ON THE POWER OF STANDARD INFORMATION FOR TRACTABILITY FOR L_2 -APPROXIMATION IN THE AVERAGE CASE SETTING

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ABSTRACT. We study multivariate approximation in the average case setting with the error measured in the weighted L_2 norm. We consider algorithms that use standard information Λ^{std} consisting of function values or general linear information Λ^{all} consisting of arbitrary continuous linear functionals. We investigate the equivalences of various notions of algebraic and exponential tractability for Λ^{std} and Λ^{all} for the absolute error criterion, and show that the power of Λ^{std} is the same as that of Λ^{all} for all notions of algebraic and exponential tractability without any condition. Specifically, we solve Open Problems 116-118 and almost solve Open Problem 115 as posed by E.Novak and H.Woźniakowski in the book: Tractability of Multivariate Problems, Volume III: Standard Information for Operators, EMS Tracts in Mathematics, Zürich, 2012.

1. INTRODUCTION

In this paper, we study multivariate approximation $APP = \{APP_d\}_{d \in \mathbb{N}}$ in the average case setting, where

$$APP_d: F_d \to G_d$$
 with $APP_d f = f$

is the continuous embedding operator, F_d is a separable Banach function space on D_d equipped with a zero-mean Gaussian measure μ_d , G_d is a weighted L_2 space on D_d , D_d is a Lebesgue measurable subset of \mathbb{R}^d , and the dimension d is large or even huge. We consider algorithms that use finitely many information evaluations. Here information evaluation means continuous linear functional on F_d (general linear information) or function value at some point (standard information). We use Λ^{all} and Λ^{std} to denote the class of all continuous linear functionals and the class of all function values, respectively.

For a given error threshold $\varepsilon \in (0, 1)$, the information complexity $n(\varepsilon, d)$ is defined to be the minimal number of information evaluations for which the average case error of some algorithm is at most ε . Tractability is aimed at studying how the information complexity $n(\varepsilon, d)$ depends on ε and d. There are two kinds of tractability based on polynomial convergence and exponential convergence. The algebraic tractability (ALG-tractability) describes how the information complexity $n(\varepsilon, d)$ behaves as a function of d and ε^{-1} , while the exponential tractability (EXP-tractability) does as one of d and $(1 + \ln \varepsilon^{-1})$. The existing

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notions of tractability mainly include strong polynomial tractability (SPT), polynomial tractability (PT), quasi-polynomial tractability (QPT), weak tractability (WT), (s, t)-weak tractability ((s, t)-WT), and uniform weak tractability (UWT). In recent years the study of algebraic and exponential tractability has attracted much interest, and a great number of interesting results have been obtained (see [22, 23, 24, 34, 6, 35, 29, 30, 5, 4, 15, 27, 36, 9, 1, 19, 28] and the references therein).

This paper is devoted to investigating the equivalences of various notions of algebraic and exponential tractability for Λ^{std} and Λ^{all} in the average case setting (see [24, Chapter 24]). The class Λ^{std} is much smaller and much more practical, and is much more difficult to analyze than the class Λ^{all} . Hence, it is very important to study the power of Λ^{std} compared to Λ^{all} . There are many paper devoted to this field. For example, for the randomized setting, see [24, 33, 16, 11, 2, 16, 3, 20]; for the average case setting, see [24, 7, 18, 38]; for the worst case setting, see [24, 32, 8, 17, 25, 26, 13, 14, 10, 21, 12].

In [7, 24, 38] the authors obtained the equivalences of various notions of algebraic and exponential tractability in the average case setting for Λ^{std} and Λ^{all} for the normalized error criterion without any condition. Meanwhile, for the absolute error criterion under some conditions on the initial error, the equivalences of ALG-SPT, ALG-PT, ALG-QPT, ALG-WT were also obtained in [24]. Xu obtained in [38] the equivalences of ALG-PT, ALG-QPT, ALG-WT for Λ^{all} and Λ^{std} under much weaker conditions. This gives a partial solution to Open problems 116-118 in [24]. Xu also obtained in [38] the equivalences of ALG-(s, t)-WT, ALG-UWT, and various notions of EXP-tractability under some conditions on the initial error.

In this paper we obtain the equivalences of various notions of algebraic and exponential tractability for Λ^{all} and Λ^{std} in the average case setting for the absolute error criterion without any condition, which means the above conditions are unnecessary. This completely solves Open problems 116-118 in [24]. We also give an almost complete solution to Open Problem 115 in [24].

This paper is organized as follows. In Subsections 2.1 we introduce the approximation problem in the average case setting. The various notions of algebraic and exponential tractability are given in Subsection 2.2. Our main results Theorems 2.1-2.5 are stated in Subsection 2.3. In Section 3, we give the proofs of Theorems 2.1 and 2.2. After that, in Section 4 we show the equivalences of the notions of algebraic tractability for the absolute error criterion without any condition. The equivalence results for the notions of exponential tractability for the absolute error criterion are proved in Section 5.

2. Preliminaries and Main Results

2.1. Average case setting.

For $d \in \mathbb{N}$, let F_d be a separable Banach space of *d*-variate real-valued functions on D_d equipped with a zero-mean Gaussian measure μ_d , $G_d = L_2(D_d, \rho_d(x)dx)$ be a weighted L_2 space, where D_d is a Borel measurable subset of \mathbb{R}^d with positive Lebesgue measure, ρ_d is a probability density function on D_d . We consider the multivariate approximation problem APP = {APP_d}_{d\in\mathbb{N}} in the average case setting which is defined via the continuous linear operator

(2.1)
$$\operatorname{APP}_d: F_d \to G_d \text{ with } \operatorname{APP}_d f = f.$$

We suppose that function value at some point $x \in D_d$ is well defined continuous linear functional on F_d . That is, we suppose that $\Lambda^{\text{std}} \subset \Lambda^{\text{all}} = (F_d)^*$, where $(F_d)^*$ is the dual space of F_d . It is well known that, in the average case setting with the average being with respect to a zero-mean Gaussian measure, adaptive choice of the above information evaluations do not essentially help, see [31]. Hence, we can restrict our attention to nonadaptive algorithms, i.e., algorithms $A_{n,d}f$ of the form

(2.2)
$$A_{n,d}f = \phi_{n,d}(L_1(f), L_2(f), \dots, L_n(f)),$$

where $L_i \in \Lambda$, $i = 1, ..., n, \Lambda \in {\Lambda^{\text{all}}, \Lambda^{\text{std}}}$, and $\phi_{n,d} : \mathbb{R}^n \to G_d$ is an arbitrary measurable mapping from \mathbb{R}^n to G_d . The average case error for the algorithm $A_{n,d}$ of the form (2.2) is defined as

$$e^{\operatorname{avg}}(A_{n,d}) := \left(\int_{F_d} \|\operatorname{APP}_d f - A_{n,d} f\|_{G_d}^2 \ \mu_d(df)\right)^{1/2}$$

The *n*th minimal average case error for $\Lambda \in {\{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}}$ is defined by

$$e^{\operatorname{avg}}(n,d;\Lambda) := \inf_{A_{n,d} \text{ with } L_i \in \Lambda} e^{\operatorname{avg}}(A_{n,d}),$$

where the infimum is taken over all algorithms of the form (2.2).

For n = 0, we use $A_{0,d} = 0$. We obtain the so-called initial error $e^{\text{avg}}(0,d)$ defined by

$$e^{\text{avg}}(0,d) := e^{\text{avg}}(0,d;\Lambda^{\text{all}}) = e^{\text{avg}}(0,d;\Lambda^{\text{std}}) = \left(\int_{F_d} \|\text{APP}_d f\|_{G_d}^2 \ \mu_d(df)\right)^{1/2}.$$

We set

$$\Gamma_d := (e^{\operatorname{avg}}(0,d;\Lambda^{\operatorname{all}}))^2 = (e^{\operatorname{avg}}(0,d;\Lambda^{\operatorname{std}}))^2.$$

It follows from [31, Chapter 6] and [24] that $e^{\operatorname{avg}}(n, d; \Lambda^{\operatorname{all}})$ are described through the eigenvalues and the eigenvectors of the covariance operator $C_{\nu_d}: G_d \to G_d$ of the induced measure $\nu_d = \mu_d S_d^{-1}$ of μ_d . Here, μ_d is a zero-mean Gaussian measure of F_d , so that ν_d is a zero-mean Gaussian measure on the Borel sets of G_d . The operator C_{ν_d} is self-adjoint, non-negative definite, and the trace of C_{ν_d} is finite. Let $\{(\lambda_{k,d}, \eta_{k,d})\}_{k=1}^{\infty}$ denote the eigenpairs of C_{ν_d} satisfying

$$\lambda_{1,d} \ge \lambda_{2,d} \ge \dots \lambda_{n,d} \dots \ge 0$$

That is, $\{\eta_{k,d}\}_{k=1}^{\infty}$ is an orthonormal basis in G_d , and

$$C_{\nu_d} \eta_{k,d} = \lambda_{k,d} \eta_{k,d}, \ k \in \mathbb{N}.$$

From [31, 24] we get that the *n*th minimal average case error is

$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{all}}) = \left(\sum_{k=n+1}^{\infty} \lambda_{k,d}\right)^{1/2},$$

and it is achieved by the optimal algorithm

$$A_{n,d}^*f = \sum_{k=1}^n \langle f, \eta_{k,d} \rangle_{G_d} \eta_{k,d}.$$

That is,

(2.3)
$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{all}}) = \left(\int_{F_d} \|f - A_{n,d}^*f\|_{G_d}^2 \mu_d(df)\right)^{1/2} = \left(\sum_{k=n+1}^\infty \lambda_{k,d}\right)^{1/2}.$$

