Asymptotic analysis in multivariate average case approximation with Gaussian kernels

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Abstract

We consider tensor product random fields Y_d , $d \in \mathbb{N}$, whose covariance functions are Gaussian kernels with a given sequence of length scale parameters. The average case approximation complexity $n^{Y_d}(\varepsilon)$ is defined as the minimal number of evaluations of arbitrary linear functionals needed to approximate Y_d , with relative 2-average error not exceeding a given threshold $\varepsilon \in (0,1)$. We investigate the growth of $n^{Y_d}(\varepsilon)$ for arbitrary fixed $\varepsilon \in (0,1)$ and $d \to \infty$. Namely, we find criteria of boundedness for $n^{Y_d}(\varepsilon)$ on d and of tending $n^{Y_d}(\varepsilon) \to \infty$, $d \to \infty$, for any fixed $\varepsilon \in (0,1)$. In the latter case we obtain necessary and sufficient conditions for the following logarithmic asymptotics

$$\ln n^{Y_d}(\varepsilon) = a_d + q(\varepsilon)b_d + o(b_d), \quad d \to \infty,$$

with any $\varepsilon \in (0,1)$. Here $q:(0,1) \to \mathbb{R}$ is a non-decreasing function, $(a_d)_{d \in \mathbb{N}}$ is a sequence and $(b_d)_{d \in \mathbb{N}}$ is a positive sequence such that $b_d \to \infty$, $d \to \infty$. We show that only special quantiles of self-decomposable distribution functions appear as functions q in a given asymptotics. These general results apply to $n^{Y_d}(\varepsilon)$ under particular assumptions on the length scale parameters.

Keywords and phrases: average case approximation, multivariate problems, random fields, Gaussian kernels, asymptotic analysis, tractability.

1 Introduction and problem setting

We consider a multivariate approximation problem in average case setting for special random fields with arbitrary large parametric dimension.

Let $X = \{X(t), t \in \mathbb{R}\}$ be a random process defined on some probability space. Here and below \mathbb{R} denotes the set of real numbers. Suppose that the process has zero mean and the following covariance function

$$\mathcal{K}_{\sigma}(t,s) = \exp\left\{-\frac{(t-s)^2}{2\sigma^2}\right\}, \quad t,s \in \mathbb{R},$$

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where $\sigma > 0$ is a length scale parameter. The process is usually considered as a random element of the space $L_2(\mathbb{R}, \mu)$, where μ is the standard Gaussian measure on \mathbb{R} . Covariance operator acts as follows

$$K_{\sigma}f(t) = \int_{\mathbb{R}} \mathcal{K}_{\sigma}(t,s)f(s)\mu(\mathrm{d}s) = \int_{\mathbb{R}} \mathcal{K}_{\sigma}(t,s)f(s)\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\mathrm{d}s, \quad t \in \mathbb{R}.$$
 (1)

We consider d-variate version of X with arbitrary large $d \in \mathbb{N}$ (set of positive integers). Namely, we consider a zero-mean random field $Y_d = \{Y_d(t), t \in \mathbb{R}^d\}$ with the following covariance function

$$\mathcal{K}^{Y_d}(t,s) = \prod_{j=1}^d \mathcal{K}_{\sigma_j}(t_j, s_j) = \exp\left\{-\sum_{j=1}^d \frac{(t_j - s_j)^2}{2\sigma_j^2}\right\},\tag{2}$$

where $t = (t_1, \ldots, t_d)$ and $s = (s_1, \ldots, s_d)$ are from \mathbb{R}^d . Here $(\sigma_j)_{j \in \mathbb{N}}$ is a given sequence of length scale parameters, which are generally have different values. If every \mathcal{K}_{σ_j} corresponds to a zero-mean process $X_j = \{X_j(t), t \in \mathbb{R}\}$ (defined on some probability space), $j \in \mathbb{N}$, then Y_d is called tensor product of X_1, \ldots, X_d (see [10]). Function (2) is well known as Gaussian kernel, which is often used in numerical computation and statistical learning (see [3], [8], [18], [20], [23]).

