RANDOMIZED RUNGE-KUTTA METHOD – STABILITY AND CONVERGENCE UNDER INEXACT INFORMATION

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ABSTRACT. We deal with optimal approximation of solutions of ODEs under local Lipschitz condition and inexact discrete information about the right-hand side functions. We show that the randomized two-stage Runge-Kutta scheme is the optimal method among all randomized algorithms based on standard noisy information. We perform numerical experiments that confirm our theoretical findings. Moreover, for the optimal algorithm we rigorously investigate properties of regions of absolute stability.

Key words: noisy information, randomized Runge-Kutta algorithm, minimal error, meansquare stability, asymptotic stability, stability in probability

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This paper is devoted to the problem of optimal approximation of solutions of ordinary differential equations (ODEs) of the following form

$$\begin{cases} z'(t) = f(t, z(t)), \ t \in [a, b], \\ z(a) = \eta, \end{cases}$$
(1)

where $-\infty < a < b < \infty$, $d \in \mathbb{N}$, $\eta \in \mathbb{R}^d$, $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$. We consider the case when f is only locally Lipschitz. Due to the low regularity of the problem, we focus on the class of randomized algorithms. Moreover, we assume that we have access to f only through its noisy evaluations. We aim at defining an algorithm that approximates z optimally, i.e. with the minimal possible error. Moreover, we want to investigate stability properties of the optimal scheme.

Approximation of solutions of ODEs via randomized algorithms and under exact information about right-hand side functions is a problem well studied in the literature, see, for example, [1, 4, 5, 7, 8, 14, 24, 25]. However, there are still few papers on approximate solving (even via deterministic algorithms) of ODEs when the available information is corrupted, see [11]. Inexact information has been mainly investigated in the context of function integration and approximation ([16, 17]), approximate solving of PDEs ([27, 28]), stochastic integration and SDEs ([13, 18, 19]). Such analysis, under noisy information, seems to be important from the point of view of applications and stability issues, see [22] and Remarks 1, 5. We also refer to [9, 10, 15] for further discussion and examples.

In this paper we extend the results concerning randomized Runge-Kutta scheme (known from [14]) in three directions. Firstly, we investigate the error and optimality of the randomized Runge-Kutta method in the case when f is only locally Lipschitz. Secondly, we allow noisy evaluations of f. This means that the (randomized) evaluations of f might be corrupted by some noise at level of at most $\delta \in [0, 1]$, which corresponds to the precision level. Finally, we rigorously prove fundamental properties of regions of stability, such as openness, boundedness, symmetry. We consider three types of such regions due to the three types of convergence of underlying sequences of random variables: mean-square, with probability 1, and in probability. For the stability analysis we adopt the approach used in [6] in the context of stochastic differential equations.

The novelty and main results of the paper can be summarized as follows:

- We present upper bound on the L^p(Ω)-error for the randomized Runge-Kutta method in the presence of informational noise and under local Lipschitz condition (Theorem 1). We emphasise a strong connection between the error analysis under inexact information and 0-stability of the method (Remark 5).
- We show respective lower bound and then we justify that the randomized Runge-Kutta scheme is optimal in the class of locally Lipschitz right-hand side functions f, among all algorithms based on randomized inexact information about f (Theorem 2).
- We rigorously prove properties of regions of stability for the randomized Runge-Kutta scheme (Theorems 3, 4 and the equality (111)). According to our best knowledge this is a first attempt in this direction.

The paper is organized into seven sections. Section 1 contains problem definition and description of the used model of computation under noisy information. Upper bounds on the error of randomized Runge-Kutta methods are established in Section 2. Corresponding lower bound and optimality are discussed in Section 3. In Section 4 we report the results of numerical experiments performed for two exemplary equations, where one of them is the SIR model. Properties of regions of stability for the randomized Runge-Kutta method are investigated in Section 6. Finally, Appendix consists of some auxiliary results that are used in the paper.

1. Preliminaries

Let $\|\cdot\|$ be the first norm in \mathbb{R}^d , i.e. $\|x\| = \sum_{k=1}^d |x_k|$ for $x \in \mathbb{R}^d$. By $\{e_k\}_{k=1}^d$ we denote the canonical base in \mathbb{R}^d . For $x \in \mathbb{R}^d$ and $r \in [0, \infty)$ we denote by $B(x, r) = \{y \in \mathbb{R}^d \mid \|y - x\| \leq r\}$ the closed ball in \mathbb{R}^d . Moreover, we write $\mathbb{C}_- = \{z \in \mathbb{C} : \Re(z) < 0\}$. Let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space. For a random variable $X : \Omega \to \mathbb{R}$, defined on $(\Omega, \Sigma, \mathbb{P})$, we denote by $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$, $p \in [2, \infty)$. For a Polish space E by $\mathcal{B}(E)$ we denote the Borel σ -field on E.

Let $K, L \in (0, \infty)$, $\rho \in (0, 1]$. We will consider a class $F^{\rho} = F^{\rho}(K, L)$ of pairs (η, f) defined by the following conditions:

(A0) $\|\eta\| \leq K$,

(A1) $f \in C([a,b] \times \mathbb{R}^d)$, (A2) $||f(t,x)|| \leq K(1+||x||)$, for all $(t,x) \in [a,b] \times \mathbb{R}^d$. Take $\bar{R} = \bar{R}(a,b,K)$ as

$$\bar{R} = \max\left\{K(1+b-a)\left(1+e^{K(b-a)}(1+K(b-a))\right), \\ K+(b-a)(1+K) + \left(\frac{1}{K}+1\right)(1+K(b-a))\left(e^{K(b-a)(1+K(b-a))}(1+K)-1\right)\right\}.$$
 (2)

By Lemma 1 below we will see that it is sufficient for our analysis to assume that f satisfies Lipschitz condition only in the ball $B(\eta, \bar{R})$. Namely, in addition to (A0), (A1), (A2) we assume that

- (A3) $||f(t,x) f(s,x)|| \le L|t-s|^{\varrho}$ for all $t,s \in [a,b], x \in B(\eta,\bar{R}),$
- (A4) $||f(t,x) f(t,y)|| \le L||x y||$ for all $t \in [a,b], x, y \in B(\eta, \bar{R}).$

The numbers a, b, d, ϱ, K, L will be called parameters of the class F^{ϱ} . Except for a, b, and d the parameters are, in general, not known and the algorithms presented later on will not use them as input parameters.

We wish to approximate solution of (1) for $f \in F^{\varrho}$ by an algorithm that is based on noisy information about f. Namely, we assume that access to the function f is possible only through its noisy evaluations

$$f(t,y) = f(t,y) + \delta_f(t,y), \ (t,y) \in [a,b] \times \mathbb{R}^d,$$
(3)

where $\tilde{\delta}_f(t, y)$ is an error function corrupting the exact value f(t, y), such that $\|\tilde{\delta}_f(t, y)\| \leq \delta$. We refer to δ as to precision parameter. Moreover, we allow randomized choice of the evaluation points (t, y). We now describe model of computation in all details.

In order to define model of computation under randomized inexact information about f we need to introduce the following auxiliary class

$$\mathcal{K}(\delta) = \{ \tilde{\delta} : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d : \tilde{\delta} - \text{Borel measurable}, \ \|\tilde{\delta}(t, y)\| \le \delta \text{ for all } t \in [a, b], y \in \mathbb{R}^d \},\$$

where we assume for the precision parameter δ that $\delta \in [0, 1]$. Note that the constant mappings $\tilde{\delta}(t, y) = \pm \delta e_1$ belong to $\mathcal{K}(\delta)$. (This is important fact when establishing lower bounds, see [18] and Section 4 below.) Moreover, for $(\eta, f) \in F^{\varrho}$ let

$$V_{(\eta,f)}(\delta) = B(\eta,\delta) \times V_f(\delta), \tag{4}$$

where

$$V_f(\delta) = \{ \tilde{f} \colon \exists_{\tilde{\delta}_f \in \mathcal{K}(\delta)} \; \tilde{f} = f + \tilde{\delta}_f \}.$$
(5)

It holds that $V_{(\eta,f)}(\delta) \subset V_{(\eta,f)}(\delta')$ for $0 \le \delta \le \delta' \le 1$ and $V_{(\eta,f)}(0) = \{\eta\} \times \{f\}.$

For $(\eta, f) \in F^{\varrho}$ let $(\tilde{\eta}, \tilde{f}) \in V_{(\eta, f)}(\delta)$. A vector of noisy information about (η, f) is as follows

$$\mathcal{N}(\tilde{\eta}, f) = [f(t_0, y_0), \dots, f(t_{i-1}, y_{i-1}), f(\theta_0, z_0), \dots, f(\theta_{i-1}, z_{i-1}), \tilde{\eta}],$$
(6)

where $i \in \mathbb{N}$ and $(\theta_0, \theta_1, \dots, \theta_{i-1})$ is a random vector on $(\Omega, \Sigma, \mathbb{P})$. For Borel measurable mappings $\psi_j : \mathbb{R}^{(2j+1)d} \to \mathbb{R}^d \times \mathbb{R}^d$, $j \in \{0, \dots, i-1\}$, we set

$$(y_0, z_0) = \psi_0(\tilde{\eta}),\tag{7}$$

and

$$(y_j, z_j) = \psi_j \Big(\tilde{f}(t_0, y_0), \dots, \tilde{f}(t_{j-1}, y_{j-1}), \tilde{f}(\theta_0, z_0), \dots, \tilde{f}(\theta_{j-1}, z_{j-1}), \tilde{\eta} \Big)$$
(8)

for $j \in \{1, \ldots, i-1\}$. The total number of noisy valuations of f is l = 2i. Note that $\mathcal{N}(\tilde{\eta}, \tilde{f}) : \Omega \to \mathbb{R}^{(2i+1)d}$ is a random vector.

Any algorithm \mathcal{A} using $\mathcal{N}(\tilde{\eta}, \tilde{f})$ that computes the approximation to z is given by

$$\mathcal{A}(\tilde{\eta}, f, \delta) = \varphi(\mathcal{N}(\tilde{\eta}, f)), \tag{9}$$

where

$$\varphi: \mathbb{R}^{(2i+1)d} \to D([a,b]; \mathbb{R}^d) \tag{10}$$

is a Borel measurable function. In the Skorokhod space $D([a, b]; \mathbb{R}^d)$ we consider the Borel σ -field $\mathcal{B}(D([a, b]; \mathbb{R}^d))$ that coincides with the σ -field generated by coordinate mappings, see Theorem 7.1 in [21]. This assures that $\mathcal{A}(\tilde{\eta}, \tilde{f}, \delta) : \Omega \to D([a, b]; \mathbb{R}^d)$ is Σ -to- $\mathcal{B}(D([a, b]; \mathbb{R}^d))$ measurable and, by Theorem 7.1 in [21], for all $t \in [a, b]$ the mapping

$$\Omega \ni \omega \mapsto \mathcal{A}(\tilde{\eta}, \tilde{f}, \delta)(\omega)(t) \in \mathbb{R}^d$$
(11)

is Σ -to- $\mathcal{B}(\mathbb{R}^d)$ -measurable. For a given $n \in \mathbb{N}$ we denote by Φ_n a class of all algorithms of the form (9) for which the total number of evaluations l is at most n.