The trace of C_{ν_d} is just the square of the initial error $e^{\operatorname{avg}}(0,d)$ given by

$$\operatorname{trace}(C_{\nu_d}) = \Gamma_d = (e^{\operatorname{avg}}(0,d))^2 = \int_{G_d} \|g\|_{G_d}^2 \nu_d(dg) = \sum_{k=1}^\infty \lambda_{k,d} < \infty.$$

The information complexity can be studied using either the absolute error criterion (ABS) or the normalized error criterion (NOR). In the average case setting for $\star \in \{ABS, NOR\}$ and $\Lambda \in \{\Lambda^{all}, \Lambda^{std}\}$, we define the information complexity $n^{\star}(\varepsilon, d; \Lambda)$ as

(2.4)
$$n^{\star}(\varepsilon, d; \Lambda) := n^{\operatorname{avg}, \star}(\varepsilon, d; \Lambda) := \inf\{n \mid e^{\operatorname{avg}}(n, d; \Lambda) \le \varepsilon \operatorname{CRI}_d\},$$

where

$$\operatorname{CRI}_{d} := \begin{cases} 1, & \text{for } \star = \operatorname{ABS}, \\ e^{\operatorname{avg}}(0, d), & \text{for } \star = \operatorname{NOR} \end{cases} = \begin{cases} 1, & \text{for } \star = \operatorname{ABS}, \\ (\Gamma_{d})^{1/2}, & \text{for } \star = \operatorname{NOR}. \end{cases}$$

Since $\Lambda^{\text{std}} \subset \Lambda^{\text{all}}$, we get

$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{all}}) \le e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}).$$

It follows that for $\star \in \{ABS, NOR\},\$

(2.5)
$$n^{\star}(\varepsilon, d; \Lambda^{\text{all}}) \le n^{\star}(\varepsilon, d; \Lambda^{\text{std}}).$$

2.2. Notions of tractability.

In this subsection we briefly recall the various tractability notions in the average case setting. First we introduce all notions of algebraic tractability. Let $APP = \{APP_d\}_{d\in\mathbb{N}}, \star \in \{ABS, NOR\}, \text{ and } \Lambda \in \{\Lambda^{all}, \Lambda^{std}\}$. In the average case setting for the class Λ , and for error criterion \star , we say that APP is

• Algebraic strongly polynomially tractable (ALG-SPT) if there exist C > 0 and non-negative number p such that

(2.6)
$$n^{\star}(\varepsilon, d; \Lambda) \leq C\varepsilon^{-p}$$
, for all $\varepsilon \in (0, 1)$.

The exponent ALG- $p^*(\Lambda)$ of ALG-SPT is defined as the infimum of p for which (2.6) holds;

 \bullet Algebraic polynomially tractable (ALG-PT) if there exist C>0 and non-negative numbers p,q such that

$$n^{\star}(\varepsilon, d; \Lambda) \leq C d^{q} \varepsilon^{-p}$$
, for all $d \in \mathbb{N}, \ \varepsilon \in (0, 1)$

 \bullet Algebraic quasi-polynomially tractable (ALG-QPT) if there exist C>0 and non-negative number t such that

(2.7)
$$n^*(\varepsilon, d; \Lambda) \le C \exp(t(1 + \ln d)(1 + \ln \varepsilon^{-1})), \text{ for all } d \in \mathbb{N}, \ \varepsilon \in (0, 1).$$

The exponent ALG- $t^*(\Lambda)$ of ALG-QPT is defined as the infimum of t for which (2.7) holds;

• Algebraic uniformly weakly tractable (ALG-UWT) if

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{\star}(\varepsilon,d;\Lambda)}{\varepsilon^{-\alpha}+d^{\beta}}=0, \text{ for all } \alpha,\beta>0;$$

• Algebraic weakly tractable (ALG-WT) if

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{\star}(\varepsilon,d;\Lambda)}{\varepsilon^{-1}+d}=0;$$

• Algebraic (s, t)-weakly tractable (ALG-(s, t)-WT) for fixed s, t > 0 if

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\star}(\varepsilon,d;\Lambda)}{\varepsilon^{-s}+d^t} = 0.$$

Clearly, ALG-(1, 1)-WT is the same as ALG-WT. If APP is not ALG-WT, then APP is called intractable.

If the *n*th minimal error decays faster than any polynomial and is exponentially convergent, then we should study tractability with ε^{-1} being replaced by $(1 + \ln \frac{1}{\varepsilon})$, which is called exponential tractability. Recently, there have been many papers studying exponential tractability (see [5, 4, 37, 28, 15, 9, 1, 19]).

In the definitions of ALG-SPT, ALG-PT, ALG-QPT, ALG-UWT, ALG-WT, and ALG-(s, t)-WT, if we replace $\frac{1}{\varepsilon}$ by $(1 + \ln \frac{1}{\varepsilon})$, we get the definitions of exponential strong polynomial tractability (EXP-SPT), exponential polynomial tractability (EXP-PT), exponential quasi-polynomial tractability (EXP-QPT), exponential uniform weak tractability (EXP-UWT), exponential weak tractability (EXP-WT), and exponential (s, t)-weak tractability (EXP-(s, t)-WT), respectively. We now give the above notions of exponential tractability in detail.

Let $APP = \{APP_d\}_{d \in \mathbb{N}}, \star \in \{ABS, NOR\}, \text{ and } \Lambda \in \{\Lambda^{all}, \Lambda^{std}\}$. In the average case setting for the class Λ , and for error criterion \star , we say that APP is

• Exponential strongly polynomially tractable (EXP-SPT) if there exist C > 0 and non-negative number p such that

(2.8)
$$n^{\star}(\varepsilon, d; \Lambda) \leq C(\ln \varepsilon^{-1} + 1)^p$$
, for all $\varepsilon \in (0, 1)$.

The exponent EXP- $p^*(\Lambda)$ of EXP-SPT is defined as the infimum of p for which (2.8) holds;

• Exponential polynomially tractable (EXP-PT) if there exist C > 0 and non-negative numbers p, q such that

$$n^{\star}(\varepsilon, d; \Lambda) \leq C d^q (\ln \varepsilon^{-1} + 1)^p$$
, for all $d \in \mathbb{N}, \ \varepsilon \in (0, 1);$

• Exponential quasi-polynomially tractable (EXP-QPT) if there exist C > 0 and non-negative number t such that

(2.9)
$$n^{\star}(\varepsilon, d; \Lambda) \leq C \exp(t(1 + \ln d)(1 + \ln(\ln \varepsilon^{-1} + 1))), \text{ for all } d \in \mathbb{N}, \ \varepsilon \in (0, 1).$$

The exponent EXP- $t^*(\Lambda)$ of EXP-QPT is defined as the infimum of t for which (2.9) holds;

• Exponential uniformly weakly tractable (EXP-UWT) if

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{\star}(\varepsilon,d;\Lambda)}{(1+\ln\varepsilon^{-1})^{\alpha}+d^{\beta}}=0, \text{ for all } \alpha,\beta>0;$$

• Exponential weakly tractable (EXP-WT) if

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{\star}(\varepsilon,d;\Lambda)}{1+\ln\varepsilon^{-1}+d}=0;$$

• Exponential (s, t)-weakly tractable (EXP-(s, t)-WT) for fixed s, t > 0 if

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^*(\varepsilon,d;\Lambda)}{(1+\ln\varepsilon^{-1})^s+d^t}=0.$$

2.3. Main results.

We shall give main results of this paper in this subsection. We remark that for multivariate approximation problem results and proofs in the average case setting are in full analogy with ones in the randomized setting (see [20]). For the convenience of the reader, we provide details of all proofs.

The authors in [7, 24, 38] used the mean value theorem and iterated Monte Carlo methods to obtain the relation between $e^{\text{avg}}(n, d; \Lambda^{\text{std}})$ and $e^{\text{avg}}(n, d; \Lambda^{\text{all}})$. We use the mean value theorem and the method used in [10, 20] to get an inequality between $e^{\text{avg}}(n, d; \Lambda^{\text{std}})$ and $e^{\text{avg}}(n, d; \Lambda^{\text{all}})$. See the following theorem.

Theorem 2.1. Let $\delta \in (0,1)$, $m, n \in \mathbb{N}$ be such that

$$m = \left\lfloor \frac{n}{48(\sqrt{2}\ln(2n) - \ln\delta)} \right\rfloor$$

Then we have

(2.10)
$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}) \le \left(1 + \frac{4m}{n}\right)^{\frac{1}{2}} \frac{1}{\sqrt{1-\delta}} e^{\operatorname{avg}}(m,d;\Lambda^{\operatorname{all}}),$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Based on Theorem 2.1, we obtain two relations between the information complexities $n^{\star}(\varepsilon, d; \Lambda^{\text{std}})$ and $n^{\star}(\varepsilon, d; \Lambda^{\text{all}})$ for $\star \in \{\text{ABS, NOR}\}$.

Theorem 2.2. For $\star \in \{ABS, NOR\}$ and $\omega > 0$, we have

(2.11)
$$n^{\star}(\varepsilon, d; \Lambda^{\text{std}}) \le C_{\omega} \left(n^{\star}(\frac{\varepsilon}{4}, d; \Lambda^{\text{all}}) + 1 \right)^{1+\omega}$$

where C_{ω} is a positive constant depending only on ω . Similarly, for sufficiently small $\omega, \delta > 0$ and $\star \in \{ABS, NOR\}$, we have

(2.12)
$$n^{\star}(\varepsilon, d; \Lambda^{\text{std}}) \le C_{\omega, \delta} \left(n^{\star}(\frac{\varepsilon}{A_{\delta}}, d; \Lambda^{\text{all}}) + 1 \right)^{1+\omega},$$

where $A_{\delta} := \left(1 + \frac{1}{12 \ln \frac{1}{\delta}}\right)^{\frac{1}{2}} \frac{1}{\sqrt{1-\delta}}$, $C_{\omega,\delta}$ is a positive constant depending only on ω and δ .

