big(\frac{q_1}{q_1'}\Big)^{\alpha+\frac{1}{2}} \frac{y(q_1')}{y(q_1)}.$$

As is well known, the function y satisfies the differential equation

$$y'' + A(t)y = 0 \quad \text{with } A(t) = 1 - \frac{(\alpha+2)^2 - 1/4}{t^2}. \tag{5.23}$$

Note that by (3.7), $q_1 > ((\alpha+2)^2 - 1/4)^{\frac{1}{2}}$. Thus, we have

$$A(t) > 0, \quad A'(t) > 0, \quad \forall\, t \geq q_1.$$

As in the proof of the claim (5.17), the differential equation (5.23) allows us to construct a decreasing function on $[q_1, \infty)$. Indeed, let

$$\psi(t) := y^2 + \frac{1}{A(t)}\Big(\frac{dy}{dt}\Big)^2.$$

Then

$$\psi'(t) = -\frac{A'(t)}{A(t)^2}\Big(\frac{dy}{dt}\Big)^2 < 0, \quad t \geq q_1.$$

Since $q_1 = q_{\alpha+1,1} < q_1' = q_{\alpha+3,1}$, it follows that

$$\psi(q_1') < \psi(q_1). \tag{5.24}$$

However, using the relations $J_{\alpha+2}'(q_1') = \frac{\alpha+2}{q_1'} J_{\alpha+2}(q_1')$ and $J_{\alpha+2}'(q_1) = -\frac{\alpha+2}{q_1} J_{\alpha+2}(q_1)$, we obtain

$$\psi(q_1') = y^2(q_1')\Big(1 + \frac{(\alpha+\frac{5}{2})^2}{q_1'^2 - (\alpha+2)^2 + \frac{1}{4}}\Big), \quad \psi(q_1) = y^2(q_1)\Big(1 + \frac{(\alpha+\frac{3}{2})^2}{q_1^2 - (\alpha+2)^2 + \frac{1}{4}}\Big).$$

Thus, using (5.24), we can estimate the function $\rho(\alpha)$ as follows:

$$\rho(\alpha) < \tilde{\rho}(\alpha) := \Big(1 + \frac{4(\alpha+2)}{q_1'^2}\Big)\Big(\frac{1+\frac{(\alpha+\frac{3}{2})^2}{q_1^2-(\alpha+2)^2+\frac{1}{4}}}{1+\frac{(\alpha+\frac{5}{2})^2}{q_1'^2-(\alpha+2)^2+\frac{1}{4}}}\Big)^{\frac{1}{2}}\Big(\frac{q_1}{q_1'}\Big)^{\alpha+\frac{1}{2}}. \tag{5.25}$$

We then reduce to showing that $\tilde{\rho}(\alpha) \le 1$ for $\alpha \ge 0.575$. To this end, we use the following uniform estimates on the first positive zeros of Bessel functions (see [**38**]):

$$\alpha + c_1\alpha^{1/3} < q_{\alpha,1} < \alpha + c_1\alpha^{1/3} + c_2\alpha^{-1/3}, \quad \forall\, \alpha > 0, \tag{5.26}$$

where $c_1 = 1.855\cdots$ and $c_2 = 1.033\cdots$. Substituting the bounds (5.26) into the expression of $\tilde{\rho}(\alpha)$ in (5.25), one can easily verify via simple numerical calculations that $\tilde{\rho}(\alpha) < 1$ for any $\alpha \ge 0.575$. This completes the proof.

6. Proofs of Theorem 1.2 and Corollary 1.1

Proofs of Theorem 1.2. The lower estimate in Theorem 1.2 follows directly from Theorem 1.1, while the upper estimate in Theorem 1.2 is an easy consequence of Corollary 4.1 and Theorem 1.3. $\square$

Proof of Corollary 1.1. By Theorem 1.2, it suffices to show that

$$a_0^* := \frac{\int_0^{q_1} j_{\alpha+1}(t)t^{2\alpha+1}\,dt}{\int_0^{q_1} t^{2\alpha+1}\,dt} = \Big(\frac{2}{e}\Big)^{\alpha(1+O(\alpha^{-2/3}))}, \quad \alpha \to \infty, \tag{6.1}$$

where $q_1 = q_{\alpha+1,1}$.