Let $p \in [2, \infty)$. For a fixed $(\eta, f) \in F^{\varrho}$ the error of $\mathcal{A} \in \Phi_n$ is given as

$$e^{(p)}(\mathcal{A},\eta,f,\delta) = \sup_{(\tilde{\eta},\tilde{f})\in V_{(\eta,f)}(\delta)} \left\| \sup_{a\leq t\leq b} \left\| z(\eta,f)(t) - \mathcal{A}(\tilde{\eta},\tilde{f},\delta)(t) \right\| \right\|_{p},\tag{12}$$

see Remark 2. The worst-case error of the algorithm \mathcal{A} is defined by

$$e^{(p)}(\mathcal{A},\mathcal{G},\delta) = \sup_{(\eta,f)\in\mathcal{G}} e^{(p)}(\mathcal{A},\eta,f,\delta),$$
(13)

where \mathcal{G} is a certain subclass of F^{ϱ} , see [26]. Finally, we consider the *n*th minimal error defined as follows

$$e_n^{(p)}(\mathcal{G},\delta) = \inf_{\mathcal{A}\in\Phi_n} e^{(p)}(\mathcal{A},\mathcal{G},\delta).$$
(14)

Our aim is two fold:

- determine sharp bounds on the nth minimal error and to define implementable algorithm for which the infimum in (14) is asymptotically attained. We call such an algorithm the optimal one, see [26].
- investigate stability (in certain sense) of the defined optimal method.

We follow the usual convention that all constants appearing in this paper (including those in the "O", " Ω ", and " Θ " notation) will only depend on the parameters of the class F^{ϱ} and p. Furthermore, the same symbol may be used for different constants.

Remark 1. We want to underline here that proposed model of computation covers the phenomenon of lowering precision of computations. For example, in the scalar case (i.e. d = 1) we can model relative round off errors by considering the following disturbing functions $\tilde{\delta}_f$:

$$\tilde{\delta}_f(t,y) = \delta \cdot \alpha(t,y) \cdot f(t,y), \ (t,y) \in [0,T] \times \mathbb{R},$$
(15)

where α is a Borel measurable bounded function on $[0, T] \times \mathbb{R}$. This is a frequent case for efficient computations using both CPUs and GPUs. See [13], where similar model of noisy information was considered and Monte Carlo simulations were performed on GPUs. Moreover, in [18,19] the authors show results of numerical experiments (performed on CPUs) concerning approximate solving of SDEs under inexact information.

Remark 2. Due to Lemma 1 (i) for all $\omega \in \Omega$ the mapping $[a, b] \ni t \mapsto ||z(\eta, f)(t) - \mathcal{A}(\tilde{\eta}, \tilde{f}, \delta)(\omega)(t)|| \in [0, \infty)$ belongs to $D([a, b]; [0, \infty))$, and by (11) for all $t \in [a, b]$ the function $\Omega \ni \omega \mapsto ||z(\eta, f)(t) - \mathcal{A}(\tilde{\eta}, \tilde{f}, \delta)(\omega)(t)|| \in [0, \infty)$ is a Σ -to- $\mathcal{B}([0, \infty))$ measurable. Hence, $\Omega \ni \omega \mapsto \sup_{a \le t \le b} ||z(\eta, f)(t) - \mathcal{A}(\tilde{\eta}, \tilde{f}, \delta)(\omega)(t)|| \in [0, \infty)$ is Σ -to- $\mathcal{B}([0, \infty))$ measurable function and the error (12) is well-defined.

2. RANDOMIZED RUNGE-KUTTA METHOD UNDER NOISY INFORMATION

In the case of inexact information about (η, f) the randomized two-stage Runge-Kutta algorithm is defined as follows. Let $n \in \mathbb{N}$, h = (b - a)/n, and $t_j = a + jh$, $j \in \{0, \ldots, n\}$. We assume that $(\tau_j)_{j \in \mathbb{N}}$ are independent and identically distributed random variables on $(\Omega, \Sigma, \mathbb{P})$, uniformly distributed on [0, 1]. Let $(\eta, f) \in F^{\varrho}$ and $(\tilde{\eta}, \tilde{f}) \in V_{(\eta, f)}(\delta)$. Then we set

$$\begin{cases} V^{0} := \tilde{\eta}, \\ \bar{V}^{j}_{\tau} := \bar{V}^{j-1} + h\tau_{j}\tilde{f}(t_{j-1}, \bar{V}^{j-1}), \\ \bar{V}^{j} := \bar{V}^{j-1} + h\tilde{f}(\theta_{j}, \bar{V}^{j}_{\tau}) \end{cases}$$
(16)

for $j \in \{1, ..., n\}$, where $\theta_j := t_{j-1} + \tau_j h$. The final approximation of z in the interval [a, b] is obtained by taking

$$\bar{l}(t) = \bar{l}_j(t), \ t \in [t_j, t_{j+1}], \ j \in \{0, \dots, n-1\},$$
(17)

where

$$\bar{l}_j(t) = \frac{\bar{V}^{j+1} - \bar{V}^j}{t_{j+1} - t_j} (t - t_j) + \bar{V}^j.$$
(18)

The algorithm uses 2n noisy evaluations of f. Moreover, its combinatorial cost is O(n) arithmetic operations.

In the case of exact information (i.e. $\delta = 0$) we write V^j , V^j_{τ} , l, l_j instead of \bar{V}^j , \bar{V}^j_{τ} , \bar{l} , \bar{l}_j , respectively. Of course $\bar{l} = \bar{l}_n$ but we will omit the subscript n in order to simplify the notation.

Note that the algorithm (16) can be written as

$$\begin{cases} V^{0} := \tilde{\eta}, \\ \bar{V}_{\tau}^{j} := \bar{V}^{j-1} + h\tau_{j} \left[f(t_{j-1}, \bar{V}^{j-1}) + \delta_{\tau}^{j} \right], \\ \bar{V}^{j} := \bar{V}^{j-1} + h \left[f(\theta_{j}, \bar{V}_{\tau}^{j}) + \delta^{j} \right], \end{cases}$$
(19)

for $j \in \{1, \ldots, n\}$, where for the noise δ_{τ}^{j} , δ^{j} we have

$$\max_{1 \le j \le n} \max\left\{ \|\delta^j_\tau\|, \|\delta^j\| \right\} \le \delta$$
(20)

almost surely. We stress that the only source of randomness in the noise are τ_j 's, since δ^j is $\sigma(\tau_1, \ldots, \tau_j)$ -measurable for $j \in \{1, \ldots, n\}$, while δ^j_{τ} is $\sigma(\tau_1, \ldots, \tau_{j-1})$ -measurable for $j \in \{2, \ldots, n\}$ (δ^j_{τ} is deterministic).

Lemma below is a crucial result that allow us to estimate error of the randomized Runge-Kutta algorithm in the case when f is only locally Lipschitz. Similar localization technique was used in [12] for right-hand side functions that are globally Lipschitz but only locally differentiable.

Lemma 1. Let $\overline{R} = \overline{R}(a, b, K)$ be as in (2). Then for any $n \in \mathbb{N}$, $\delta \in [0, 1]$, $(\eta, f) \in F^{\varrho}$, and $(\tilde{\eta}, \tilde{f}) \in V_{(\eta, f)}(\delta)$ the following holds:

(i) There exists a unique solution $z = z(\eta, f), z \in C^1([a, b]; \mathbb{R}^d)$, of (1). Moreover, it holds that

$$z(t) \in B(\eta, R), \quad t \in [a, b], \tag{21}$$

there exists $\overline{C} = \overline{C}(a, b, K, L) \in (0, \infty)$ such that for all $s, t \in [a, b]$

$$||z'(t) - z'(s)|| \le \bar{C}|t - s|^{\varrho}, \tag{22}$$

and for all $j \in \{1, \ldots, n\}$

$$z(t_{j-1}) + h\tau_j f(t_{j-1}, z(t_{j-1})) \in B(\eta, \bar{R})$$
(23)

almost surely.

(ii) For all $j \in \{0, 1, ..., n\}$

$$V^j, \bar{V}^j \in B(\eta, \bar{R}) \tag{24}$$

almost surely.

(iii) For all $j \in \{1, \ldots, n\}$

almost surely.

Proof. Let $(\eta, f) \in F^{\varrho}$. According to suitable version of Peano's theorem (see, for example, Theorem 70.4, page 292. in [2]) under conditions (A1), (A2) the equation (1) has at least one solution $z \in C^1([a, b]; \mathbb{R}^d)$. Hence, firstly we show that there exists $R \in [0, \overline{R}]$ such that for any solution z we have $z(t) \in B(\eta, R)$ for all $t \in [a, b]$. Note that

 $V^j_{\tau}, \bar{V}^j_{\tau} \in B(\eta, \bar{R})$

$$||z(t)|| \le K(1+b-a) + K \int_{a}^{t} ||z(s)|| \, \mathrm{d}s, \ t \in [a,b]$$

and, by the Gronwall's lemma,

$$\sup_{a \le t \le b} \|z(t)\| \le C_1,$$
(26)

where

$$C_1 = C_1(a, b, K) = e^{K(b-a)}K(1+b-a)$$

Moreover,

$$\sup_{a \le t \le b} \|z(t) - \eta\| \le C_2,$$

with

$$C_2 = C_2(a, b, K) = C_1 + K \le \bar{R}$$

where \bar{R} is defined in (2). Hence, for all z being the solution of the problem (1) and it holds

$$z(t) \in B(\eta, C_2) \subset B(\eta, \overline{R}), \ t \in [a, b].$$

$$(27)$$

Now, let z and \tilde{z} are two solutions of (1). Due to (27) we have that $z(t), \tilde{z}(t) \in B(\eta, \bar{R})$ for all $t \in [a, b]$. Therefore, by (A4) we have for all $t \in [a, b]$

$$||z(t) - \tilde{z}(t)|| \le \int_{a}^{t} ||f(s, z(s)) - f(s, \tilde{z}(s))|| \, \mathrm{d}s \le L \int_{a}^{t} ||z(s) - \tilde{z}(s)|| \, \mathrm{d}s.$$

This implies that for all $t \in [a, b]$ we have $z(t) = \tilde{z}(t)$, and the uniqueness in (i) follows. For the unique solution z of (1), by (A2) and (26), we have for all $t, s \in [a, b]$

$$||z(t) - z(s)|| \le \int_{\min\{t,s\}}^{\max\{t,s\}} ||f(u,z(u))|| du \le K(1+C_1)|t-s|.$$

Hence, by (A3) and (A4) we have for all $t, s \in [a, b]$

$$|z'(t) - z'(s)|| \le L|t - s|^{\varrho} + L||z(t) - z(s)|| \le \bar{C}|t - s|^{\varrho}$$
(28)

with $\bar{C} = L(1 + K(1 + C_1)(1 + b - a))$. This ends the proof of (22). Moreover, for all $n \in \mathbb{N}$, $j \in \{1, \ldots, n\}$, and almost surely

$$\|z(t_{j-1}) + h\tau_j f(t_{j-1}, z(t_{j-1})) - \eta\| \le \|z(t_{j-1}) - \eta\| + hK(1 + \|z(t_{j-1})\|) \le C_3$$

where

$$C_3 = C_3(a, b, K) = C_2 + (b - a)K(1 + C_1)$$

(25)