In the average case setting, for the normalized error criterion, [24, Theorems 24.10, 24.12, and 24.6] gives the equivalences of ALG-PT (ALG-SPT), ALG-QPT, ALG-WT for Λ^{all} and Λ^{std} , and shows that the exponents of ALG-SPT and ALG-QPT for Λ^{all} and Λ^{std} are same; [38, Theorems 3.4 and 3.5] gives the equivalences of ALG-(s, t)-WT, ALG-UWT for Λ^{all} and Λ^{std} .

For the absolute error criterion, [24, Theorems 24.11, 24.13, and 24.6] gives the equivalences of ALG-PT (ALG-SPT), ALG-QPT, ALG-WT for Λ^{all} and Λ^{std} under some conditions on the initial error. Novak and Woźniakowski posed Open problems 116-118 in [24] which ask whether the above conditions are necessary. Xu obtained in [38, Theorems 3.1-3.5] the equivalences of ALG-PT, ALG-QPT, ALG-WT, ALG-(s, t)-WT, ALG-UWT for Λ^{all} and Λ^{std} under much weaker conditions. This gives a partial solution to Open problems 116-118 in [24].

In this paper we obtain the equivalences of ALG-SPT, ALG-PT, ALG-QPT, ALG-WT, ALG-(s, t)-WT, ALG-UWT for Λ^{all} and Λ^{std} in the average case setting for the absolute error criterion without any condition, which means the above conditions are unnecessary. This solves Open problems 116-118 in [24]. See the following theorem.

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Theorem 2.3. Consider the problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ in the average case setting for the absolute error criterion. Then

• ALG-SPT, ALG-PT, ALG-QPT, ALG-WT, ALG-(s,t)-WT, ALG-UWT for Λ^{all} is equivalent to ALG-SPT, ALG-PT, ALG-QPT, ALG-WT, ALG-(s,t)-WT, ALG-UWT for Λ^{std} ;

• The exponents $ALG-p^{ABS}(\Lambda)$ of ALG-SPT for Λ^{all} and Λ^{std} are same, and the exponents $ALG-t^{ABS}(\Lambda)$ of ALG-QPT for Λ^{all} and Λ^{std} are also same.

For exponential convergence in the average case setting, we first give an almost complete solution to Open Problem 115 in [24].

In the average case setting for the normalized error criterion, Xu obtained in [38, Theorems 4.1-4.5] the equivalences of EXP-SPT, EXP-PT, EXP-QPT, EXP-WT, EXP-(s, t)-WT, EXP-UWT for $\Lambda^{\rm all}$ and $\Lambda^{\rm std}$, however, he did not show that the exponents of EXP-SPT and EXP-QPT for $\Lambda^{\rm all}$ and $\Lambda^{\rm std}$ are same.

For the absolute error criterion, Xu also obtained in [38, Theorems 4.1-4.5] the equivalences of EXP-SPT, EXP-PT, EXP-QPT, EXP-WT, EXP-(s, t)-WT, EXP-UWT for Λ^{all} and Λ^{std} under weak conditions on the initial error.

In this paper we obtain the equivalences of EXP-SPT, EXP-PT, EXP-QPT, EXP-WT, EXP-(s, t)-WT, EXP-UWT for Λ^{all} and Λ^{std} in the average case setting for the absolute error criterion without any condition, which means the above conditions are unnecessary. We also show that the exponents of EXP-SPT and EXP-QPT for Λ^{all} and Λ^{std} are same for the normalized or absolute error criterion. See the following theorem.

Theorem 2.4. Consider the problem $APP = {APP_d}_{d \in \mathbb{N}}$ in the average case setting. Then

• for the absolute error criterion, EXP-SPT, EXP-PT, EXP-QPT, EXP-WT, EXP-(s, t)-WT, EXP-UWT for Λ^{all} is equivalent to EXP-SPT, EXP-PT, EXP-QPT, EXP-WT, EXP-WT, EXP-UWT for Λ^{std} ;

• for $\star \in \{ABS, NOR\}$, the exponents EXP- $p^{\star}(\Lambda)$ of EXP-SPT for Λ^{all} and Λ^{std} are same, and the exponents EXP- $t^{\star}(\Lambda)$ of EXP-QPT for Λ^{all} and Λ^{std} are also same.

Combining the obtained results in [24, 38] with Theorems 2.3 and 2.4 we obtain the following corollary.

Corollary 2.5. Consider the approximation problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ for the absolute or normalized error criterion in the average case setting. Then

• ALG-SPT, ALG-PT, ALG-QPT, ALG-WT, ALG-(s, t)-WT, ALG-UWT for Λ^{all} is equivalent to ALG-SPT, ALG-PT, ALG-QPT, ALG-WT, ALG-(s, t)-WT, ALG-UWT for Λ^{std} ;

• EXP-SPT, EXP-PT, EXP-QPT, EXP-WT, EXP-(s, t)-WT, EXP-UWT for Λ^{all} is equivalent to EXP-SPT, EXP-PT, EXP-QPT, EXP-WT, EXP-(s, t)-WT, EXP-UWT for Λ^{std} ;

• the exponents of SPT and QPT are the same for Λ^{all} and Λ^{std} , i.e., for $\star \in \{\text{ABS}, \text{NOR}\},\$

$$\begin{split} \mathrm{ALG}-p^{\star}(\Lambda^{\mathrm{all}}) &= \mathrm{ALG}-p^{\star}(\Lambda^{\mathrm{std}}), \qquad \mathrm{ALG}-t^{\star}(\Lambda^{\mathrm{all}}) = \mathrm{ALG}-t^{\star}(\Lambda^{\mathrm{std}}), \\ \mathrm{EXP}-p^{\star}(\Lambda^{\mathrm{all}}) &= \mathrm{EXP}-p^{\star}(\Lambda^{\mathrm{std}}), \qquad \mathrm{EXP}-t^{\star}(\Lambda^{\mathrm{all}}) = \mathrm{EXP}-t^{\star}(\Lambda^{\mathrm{std}}). \end{split}$$

3. Proofs of Theorems 2.1 and 2.2

Let us keep the notation of Subsection 2.1. For any $m \in \mathbb{N}$, we define the functions $h_{m,d}(x)$ and $\omega_{m,d}$ on D_d by

$$h_{m,d}(x) := \frac{1}{m} \sum_{j=1}^{m} |\eta_{j,d}(x)|^2, \quad \omega_{m,d}(x) := h_{m,d}(x) \,\rho_d(x),$$

where $\{\eta_{j,d}\}_{j=1}^{\infty}$ is an orthonormal basis in $G_d = L_2(D_d, \rho_d(x)dx)$. Then $\omega_{m,d}$ is a probability density function on D_d , i.e., $\int_{D_d} \omega_{m,d}(x) dx = 1$. We define the corresponding probability measure $\mu_{m,d}$ by

$$\mu_{m,d}(A) = \int_A \omega_{m,d}(x) \, dx,$$

where A is a Borel subset of D_d . We use the convention that $\frac{0}{0} := 0$. Then $\{\tilde{\eta}_{j,d}\}_{j=1}^{\infty}$ is an orthonormal system in $L_2(D_d, \mu_{m,d})$, where

$$\tilde{\eta}_{j,d} := \frac{\eta_{j,d}}{\sqrt{h_{m,d}}}.$$

For $\mathbf{X} = (x^1, \dots, x^n) \in D_d^n$, we use the following matrices (3.1)

$$\widetilde{L}_m = \widetilde{L}_m(\mathbf{X}) = \begin{pmatrix} \widetilde{\eta}_{1,d}(x^1) & \widetilde{\eta}_{2,d}(x^1) & \cdots & \widetilde{\eta}_{m,d}(x^1) \\ \widetilde{\eta}_{1,d}(x^2) & \widetilde{\eta}_{2,d}(x^2) & \cdots & \widetilde{\eta}_{m,d}(x^2) \\ \vdots & \vdots & & \vdots \\ \widetilde{\eta}_{1,d}(x^n) & \widetilde{\eta}_{2,d}(x^n) & \cdots & \widetilde{\eta}_{m,d}(x^n) \end{pmatrix} \quad \text{and} \quad \widetilde{H}_m = \frac{1}{n} \widetilde{L}_m^* \widetilde{L}_m,$$

where A^* is the conjugate transpose of a matrix A. Note that

$$\widetilde{N}(m) := \sup_{x \in D_d} \sum_{k=1}^m |\widetilde{\eta}_{k,d}(x)|^2 = m.$$

According to [10, Propositions 5.1 and 3.1] we have the following results.

Lemma 3.1. Let $n, m \in \mathbb{N}$. Let $x^1, \ldots, x^n \in D_d$ be drawn independently and identically distributed at random with respect to the probability measure $\mu_{m,d}$. Then it holds that

$$\mathbb{P}(\|\widetilde{H}_m - I_m\| > \frac{1}{2}) \le (2n)^{\sqrt{2}} \exp\left(-\frac{n}{48m}\right),$$

where \tilde{L}_m , \tilde{H}_m are given by (3.1), I_m is the identity matrix of order m, and ||L|| denotes the spectral norm (i.e. the largest singular value) of a matrix L. Furthermore, if $||\tilde{H}_m - I_m|| \leq 1/2$, then

(3.2)
$$\|(\tilde{L}_m^*\tilde{L}_m)^{-1}\| \le \frac{2}{n}.$$

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Remark 3.2. From Lemma 3.1 we immediately obtain

(3.3)
$$\mathbb{P}(\|\widetilde{H}_m - I_m\| \le 1/2) \ge 1 - \delta$$

if

(3.4)
$$m = \left\lfloor \frac{n}{48(\sqrt{2}\ln(2n) - \ln \delta)} \right\rfloor \ge 1,$$

holds, where |x| denotes the largest integer not exceeding x.