To show (6.1), we use the following known formula (see [**6**, 7.14.1 (7)]):

$$\int_0^z t^{2\alpha+1} j_{\alpha+1}(t)\,dt = 2\alpha z^{\alpha+2} j_{\alpha+1}(z) S_{\alpha-1,\alpha}(z) - (2\alpha+2) z^{\alpha+1} j_\alpha(z) S_{\alpha,\alpha+1}(z), \tag{6.2}$$

where $S_{\mu,\nu}(z)$ denotes the Lommel function. We then obtain

$$a_0^* = -(2\alpha+2)^2 q_1^{-\alpha-1} j_\alpha(q_1) S_{\alpha,\alpha+1}(q_1).$$

Note that by (3.7), $\alpha + 1 < q_1 = \alpha + 1 + O((\alpha+1)^{1/3})$ as $\alpha \to \infty$.

We will also use the following estimate of the Lommel function:

$$S_{\alpha,\alpha+1}(z) = z^{\alpha-1}(1 + O(z^{-1})) \quad \text{uniformly for } z > \alpha, \quad \text{as } \alpha \to \infty. \tag{6.3}$$

Since we are unable to find this estimate in literature, we decide to include a proof here. Indeed, using [**34**, Th. 1.1], we have that for an integer $N > \alpha$,

$$S_{\alpha,\alpha+1}(z) = z^{\alpha-1}\Big(\sum_{k=0}^{N-1} \frac{\prod_{v=1}^k (\alpha - v + 1)v}{(z/2)^{2k}} + r_N(z)\Big), \quad z > 0. \tag{6.4}$$

Here $r_N(z) \equiv 0$ if α is a nonnegtive integer, and $r_N(z)$ can be estimated by an integral of the Macdonald function otherwise:

$$|r_N(z)| \le \frac{2^{\alpha+1} z^{-2N}}{\Gamma(-\alpha)} \int_0^\infty t^{2N-\alpha} K_{\alpha+1}(t)\,dt = \frac{\Gamma(N+1)\Gamma(N-\alpha)}{\Gamma(-\alpha)(z/2)^{2N}}. \tag{6.5}$$

Letting $\alpha \to \infty$ and setting $N = [\alpha+2]$ in (6.5), we obtain that for $z > \alpha$,

$$r_N(z) = O((2/e)^{2\alpha}(z/2)^{-2(N-\alpha)}) = O(z^{-1}).$$

Since each fraction in (6.4) is decreasing in $k \geq 1$, (6.3) then follows.

Now using (6.2) and (6.3), we obtain that for $\alpha \to \infty$

$$\begin{aligned} a_0^* &= -\frac{(2\alpha+2)^2}{q_1^2}\, j_\alpha(q_1)(1+O(q_1^{-1})) \\ &\sim -4j_\alpha(q_1) = -\frac{2^{\alpha+2}\Gamma(\alpha+1)}{q_1^\alpha}\, J'_{\alpha+1}(q_1), \end{aligned}$$

where the last step uses the formula $J_\alpha(q_1) = J'_{\alpha+1}(q_1)$. On the other hand, however, according to [**37**, Eq. 10.19.12], we have

$$-J'_{\alpha+1}(q_1) = -J'_{\alpha+1}(\alpha+1+O((\alpha+1)^{1/3})) = c\alpha^{-2/3} + O(\alpha^{-4/3}),$$

where c is a positive constant independent of α that can be expressed explicitly in terms of the Airy function. Thus, using Stirling's formula $\Gamma(\alpha+1) \sim (2\pi\alpha)^{1/2}(\alpha/e)^\alpha$, we obtain

$$a_0^* \sim C\alpha^{-1/6}(2\alpha/e)^\alpha(\alpha+O(\alpha^{1/3}))^{-\alpha} = (2/e)^{\alpha(1+O(\alpha^{-2/3}))} \quad \text{as } \alpha \to \infty.$$

This proves (6.1) and hence completes the proof of the corollary. $\square$

7. Applications in the Remez-type problem

In this section, we give an application of our results on Nikolskii constants in a Remez type problem, which appears frequently in approximation theory and number theory.

Consider a Lebesgue-measurable set $E \subset \mathbb{R}^d$ for which there exists function $f \in \mathcal{E}_1^d \backslash \{0\}$ such that

$$\int_E |f(x)|\, dx \geq \frac{1}{2}\int_{\mathbb{R}^d} |f(x)|\, dx. \tag{7.1}$$

We define the Remez constant α_d^* to be the infimum of the Lebesgue measure $|E|$ over all measurable $E \subset \mathbb{R}^d$ with the above mentioned property. In the case of $d = 1$, the exact value of the constant α_d^* was founded in [**33**], where it was proved that $\alpha_1^* = \pi$ and the corresponding extremal function is $\frac{\cos x}{1-(2x/\pi)^2}$. The exact value of α_d^* for $d \geq 2$ remains unknown. Note that the Remez constant plays an important role in L_1-approximation of functions with small support and, in particular, in the study of sparse representations (compressed sensing).