For any $n \in \mathbb{N}$, $\delta \in [0, 1]$, $(\tilde{\eta}, \tilde{f}) \in V_{(\eta, f)}(\delta)$, and $j \in \{0, 1, \dots, n\}$ we get

$$\|V^{\circ}\| \leq K+1,$$

$$\|\bar{V}_{\tau}^{j}\| \leq \|\bar{V}^{j-1}\| + h\left(\|f(t_{j-1},\bar{V}^{j-1})\| + \|\delta_{\tau}^{j}\|\right) \leq (1+hK)\|\bar{V}^{j-1}\| + h(K+1),$$
(29)
therefore

$$\|\bar{V}^{j}\| \le \|\bar{V}^{j-1}\| + h\left(\|f(\theta_{j},\bar{V}_{\tau}^{j})\| + 1\right) \le (1+K_{1}h)\|\bar{V}^{j-1}\| + K_{2}h$$

where

and

$$K_1 = K_1(a, b, K) := K(1 + (b - a)K), \ K_2 = K_2(a, b, K) := (K + 1)(1 + (b - a)K).$$

Hence, we get for all $0 \le j \le n$

$$\|\bar{V}^{j}\| \le (1+K_{1}h)^{j}(K+1) + \frac{(1+K_{1}h)^{j}-1}{K_{1}h}K_{2}h \le C_{4},$$
(30)

where

$$C_4 = C_4(a, b, K) := \frac{K+1}{K} \Big((K+1)e^{K_1(b-a)} - 1 \Big).$$

By (29) we obtain for all $1 \le j \le n$

$$\|\bar{V}_{\tau}^{j}\| \le C_5,\tag{31}$$

with

$$C_5 = C_5(a, b, K) := C_4(1 + (b - a)K) + (b - a)(K + 1).$$

Therefore

$$\max_{0 \le j \le n} \|\bar{V}^{j} - \eta\| \le \max_{0 \le j \le n} \|\bar{V}^{j}\| + \|\eta\| \le C_{6},$$
(32)

$$\max_{1 \le j \le n} \|\bar{V}_{\tau}^{j} - \eta\| \le C_{7},\tag{33}$$

where

$$C_6 = C_6(a, b, K) = C_4 + K, \ C_7 = C_7(a, b, K) = C_5 + K.$$
(34)

Note that $\overline{R} = \max\{C_2, C_3, C_6, C_7\} = \max\{C_3, C_7\}$ and the inclusions in (i)-(iii) follow.

We are ready to prove the following theorem that states upper error bounds for randomized Runge-Kutta algorithm under noisy information about the right-hand side function.

Theorem 1. Let $p \in [2, \infty)$. There exists C, depending only on the parameters of the class F^{ϱ} and p, such that for all $n \geq \lfloor b - a \rfloor + 1$, $\delta \in [0, 1]$, $(\eta, f) \in F^{\varrho}$, $(\tilde{\eta}, \tilde{f}) \in V_{(\eta, f)}(\delta)$ we have

$$\left\|\sup_{a\leq t\leq b} \left\| z(\eta,f)(t) - \bar{l}(\tilde{\eta},\tilde{f},\delta)(t) \right\| \right\|_p \leq C \left(h^{\varrho+1/2} + \delta \right).$$
(35)

Proof. We define

$$\bar{z}_j(t) = \frac{z(t_{j+1}) - z(t_j)}{h}(t - t_j) + z(t_j),$$
(36)

for $t \in [t_j, t_{j+1}], j \in \{0, \dots, n-1\}$. Then

$$\begin{aligned} \left\| \sup_{a \le t \le b} \| z(t) - \bar{l}(t) \| \right\|_{p} &\le \max_{0 \le j \le n-1} \sup_{t_{j} \le t \le t_{j+1}} \| z(t) - \bar{z}_{j}(t) \| \\ &+ \left\| \max_{0 \le j \le n-1} \sup_{t_{j} \le t \le t_{j+1}} \| \bar{z}_{j}(t) - \bar{l}_{j}(t) \| \right\|_{p}, \end{aligned}$$
(37)

where, from (18), (36), it holds

$$\left\| \max_{0 \le j \le n-1} \sup_{t_j \le t \le t_{j+1}} \| \bar{z}_j(t) - \bar{l}_j(t) \| \right\|_p \le 3 \left\| \max_{0 \le j \le n} \| z(t_j) - V^j \| \right\|_p$$

$$+3 \left\| \max_{0 \le j \le n} \| V^j - \bar{V}^j \| \right\|_p.$$
(38)

In the case of global Lipschitz condition the first term $\left\|\max_{0\leq j\leq n} \|z(t_j) - V^j\|\right\|_p$ was estimated in Theorem 5.2 in [14]. Under local Lipschitz assumptions (A3), (A4) and with the help of Lemma 1 we can estimate it essentially in the same fashion as in [14], and we get the same upper bound

$$\left\| \max_{0 \le j \le n} \| z(t_j) - V^j \| \right\|_p \le C h^{\varrho + 1/2}.$$
(39)

However, for the convenience of the reader we present complete justification of (39), where we explicitly point out the use of Lemma 1.

For $k \in \{1, \ldots, n\}$ we have

$$z(t_k) - V^k = S_1^k + S_2^k + S_3^k, (40)$$

where

$$S_1^k = \sum_{j=1}^k \Bigl(\int_{t_{j-1}}^{t_j} z'(s) ds - hz'(\theta_j) \Bigr),$$
(41)

$$S_2^k = h \sum_{j=1}^{\kappa} \Big(f(\theta_j, z(\theta_j)) - f(\theta_j, z(t_{j-1}) + h\tau_j f(t_{j-1}, z(t_{j-1}))) \Big), \tag{42}$$

$$S_3^k = h \sum_{j=1}^{\kappa} \Big(f(\theta_j, z(t_{j-1}) + h\tau_j f(t_{j-1}, z(t_{j-1}))) - f(\theta_j, V_{\tau}^j) \Big).$$
(43)

By (22) in Lemma 1 and Theorem 3.1 in [14] we have that there exists $C_1 > 0$ such that for all $n \ge \lfloor b - a \rfloor + 1$

$$\left\|\max_{1\le k\le n} \|S_1^k\|\right\|_p = \left\|\max_{1\le k\le n} \left\|\int_a^{t_k} z'(s)ds - h\sum_{j=1}^k z'(\theta_j)\right\|\right\|_p \le C_1 h^{\varrho+1/2}.$$
(44)

Furthermore, by Lemma 1 and (A4) we get for $k \in \{1, ..., n\}$

$$||S_{2}^{k}|| \leq hL \sum_{j=1}^{k} ||z(\theta_{j}) - z(t_{j-1}) - h\tau_{j}f(t_{j-1}, z(t_{j-1}))||$$

$$\leq hL \sum_{j=1}^{k} \int_{t_{j-1}}^{\theta_{j}} ||z'(s) - z'(t_{j-1})|| ds \leq C_{2}h^{\varrho+1}.$$
 (45)

Moreover, by (19) (with $\delta = 0$), Lemma 1 and (A4) we have for $k \in \{1, \ldots, n\}$

$$||S_3^k|| \le hL \sum_{j=1}^k ||z(t_{j-1}) + h\tau_j f(t_{j-1}, z(t_{j-1})) - V_\tau^j||$$

$$\le hL(1 + L(b-a)) \sum_{j=1}^k ||z(t_{j-1}) - V^{j-1}|| \le hC_3 \sum_{j=1}^k \max_{0 \le i \le j-1} ||z(t_i) - V^i||.$$
(46)

From (40), (44), (45), and (46) we have for $k \in \{1, \ldots, n\}$ that

$$\left\|\max_{1 \le i \le k} \|z(t_i) - V^i\|\right\|_p \le \left\|\max_{1 \le i \le n} \|S_1^i\|\right\|_p + \left\|\max_{1 \le i \le n} \|S_2^i\|\right\|_p + hC_3 \sum_{j=1}^{k-1} \left\|\max_{0 \le i \le j} \|z(t_i) - V^i\|\right\|_p$$

$$\leq C_4 h^{\varrho+1/2} + hC_3 \sum_{j=1}^{k-1} \left\| \max_{0 \leq i \leq j} \| z(t_i) - V^i \| \right\|_p.$$
(47)

Using the weighted version of discrete Gronwall's lemma (see, for example, Lemma 2.1. in [14]) we obtain (39).

We now establish upper bound on $\left\|\max_{0\leq j\leq n} \|V^j - \bar{V}^j\|\right\|_p$. We have $\|V^0 - \bar{V}^0\| \leq \delta$,

and by Lemma 1 (ii), (iii) it holds for $j \in \{1, \ldots, n\}$

$$\|V^{j} - \bar{V}^{j}\| \le \|V^{j-1} - \bar{V}^{j-1}\| + hL\|V^{j}_{\tau} - \bar{V}^{j}_{\tau}\| + h\delta,$$
(48)

where

$$\|V_{\tau}^{j} - \bar{V}_{\tau}^{j}\| \le \|V^{j-1} - \bar{V}^{j-1}\| + h\delta + hL\|V^{j-1} - \bar{V}^{j-1}\|.$$
(49)

Hence

$$\|V^{j} - \bar{V}^{j}\| \le (1 + C_{5}h) \cdot \|V^{j-1} - \bar{V}^{j-1}\| + h\delta C_{6},$$
(50)

and for all $j \in \{0, 1, ..., n\}$

$$\|V^{j} - \bar{V}^{j}\| \le (1 + C_{5}h)^{j} \|V^{0} - \bar{V}^{0}\| + \frac{(1 + C_{5}h)^{j} - 1}{C_{5}h} \cdot h\delta C_{6} \le C_{7}\delta.$$
 (51)

Therefore,

$$\max_{0 \le j \le n} \|V^j - \bar{V}^j\| \le C_7 \delta,$$
(52)

almost surely and by (39) we get

$$\left\| \max_{0 \le j \le n-1} \sup_{t_j \le t \le t_{j+1}} \| \bar{z}_j(t) - \bar{l}_j(t) \| \right\|_p \le C_8(h^{\varrho+1/2} + \delta).$$
(53)

We now show the upper bound on $\max_{0 \le j \le n-1} \sup_{t_j \le t \le t_{j+1}} ||z(t) - \bar{z}_j(t)||$, which is the deterministic term in the error estimate (37).

For every $t \in [t_j, t_{j+1}], j \in \{0, \dots, n-1\}$ we obtain, by applying the mean value-theorem component-wise,

$$z(t) = \sum_{k=1}^{d} z_k(t) e_k = \sum_{k=1}^{d} \left(z_k(t_j) + z'_k(\alpha^t_{k,j})(t-t_j) \right) e_k,$$

$$\bar{z}_j(t) = \sum_{k=1}^{d} \bar{z}_{j,k}(t) e_k = \sum_{k=1}^{d} \left(z'_k(\beta_{k,j})(t-t_j) + z_k(t_j) \right) e_k$$
(54)

for some $\alpha_{k,j}^t \in [t_j, t] \subset [t_j, t_{j+1}]$ and $\beta_{k,j} \in [t_j, t_{j+1}]$. Thereby, for $t \in [t_j, t_{j+1}]$

$$z(t) - \bar{z}_j(t) = \sum_{k=1}^d \left(z'_k(\alpha^t_{k,j}) - z'_k(\beta_{k,j}) \right) (t - t_j) e_k.$$
(55)

Since $||e_k|| = 1$ for $k \in \{1, ..., d\}$, by (22) we get for $j \in \{0, ..., n-1\}$, $t \in [t_j, t_{j+1}]$

$$||z(t) - \bar{z}_j(t)|| \le h \sum_{k=1}^d |z'_k(\alpha^t_{k,j}) - z'_k(\beta_{k,j})| \le d\bar{C}h^{\varrho+1}.$$
(56)

Combining (37), (53), and (56) we get the thesis.