Now let $m, n \in \mathbb{N}$ satisfy (3.4), x^1, \ldots, x^n be independent and identically distributed sample points from D_d that are distributed according to the probability measure $\mu_{m,d}$, and $\widetilde{L}_m, \widetilde{H}_m$ be given by (3.1). We consider the conditional distribution given the event $\|H_m - I_m\| \leq 1/2$ and the conditional expectation

$$\mathbb{E}(X \mid \|\widetilde{H}_m - I_m\| \le 1/2) = \frac{\int_{\|\widetilde{H}_m - I_m\| \le 1/2} X(x^1, \dots, x^n) \, d\mu_{m,d}(x^1) \dots \, d\mu_{m,d}(x^n)}{\mathbb{P}(\|\widetilde{H}_m - I_m\| \le 1/2)}$$

of a random variable X.

If $\|\widetilde{H}_m - I_m\| \leq 1/2$ for some $\mathbf{X} = (x^1, \dots, x^n) \in D_d^n$, then $\widetilde{L}_m = \widetilde{L}_m(\mathbf{X})$ has the full rank. The algorithm is a weighted least squares estimator

(3.5)
$$S_{\mathbf{X}}^{m} f = \underset{g \in V_{m}}{\operatorname{arg\,min}} \frac{|f(x^{i}) - g(x^{i})|^{2}}{h_{m,d}(x^{i})}$$

which has a unique solution, where $V_m := \operatorname{span}\{\eta_{1,d}, \ldots, \eta_{m,d}\}$. It follows that $S_{\mathbf{X}}^m f = f$ whenever $f \in V_m$.

Algorithm Weighted least squares regression.		
Input:	$\mathbf{X} = (x^1, \dots, x^n) \in D_d^n$	set of distinct sampling nodes,
	$\tilde{\mathbf{f}} = \left(\frac{f(x^1)}{\sqrt{h_{m,d}(x^1)}}, \dots, \frac{f(x^n)}{\sqrt{h_{m,d}(x^n)}}\right)^T$	weighted samples of f evaluted
		at the nodes from X,
	$m \in \mathbb{N}$	m < n such that the matrix
		$\widetilde{L}_m := \widetilde{L}_m(\mathbf{X})$ from (3.1) has
		full (column) rank.
Solve the over-determined linear system		

$$\widetilde{L}_m(\widetilde{c}_1,\cdots,\widetilde{c}_m)^T = \widetilde{\mathbf{f}}$$

via least square, i.e., compute

$$(\widetilde{c}_1,\cdots,\widetilde{c}_m)^T = (\widetilde{L}_m^*\widetilde{L}_m)^{-1}\widetilde{L}_m^*\widetilde{\mathbf{f}}.$$

Output: $\widetilde{c} = (\widetilde{c}_1, \cdots, \widetilde{c}_m)^T \in \mathbb{C}^m$ coefficients of the approximant $S_X^m(f) := \sum_{k=1}^m \widetilde{c}_k \eta_{k,d}$ which is the unique solution of (3.5).

Proof of Theorem 2.1.

We use the above notation. Let $m, n \in \mathbb{N}$ satisfy (3.4), x^1, \ldots, x^n be independent and identically distributed sample points from D_d that are distributed according to the probability measure $\mu_{m,d}$, $\|\widetilde{H}_m - I_m\| \leq 1/2$, and $S_X^m(f)$ be defined as above. We estimate $\|f - S_X^m(f)\|_{G_d}^2$ for $f \in F_d$. We set

$$H_d = L_2(D_d, \mu_{m,d}).$$

We recall that $\{\eta_{j,d}\}_{j=1}^{\infty}$ is an orthonormal basis in $G_d = L_2(D_d, \rho_d(x)dx)$, and hence $\{\tilde{\eta}_{j,d}\}_{j=1}^{\infty}$ is an orthonormal system in $H_d = L_2(D_d, \mu_{m,d})$, where

$$\tilde{\eta}_{j,d} := \frac{\eta_{j,d}}{\sqrt{h_{m,d}}}, \quad \langle \tilde{\eta}_{j,d}, \tilde{\eta}_{k,d} \rangle_{H_d} = \langle \eta_{j,d}, \eta_{k,d}, \rangle_{G_d} = \delta_{i,j}$$

For $f \in F_d \subset G_d$, we have

$$f = \sum_{k=1}^{\infty} \langle f, \eta_{k,d} \rangle_{G_d} \eta_{k,d}.$$

We note that $f - A^*_{m,d}(f)$ is orthogonal to the space V_m with respect to the inner product $\langle \cdot, \cdot \rangle_{G_d}$, and

$$A_{m,d}^{*}(f) - S_{\mathbf{X}}^{m}(f) = S_{\mathbf{X}}^{m}(f - A_{m,d}^{*}(f)) \in V_{m} := \operatorname{span}\{\eta_{1,d}, \dots, \eta_{m,d}\},\$$

where

$$A_{m,d}^*(f) = \sum_{k=1}^m \langle f, \eta_{k,d} \rangle_{G_d} \eta_{k,d}.$$

It follows that

$$\begin{split} \|f - S_{\mathbf{X}}^{m}(f)\|_{G_{d}}^{2} &= \|f - A_{m,d}^{*}(f)\|_{G_{d}}^{2} + \|S_{\mathbf{X}}^{m}(f - A_{m,d}^{*}(f))\|_{G_{d}}^{2} \\ &= \|g\|_{G_{d}}^{2} + \|S_{\mathbf{X}}^{m}(g)\|_{G_{d}}^{2}, \end{split}$$

where $g := f - A^*_{m,d}(f)$. We recall that

$$S_{\mathbf{X}}^{m}(g) = \sum_{k=1}^{m} \widetilde{c}_{k} \eta_{k,d}, \, \widetilde{c} = (\widetilde{c}_{1}, \dots, \widetilde{c}_{m})^{T} = (\widetilde{L}_{m}^{*} \widetilde{L}_{m})^{-1} (\widetilde{L}_{m})^{*} \widetilde{\mathbf{g}},$$

where

$$\widetilde{\mathbf{g}} := (\widetilde{g}(x^1), \cdots, \widetilde{g}(x^n))^T, \quad \widetilde{g} := \frac{g}{\sqrt{h_{m,d}}}.$$

Since $\{\eta_{k,d}\}_{k=1}^{\infty}$ is an orthonormal system in G_d , we get

$$\begin{split} \|S_{\mathbf{X}}^{m}(g)\|_{G_{d}}^{2} &= \|\widetilde{c}\|_{2}^{2} = \|((\widetilde{L}_{m})^{*}\widetilde{L}_{m})^{-1}(\widetilde{L}_{m})^{*}\widetilde{\mathbf{g}}\|_{2}^{2} \\ &\leq \|((\widetilde{L}_{m})^{*}\widetilde{L}_{m})^{-1}\| \cdot \|(\widetilde{L}_{m})^{*}\widetilde{\mathbf{g}}\|_{2}^{2} \\ &\leq \frac{4}{n^{2}}\|(\widetilde{L}_{m})^{*}\widetilde{\mathbf{g}}\|_{2}^{2}, \end{split}$$

where $\|\cdot\|_2$ is the Euclidean norm of a vector. We have

$$\begin{split} \|(\widetilde{L}_m)^*\widetilde{\mathbf{g}}\|_2^2 &= \sum_{k=1}^m \left|\sum_{j=1}^n \overline{\widetilde{\eta}_{k,d}(x^j)} \cdot \widetilde{g}(x^j)\right|^2 \\ &= \sum_{k=1}^m \sum_{j=1}^n \sum_{i=1}^n \overline{\widetilde{\eta}_{k,d}(x^j)} \widetilde{g}(x^j) \widetilde{\eta}_{k,d}(x^i) \overline{\widetilde{g}(x^i)}. \end{split}$$

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It follows that

$$J = \int_{\|\widetilde{H}_m - I_m\| \le \frac{1}{2}} \|(\widetilde{L}_m)^* \widetilde{g}\|_2^2 d\mu_{m,d}(x^1) \dots d\mu_{m,d}(x^n)$$

$$\leq \int_{D_d^n} \|(\widetilde{L}_m)^* \widetilde{g}\|_2^2 d\mu_{m,d}(x^1) \dots d\mu_{m,d}(x^n)$$

$$\leq \sum_{k=1}^m \sum_{i,j=1}^n \int_{D_d^n} \overline{\widetilde{\eta}_{k,d}(x^j)} \widetilde{g}(x^j) \widetilde{\eta}_{k,d}(x^i) \overline{\widetilde{g}(x^i)} d\mu_{m,d}(x^1) \dots d\mu_{m,d}(x^n),$$

Noting that for $i \neq j$ and $1 \leq k \leq m$,

$$\int_{D_d^n} \overline{\widetilde{\eta}_{k,d}(x^j)} \widetilde{g}(x^j) \widetilde{\eta}_{k,d}(x^i) \overline{\widetilde{g}(x^i)} \, d\mu_{m,d}(x^1) \dots d\mu_{m,d}(x^n)$$
$$= |\langle \widetilde{g}, \widetilde{\eta}_{k,d} \rangle_{H_d}|^2 = |\langle g, \eta_{k,d} \rangle_{G_d}|^2 = 0,$$

and $h_{m,d}(x) = \frac{1}{m} \sum_{k=1}^{m} |\eta_{k,d}(x)|^2$, we continue to get

$$\begin{split} J &\leq n \sum_{k=1}^{m} \| \widetilde{\eta}_{k,d} \cdot \widetilde{g} \|_{H_d}^2 \\ &= n \sum_{k=1}^{m} \int_{D_d^n} |\widetilde{g}(x) \widetilde{\eta}_{k,d}(x)|^2 \rho_d(x) h_{m,d}(x) \, dx \\ &= n \sum_{k=1}^{m} \int_{D_d^n} \frac{|g(x) \eta_{k,d}(x)|^2}{h_{m,d}(x)} \rho_d(x) \, dx \\ &= n \int_{D_d^n} m |g(x)|^2 \rho_d(x) \, dx \\ &= n m \cdot \|g\|_{G_d}^2. \end{split}$$