For every $d \in \mathbb{N}$ the random field Y_d is considered as random element of the space $L_2(\mathbb{R}^d, \mu_d)$, where μ_d is the standard Gaussian measure on \mathbb{R}^d . So the space is equipped with the inner product

$$\langle f, g \rangle_{2,d} = \int_{\mathbb{R}^d} f(x)g(x)\mu_d(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)g(x)\frac{1}{(2\pi)^{d/2}} \exp\left\{-\frac{1}{2}\sum_{j=1}^d x_j^2\right\} \mathrm{d}x,$$

and the norm

$$||f||_{2,d} = \left(\int_{\mathbb{R}^d} f(x)^2 \,\mu_d(\mathrm{d}x)\right)^{1/2} = \left(\int_{\mathbb{R}^d} f(x)^2 \,\frac{1}{(2\pi)^{d/2}} \exp\left\{-\frac{1}{2} \sum_{j=1}^d x_j^2\right\} \mathrm{d}x\right)^{1/2},$$

where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ in the integrals. The covariance operator K^{Y_d} of Y_d acts as follows

$$K^{Y_d} f(t) = \int_{\mathbb{R}^d} \mathcal{K}^{Y_d}(t, s) f(s) \mu_d(\mathrm{d}s) = \int_{\mathbb{R}^d} \mathcal{K}^{Y_d}(t, s) f(s) \frac{1}{(2\pi)^{d/2}} \exp\left\{-\frac{1}{2} \sum_{j=1}^d s_j^2\right\} \mathrm{d}s,$$

where $t = (t_1, \ldots, t_d)$ and $s = (s_1, \ldots, s_d)$ are from \mathbb{R}^d .

We consider the average case approximation complexity (approximation complexity for short) of Y_d , $d \in \mathbb{N}$:

$$n^{Y_d}(\varepsilon) := \min\{n \in \mathbb{N} : e^{Y_d}(n) \leqslant \varepsilon e^{Y_d}(0)\},\tag{3}$$

where $\varepsilon \in (0,1)$ is a given error threshold, and

$$e^{Y_d}(n) := \inf \left\{ \left(\mathbb{E} \left\| Y_d - Y_d^{(n)} \right\|_{2,d}^2 \right)^{1/2} : Y_d^{(n)} \in \mathcal{A}_n^{Y_d} \right\}$$

is the smallest 2-average error among all linear approximations of Y_d having rank $n \in \mathbb{N}$ (\mathbb{E} is the expectation). The corresponding classes of linear algorithms are

$$\mathcal{A}_n^{Y_d} := \left\{ \sum_{m=1}^n \langle Y_d, \psi_m \rangle_{2,d} \, \psi_m : \psi_m \in L_2(\mathbb{R}^d, \mu_d) \right\}.$$

We will deal with the *normalized error*, i.e. we take into account the quantity:

$$e^{Y_d}(0) := (\mathbb{E} \|Y_d\|_{2,d}^2)^{1/2} < \infty,$$

which is the approximation error of Y_d by zero element.

For a given sequence $(\sigma_j)_{j\in\mathbb{N}}$ of length scale parameters in (2) the quantity $n^{Y_d}(\varepsilon)$ is considered as a function depending on two variables $d\in\mathbb{N}$ and $\varepsilon\in(0,1)$. There are a lot of results in this direction concerning the tractability (see [15]). They provide necessary and sufficient conditions on $(\sigma_j)_{j\in\mathbb{N}}$ to have upper bounds of given forms for the approximation complexity. The results within the described average case setting can be find in the papers [4], [5], and [12]. The other setting of the worst case was considered in [6], [16], and [21]. We will investigate $n^{Y_d}(\varepsilon)$ in the different way. Namely, we are interested in the asymptotic behaviour of $n^{Y_d}(\varepsilon)$ for arbirarily small fixed ε and $d\to\infty$. We are not aware of any asymptotic results in this way specially for random fields with covariance functions (2). There exist a suitable general methods and results from [11], but their application requires an additional analysis. So in fact we will do such analysis in this paper.