3. Lower bounds and optimality of the randomized Runge-Kutta algorithm

This section is devoted to lower bounds on the worst-case error of any algorithm from the class Φ_n . They will allow us to conclude that the randomized Runge-Kutta \bar{l} is asymptotically optimal within this setting.

Lemma 2. Let $p \in [2, \infty)$ and $\varrho \in (0, 1]$, then

$$e_n^{(p)}(F^{\varrho},\delta) = \Omega(\max\{n^{-(\varrho+1/2)},\delta\})$$

as $n \to \infty$ and $\delta \to 0+$.

Proof. Firstly, for the exact randomized information the following lower bound holds

$$e_n^{(p)}(F^{\varrho}, \delta) \ge e_n^{(p)}(F^{\varrho}, 0) = \Omega(n^{-(\varrho+1/2)}), \ n \to \infty.$$
 (57)

This follows from reducing an integration problem of Hölder continuous functions to the solution of initial value problem, see [5] and [20] for the details.

Note that for any algorithm $\mathcal{A} \in \Phi_n$ and any $(\eta_1, f_1), (\eta_2, f_2) \in F^{\varrho}$, such that $V_{(\eta_1, f_1)}(\delta) \cap V_{(\eta_2, f_2)}(\delta) \neq \emptyset$, we have

$$e^{(p)}(\mathcal{A}, F^{\varrho}, \delta) \ge \frac{1}{2} \sup_{a \le t \le b} \| z(\eta_1, f_1)(t) - z(\eta_2, f_2)(t) \|.$$
(58)

Hence, let us take $(\eta_1, f_1) = (0e_1, +\delta e_1), (\eta_2, f_2) = (0e_1, -\delta e_1)$ that belong to F^{ϱ} if $\delta \in [0, \min\{K, 1\}]$. Then $(0e_1, 0e_1) \in V_{(\eta_1, f_1)}(\delta) \cap V_{(\eta_2, f_2)}(\delta)$ and

$$e^{(p)}(\mathcal{A}, F^{\varrho}, \delta) \ge \delta \sup_{a \le t \le b} \|(t-a)e_1\| = (b-a)\delta,$$

which implies the following

$$e_n^{(p)}(F^{\varrho},\delta) \ge (b-a)\delta.$$
(59)

By (57) and (59) we get the thesis.

Lemma 2, together with Theorem 1 immediately imply the following theorem of optimality of randomized Runge-Kutta algorithm.

Theorem 2. Let $p \in [2, \infty)$ and $\varrho \in (0, 1]$, then

 $e_n^{(p)}(F^{\varrho},\delta) = \Theta(\max\{n^{-(\varrho+1/2)},\delta\})$

as $n \to \infty$ and $\delta \to 0+$. The optimal algorithm is the randomized Runge-Kutta algorithm \bar{l} .

Remark 3. If we restrict considerations to *deterministic algorithms*, then the following sharp bounds on the *n*th minimal error hold

$$e_n^{(p)}(F^{\varrho},\delta) = \Theta(\max\{n^{-\varrho},\delta\}),\tag{60}$$

as $n \to \infty$ and $\delta \to 0+$. The classical Euler scheme, based on the equidistant mesh, is the optimal one.

4. Numerical experiments

In order to support the obtained theoretical results we conducted several numerical experiments. The worst case noise was simulated in two ways: by using two constant noises equal to δ and $-\delta$ (as in the proof of the lower bounds), and then taking the worst of them, and by generating 100 of repetitions of random noise from the uniform distribution on $[-\delta, \delta]$ for each step of the algorithm, and then by taking the worst of them. The error was approximated in L^2 norm at the terminal point by M repetitions of the randomized Runge-Kutta scheme (with M equal to 1000 for the constant noise and 100 for the random noise).

Example 1. As the first problem we consider the following scalar ODE

$$\begin{cases} z'(t) = 1 + z(t) \cos\left(10(2-t)^{1/\gamma} |z(t)|^{3/2}\right), & t \in [0,2] \\ z(0) = -1 \end{cases}$$
(61)

for different values of $\gamma \in (0, +\infty)$. Note that the right-hand side function f in (61) satisfies the assumptions (A1)–(A4) with $\rho = 1/\gamma$. The results for exact information ($\delta = 0$), $\gamma \in \{2, 5, 10\}$ and n varying for 100 to 50000 are presented in Figure 1 (left graph). The results are printed in the logarithmic scale as the relation $\lg(err)$ versus $\lg(n)$. We have added also the slope of the relations. Note that we get a little better behavior that it follows from the theoretical results, which may be due to the fact that the the right-hand side function f = f(t, y) is Lipschitz continuous with respect to time variable t on every interval $[0, \beta]$ with $\beta < 2$.



FIGURE 1. lg(err) vs lg(n) for Example 1 with $\delta = 0$ (left graph) and $\delta = h^{\rho+1/2}$ (right graph)

In Figure 2 the relation $-\lg(err)$ versus $\lg(n)$ for $\gamma = 3$ and different values of δ is presented. We have also run the test with varying $\delta \simeq h^{\varrho+1/2}$. The result is presented in Figure 1 (right graph). As we can see, the error decreases proportionally to δ , which confirms the theoretical results.

Example 2. Below we recall the well-known SIR model that models the spread of disease

$$[S'(t), I'(t), R'(t)] = [-\beta S(t)I(t), \beta S(t)I(t) - \gamma I(t), \gamma I(t)], \ t \in [0, 30].$$
(62)

In numerical experiments we set $[S(0), I(0), R(0)] = [50, 1, 0], \beta = \frac{1}{768}, \gamma = \frac{1}{120}$. The righthand side function in (62) does not belong to the class F^{ϱ} , since it is not globally of at most linear growth. However, we still achieve the desired empirical convergence rate $O(h^{3/2})$. This suggests possibility of weakening the assumption (A2) in the future investigations.

We have made similar simulation as for Example 1. In Figure 3 we present the relation $\lg(err)$ versus $\lg(n)$ for exact information (left graph) and for inexact information with the precision parameter $\delta \simeq h^{3/2}$ (right graph). We can see on both graphs that the error is proportional to $n^{-3/2}$. In Figure 4 we present also the relation $-\lg(err)$ versus $\lg(n)$ for different values of δ .



FIGURE 2. $-\lg(err)$ vs $\lg(n)$ for Example 1, $\gamma = 3$ (left graph – constant noise, right graph – random noise)



FIGURE 3. $\lg(err)$ vs $\lg(n)$ for Example 2 with $\delta = 0$ (left graph) and $\delta = h^{3/2}$ (right graph)

5. Three types of regions of stability for the randomized Runge-Kutta method

In the case when the information is exact we investigate absolute stability of the randomized Runge-Kutta method $(V^j)_{j \in \{0,1,\dots,n\}}$. Since now the algorithm is random we have to generalize definitions concerning the absolute stability known for deterministic methods, see, for example, [3].

Let us consider the well-known test problem

$$\begin{cases} z'(t) = \lambda z(t), \ t \ge 0, \\ z(0) = \eta \end{cases}$$
(63)

with $\lambda \in \mathbb{C}$, $\eta \neq 0$. The exact solution of (63) is $z(t) = \eta \exp(\lambda t)$ and

$$\lim_{t \to \infty} z(t) = 0 \text{ iff } \Re(\lambda) < 0.$$
(64)



FIGURE 4. $-\lg(err)$ vs $\lg(n)$ for Example 2 (left graph – constant noise, right graph – random noise)

For a fixed step-size h > 0 we apply the algorithm $(V^j)_{\in \{0,1,\dots,n\}}$ based on the mesh $t_j = jh$, $j \in \mathbb{N}_0$, to the test problem (63). As a result we obtain the following recurrence

$$V^{0} = \eta \text{ and } V^{j} = p_{j}(h\lambda) \cdot V^{j-1} \text{ for } j \in \{1, \dots, n\},$$
 (65)

where

$$p_j(z) = \tau_j \cdot z^2 + z + 1, \ z \in \mathbb{C}$$

$$(66)$$

is a second-degree polynomial with random coefficient τ_j . Let us substitute $z = h\lambda$. For any $z \in \mathbb{C}$, $(p_j(z))_{j \in \mathbb{N}}$ is a sequence of complex-valued, independent, and identically distributed random variables on $(\Omega, \Sigma, \mathbb{P})$. Solving (65) we get that

$$V^k = \eta \cdot \prod_{j=1}^k p_j(z). \tag{67}$$

We consider three sets

$$\mathcal{R}^{MS} = \{ z \in \mathbb{C} \colon V^k \to 0 \text{ in } L^2(\Omega) \text{ as } k \to \infty \}, \\ \mathcal{R}^{AS} = \{ z \in \mathbb{C} \colon V^k \to 0 \text{ almost surely as } k \to \infty \}, \\ \mathcal{R}^{SP} = \left\{ z \in \mathbb{C} \colon V^k \to 0 \text{ in probability as } k \to \infty \right\},$$
(68)

where we call \mathcal{R}^{MS} the region of mean-square stability, \mathcal{R}^{AS} – the region of asymptotic stability, while \mathcal{R}^{SP} – the region of stability in probability. Of course we have that

$$\mathcal{R}^{MS} \cup \mathcal{R}^{AS} \subset \mathcal{R}^{SP},\tag{69}$$

but we will show more accurate inclusions. If in $(V^j)_{j \in \{0,1,\dots,n\}}$ we set $\tau_j := 1/2$ for all $j \in \mathbb{N}$ then we arrive at the well-known (deterministic) midpoint scheme. The well-known region of absolute stability of this algorithm is

$$\mathcal{R}^{Mid} = \left\{ z \in \mathbb{C} : \left| \frac{1}{2} z^2 + z + 1 \right|^2 < 1 \right\}.$$
 (70)

We use \mathcal{R}^{Mid} as a reference set for \mathcal{R}^{MS} , \mathcal{R}^{AS} , and \mathcal{R}^{SP} . We also investigate the intervals of absolute stability

$$\mathcal{I}^{\diamond} = \mathcal{R}^{\diamond} \cap \{ z \in \mathbb{C} \colon \Im(z) = 0 \}, \ \diamond \in \{ MS, AS, SP \}.$$

$$\tag{71}$$

In [6] regions \mathcal{R}^{MS} , \mathcal{R}^{AS} were defined in order to investigate stability of numerical schemes for stochastic differential equations. Here we adopt this methodology for randomized Runge-Kutta algorithm in the context of deterministic ordinary differential equations. According to our best knowledge this is the first attempt in that direction. Moreover, we also investigate properties of the region \mathcal{R}^{SP} , which was not the case in [6].