Hence, we have

(3.6)
$$\int_{\|\widetilde{H}_m - I_m\| \le \frac{1}{2}} \|f - S_X^m(f)\|_{G_d}^2 d\mu_{m,d}(x^1) \dots d\mu_{m,d}(x^n) \le \|g\|_{G_d}^2 + \frac{4}{n^2} J \le (1 + \frac{4m}{n}) \|g\|_{G_d}^2 = (1 + \frac{4m}{n}) \|f - A_{m,d}^*(f)\|_{G_d}^2.$$

By Fubini's theorem, (3.3), (3.6), and (2.3) we have

$$\begin{split} & \mathbb{E}\Big(\int_{F_d} \|f - S_X^m(f)\|_{G_d}^2 \mu_d(df) \ \Big| \ \|\widetilde{H}_m - I_m\| \le 1/2\Big) \\ &= \int_{F_d} \mathbb{E}\Big(\|f - S_X^m(f)\|_{G_d}^2 \ \Big| \ \|\widetilde{H}_m - I_m\| \le 1/2\Big) \,\mu_d(df) \\ &= \frac{\int_{F_d} \int_{\|\widetilde{H}_m - I_m\| \le \frac{1}{2}} \|f - S_X^m f\|_{G_d}^2 d\mu_{m,d}(x^1) \dots d\mu_{m,d}(x^n) \,\mu_d(df)}{\mathbb{P}(\|\widetilde{H}_m - I_m\| \le \frac{1}{2})} \\ &\le \left(1 + \frac{4m}{n}\right) \frac{1}{1 - \delta} \, \int_{F_d} \|f - A_{m,d}^*(f)\|_{G_d}^2 \mu_d(df) \\ &= \left(1 + \frac{4m}{n}\right) \frac{1}{1 - \delta} \, (e^{\operatorname{avg}}(m, d; \Lambda^{\operatorname{all}}))^2. \end{split}$$

By the mean value theorem, we conclude that there are sample points X^* = $\{x^{1*},\ldots,x^{n*}\}$ such that $\|\widetilde{H}_m^*-I_m\|\leq \frac{1}{2}$ and

$$\int_{F_d} \|f - S_{X^*}^m(f)\|_{G_d}^2 \mu(df) = \mathbb{E}\Big(\int_{F_d} \|f - S_X^m(f)\|_{G_d}^2 \mu_d(df) \mid \|\widetilde{H}_m - I_m\| \le 1/2\Big).$$

We obtain that

$$(e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}))^2 \leq \int_{F_d} \|f - S^m_{X^*}(f)\|^2_{G_d} \mu(df)$$
$$\leq \left(1 + \frac{4m}{n}\right) \frac{1}{1 - \delta} \left(e^{\operatorname{avg}}(m,d;\Lambda^{\operatorname{all}})\right)^2.$$

This completes the proof of Theorem 2.1.

We stress that Theorem 2.1 is not constructive since we do not know how to choose the sample points $X^* = \{x^{1*}, \ldots, x^{n*}\}$. We only know that there exist $X^* = \{x^{1*}, \dots, x^{n*}\}$ for which the average case error of the weighted least squares algorithm $S_{X^*}^m$ enjoys the average case error bound of Theorem 2.1.

Proof of Theorem 2.2.

Applying Theorem 2.1 with $\delta = \frac{1}{2\sqrt{2}}$, we obtain

(3.7)
$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}) \le \left(1 + \frac{4m}{n}\right)^{\frac{1}{2}} \left(\frac{2^{\sqrt{2}}}{2^{\sqrt{2}}-1}\right)^{\frac{1}{2}} e^{\operatorname{avg}}(m,d;\Lambda^{\operatorname{all}}),$$

where $m, n \in \mathbb{N}$, and

$$m = \left\lfloor \frac{n}{48\sqrt{2}\ln(4n)} \right\rfloor$$

Since $1 + \frac{4m}{n} \le 1 + \frac{1}{12\sqrt{2}\ln(4n)} \le 2$, by (3.7) we get $e^{\operatorname{avg}}(n, d; \Lambda^{\operatorname{std}}) \le 4e^{\operatorname{avg}}(m, d; \Lambda^{\operatorname{all}}).$ (3.8)

It follows that

$$n^*(\varepsilon \ d \cdot \Lambda^{\text{std}})$$

$$n^{\star}(\varepsilon, d; \Lambda^{\text{std}}) = \min \left\{ n \mid e^{\text{avg}}(n, d; \Lambda^{\text{std}}) \leq \varepsilon \text{CRI}_d \right\}$$
$$\leq \min \left\{ n \mid 4e^{\text{avg}}(m, d; \Lambda^{\text{all}}) \leq \varepsilon \text{CRI}_d \right\}$$
$$= \min \left\{ n \mid e^{\text{avg}}(m, d; \Lambda^{\text{all}}) \leq \frac{\varepsilon}{4} \text{CRI}_d \right\}.$$

We note that

(3.9)

$$m = \left\lfloor \frac{n}{48\sqrt{2}\ln(4n)} \right\rfloor \ge \frac{n}{48\sqrt{2}\ln(4n)} - 1.$$

This inequality is equivalent to

(3.10)
$$4n \le 192\sqrt{2}(m+1)\ln(4n).$$

Taking logarithm on both sides of (3.10), and using the inequality $\ln x \leq \frac{1}{2}x$ for $x \ge 1$, we get

$$\ln(4n) \le \ln(m+1) + \ln(192\sqrt{2}) + \ln\ln(4n),$$

and

$$\frac{1}{2}\ln(4n) \le \ln(4n) - \ln\ln(4n) \le \ln(m+1) + \ln(192\sqrt{2}).$$

It follows from (3.10) that

(3.11)
$$n \le 96\sqrt{2}(m+1)(\ln(m+1) + \ln(192\sqrt{2})).$$

By (3.9) and (3.11) we obtain

(3.12)

$$n^{\star}(\varepsilon, d; \Lambda^{\mathrm{std}}) \leq 96\sqrt{2} \Big(n^{\star}(\frac{\varepsilon}{4}, d; \Lambda^{\mathrm{all}}) + 1 \Big) \Big(\ln \left(n^{\star}(\frac{\varepsilon}{4}, d; \Lambda^{\mathrm{all}}) + 1 \right) + \ln(192\sqrt{2}) \Big).$$

Since for any $\omega > 0$,

$$\sup_{x \ge 1} \frac{96\sqrt{2}(\ln x + \ln(192\sqrt{2}))}{x^{\omega}} = C_{\omega} < +\infty,$$

we obtain (2.11).

For sufficiently small $\delta > 0$ and $m, n \in \mathbb{N}$ satisfying

$$m = \left\lfloor \frac{n}{48(\sqrt{2}\ln(2n) - \ln\delta)} \right\rfloor,\,$$

by Theorem 2.1 we have

$$\begin{split} e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}) &\leq \left(1 + \frac{4m}{n}\right)^{\frac{1}{2}} \frac{1}{\sqrt{1-\delta}} e^{\operatorname{avg}}(m,d;\Lambda^{\operatorname{all}}) \\ &\leq \left(1 + \frac{1}{12\left(\sqrt{2}\ln(2n) + \ln\frac{1}{\delta}\right)}\right)^{\frac{1}{2}} \frac{1}{\sqrt{1-\delta}} e^{\operatorname{avg}}(m,d;\Lambda^{\operatorname{all}}) \\ &\leq \left(1 + \frac{1}{12\ln\frac{1}{\delta}}\right)^{\frac{1}{2}} \frac{1}{\sqrt{1-\delta}} e^{\operatorname{avg}}(m,d;\Lambda^{\operatorname{all}}) = A_{\delta} e^{\operatorname{avg}}(m,d;\Lambda^{\operatorname{all}}), \end{split}$$

where $A_{\delta} = \left(1 + \frac{1}{12 \ln \frac{1}{\delta}}\right)^{\frac{1}{2}} \frac{1}{\sqrt{1-\delta}}$. Using the same method used in the proof of (3.9), we have

$$n^{\star}(\varepsilon, d; \Lambda^{\mathrm{std}}) \leq \min\left\{n \mid e^{\mathrm{avg}}(m, d; \Lambda^{\mathrm{all}}) \leq \frac{\varepsilon}{A_{\delta}} \mathrm{CRI}_{d}\right\}.$$

We note that

$$n \le 48\left(\sqrt{2}\ln(2n) + \ln\frac{1}{\delta}\right)(m+1).$$

Taking logarithm on both sides, and using the inequalities $\ln x \le \frac{x}{4}$ for $x \ge 9$ and $a+b \le ab$ for $a,b \ge 2$, we get

$$\ln n \le \ln 48 + \ln \left(\sqrt{2}\ln(2n) + \ln \frac{1}{\delta}\right) + \ln(m+1)$$

$$\le \ln 48 + \ln(\sqrt{2}\ln(2n)) + \ln \ln \frac{1}{\delta} + \ln(m+1)$$

$$\le \ln 48 + \frac{\sqrt{2}}{4}\ln(2n) + \ln \ln \frac{1}{\delta} + \ln(m+1).$$

Since

$$\frac{\sqrt{2}}{4}\ln(2n) \le \ln n - \frac{\sqrt{2}}{4}\ln(2n)$$
 for $n \ge 9$,

we get

$$\sqrt{2}\ln(2n) \le 4\left(\ln 48 + \ln\ln\frac{1}{\delta} + \ln(m+1)\right).$$

It follows that

$$n \le 48 \left(4 \left(\ln 48 + \ln \ln \frac{1}{\delta} + \ln(m+1) \right) + \ln \frac{1}{\delta} \right) (m+1).$$

We conclude that for sufficiently small $\delta > 0$,

(3.13)
$$n^{\star}(\varepsilon, d; \Lambda^{\text{std}}) \leq 48 \Big(4 \Big(\ln 48 + \ln \ln \frac{1}{\delta} + \ln \Big(n^{\star} \Big(\frac{\varepsilon}{A_{\delta}}, d; \Lambda^{\text{all}} \Big) + 1 \Big) \Big) \\ + \ln \frac{1}{\delta} \Big) \Big(n^{\star} \Big(\frac{\varepsilon}{A_{\delta}}, d; \Lambda^{\text{all}} \Big) + 1 \Big).$$

Since for sufficiently small $\omega, \delta > 0$, there holds

$$\sup_{x \ge 1} \frac{48(4(\ln 48 + \ln \ln \frac{1}{\delta} + \ln x) + \ln \frac{1}{\delta})}{x^{\omega}} = C_{\omega,\delta} < +\infty,$$

we get (2.12).