We will use the following notation. Let \mathbb{N}_0 denote the set of non-negative integers. We write $a_n \sim b_n$ if $a_n/b_n \to 1$, $n \to \infty$. The indicator $\mathbb{1}(A)$ equals one if A is true and zero if A is false. For any function f we will denote by $\mathcal{C}(f)$ the set of all its continuity points and by f^{-1} the generalized inverse function $f^{-1}(y) := \inf\{x \in \mathbb{R} : f(x) \ge y\}$, where y is from the range of f. By distribution function F we mean a non-decreasing function F on \mathbb{R} that is right-continuous on \mathbb{R} , $\lim_{x \to -\infty} F(x) = 0$, and $\lim_{x \to \infty} F(x) = 1$.

2 Preliminaries

The quantity $n^{Y_d}(\varepsilon)$ can be described in terms of the eigenvalues of the covariance operator K^{Y_d} . Let $(\lambda_m^{Y_d})_{m\in\mathbb{N}}$ denote the sequence of eigenvalues and $(\psi_m^{Y_d})_{m\in\mathbb{N}}$ the corresponding sequence of orthonormal eigenvectors of K^{Y_d} . The family $(\lambda_m^{Y_d})_{m\in\mathbb{N}}$ is assumed to be ranked in non-increasing order. We have therefore $K^{Y_d}\psi_m^{Y_d}(t) = \lambda_m^{Y_d}\psi_m^{Y_d}(t)$, $m\in\mathbb{N}$, $t\in\mathbb{R}^d$. We denote by Λ^{Y_d} the trace of K^{Y_d} , i.e. $\Lambda^{Y_d} := \sum_{m=1}^{\infty} \lambda_m^{Y_d}$.

It is well known (see [2], [19], [22]) that for any $n \in \mathbb{N}$ the following n-rank random field

$$\widetilde{Y}_d^{(n)}(t) := \sum_{k=1}^n \langle Y_d, \psi_k^{Y_d} \rangle_{2,d} \, \psi_k^{Y_d}(t), \quad t \in \mathbb{R}^d, \tag{4}$$

minimizes the 2-average case error. Hence formula (3) is reduced to

$$n^{Y_d}(\varepsilon) = \min \left\{ n \in \mathbb{N} : \mathbb{E} \left\| Y_d - \widetilde{Y}_d^{(n)} \right\|_{2,d}^2 \leqslant \varepsilon^2 \mathbb{E} \left\| Y_d \right\|_{2,d}^2 \right\}, \quad d \in \mathbb{N}, \ \varepsilon \in (0,1).$$

Due to (4) and $\mathbb{E}\langle Y_d, \psi_m^{Y_d} \rangle_{2,d}^2 = \langle \psi_m^{Y_d}, K^{Y_d} \psi_m^{Y_d} \rangle_{2,d} = \lambda_m^{Y_d}, m \in \mathbb{N}$, we have the needed representation:

$$n^{Y_d}(\varepsilon) = \min \Big\{ n \in \mathbb{N} : \sum_{m=n+1}^{\infty} \lambda_m^{Y_d} \leqslant \varepsilon^2 \Lambda^{Y_d} \Big\}, \quad d \in \mathbb{N}, \ \varepsilon \in (0,1).$$

We now consider the sequence $(\lambda_m^{Y_d})_{m\in\mathbb{N}}$. It has the following description. Let $(\lambda_{\sigma,k})_{k\in\mathbb{N}}$ denote the sequence of eigenvalues (ranked in non-increasing order) of the covariance operator K_{σ} defined by (1). This sequence is known (see [16], [18], and [24]):

$$\lambda_{\sigma,k} = (1 - \omega) \omega^{k-1}, \quad k \in \mathbb{N}, \quad \text{where} \quad \omega := \left(1 + \frac{\sigma^2}{2} \left(1 + \sqrt{1 + \frac{4}{\sigma^2}}\right)\right)^{-1}.$$
 (5)