5.1. Region \mathcal{R}^{MS} of mean-square stability. For all $z \in \mathbb{C}$ and $j \in \mathbb{N}$

$$|p_j(z)|^2 = 1 + 2\Re(z) + |z|^2 + 2\tau_j \Big(2(\Re(z))^2 - |z|^2 + |z|^2 \Re(z) \Big) + \tau_j^2 |z|^4,$$
(72)

and hence

$$\mathbb{E}|p_j(z)|^2 = 1 + 2\Re(z)\left(1 + \frac{1}{2}|z|^2\right) + 2(\Re(z))^2 + \frac{1}{3}|z|^4.$$
(73)

By (67) we get that

$$\|V^k\|_2 = |\eta| \cdot (\mathbb{E}|p_1(z)|^2)^{k/2}.$$
(74)

Hence, we can write that

$$\mathcal{R}^{MS} = \{ z \in \mathbb{C} \colon \mathbb{E} | p_1(z) |^2 < 1 \}.$$
(75)

Theorem 3.

- (i) The sets \mathcal{R}^{MS} , \mathcal{R}^{Mid} are open and symmetric with respect to the real axis.
- (ii) There exists $r_0 \in (0, \infty)$ such that $\mathcal{R}^{MS} \subset \mathcal{R}^{Mid} \subset \mathbb{C}_- \cap \{z \in \mathbb{C} : |z| < r_0\}.$
- (iii) $\mathcal{I}^{MS} = (x_0, 0)$ with $x_0 = -1 (\sqrt{2} 1)^{-1/3} + (\sqrt{2} 1)^{1/3}$, while $\mathcal{I}^{Mid} := \mathcal{R}^{Mid} \cap \{z \in \mathbb{C} : \Im(z) = 0\} = (-2, 0)$, and $\mathcal{I}^{MS} \subset \mathcal{I}^{Mid}$.

Proof. It holds

$$\mathcal{R}^{MS} = (\phi^{MS})^{-1}((-\infty, 0)), \tag{76}$$

$$\mathcal{R}^{Mid} = (\phi^{Mid})^{-1}((-\infty, 0)), \tag{77}$$

where the functions $\phi^{MS}, \phi^{Mid} : \mathbb{C} \to \mathbb{R}$ are given as follows

$$\phi^{MS}(z) = 2\Re(z)\left(1 + \frac{1}{2}|z|^2\right) + 2(\Re(z))^2 + \frac{1}{3}|z|^4,$$

$$\phi^{Mid}(z) = 2\Re(z)\left(1 + \frac{1}{2}|z|^2\right) + 2(\Re(z))^2 + \frac{1}{4}|z|^4.$$
 (78)

Note that ϕ^{MS} , ϕ^{Mid} are continuous, thereby \mathcal{R}^{MS} , \mathcal{R}^{Mid} are open. Since for all $z \in \mathbb{C}$ we have that $\phi^{MS}(z) = \phi^{MS}(\bar{z})$, $\phi^{Mid}(z) = \phi^{Mid}(\bar{z})$, the conclusion in (i) follows. Moreover $\phi^{Mid}(z) \leq \phi^{MS}(z)$ for all $z \in \mathbb{C}$, and hence $\mathcal{R}^{MS} \subset \mathcal{R}^{Mid}$. Furthermore, for any $z \in \mathcal{R}^{Mid}$ we have

$$\Re(z) < \frac{-2(\Re(z))^2 - \frac{1}{4}|z|^4}{2\left(1 + \frac{1}{2}|z|^2\right)} < 0,$$
(79)

which implies that

0

$$\mathcal{R}^{Mid} \subset \mathbb{C}_{-}.$$
 (80)

Let us consider any $z \in \mathcal{R}^{Mid}$. Then, by the definition of \mathcal{R}^{Mid} we obtain what follows

$$1 > \left|\frac{1}{2}z^{2} + z + 1\right| \ge \left|\frac{1}{2}z^{2} + z\right| - 1 = |z| \cdot \left|\frac{1}{2}z + 1\right| - 1 \ge |z| \cdot \left(\frac{1}{2}|z| - 1\right) - 1, \tag{81}$$

leading to

$$> \frac{1}{2}|z|^{2} - |z| - 2 = \frac{1}{2} \cdot \left(|z| - \left(1 + \sqrt{5}\right)\right) \cdot \left(|z| - \left(1 - \sqrt{5}\right)\right).$$
(82)

Since $|z| - (1 - \sqrt{5}) > 0$ for all $z \in \mathbb{C}$, we conclude that $|z| < 1 + \sqrt{5}$. Thus,

$$\mathcal{R}^{Mid} \subset \left\{ z \in \mathbb{C} \colon |z| < r_0 \right\},\tag{83}$$

where $r_0 = 1 + \sqrt{5}$. Inclusion (80), combined with (83), leads to (ii). By (75) and Cardano's formula we get that

$$\mathcal{I}^{MS} = \{ x \in (-\infty, 0) \colon x^3 + 3x^2 + 6x + 6 > 0 \} = (x_0, 0), \tag{84}$$

where

$$x_0 = -1 - \frac{1}{(\sqrt{2} - 1)^{1/3}} + (\sqrt{2} - 1)^{1/3}.$$
(85)

It is well-known that for the deterministic mid-point method $\mathcal{I}^{Mid} = (-2, 0)$, however, for the convenience of the reader we provide a short justification. Namely, from (72) with $\tau_i = 1/2$ we get that

$$\mathcal{I}^{Mid} = \{ x \in (-\infty, 0) \colon x^3 + 4x^2 + 8x + 8 > 0 \} = (-2, 0).$$
(86)

Since $x_0 > -2$ we get the inclusion $\mathcal{I}^{MS} \subset \mathcal{I}^{Mid}$. This proves (iii).

5.2. Region \mathcal{R}^{AS} of asymptotic stability. The following result is a rearrangement of Lemma 5.1 in [6].

Lemma 3. Given a sequence of real-valued, independent and identically distributed random variables $\{Z_n\}_{n\in\mathbb{N}_0}$ with $\mathbb{P}(Z_1>0)=1$, consider the sequence of random variables $\{Y_n\}_{n\in\mathbb{N}}$ defined by

$$Y_n = \left(\prod_{i=0}^{n-1} Z_i\right) Y_0,\tag{87}$$

where Y_0 is independent of $\{Z_n\}_{n\in\mathbb{N}_0}$ and $\mathbb{P}(Y_0>0)=1$. The following holds:

(i) if $\ln(Z_1)$ is integrable, then

$$\mathbb{E}(\ln(Z_1)) < 0 \Rightarrow \lim_{n \to \infty} Y_n = 0, \text{ with probability } 1 \Rightarrow \mathbb{E}(\ln(Z_1)) \le 0.$$
(88)

(ii) if $\ln(Z_1)$ is square-integrable, then

$$\lim_{n \to \infty} Y_n = 0, \text{ with probability } 1 \Leftrightarrow \mathbb{E}(\ln(Z_1)) < 0.$$
(89)

In our case we set $Z_j := |p_j(z)|$ for a chosen $z \in \mathbb{C}$, where p_j defined as in (66). Recall from (68) that

$$\mathcal{R}^{AS} = \left\{ z \in \mathbb{C} \colon \lim_{k \to \infty} \left(|\eta| \cdot \prod_{j=1}^{k} |p_j(z)| \right) = 0 \text{ with probability } 1 \right\}.$$

Let us observe that $|p_1(z)| = |\tau_1 \cdot z^2 + z + 1| = \sqrt{f_{a,b}(\tau_1)}$ for z = a + bi, $a, b \in \mathbb{R}$, where function $f_{a,b}$ is defined as in Appendix and τ_1 is uniformly distributed over [0, 1]. From Fact 1 in Appendix it follows that, for all $z \in \mathbb{C}$, $\mathbb{P}(|p_1(z)| > 0) = 1$, whereas Fact 3 implies that the random variable $\ln(|p_1(z)|) = \frac{1}{2} \ln(f_{a,b}(\tau_1))$ is square-integrable. Hence, by Lemma 3(ii) we obtain 10

$$\mathcal{R}^{AS} = \left\{ z \in \mathbb{C} \colon \mathbb{E}(\ln |p_1(z)|) < 0 \right\}.$$
(90)

Theorem 4.

- (i) The set \mathcal{R}^{AS} is open and symmetric with respect to the real axis.
- (ii) It holds that $\mathcal{I}^{Mid} \subset [-2,0) \subset \mathcal{I}^{AS} \subset (-\sqrt{2e},0)$. (iii) There exists $r_0 \in (0,\infty)$ such that $\mathcal{R}^{AS} \subset \mathbb{C}_- \cap \{z \in \mathbb{C} : |z| < r_0\}\}$.

Proof. In this proof we will refer many times to the family of functions $\{f_{a,b}: (a,b) \in \mathbb{R}^2\}$ defined by (114) and the function F linked to this family via (116).

Let us notice that \mathcal{R}^{AS} given by (90) is isomorphic with the following set (we will use the same name for both sets):

$$\mathcal{R}^{AS} = \left\{ (a,b) \in \mathbb{R}^2 \colon F(a,b) < 0 \right\}.$$
(91)

Since function $F \colon \mathbb{R}^2 \to \mathbb{R}$ is continuous (see Proposition 1), the set $\mathcal{R}^{AS} = F^{-1}((-\infty, 0))$ is open. Moreover, F(a, b) = F(a, -b) for all $(a, b) \in \mathbb{R}^2$ (see the proof of Proposition 1), which implies symmetry of \mathcal{R}^{AS} with respect to the abscissa. This proves (i).

By (91),

$$\mathcal{I}^{AS} = \left\{ x \in \mathbb{R} \colon F(x,0) < 0 \right\}.$$
(92)

Fact 2 gives the formula for F(x, 0):

$$F(x,0) = \frac{x^2 + x + 1}{x^2} \ln\left(x^2 + x + 1\right) - \frac{x+1}{x^2} \ln|x+1| - 1$$
(93)

for $x \in (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$, F(-1, 0) = -1 and F(0, 0) = 0.

For $x \in (-1,0)$ we have $1 > x^2 + x + 1 > x + 1 > 0$. Function $(0,1) \ni t \mapsto t \ln t - t \in \mathbb{R}$ is decreasing. As a result,

$$(x^{2} + x + 1)\ln(x^{2} + x + 1) - (x^{2} + x + 1) < (x + 1)\ln(x + 1) - (x + 1),$$
(94)

which is equivalent to F(x,0) < 0. We conclude that $(-1,0) \subset \mathcal{I}^{AS}$.

Let us consider a function $g: (-\infty, -1] \to \mathbb{R}$ given by $g(x) = x^2 \cdot F(x, 0)$, that is

$$g(x) = (x^2 + x + 1) \ln (x^2 + x + 1) - (x + 1) \ln (-(x + 1)) - x^2$$
(95)

for x < -1 and g(-1) = -1. Function g is continuous in $(-\infty, -1]$ (since F is continuous) and convex because its second derivative

$$g''(x) = 2\ln\left(x^2 + x + 1\right) + \frac{(2x+1)^2}{x^2 + x + 1} + \frac{1}{-(x+1)}$$
(96)

is positive for $x \in (-\infty, -1)$ (as each term of the above sum is positive). From Jensen's inequality it follows that

$$g(x) = g\left((x+2) \cdot (-1) + (1-x) \cdot (-2)\right) \le (x+2) \cdot g(-1) + (1-x) \cdot g(-1) < 0 \tag{97}$$

for all $x \in [-2, -1]$. Hence, $[-2, -1] \subset \mathcal{I}^{AS}$.