Theorem 2.2 is proved.

4. Equivalence results of algebraic tractability

First we consider the equivalences of ALG-PT and ALG-SPT for Λ^{std} and Λ^{all} in the average case setting. The equivalent results for the normalized error criterion can be found in [7] and [24, Theorem 24.10]. For the absolute error criterion, [24, Theorem 24.11] shows the equivalence of ALG-PT under the condition

(4.1)
$$\Gamma_d \leq Cd^v$$
 for all $d \in \mathbb{N}$, some $C > 0$, and some $v \geq 0$,

and the equivalence of ALG-SPT under the condition (4.1) with v = 0. Xu obtained in [38, Theorem 3.1] the equivalence of ALG-PT under the weaker condition

(4.2)
$$\Gamma_d \leq \exp(Cd^v)$$
 for all $d \in \mathbb{N}$, some $C > 0$, and some $v \geq 0$.

We obtain the following equivalent results of ALG-PT and ALG-SPT without any condition. Hence, the condition (4.1) or (4.2) is unnecessary. This solves Open Problem 117 as posed by Novak and Woźniakowski in [24].

Theorem 4.1. We consider the problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ in the average case setting for the absolute error criterion. Then,

• ALG-PT for Λ^{all} is equivalent to ALG-PT for Λ^{std}

• ALG-SPT for Λ^{all} is equivalent to ALG-SPT for Λ^{std} . In this case, the exponents of ALG-SPT for Λ^{all} and Λ^{std} are the same.

Proof. It follows from (2.5) that ALG-PT (ALG-SPT) for Λ^{std} means ALG-PT (ALG-SPT) for Λ^{all} . It suffices to show that ALG-PT (ALG-SPT) for Λ^{all} means that ALG-PT (ALG-SPT) for Λ^{std} .

Suppose that ALG-PT holds for Λ^{all} . Then there exist $C \ge 1$ and non-negative p, q such that

(4.3)
$$n^{ABS}(\varepsilon, d; \Lambda^{all}) \le Cd^q \varepsilon^{-p}, \text{ for all } d \in \mathbb{N}, \ \varepsilon \in (0, 1).$$

It follows from (2.11) and (4.3) that

$$n^{\text{ABS}}(\varepsilon, d; \Lambda^{\text{std}}) \leq C_{\omega} \left(C d^{q} (\frac{\varepsilon}{4})^{-p} + 1 \right)^{1+\omega} \\ \leq C_{\omega} (2C4^{p})^{1+\omega} d^{q(1+\omega)} \varepsilon^{-p(1+\omega)},$$

which means that ALG-PT holds for Λ^{std} .

If ALG-SPT holds for Λ^{all} , then (4.3) holds with q = 0. We obtain

$$n^{\text{ABS}}(\varepsilon, d; \Lambda^{\text{std}}) \le C_{\omega}(2C4^p)^{1+\omega} \varepsilon^{-p(1+\omega)},$$

which means that ALG-SPT holds for Λ^{std} . Furthermore, since ω can be arbitrary small, we get

$$\mathrm{ALG}_p^{\mathrm{ABS}}(\Lambda^{\mathrm{std}}) \leq \mathrm{ALG}_p^{\mathrm{ABS}}(\Lambda^{\mathrm{all}}) \leq \mathrm{ALG}_p^{\mathrm{ABS}}(\Lambda^{\mathrm{std}}),$$

which means that the exponents of ALG-SPT for Λ^{all} and Λ^{std} are the same. This completes the proof of Theorem 4.1.

Next we consider the equivalence of ALG-QPT for Λ^{std} and Λ^{all} in the average case setting. The result for the normalized error criterion can be found in [24, Theorem 24.12]. For the absolute error criterion, [24, Theorem 24.13] shows the equivalence of ALG-QPT under the condition

$$\limsup_{d\to\infty}\,\Gamma_d<\infty.$$

Xu obtained in [38, Theorem 3.2] the equivalence of ALG-QPT under the weaker condition (4.2).

We obtain the following equivalent results of ALG-QPT without any condition. Hence, the condition (4.2) is unnecessary. This solves Open Problem 118 as posed by Novak and Woźniakowski in [24].

Theorem 4.2. We consider the problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ in the average case setting for the absolute error criterion. Then, ALG-QPT for Λ^{all} is equivalent to ALG-QPT for Λ^{std} . In this case, the exponents of ALG-QPT for Λ^{all} and Λ^{std} are the same.

Proof. Similar to the proof of Theorem 4.1, it is enough to prove that ALG-QPT for Λ^{all} implies ALG-QPT for Λ^{std} .

Suppose that ALG-QPT holds for Λ^{all} . Then there exist $C \ge 1$ and non-negative t such that

(4.4)
$$n^{\text{ABS}}(\varepsilon, d; \Lambda^{\text{all}}) \le C \exp(t(1 + \ln d)(1 + \ln \varepsilon^{-1})), \text{ for all } d \in \mathbb{N}, \ \varepsilon \in (0, 1).$$

For sufficiently small $\delta > 0$ and $\omega > 0$, it follows from (2.12) and (4.4) that

$$n^{\operatorname{ran},\star}(\varepsilon,d;\Lambda^{\operatorname{std}}) \leq C_{\omega,\delta} \left(n^{\operatorname{wor},\star} \left(\frac{\varepsilon}{A_{\delta}},d;\Lambda^{\operatorname{all}}\right) + 1 \right)^{1+\omega} \\ \leq C_{\omega,\delta} \left(C \exp\left(t(1+\ln d)\left(1+\ln\left(\frac{\varepsilon}{A_{\delta}}\right)^{-1}\right)\right) + 1 \right)^{1+\omega} \\ \leq C_{\omega,\delta}(2C)^{1+\omega} \exp\left((1+\omega)t(1+\ln d)(1+\ln A_{\delta}+\ln \varepsilon^{-1})\right) \\ \leq C_{\omega,\delta}(2C)^{1+\omega} \exp\left((1+\omega)t(1+\ln A_{\delta})(1+\ln d)(1+\ln \varepsilon^{-1})\right),$$

where $t^* = (1 + \omega)(1 + \ln A_{\delta})t$, $A_{\delta} = \left(1 + \frac{1}{12\ln\frac{1}{\delta}}\right)^{\frac{1}{2}} \frac{1}{\sqrt{1-\delta}}$. This implies that ALG-QPT holds for Λ^{std} . Furthermore, taking the infimum over t for which (4.4) holds, and noting that $\lim_{(\delta,\omega)\to(0,0)} (1+\omega)(1+\ln A_{\delta}) = 1$, we get that

$$ALG-t^{ABS}(\Lambda^{std}) \leq ALG-t^{ABS}(\Lambda^{all}).$$

It follows from (2.5) that

$$\mathrm{ALG}\text{-}t^{\mathrm{ABS}}(\Lambda^{\mathrm{std}}) \leq \mathrm{ALG}\text{-}t^{\mathrm{ABS}}(\Lambda^{\mathrm{all}}) \leq \mathrm{ALG}\text{-}t^{\mathrm{ABS}}(\Lambda^{\mathrm{std}}),$$

which means that the exponents ALG- $t^{ABS}(\Lambda^{all})$ and ALG- $t^{ABS}(\Lambda^{std})$ are equal if ALG-QPT holds for Λ^{all} . This completes the proof of Theorem 4.2.

Now we consider the equivalence of ALG-WT for Λ^{std} and Λ^{all} in the average case setting . The result for the normalized error criterion can be found in [24, Theorem 24.6]. For the absolute error criterion, [24, Theorem 24.6] shows the equivalence of ALG-WT under the condition

$$\lim_{d \to \infty} \frac{\ln \max(\Gamma_d, 1)}{d} = 0.$$

Xu obtained in [38, Theorem 3.3] the equivalence of ALG-QPT under the much weaker condition.

(4.5)
$$\lim_{d \to \infty} \frac{\ln \left(1 + \ln \max(\Gamma_d, 1)\right)}{d} = 0.$$

We obtain the following equivalent results of ALG-WT without any condition. Hence, the condition (4.5) is unnecessary. This solves Open Problem 116 as posed by Novak and Woźniakowski in [24].

Theorem 4.3. We consider the problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ in the average case setting for the absolute error criterion. Then, ALG-WT for Λ^{all} is equivalent to ALG-WT for Λ^{std} .

Proof. The proof is identical to the proof of Theorem 4.4 with s = t = 1 for the absolute error criterion. We omit the details.