In particular, we have $\lambda_{\sigma,1} = 1 - \omega$ and $\sum_{k \in \mathbb{N}} \lambda_{\sigma,k} = 1$. It is well known (see [13] and [16]) that, due to the tensor product structure (2) with given σ_j , $j \in \mathbb{N}$, $(\lambda_m^{Y_d})_{m \in \mathbb{N}}$ is the sequence of numbers

$$\prod_{j=1}^{d} \lambda_{\sigma_{j}, k_{j}} = \prod_{j=1}^{d} (1 - \omega_{j}) \, \omega_{j}^{k_{j}-1}, \quad k_{1}, k_{2}, \dots, k_{d} \in \mathbb{N},$$

ranked in non-increasing order (see [15]). Here, according to (5), we set

$$\omega_j := \left(1 + \frac{\sigma_j^2}{2} \left(1 + \sqrt{1 + \frac{4}{\sigma_j^2}}\right)\right)^{-1}, \quad j \in \mathbb{N}.$$

$$(6)$$

Observe that

$$\Lambda^{Y_d} = \prod_{i=1}^d \sum_{k \in \mathbb{N}} \lambda_{\sigma_j, k} = 1, \quad d \in \mathbb{N}.$$

Thus each of the sequences $(\sigma_j)_{j\in\mathbb{N}}$ and $(\omega_j)_{j\in\mathbb{N}}$ fully determines $(\lambda_m^{Y_d})_{m\in\mathbb{N}}$ and hence $n^{Y_d}(\varepsilon)$ for any $d\in\mathbb{N}$ and $\varepsilon\in(0,1)$.

3 General results

Before proceeding to the asymptotic analysis of the quantity $n^{Y_d}(\varepsilon)$, we find criteria of its boundedness and unboundedness on d for any fixed $\varepsilon \in (0,1)$. The following propositions show that for any fixed $\varepsilon \in (0,1)$ either the quantity $n^{Y_d}(\varepsilon)$ is a bounded function on $d \in \mathbb{N}$ or it tends to infinity as $d \to \infty$.

Proposition 1 The following conditions are equivalent:

- (i) $\sup_{d\in\mathbb{N}} n^{Y_d}(\varepsilon) < \infty$ for every $\varepsilon \in (0,1)$;
- (ii) $\sum_{i=1}^{\infty} \omega_i < \infty$;
- (iii) $\sum_{j=1}^{\infty} \sigma_j^{-2} < \infty$.

Proof of Proposition 1. By Proposition 5 from [11], the relation $\sup_{d\in\mathbb{N}} n^{Y_d}(\varepsilon) < \infty$, $\varepsilon \in (0,1)$, is equivalent to convergence of the following series

$$\sum_{j=1}^{\infty} \sum_{k=2}^{\infty} \frac{\lambda_{\sigma_j,k}}{\lambda_{\sigma_j,1}} = \sum_{j=1}^{\infty} \sum_{k=2}^{\infty} \omega_j^{k-1} = \sum_{j=1}^{\infty} \frac{\omega_j}{1 - \omega_j} = \sum_{j=1}^{\infty} \frac{2}{\sigma_j^2 + \sqrt{\sigma_j^4 + 4\sigma_j^2}}.$$
 (7)

Since $\omega_j \in (0,1)$, the convergence of $\sum_{j=1}^{\infty} \frac{\omega_j}{1-\omega_j}$ implies the convergence of $\sum_{j=1}^{\infty} \omega_j$. Next, if $\sum_{j=1}^{\infty} \omega_j < \infty$, then $\omega_j \to 0$ and hence $\frac{\omega_j}{1-\omega_j} \sim \omega_j$, $j \to \infty$. So we have $\sum_{j=1}^{\infty} \frac{\omega_j}{1-\omega_j} < \infty$. It is easily seen that the convergence of $\sum_{j=1}^{\infty} \sigma_j^{-2}$ implies the convergence of (7). Next, if (7)