We have already shown that $[-2,0) \subset \mathcal{I}^{AS}$. The inclusion $\mathcal{I}^{AS} \subset (-\sqrt{2e},0)$ follows from the Fact that $\mathcal{R}^{AS} \subset \mathbb{C}^-$, which will be proved later in (iii), and the well-known log sum inequality:

$$\sum_{i=1}^{n} a_i \ln\left(\frac{a_i}{b_i}\right) \ge a \ln\left(\frac{a}{b}\right)$$

for any $n \in \mathbb{Z}_+$ and $a_1, \ldots, a_n, b_1, \ldots, b_n > 0$, $a = a_1 + \ldots + a_n$, $b = b_1 + \ldots, b_n$. In fact, for $x \in \mathcal{I}^{AS} \cap (-\infty, -1)$ we have

$$x^{2} > (x^{2} + x + 1) \ln (x^{2} + x + 1) - (x + 1) \ln (-(x + 1)) \ge x^{2} \ln \left(\frac{x^{2}}{2}\right),$$
(98)

where the former inequality follows from the condition F(x,0) < 0 and the latter is log sum inequality with $a_1 = x^2 + x + 1$, $a_2 = -(x+1)$ and $b_1 = b_2 = 1$. We conclude that $1 > \ln\left(\frac{x^2}{2}\right)$, which leads to $x > -\sqrt{2e}$ and the proof of (ii) is completed.

Now we will show that $\mathcal{R}^{AS} \subset \mathbb{C}_{-}$. This inclusion is equivalent to the condition F(a, b) > 0for all $a \ge 0$ and $b \in \mathbb{R}$. Since the difference

$$f_{a,b}(t) - f_{0,b}(t) = \left(a^4 + 2a^2b^2\right)t^2 + 2\left(a^3 + a^2 + ab^2\right)t + a^2 + 2a \tag{99}$$

is non-negative for all $t \in [0, 1]$, $a \in [0, \infty)$ and $b \in \mathbb{R}$, we have

$$F(a,b) = \frac{1}{2}\mathbb{E}\left(\ln\left(f_{a,b}(\tau)\right)\right) \ge \frac{1}{2}\mathbb{E}\left(\ln\left(f_{0,b}(\tau)\right)\right) = F(0,b).$$
(100)

Thus, it suffices to show that $F(0, b) \ge 0$ for all $b \in \mathbb{R}$.

As stated in Fact 2, F(0,0) = 0. Hereinafter we assume that $b \neq 0$. Then $f_{0,b} \colon \mathbb{R} \ni t \mapsto$ $b^4t^2 - 2b^2t + b^2 + 1 \in \mathbb{R}$ is a quadratic function and its global minimum b^2 is achieved for the argument b^{-2} . When $|b| \ge 1$, we have $\ln(f_{0,b}(t)) \ge \ln(b^2) \ge \ln 1 = 0$ for all $t \in [0,1]$ and as a result $F(0,b) = \frac{1}{2} \cdot \mathbb{E} (\ln (f_{0,b}(\tau))) \ge 0.$

Now we will investigate the remaining case 0 < |b| < 1. Recall that $\ln(x) \ge 1 - \frac{1}{x}$ for x > 0,

$$\operatorname{arctg}(x) - \operatorname{arctg}(y) = \operatorname{arctg}\left(\frac{x-y}{1+xy}\right)$$
 (101)

for y < x < 0 and $0 < \operatorname{arctg}(x) < x$ for x > 0. Hence, for x > 0 it follows that

$$2F(0,b) = \mathbb{E}\left(\ln\left(f_{0,b}(\tau)\right)\right) \ge 1 - \mathbb{E}\left(\frac{1}{f_{0,b}(\tau)}\right) = 1 - \frac{\operatorname{arctg}\left(|b|^3\right)}{|b|^3} > 0.$$
(102)

This completes the proof of inclusion $\mathcal{R}^{AS} \subset \mathbb{C}_{-}$.

From (ii) we know that \mathcal{I}^{AS} is bounded. The boundedness of \mathcal{R}^{AS} follows from the following observation: if $z \in \mathcal{R}^{AS}$ and $|z| \ge 4$, then $-|z| \in \mathcal{I}^{AS}$.

Let us consider $z = a + bi \in \mathcal{R}^{AS}$, where $a, b \in \mathbb{R}$, such that $|z| \ge 4$. For $t \in [0, 1]$ we have

$$\begin{aligned} \left| tz^2 + z + 1 \right|^2 &- \left| t |z|^2 - \left| z \right| + 1 \right|^2 \\ &= 2 \left(\sqrt{a^2 + b^2} + a \right) + 2t \left[\left(a^2 + b^2 \right) \sqrt{a^2 + b^2} + a \left(a^2 + b^2 \right) - 2b^2 \right] \ge 0. \end{aligned}$$

We need to provide justification for the last inequality. Firstly, let us observe that

$$\sqrt{a^2 + b^2} + a \ge |a| + a \ge 0.$$

Secondly, let us notice that a < 0 because $z \in \mathcal{R}^{AS} \subset \mathbb{C}_{-}$ and choose $\alpha \in \mathbb{R}$ such that $b = \alpha \cdot a$. Then $|z| = |a|\sqrt{1 + \alpha^2}$ and

$$(a^{2}+b^{2})\sqrt{a^{2}+b^{2}}+a(a^{2}+b^{2})-2b^{2}=a^{2}\alpha^{2}\left(|z|\frac{\sqrt{1+\alpha^{2}}}{\sqrt{1+\alpha^{2}}+1}-2\right)\geq0,$$

since $|z| \ge 4$ and $\frac{\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}+1} \ge \frac{1}{2}$ for all $\alpha \in \mathbb{R}$. Hence, for $z \in \mathcal{R}^{AS}$ such that $|z| \ge 4$ the following holds:

$$z \in \mathcal{R}^{AS} \iff \mathbb{E}\left(\ln\left|\tau z^2 + z + 1\right|\right) < 0 \implies \mathbb{E}\left(\ln\left|\tau|z|^2 - |z| + 1\right|\right) < 0 \iff -|z| \in \mathcal{I}^{AS}.$$

This concludes the proof. 5.3. Region \mathcal{R}^{SP} of stability in probability. Below we prove analogous result to Lemma 3, but now we deal with convergence in probability.

Lemma 4. Given a sequence of real-valued, independent and identically distributed random variables $\{Z_n\}_{n\in\mathbb{N}_0}$ with $\mathbb{P}(Z_1>0)=1$, consider the sequence of random variables $\{Y_n\}_{n\in\mathbb{N}}$ defined by

$$Y_n = \left(\prod_{i=0}^{n-1} Z_i\right) y_0,$$
 (103)

where $y_0 \in \mathbb{R}_+$. The following holds:

(i) if $\ln(Z_1)$ is integrable, then

$$\mathbb{E}(\ln(Z_1)) < 0 \Rightarrow \lim_{n \to \infty} Y_n = 0 \text{ in probability } \Rightarrow \mathbb{E}(\ln(Z_1)) \le 0.$$
(104)

(ii) if $\ln(Z_1)$ is square-integrable, then

$$\lim_{n \to \infty} Y_n = 0 \text{ in probability } \Leftrightarrow \mathbb{E}(\ln(Z_1)) < 0.$$
(105)

Proof. By Lemma 3 we obtain

$$\mathbb{E}(\ln(Z_1)) < 0 \quad \Rightarrow \quad Y_n \xrightarrow{a.s.} 0 \quad \Rightarrow \quad Y_n \xrightarrow{\mathbb{P}} 0. \tag{106}$$

To prove the second implication in (i), let us suppose that $Y_n \xrightarrow{\mathbb{P}} 0$ and $\mathbb{E}(\ln(Z_1)) > 0$. By Riesz theorem, there exists a subsequence $(Y_{n_k})_{k=0}^{\infty}$ of the sequence $(Y_n)_{n=0}^{\infty}$ such that $Y_{n_k} \xrightarrow{a.s} 0$. On the other hand, by the strong law of large numbers,

$$\frac{S_{n_k}}{n_k} = \frac{1}{n_k} \sum_{i=0}^{n_k-1} \ln\left(Z_i\right) \xrightarrow{a.s.} \mathbb{E}\left(\ln\left(Z_1\right)\right) > 0.$$
(107)

Thus, $n_k \cdot \frac{S_{n_k}}{n_k} \xrightarrow{a.s.} \infty$ and $Y_{n_k} = y_0 \cdot \exp\left(n_k \cdot \frac{S_{n_k}}{n_k}\right) \xrightarrow{a.s.} \infty$. This contradiction proves (i).

To prove part (ii), it suffices to show that the case $Y_n \xrightarrow{\mathbb{P}} 0$ and $\mathbb{E}(\ln(Z_1)) = 0$ is impossible. Let us consider this case. Then, by the central limit theorem,

$$\mathbb{P}\left(\sum_{i=0}^{n-1}\ln\left(Z_{i}\right) > \sqrt{n} \cdot \sigma\right) = \mathbb{P}\left(\frac{\frac{1}{n}\sum_{i=0}^{n-1}\ln\left(Z_{i}\right)}{\frac{\sigma}{\sqrt{n}}} > 1\right) \xrightarrow{n \to \infty} 1 - \Phi(1), \quad (108)$$

where $\sigma = \sqrt{\operatorname{Var}(\ln(Z_1))} > 0$ and Φ denotes the CDF of the standard normal distribution. For all $n \in \mathbb{Z}_+$ we have $\sqrt{n} \cdot \sigma > 0$. As a result,

$$\mathbb{P}\left(\sum_{i=0}^{n-1}\ln\left(Z_{i}\right) > \sqrt{n} \cdot \sigma\right) \leq \mathbb{P}\left(\sum_{i=0}^{n-1}\ln\left(Z_{i}\right) > 0\right) = \mathbb{P}\left(Y_{n} > y_{0}\right) \xrightarrow{n \to \infty} 0, \quad (109)$$

because $Y_n \xrightarrow{\mathbb{P}} 0$. From (108) and (109) it follows that $1 - \Phi(1) = 0$, which is a contradiction. Hence, the proof of the lemma is completed.

From Lemmas 3, 4 we get the following.

Corollary 1. Under the assumptions of Lemma 4, if $\ln(Z_1)$ is square-integrable, then

$$Y_n \xrightarrow{a.s.} 0 \iff \mathbb{E}(\ln(Z_1)) < 0 \iff Y_n \xrightarrow{\mathbb{P}} 0.$$
(110)



FIGURE 5. Stability regions \mathcal{R}^{AS} , \mathcal{R}^{MS} and \mathcal{R}^{Mid}

Corollary 1, (68), and (90) imply that

$$\mathcal{R}^{SP} = \{ z \in \mathbb{C} \colon \mathbb{E}(\ln |p_1(z)|) < 0 \} = \mathcal{R}^{AS}.$$
(111)

Hence, for the randomized Runge-Kutta scheme the notions of asymptotic stability and stability in probability coincide. Furthermore, by (69), (111), and Theorem 3 (ii) we have that

$$\mathcal{R}^{MS} \subset \mathcal{R}^{AS} \cap \mathcal{R}^{Mid}.$$
(112)

Remark 4. The sets \mathcal{R}^{MS} , \mathcal{R}^{AS} , \mathcal{R}^{Mid} are open (so Borel) and, since they are also bounded, their Lebesgue measure is well defined and finite. Below we present estimates for areas of \mathcal{R}^{MS} , \mathcal{R}^{AS} and \mathcal{R}^{Mid}

$$area(\mathcal{R}^{MS}) \approx 3.92 \le area(\mathcal{R}^{AS}) = area(\mathcal{R}^{SP}) \approx 5.38 \le area(\mathcal{R}^{Mid}) \approx 5.87$$

however

$$\mathcal{I}^{MS} \subset \mathcal{I}^{Mid} \subset \mathcal{I}^{AS} = \mathcal{I}^{SP}$$

In Figure 5 we show the pictures of \mathcal{R}^{MS} , \mathcal{R}^{AS} , and \mathcal{R}^{Mid} obtained by the Maple package.