Finally, we consider the equivalences of ALG-(s, t)-WT and ALG-UWT for Λ^{std} and Λ^{all} in the average case setting. The results for the normalized error criterion can be found in [38, Theorems 3.4 and 3.5]. For the absolute error criterion, [38, Theorem 3.4] shows the equivalence of ALG-(s, t)-WT under the condition

(4.6)
$$\lim_{d \to \infty} \frac{\ln\left(1 + \ln \max(\Gamma_d, 1)\right)}{d^t} = 0$$

[38, Theorem 3.5] shows the equivalence of ALG-UWT under the condition

(4.7)
$$\lim_{d \to \infty} \frac{\ln\left(1 + \ln \max(\Gamma_d, 1)\right)}{d^t} = 0 \text{ for all } t > 0.$$

We obtain the following equivalent results of ALG-(s, t)-WT for fixed s, t > 0and ALG-UWT for the absolute error criterion without any condition. Hence, the condition (4.6) or (4.7) is unnecessary.

Theorem 4.4. We consider the problem $APP = \{APP_d\}_{d\in\mathbb{N}}$ in the average case setting for the absolute error criterion. Then for fixed s, t > 0, ALG-(s, t)-WT for Λ^{all} is equivalent to ALG-(s, t)-WT for Λ^{std} .

Proof. Again it is enough to prove that ALG-(s, t)-WT for Λ^{all} implies ALG-(s, t)-WT for Λ^{std} .

Suppose that ALG-(s, t)-WT holds for Λ^{all} . Then we have

(4.8)
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\text{ABS}}(\varepsilon,d;\Lambda^{\text{all}})}{\varepsilon^{-s}+d^t} = 0$$

It follows from (2.11) that for $\omega > 0$,

$$\frac{\ln n^{\text{ABS}}(\varepsilon, d; \Lambda^{\text{std}})}{\varepsilon^{-s} + d^{t}} \leq \frac{\ln \left(C_{\omega} \left(n^{\text{ABS}}(\varepsilon/4, d; \Lambda^{\text{all}}) + 1 \right)^{1+\omega} \right)}{\varepsilon^{-s} + d^{t}}$$
$$\leq \frac{\ln(C_{\omega} 2^{1+\omega})}{\varepsilon^{-s} + d^{t}} + \frac{4^{s}(1+\omega) \ln n^{\text{ABS}}(\varepsilon/4, d; \Lambda^{\text{all}})}{(\varepsilon/4)^{-s} + d^{t}}.$$

Since $\varepsilon^{-1} + d \to \infty$ is equivalent to $\varepsilon^{-s} + d^t \to \infty$, by (4.8) we get that

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln(C_{\omega}2^{1+\omega})}{\varepsilon^{-s}+d^t} = 0 \quad \text{and} \quad \lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{\text{ABS}}(\varepsilon/4,d;\Lambda^{\text{all}})}{(\varepsilon/4)^{-s}+d^t} = 0.$$

We obtain

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{\text{ABS}}(\varepsilon,d;\Lambda^{\text{std}})}{\varepsilon^{-s}+d^t}=0,$$

which implies ALG-(s, t)-WT for Λ^{std} . The proof of Theorem 4.4 is finished. \Box

Theorem 4.5. We consider the problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ in the average case setting for the absolute error criterion. Then ALG-UWT for Λ^{all} is equivalent to ALG-UWT for Λ^{std} .

Proof. By definition we know that APP is ALG-UWT if and only if APP is ALG-(s, t)-WT for all s, t > 0. Since by Theorem 4.4 ALG-(s, t)-WT for Λ^{std} is equivalent to ALG-(s, t)-WT for Λ^{all} for all s, t > 0, we get the equivalence of ALG-UWT for Λ^{std} and Λ^{all} . Theorem 4.5 is proved.

Proof of Theorem 2.3.

Theorem 2.3 follows from Theorems 4.1-4.5 immediately.

5. Equivalence results of exponential tractability

First we consider exponential convergence. Assume that there exist two constants $A \ge 1$ and $q \in (0, 1)$ such that

(5.1)
$$\sqrt{\lambda_{n,d}} \le Aq^n e^{\operatorname{avg}}(0,d;\Lambda^{\operatorname{all}}) = Aq^n \sqrt{\Gamma_d}.$$

It follows that

$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{all}}) \le \frac{A}{1-q}q^{n+1}\sqrt{\Gamma_d}.$$

Novak and Woźniakowski proved in [24, Corollary 24.5] that there exist two constants $C_1 \ge 1$ and $q_1 \in (q, 1)$ independent of d and n such that

(5.2)
$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}) \le \frac{C_1 A}{1-q} q_1^{\sqrt{n}} \sqrt{\Gamma_d}$$

If A, q in (5.1) are independent of d, then

$$n^{\mathrm{NOR}}(\varepsilon, d; \Lambda^{\mathrm{all}}) \le C_2(\ln \varepsilon^{-1} + 1),$$

and

$$n^{\mathrm{NOR}}(\varepsilon, d; \Lambda^{\mathrm{std}}) \le C_3 (\ln \varepsilon^{-1} + 1)^2$$

Novak and Woźniakowski posed the following Open Problem 115:

- (1) Verify if the upper bound in (5.2) can be improved.
- (2) Find the smallest p for which there holds

$$n^{\mathrm{NOR}}(\varepsilon, d; \Lambda^{\mathrm{std}}) \le C_4 (\ln \varepsilon^{-1} + 1)^p.$$

We know that $p \leq 2$, and if (5.1) is sharp then $p \geq 1$.

The following theorem gives a confirmative solution to Open Problem 115 (1). We improve enormously the upper bound $q_1^{\sqrt{n}}$ in (5.2) to $q_2^{\frac{n}{\ln(4n)}}$ in (5.5), where $q_1, q_2 \in (q, 1)$.

Theorem 5.1. Let $m, n \in \mathbb{N}$ and

(5.3)
$$m = \left\lfloor \frac{n}{48\sqrt{2}\ln(4n)} \right\rfloor.$$

Then we have

(5.4)
$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}) \le 4e^{\operatorname{avg}}(m,d;\Lambda^{\operatorname{all}}).$$

Specifically, if (5.1) holds, then we have

(5.5)
$$e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}) \leq \frac{4A}{1-q} q_2^{\frac{n}{\ln(4n)}} e^{\operatorname{avg}}(0,d;\Lambda^{\operatorname{all}}),$$

where $q_2 = q^{\frac{1}{48\sqrt{2}}} \in (q, 1).$

Proof. Inequality (5.4) is just (3.8), which has been proved. If (5.1) holds, then by (5.3) and (5.4) we get

$$\begin{split} e^{\operatorname{avg}}(n,d;\Lambda^{\operatorname{std}}) &\leq \frac{4A}{1-q} q^{\left\lfloor \frac{n}{48\sqrt{2}\ln(4n)} \right\rfloor + 1} \sqrt{\Gamma_d} \\ &\leq \frac{4A}{1-q} q^{\frac{n}{48\sqrt{2}\ln(4n)}} \sqrt{\Gamma_d} \\ &= \frac{4A}{1-q} q^{\frac{n}{\ln(4n)}} e^{\operatorname{avg}}(0,d;\Lambda^{\operatorname{all}}). \end{split}$$

This completes the proof of Theorem 5.1.

Now we consider the equivalences of various notions of exponential tractability for Λ^{std} and Λ^{all} for the absolute error criterion in the average case setting.

First we consider the equivalences of EXP-PT and EXP-SPT for Λ^{std} and Λ^{all} . The results for the normalized error criterion can be found in [38, Theorem 4.1]. For the absolute error criterion, [38, Theorem 4.1] shows the equivalences of EXP-PT and EXP-SPT under the condition (4.2).

We obtain the following equivalent results of EXP-PT and EXP-SPT without any condition.

Theorem 5.2. We consider the problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ in the average case setting. Then

• for the absolute error criterion, EXP-PT (EXP-SPT) for Λ^{all} is equivalent to EXP-PT (EXP-SPT) for Λ^{std} ;

• if EXP-SPT holds for Λ^{all} for the absolute or normalized error criterion, then the exponents of EXP-SPT for Λ^{all} and Λ^{std} are the same.

Proof. Again, it is enough to prove that EXP-PT for Λ^{all} implies EXP-PT for Λ^{std} for the absolute error criterion.

Suppose that EXP-PT holds for $\Lambda^{\rm all}.$ Then there exist $C\geq 1$ and non-negative p,q such that

(5.6)
$$n^{\text{ABS}}(\varepsilon, d; \Lambda^{\text{all}}) \le C d^q (\ln \varepsilon^{-1} + 1)^p$$
, for all $d \in \mathbb{N}, \ \varepsilon \in (0, 1)$.

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It follows from (2.11) and (5.6) that

$$n^{\text{ABS}}(\varepsilon, d; \Lambda^{\text{std}}) \leq C_{\omega} \left(C d^q \left(\ln\left(\frac{\varepsilon}{4}\right)^{-1} + 1\right)^p + 1 \right)^{1+\omega} \\ \leq C_{\omega} (2C)^{1+\omega} (1+\ln 4)^{p(1+\omega)} d^{q(1+\omega)} (\ln \varepsilon^{-1} + 1)^{p(1+\omega)},$$

which means that EXP-PT holds for Λ^{std} .