Using our results on Nikolskii constants, we may give an asymptotic estimate of the constant α_d^* as $d \to \infty$. To be precise, we first recall that $\mathcal{N}(\mathbb{R}^d)_1 = \mathcal{N}(\mathbb{R}^d)_{1,\infty} = \frac{V_d}{(2\pi)^d}\mathcal{L}^*(d)$, the constant $\mathcal{I}_\alpha$ is given in (1.10), $V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)} = \mathrm{Vol}(B_d)$, and $q_{\alpha,1}$ is the first positive zero of the Bessel function J_α. We will need the following known result:

Theorem 7.1 ([**21**]). *For $d \geq 1$, we have*

$$\begin{aligned} r_d := \inf\Big\{ r > 0 : \quad &\exists\, f \in \mathcal{E}_1^d \setminus \{0\} \text{ such that } \widehat{f}(0) \geq 0 \text{ and } f(x) \leq 0 \text{ for all } |x| \geq r \Big\} \\ = 2q_{\frac{d}{2}-1,1} & \end{aligned}$$

with the extremal function given by

$$f_d(x) := \frac{(j_{d/2-1}(|x|/2))^2}{1-(|x|/r_d)^2)}. \tag{7.2}$$

As a consequence, we have

Corollary 7.1. *For $d \geq 2$,*

$$\frac{(2\pi)^d}{2V_d\mathcal{I}_{d/2-1}} \leq \alpha_d^* \leq (2q_{d/2-1,1})^d V_d. \tag{7.3}$$

In particular, we have

$$(\sqrt{e/2})^{1+o(1)} \leq \Big(\frac{V_d\alpha_d^*}{(2\pi)^d}\Big)^{1/d} \leq e^{1+o(1)} \quad \textit{as } d \to \infty. \tag{7.4}$$

Note that the lower estimate here improves significantly the estimate $\big(\frac{V_d\alpha_d^*}{(2\pi)^d}\big)^{1/d} \geq 1$ given in [**8**].

Proof. Let $E \subset \mathbb{R}^d$ and $f \in \mathcal{E}_1^d \setminus \{0\}$ be such that (7.1) is satisfied. Then

$$\frac{1}{2}\int_{\mathbb{R}^d} |f(x)|\,dx \leq \int_E |f(x)|\,dx \leq |E|\,\|f\|_\infty \leq |E|\mathcal{N}(\mathbb{R}^d)_{1,\infty}\|f\|_1.$$

This in particular implies that

$$\alpha_d^* \geq \frac{1}{2\mathcal{N}(\mathbb{R}^d)_1},$$

which further implies the lower estimate $\alpha_d^* \geq \frac{(2\pi)^d}{2V_d\mathcal{I}_{d/2-1}}$ because

$$\frac{(2\pi)^d}{V_d}\mathcal{N}(\mathbb{R}^d)_1 = \mathcal{L}^*(d) \leq \mathcal{I}_{d/2-1}.$$

To show the corresponding upper estimate, let f_d be the function given in (7.2). Then

$$0 = \int_{\mathbb{R}^d} f_d(x)\,dx = \int_{|x|\leq r_d} |f_d(x)|\,dx - \int_{|x|\geq r_d} |f_d(x)|\,dx,$$

and hence,

$$\int_{|x|\leq r_d} |f_d(x)|\,dx = \frac{1}{2}\int_{\mathbb{R}^d} |f_d(x)|\,dx.$$

This together with (7.1) implies the upper estimate:

$$\alpha_d^* \leq (r_d)^d V_d \leq (2q_{\frac{d}{2}-1,1})^d V_d.$$

Finally, we prove (7.4). Note that the lower asymptotic estimate in (7.4) follows directly from Corollary 1.1. The upper estimate as $d \to \infty$ follows from the upper estimate in (7.3) since

$$\frac{V_d^2(r_d)^d}{(2\pi)^d} = \frac{\pi^d(2q_{d/2-1,1})^d}{(2\pi)^d\Gamma^2(d/2+1)} \sim \frac{(d/2)^d}{(d/(2e))^d} = e^d.$$

$\square$

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