Remark 5. Note that (52) the randomized Runge-Kutta method under exact information is *almost surely* 0-*stable* in the sense that there exists $K, h_0 > 0$ such that for all $f \in F^{\varrho}$, $\varepsilon \in (0,1), \delta \leq \varepsilon$ the following holds

$$\max_{0 \le j \le n} \|V^j - \bar{V}^j\| \le K\varepsilon.$$
(113)

6. Conclusions and future work

As we have seen, randomization decreases the error (under mild assumptions on right-hand side functions) and, in the case of asymptotic stability, extends the interval of absolute stability. However, all considered regions of stability are bounded. Therefore, in our future work we intend to consider randomized implicit schemes. We conjecture that at least one of the regions \mathcal{R}^{MS} , \mathcal{R}^{AS} , \mathcal{R}^{SP} contains \mathbb{C}_{-} .

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7. Appendix

Let us define a function $f_{a,b} \colon \mathbb{R} \to [0,\infty)$ for all pairs $(a,b) \in \mathbb{R}^2$ by the following formula:

$$f_{a,b}(t) = |t(a+bi)^2 + a + bi + 1|^2$$

= $(a^2 + b^2)^2 t^2 + 2(a^2 + a^3 + ab^2 - b^2)t + (a+1)^2 + b^2.$ (114)

Let us notice that $f_{0,0} \equiv 1$. For each pair $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ the function $f_{a,b}$ is quadratic and its discriminant

$$\Delta_{a,b} = -4b^2 \left(2a + a^2 + b^2\right)^2 \tag{115}$$

is non-positive. This leads to Fact 1 below.

Fact 1. The function $f_{a,b}$ has at most one real root. Moreover,

(i) there exists $t_0 \in \mathbb{R}$ such that $f_{a,b}(t_0) = 0$ if and only if

$$\begin{cases} a \neq 0 \\ b = 0 \end{cases} or \begin{cases} a \in (-2, 0) \\ b^2 = -a^2 - 2a \end{cases}$$

(ii) there exists $t_0 \in [0,1]$ such that $f_{a,b}(t_0) = 0$ if and only if

$$\begin{cases} a \in (-\infty, -1] \\ b = 0 \end{cases} \quad or \quad \begin{cases} a \in (-2, -\frac{1}{2}] \\ b^2 = -a^2 - 2a \end{cases}.$$

Let us consider the following function

$$F \colon \mathbb{R}^2 \ni (a, b) \mapsto \frac{1}{2} \mathbb{E} \left(\ln f_{a, b}(\tau) \right) \in \mathbb{R},$$
(116)

where τ is a random variable uniformly distributed over the interval [0, 1]. We will show that F is well-defined and we will express it in explicit form. Let us observe that

$$F(a,b) = \frac{1}{2} \int_{0}^{1} \ln\left(A(a,b)t^{2} + B(a,b)t + C(a,b)\right) dt,$$
(117)

where $A(a,b) = (a^2 + b^2)^2$, $B(a,b) = 2(a^2 + a^3 + ab^2 - b^2)$, $C(a,b) = (a+1)^2 + b^2$. In the case of a = b = 0 we immediately get F(0,0) = 0. Hereinafter we assume that at least one of numbers a, b is non-zero, i.e. $a^2 + b^2 > 0$. It implies that A(a,b) > 0 and $f_{a,b}$ is a quadratic function. We will usually skip arguments (a,b) when using functions A = A(a,b), B = B(a,b), C = C(a,b). The vertex of a parabola $t \mapsto At^2 + Bt + C$ has the coordinates (P,Q), where

$$P = P(a,b) = -\frac{B}{2A} = -\frac{a^2 + a^3 + ab^2 - b^2}{(a^2 + b^2)^2},$$
(118)

$$Q = Q(a,b) = \frac{4AC - B^2}{4A} = \frac{b^2 \left(2a + a^2 + b^2\right)^2}{\left(a^2 + b^2\right)^2}.$$
(119)

The proof of the following fact can be delivered by quite long but straightforward calculations. Hence, we left it to the reader.

Fact 2. On the axis b = 0, the function F can be expressed by

$$F(a,0) = \frac{a^2 + a + 1}{a^2} \ln\left(a^2 + a + 1\right) - \frac{a+1}{a^2} \ln|a+1| - 1$$
(120)

for $a \in (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$. Additionally, F(0, 0) = 0 and F(-1, 0) = -1. On the circle with center (-1, 0) and radius 1, function F has the following formula:

$$F(a,b) = \frac{2a+1}{2a} \ln|2a+1| - 1$$
(121)

for $a \in \left(-2, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right)$ and $b^2 = -a^2 - 2a$. Additionally, $F\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = F\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = -1$. For all the remaining pairs $(a, b) \in \mathbb{R}^2$ function F takes the form

$$F(a,b) = \frac{1}{2}(1-P)\ln(A+B+C) - 1 + \frac{P}{2}\ln C + \sqrt{\frac{Q}{A}} \left[\operatorname{arctg}\left((1-P)\sqrt{\frac{A}{Q}}\right) - \operatorname{arctg}\left(-P\sqrt{\frac{A}{Q}}\right) \right].$$
(122)

Proposition 1. The function F, defined in (116), is continuous in \mathbb{R}^2 .

Proof. From (117) it follows that F(a, b) = F(a, -b) for all $(a, b) \in \mathbb{R}^2$, so it suffices to check continuity of function F in $\mathbb{R} \times [0, \infty)$. Let us split $\mathbb{R} \times [0, \infty)$ into the following pairwise disjoint sub-regions associated to equations (120)–(122) of function F given in Fact 2:

$$Z_{1} = ((-\infty, -1) \cup (-1, 0) \cup (0, \infty)) \times \{0\},$$

$$Z_{2} = \left\{ (a, b) \in \mathbb{R} \times (0, \infty) \colon a \in \left(-2, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \land b^{2} = -a^{2} - 2a \right\},$$

$$Z_{3} = \left\{ (a, b) \in \mathbb{R} \times (0, \infty) \colon b^{2} \neq -a^{2} - 2a \right\}.$$

Let us notice that $\mathbb{R} \times [0, \infty) = Z_1 \cup Z_2 \cup Z_3 \cup \{X_0, X_1, X_2\}$, where $X_0 = (0, 0), X_1 = (-1, 0)$, and $X_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Restriction of F to each of the sets Z_1, Z_2, Z_3 is continuous and Z_3 is open in $\mathbb{R} \times [0, \infty)$. Hence, it is enough to show that F is continuous in each of the points X_0, X_1, X_2 and in each point belonging to Z_1 or Z_2 .

Using (120), (121) and the L'Hôpital's rule, it is easy to check that

$$\lim_{(a,b)\to(0,0),\ (a,b)\in Z_1} F(a,b) = \lim_{(a,b)\to(0,0),\ (a,b)\in Z_2} F(a,b) = 0.$$
(123)

Let us notice that $A = (a^2 + b^2)^2 \rightarrow 0$, $B = 2(a^2 + a^3 + ab^2 - b^2) \rightarrow 0$ and $C = (a+1)^2 + b^2 \rightarrow 1$, when $(a,b) \rightarrow (0,0)$. For $(a,b) \in \mathbb{Z}_3$ sufficiently close to (0,0) the following identity

holds:

$$\operatorname{arctg}\left((1-P)\sqrt{\frac{A}{Q}}\right) - \operatorname{arctg}\left(-P\sqrt{\frac{A}{Q}}\right) = \operatorname{arctg}\left(\frac{\sqrt{\frac{A}{Q}}}{1-P(1-P)\frac{A}{Q}}\right), \quad (124)$$

since $\operatorname{arctg} x - \operatorname{arctg} y = \operatorname{arctg} \left(\frac{x-y}{1+xy}\right)$ for $x, y \in \mathbb{R}$ such that $\operatorname{arctg} x - \operatorname{arctg} y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $xy \neq -1$. Let us notice that $\operatorname{arctg} \left((1-P)\sqrt{\frac{A}{Q}}\right) - \operatorname{arctg} \left(-P\sqrt{\frac{A}{Q}}\right) > 0$ because $(1-P)\sqrt{\frac{A}{Q}} > -P\sqrt{\frac{A}{Q}}$. If P > 1, then $0 > (1-P)\sqrt{\frac{A}{Q}} > -P\sqrt{\frac{A}{Q}}$ and $\operatorname{arctg} \left((1-P)\sqrt{\frac{A}{Q}}\right) - \operatorname{arctg} \left(-P\sqrt{\frac{A}{Q}}\right) < \operatorname{arctg} 0 - \lim_{x \to -\infty} \operatorname{arctg} x = \frac{\pi}{2}$.

For P < 0 we proceed analogously. If $P \in [0, 1]$, then $B^2 \leq 4A^2$ and as a result

$$\frac{A}{Q} = \frac{4A^2}{4AC - B^2} \le \frac{4A^2}{4AC - 4A^2} = \frac{A}{C - A} < 1$$

for $(a,b) \in Z_3$ sufficiently close to (0,0), since $A \to 0$ and $C \to 1$. Thus, $1 > (1-P)\sqrt{\frac{A}{Q}} > -P\sqrt{\frac{A}{Q}} > -1$ and

$$\operatorname{arctg}\left((1-P)\sqrt{\frac{A}{Q}}\right) - \operatorname{arctg}\left(-P\sqrt{\frac{A}{Q}}\right) < \operatorname{arctg}(1-\operatorname{arctg}(-1)) = \frac{\pi}{2}$$

for $(a,b) \in Z_3$ sufficiently close to (0,0). Given that $\frac{1}{1-P(1-P)\frac{A}{Q}} = \frac{2C}{2C+B} - \frac{B^2}{4A} \cdot \frac{2}{2C+B}$, we obtain

$$\sqrt{\frac{Q}{A}} \cdot \operatorname{arctg}\left(\frac{\sqrt{\frac{A}{Q}}}{1 - P(1 - P)\frac{A}{Q}}\right) = \frac{1}{1 - P(1 - P)\frac{A}{Q}} \cdot \frac{\operatorname{arctg}\left(\frac{\sqrt{\frac{A}{Q}}}{1 - P(1 - P)\frac{A}{Q}}\right)}{\frac{\sqrt{\frac{A}{Q}}}{1 - P(1 - P)\frac{A}{Q}}} = \left(C - \frac{B^2}{4A}\right) \cdot g_1(a, b),$$
(125)

where

$$g_1(a,b) = \frac{2}{2C+B} \cdot \frac{\arctan\left(\frac{\sqrt{\frac{A}{Q}}}{1-P(1-P)\frac{A}{Q}}\right)}{\frac{\sqrt{\frac{A}{Q}}}{1-P(1-P)\frac{A}{Q}}} \to 1 \text{ as } (a,b) \in Z_3 \text{ and } (a,b) \to (0,0)$$
(126)

because $\frac{\sqrt{\frac{A}{Q}}}{1-P(1-P)\frac{A}{Q}} = \frac{\sqrt{4AC-B^2}}{2C+B} \to 0$. Furthermore,

$$\frac{1}{2}(1-P)\ln(A+B+C) + \frac{P}{2}\ln C = \frac{1}{2}\ln(A+B+C) + \frac{B}{4A} \cdot \frac{A+B}{C}\ln\left(1+\frac{A+B}{C}\right)^{\frac{C}{A+B}} = g_2(a,b) + \frac{B^2}{4A} \cdot g_3(a,b),$$
(127)

where

$$g_2(a,b) = \frac{1}{2}\ln(A+B+C) + \frac{B}{4C} \cdot \ln\left(1 + \frac{A+B}{C}\right)^{\frac{C}{A+B}} \to 0$$
(128)

and

$$g_3(a,b) = \frac{1}{C} \cdot \ln\left(1 + \frac{A+B}{C}\right)^{\frac{C}{A+B}} \to 1$$
(129)

as $(a,b) \in Z_3$ and $(a,b) \to (0,0)$. We can show that $\frac{B^2}{4A}$ is bounded for $0 < a^2 + b^2 < 1$. For this purpose let us introduce polar coordinates: $a = r \cos \varphi$ and $b = r \sin \varphi$, where $r \in (0,1), \varphi \in [0,2\pi)$. Then

$$\frac{B^2}{4A} = (r\cos\varphi + \cos 2\varphi)^2 \le (r|\cos\varphi| + |\cos 2\varphi|)^2 \le (r+1)^2 < 4.$$
(130)