If EXP-SPT holds for Λ^{all} , then (5.6) holds with q = 0. We obtain

$$n^{\mathrm{ABS}}(\varepsilon, d; \Lambda^{\mathrm{std}}) \leq C_{\omega} (2C)^{1+\omega} (1+\ln 4)^{p(1+\omega)} (\ln \varepsilon^{-1} + 1)^{p(1+\omega)},$$

which means that EXP-SPT holds for Λ^{std} . Furthermore, if EXP-SPT holds for Λ^{all} for the absolute or normalized error criterion and $p^* = \text{EXP-}p^*(\Lambda^{\text{all}})$ for $\star \in \{\text{ABS, NOR}\}$, then for any $\varepsilon > 0$, there is a constant $C_{\varepsilon} \ge 1$ for which

$$n^{\star}(\varepsilon, d; \Lambda^{\mathrm{all}}) \leq C_{\varepsilon} (\ln \varepsilon^{-1} + 1)^{p^{*} + \varepsilon}$$

holds. Using the same method, we get

$$n^{\star}(\varepsilon, d; \Lambda^{\mathrm{std}}) \leq C_{\omega}(2C_{\varepsilon})^{1+\omega} (1+\ln 4)^{(p^{\star}+\varepsilon)(1+\omega)} (\ln \varepsilon^{-1}+1)^{(p^{\star}+\varepsilon)(1+\omega)},$$

Noting that ε, ω can be arbitrary small, we have for $\star \in \{ABS, NOR\}$,

$$\mathrm{EXP}_{-p^{\star}}(\Lambda^{\mathrm{std}}) \leq \mathrm{EXP}_{-p^{\star}}(\Lambda^{\mathrm{all}}) \leq \mathrm{EXP}_{-p^{\star}}(\Lambda^{\mathrm{std}}),$$

which means that the exponents of EXP-SPT for Λ^{all} and Λ^{std} are the same. This completes the proof of Theorem 5.2.

Remark 5.3. We remark that if (5.1) holds with A, q independent of d, then the problem APP is EXP-SPT for Λ^{all} in the average case setting for the normalized error criterion, and the exponent EXP- $p^{\text{NOR}}(\Lambda^{\text{all}}) \leq 1$. If (5.1) is sharp, then EXP- $p^{\text{NOR}}(\Lambda^{\text{all}}) = 1$.

Open Problem 115 (2) is equivalent to finding the exponent EXP- $p^{\text{NOR}}(\Lambda^{\text{std}})$ of EXP-SPT. By Theorem 5.2 we obtain that if (5.1) holds, then EXP- $p^{\text{NOR}}(\Lambda^{\text{std}}) \leq 1$, and if (5.1) is sharp, then EXP- $p^{\text{NOR}}(\Lambda^{\text{std}}) = 1$.

This solves Open Problem 115 (2) as posed by Novak and Woźniakowski in [24].

Next we consider the equivalence of EXP-QPT for Λ^{std} and Λ^{all} in the average case setting. The result for the normalized error criterion can be found in [38, Theorem 4.2]. For the absolute error criterion, [38, Theorem 4.2] shows the equivalence of EXP-QPT under the condition (4.2).

We obtain the following equivalent results of EXP-QPT without any condition.

Theorem 5.4. We consider the problem $APP = \{APP_d\}_{d\in\mathbb{N}}$ in the average case setting. Then, for the absolute error criterion EXP-QPT for Λ^{all} is equivalent to EXP-QPT for Λ^{std} . If EXP-QPT holds for Λ^{all} for the absolute or normalized error criterion, then the exponents of EXP-QPT for Λ^{all} and Λ^{std} are the same.

Proof. Again, it is enough to prove that EXP-QPT for Λ^{all} implies EXP-QPT for Λ^{std} for the absolute error criterion.

Suppose that EXP-QPT holds for Λ^{all} for the absolute or normalized error criterion. Then there exist $C \geq 1$ and non-negative t such that for $\star \in \{\text{ABS, NOR}\}$,

(5.7) $n^{\star}(\varepsilon, d; \Lambda^{\text{all}}) \leq C \exp(t(1+\ln d)(1+\ln(\ln \varepsilon^{-1}+1))), \text{ for all } d \in \mathbb{N}, \ \varepsilon \in (0,1).$

For sufficiently small $\omega > 0$ and $\delta > 0$, it follows from (2.12) and (5.7) that

$$n^{*}(\varepsilon, d; \Lambda^{\text{std}}) \leq C_{\omega,\delta} \left(n^{*}(\frac{\varepsilon}{A_{\delta}}, d; \Lambda^{\text{all}}) + 1 \right)^{1+\omega}$$

$$\leq C_{\omega,\delta} \left(C \exp\left(t(1+\ln d) \left(1 + \ln(\ln \varepsilon^{-1} + \ln A_{\delta} + 1) \right) \right) + 1 \right)^{1+\omega}$$

$$\leq C_{\omega,\delta} (2C)^{1+\omega} \exp\left((1+\omega)t(1+\ln d)(1+\ln(\ln A_{\delta} + 1) + \ln(\ln \varepsilon^{-1} + 1)) \right)$$

$$(5.8) \qquad \leq C_{\omega,\delta} (2C)^{1+\omega} \exp\left(t^{*}(1+\ln d)(1+\ln(\ln \varepsilon^{-1} + 1)) \right),$$

where $t^* = (1 + \omega)(1 + \ln(\ln A_{\delta} + 1))t$ and $A_{\delta} = (1 + \frac{1}{12\ln \frac{1}{\delta}})^{\frac{1}{2}} \frac{1}{\sqrt{1-\delta}}$, in the third inequality we used the fact

$$\ln(1 + a + b) \le \ln(1 + a) + \ln(1 + b), \quad a, b \ge 0.$$

The inequality (5.8) with \star = ABS implies that EXP-QPT holds for Λ^{std} for the absolute error criterion.

Next, we suppose that EXP-QPT holds for Λ^{all} for the absolute or normalized error criterion. Taking the infimum over t for which (5.7) holds, and noting that

$$\lim_{(\delta,\omega)\to(0,0)} (1+\omega)(1+\ln(\ln A_{\delta}+1)) = 1,$$

by (2.5) we obtain that

$$EXP-t^{\star}(\Lambda^{std}) \leq EXP-t^{\star}(\Lambda^{all}) \leq EXP-t^{\star}(\Lambda^{std}).$$

which means that the exponents EXP- $t^*(\Lambda^{\text{all}})$ and EXP- $t^*(\Lambda^{\text{std}})$ are equal. This completes the proof of Theorem 5.4.

Next, we consider the equivalences of EXP-(s, t)-WT and EXP-WT for Λ^{std} and Λ^{all} in the average case setting. The results for the normalized error criterion can be found in [38, Theorems 4.3 and 4.4]. For the absolute error criterion, [38, Theorem 4.3] shows the equivalence of EXP-WT under the condition(4.5). Meanwhiles, [38, Theorem 4.4] shows the equivalence of EXP-(s, t)-WT under the condition (4.6).

We obtain the following equivalent results of EXP-(s, t)-WT and EXP-WT for the absolute error criterion without any condition.

Theorem 5.5. We consider the problem $APP = \{APP_d\}_{d\in\mathbb{N}}$ in the average case setting for the absolute error criterion. Then for fixed s, t > 0, EXP-(s, t)-WT for Λ^{all} is equivalent to EXP-(s, t)-WT for Λ^{std} . Specifically, EXP-WT for Λ^{all} is equivalent to EXP-WT for Λ^{std} .

Proof. Again, it is enough to prove that EXP-(s, t)-WT for Λ^{all} implies EXP-(s, t)-WT for Λ^{std} .

Suppose that EXP-(s, t)-WT holds for Λ^{all} . Then we have

(5.9)
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\text{ABS}}(\varepsilon, d; \Lambda^{\text{all}})}{(1+\ln\varepsilon^{-1})^s + d^t} = 0.$$

It follows from (2.11) that for $\omega > 0$,

$$\frac{\ln n^{\mathrm{ABS}}(\varepsilon, d; \Lambda^{\mathrm{std}})}{(1+\ln \varepsilon^{-1})^s + d^t} \le \frac{\ln \left(C_\omega \left(n^{\mathrm{ABS}}(\varepsilon/4, d; \Lambda^{\mathrm{all}}) + 1\right)^{1+\omega}\right)}{(1+\ln \varepsilon^{-1})^s + d^t}$$
$$\le \frac{\ln (C_\omega 2^{1+\omega})}{(1+\ln \varepsilon^{-1})^s + d^t} + \frac{(1+\ln 4)^s (1+\omega) \ln n^{\mathrm{ABS}}(\varepsilon/4, d; \Lambda^{\mathrm{all}})}{(1+\ln (\varepsilon/4)^{-1})^s + d^t}.$$

Since $\varepsilon^{-1} + d \to \infty$ is equivalent to $(1 + \ln \varepsilon^{-1})^s + d^t \to \infty$, by (5.9) we get that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln(C_{\omega}2^{1+\omega})}{(1+\ln\varepsilon^{-1})^s+d^t} = 0 \quad \text{and} \quad \lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\text{ABS}}(\varepsilon/4,d;\Lambda^{\text{all}})}{(1+\ln(\varepsilon/4)^{-1})^s+d^t} = 0.$$

We obtain

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\text{ABS}}(\varepsilon, d; \Lambda^{\text{std}})}{(\ln \varepsilon^{-1})^s + d^t} = 0,$$

which implies that EXP-(s, t)-WT holds for Λ^{std} .

Specifically, EXP-WT is just EXP-(s, t)-WT with s = t = 1. This completes the proof of Theorem 5.5.

Finally, we consider the equivalences of EXP-UWT for Λ^{std} and Λ^{all} in the average case setting. The results for the normalized error criterion can be found in [38, Theorems 4.5]. For the absolute error criterion, [38, Theorem 4.5] shows the equivalence of EXP-UWT under the condition (4.7).

We obtain the following equivalent result of EXP-UWT for the absolute error criterion without any condition.

Theorem 5.6. We consider the problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ in the average case setting for the absolute error criterion. Then, EXP-UWT for Λ^{all} is equivalent to EXP-UWT for Λ^{std} .

Proof. By definition we know that APP is EXP-UWT if and only if APP is EXP-(s, t)-WT for all s, t > 0. Then Theorem 5.6 follows from Theorem 5.5 immediately.

Proof of Theorem 2.4.

Theorem 2.4 follows from Theorems 5.2 and 5.4-5.6 immediately.

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