By (122) and (124)–(130):

$$F(a,b) = g_2(a,b) + \frac{B^2}{4A} \cdot (g_3(a,b) - g_1(a,b)) + C \cdot g_1(a,b) - 1 \to 0$$

as $(a,b) \in \mathbb{Z}_3$ and $(a,b) \to (0,0)$. This combined with (123) implies continuity of F in (0,0).

Now we check the continuity of function F in point $X_1 = (-1, 0)$. From (120) it is easy to see that $F(a, b) \to -1$ as $(a, b) \in Z_1$ and $(a, b) \to X_1$. Let us notice that $A \to 1$, $B \to 0$, $C \to 0$, $P \to 0$, $Q \to 0$ and $\frac{1}{2}(1-P)\ln(A+B+C) \to 0$, when $(a, b) \in Z_3$ tends to X_1 . Moreover, $\frac{P}{2}\ln C \to 0$ because

$$\lim_{(a,b)\to(-1,0)} \frac{B\ln C}{2} = \lim_{(a,b)\to(0,0)} \left[(a-2) \left(a^2 + b^2 \right) \ln \left(a^2 + b^2 \right) + a \ln \left(a^2 + b^2 \right) \right] = 0.$$

Furthermore,

$$\sqrt{\frac{Q}{A}} \cdot \left[\operatorname{arctg} \left((1-P)\sqrt{\frac{A}{Q}} \right) - \operatorname{arctg} \left(-P\sqrt{\frac{A}{Q}} \right) \right] \to 0$$

when $(a,b) \in Z_3$ and $(a,b) \to X_1$, since $\frac{Q}{A} \to 0$ and arctangent is a bounded function. As a result, $F(a,b) \to -1$, when $(a,b) \in Z_3$ tends to X_1 and continuity of F in X_1 follows.

We check the continuity of function F in point $X_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Let us observe that $A \to 1$, $B \to -2, C \to 1, P = -\frac{B}{2A} \to 1$ and $Q = \frac{4AC-B^2}{4A} \to 0$ when $(a, b) \in Z_3$ and $(a, b) \to X_2$. Thus,

$$\frac{P}{2}\ln C \to 0 \text{ and } \sqrt{\frac{Q}{A}} \cdot \left[\operatorname{arctg}\left((1-P)\sqrt{\frac{A}{Q}} \right) - \operatorname{arctg}\left(-P\sqrt{\frac{A}{Q}} \right) \right] \to 0,$$
(131)

when $(a,b) \in Z_3$ and $(a,b) \to X_2$. Moreover, since $\sqrt{A+B+C} \cdot \ln \sqrt{A+B+C} \to 0$ and $\frac{A+\frac{B}{2}}{\sqrt{A+B+C}}$ is bounded for $(a,b) \in Z_3$ in some neighbourhood of X_2 , we get

$$\frac{1}{2}(1-P)\ln(A+B+C) = \frac{1}{A} \cdot \frac{A+\frac{B}{2}}{\sqrt{A+B+C}} \cdot \sqrt{A+B+C} \cdot \ln\sqrt{A+B+C} \to 0 \quad (132)$$

as $(a, b) \in Z_3$ tends to X_2 . From (122), (131) and (132) it follows that $F(a, b) \to -1 = F(X_2)$, when $(a, b) \to X_2$ and $(a, b) \in Z_3$. Thus, F is continuous in X_2 . Since the boundedness of $\frac{A+\frac{B}{2}}{\sqrt{A+B+C}}$ is not straightforward, we will provide a justification. To analyse this expression, it will be convenient to use the polar coordinates:

$$\begin{cases} a = r \cos \varphi - \frac{1}{2} \\ b = r \sin \varphi + \frac{\sqrt{3}}{2} \end{cases} \quad \text{with} \quad r > 0, \ \varphi \in [0, 2\pi) \end{cases}$$

Then

$$\frac{A + \frac{B}{2}}{\sqrt{A + B + C}} = \frac{r^3 + r^2 \left(2\sqrt{3}\sin\varphi - \cos\varphi\right) + r\left(\frac{5}{2} + \sin^2\varphi - \sqrt{3}\sin\varphi\cos\varphi\right) + \frac{\sqrt{3}}{2}\sin\varphi - \frac{3}{2}\cos\varphi}{\sqrt{r^2 + r \cdot 2\sqrt{3}\sin\varphi + 3}}$$

and we can observe that the above expression is bounded for $r \leq \frac{1}{2}$:

$$\frac{\left|A + \frac{B}{2}\right|}{\sqrt{A + B + C}} \le \frac{r^3 + r^2 \left(2\sqrt{3}|\sin\varphi| + |\cos\varphi|\right) + r \left(\frac{5}{2} + \sin^2\varphi + \sqrt{3}|\sin\varphi\cos\varphi|\right) + \frac{\sqrt{3}}{2}|\sin\varphi| + \frac{3}{2}|\cos\varphi|}{\sqrt{r^2 - r \cdot 2\sqrt{3}|\sin\varphi| + 3}} \le \frac{\frac{1}{8} + \frac{1}{4} \left(2\sqrt{3} + 1\right) + \frac{1}{2} \left(\frac{5}{2} + 1 + \sqrt{3}\right) + \frac{\sqrt{3}}{2} + \frac{3}{2}}{\sqrt{3 - \sqrt{3}}}.$$

We check the continuity of function F in points from Z_1 . Let us consider $(a_0, b_0) \in Z_1$. Then $a_0 \in (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$ and $b_0 = 0$. Let us observe that $A \to a_0^4$, $B \to 2a_0^2(a_0 + 1)$, $C \to (a_0 + 1)^2$, $P \to -\frac{a_0+1}{a_0^2}$ and $Q \to 0$, when $(a, b) \in Z_3$ and $(a, b) \to (a_0, 0)$. Hence,

$$\frac{1}{2}(1-P)\ln(A+B+C) \to \frac{a_0^2 + a_0 + 1}{a_0^2}\ln\left(a_0^2 + a_0 + 1\right)$$

and

$$\frac{P}{2}\ln C \to -\frac{a_0+1}{a_0^2}\ln|a_0+1|\,.$$

Since $\frac{Q}{A} \to 0$, we obtain

$$\sqrt{\frac{Q}{A}} \cdot \left[\operatorname{arctg} \left((1-P)\sqrt{\frac{A}{Q}} \right) - \operatorname{arctg} \left(-P\sqrt{\frac{A}{Q}} \right) \right] \to 0.$$

We combine the above considerations with (122), which results in the following:

$$F(a,b) \to \frac{a_0^2 + a_0 + 1}{a_0^2} \ln\left(a_0^2 + a_0 + 1\right) - \frac{a_0 + 1}{a_0^2} \ln\left|a_0 + 1\right| - 1 = F(a_0,0),$$

when $(a, b) \in Z_3$ and $(a, b) \to (a_0, 0)$. This means that F is continuous in $(a_0, 0)$ if $a_0 \neq -2$. The point (-2, 0) is special because in each its punctured neighbourhood there are points not only from Z_1 and Z_3 , but also from Z_2 . Hence, we need to calculate the following limit:

$$\lim_{(a,b)\to(-2,0),\ (a,b)\in\mathbb{Z}_2} F(a,b) = \lim_{a\to-2^+} \left[\frac{2a+1}{2a}\ln\left(-2a-1\right) - 1\right] = \frac{3}{4}\ln 3 - 1 = F(-2,0)$$

and only now we can conclude that F is continuous in (-2, 0).

We check the continuity of function F in points from Z_2 . Let us consider $(a_0, b_0) \in Z_2$. Then $a_0 \in (-2, -\frac{1}{2}) \cup (-\frac{1}{2}, 0)$ and $b_0^2 = -a_0^2 - 2a_0$. Let us observe that $A \to 4a_0^2$, $B \to 4a_0$, $C \to 1$, $P \to -\frac{1}{2a_0}$ and $Q \to 0$, when $(a, b) \in Z_3$ and $(a, b) \to (a_0, b_0)$. As a result, by (122) we obtain

$$F(a,b) \rightarrow \frac{2a_0+1}{2a_0} \ln|2a_0+1| - 1 = F(a_0,b_0),$$

when $(a,b) \in \mathbb{Z}_3$ and $(a,b) \to (a_0,b_0)$. Therefore, F is continuous in (a_0,b_0) .

Finally, we can conclude that the function F is continuous in \mathbb{R}^2 .

Fact 3. The random variable $\ln f_{a,b}(\tau)$, where τ is uniformly distributed over the interval [0,1], is square integrable for all $(a,b) \in \mathbb{R}^2$.

Proof. As stated in Fact 1, the function $[0,1] \ni t \mapsto (\ln f_{a,b}(t))^2 \in \mathbb{R}$ is continuous for all pairs $(a,b) \in \mathbb{R}^2$ but those satisfying one of the following conditions:

1°
$$\begin{cases} a \in (-\infty, -1] \\ b = 0 \end{cases}$$
 or 2° $\begin{cases} a \in (-2, -\frac{1}{2}] \\ b^2 = -a^2 - 2a \end{cases}$

In case 1° we obtain

$$\mathbb{E}\left(\ln f_{a,b}(\tau)\right)^{2} = \int_{0}^{1} \left(\ln f_{a,b}(t)\right)^{2} dt = \frac{4}{a^{2}} \left[G(-a-1) + G\left(a^{2} + a + 1\right) - 2G(0)\right] < \infty,$$

where $G(x) = x(\ln x)^2 - 2x \ln x + 2x$ is continuous on $(0, \infty)$ and can be continuously extended on $[0, \infty)$ with G(0) = 0. In case 2° we proceed similarly and arrive at

$$\int_{0}^{1} \left(\ln f_{a,b}(t)\right)^{2} dt = -\frac{1}{2a} \left[G(1) + G\left(-2a - 1\right) - 2G(0)\right] < \infty.$$

This completes the proof.

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