It is easily seen that the convergence of $\sum_{j=1}^{\infty} \sigma_j^{-2}$ implies the convergence of (7). Next, if (7) converges, then $\sigma_j \to \infty$ and $\sigma_j^2 + \sqrt{\sigma_j^4 + 4\sigma_j^2} \sim 2\sigma_j^2$, $j \to \infty$. Hence we get the convergence of $\sum_{j=1}^{\infty} \sigma_j^{-2}$. \square

Proposition 2 The following conditions are equivalent:

- (i) $\lim_{d\to\infty} n^{Y_d}(\varepsilon) = \infty$ for every $\varepsilon \in (0,1)$;
- (ii) $\sum_{j=1}^{\infty} \omega_j = \infty$;
- (iii) $\sum_{j=1}^{\infty} \sigma_j^{-2} = \infty$.

Proof of Proposition 2. According to Proposition 4 and 5 from [11], for any $\varepsilon \in (0,1)$ either $\sup_{d \in \mathbb{N}} n^{Y_d}(\varepsilon) < \infty$ or $\lim_{d \to \infty} n^{Y_d}(\varepsilon) = \infty$. Hence the latter is equivalent to divergence of each of the series $\sum_{j=1}^{\infty} \omega_j$ and $\sum_{j=1}^{\infty} \sigma_j^{-2}$ by Proposition 1. \square

It is known (see [11]) that for wide class of tensor product random fields the quantity $n^{Y_d}(\varepsilon)$ has the logarithmic asymptotics of the form (8) below. Our next theorem shows that the function q can be only a special quantile of self-decomposable distribution function in such asymptotics (see [7], [17], or [11], Appendix). Recall that self-decomposable distribution functions are completely described by the triplet (c, v, L) from spectral representation of their characteristic functions, where $c \in \mathbb{R}$ is a shift parameter, v > 0 is the Gaussian component, L is the Lévy spectral function (see [11], Appendix).

Theorem 1 Let $(a_d)_{d\in\mathbb{N}}$ be a sequence, $(b_d)_{d\in\mathbb{N}}$ be a positive sequence such that $b_d \to \infty$, $d \to \infty$. Let a non-increasing function $q:(0,1)\to\mathbb{R}$ and a distribution function G satisfy the equation $q(\varepsilon)=G^{-1}(1-\varepsilon^2)$ for all $\varepsilon\in\mathcal{C}(q)$. Suppose that the following asymptotics holds

$$\forall \varepsilon \in \mathcal{C}(q) \quad \ln n^{Y_d}(\varepsilon) = a_d + q(\varepsilon)b_d + o(b_d), \quad d \to \infty.$$
 (8)

Then G is self-decomposable with zero Lévy spectral function on $(-\infty, 0)$.

Proof of Theorem 1. Due to Theorem 1 from [11], the condition (8) is equivalent to the convergence

$$\lim_{d \to \infty} G_d(x) = G(x), \quad x \in \mathcal{C}(G), \tag{9}$$

where

$$G_d(x) := \sum_{m \in \mathbb{N}} \lambda_m^{Y_d} \mathbb{1}\left(\lambda_m^{Y_d} \geqslant e^{-a_d - b_d x}\right), \quad x \in \mathbb{R}, \quad d \in \mathbb{N}.$$

$$(10)$$

It was shown in [11] that G_d , $d \in \mathbb{N}$, can be considered as the following distribution functions

$$G_d(x) = \mathbb{P}\left(\frac{\sum_{j=1}^d U_j - a_d}{b_d} \leqslant x\right), \quad x \in \mathbb{R}, \quad d \in \mathbb{N},$$

where U_j , $j \in \mathbb{N}$, are independent random variables on some probability space with the measure \mathbb{P} . Here U_j , $j \in \mathbb{N}$, have the following distribution

$$\mathbb{P}(U_j = |\ln \lambda_{\sigma_j,k}|) = \lambda_{\sigma_j,k}, \quad k \in \mathbb{N}, \quad j \in \mathbb{N}.$$

Now we center these variables in the following way: $\hat{U}_j := U_j - |\ln \lambda_{\sigma_j,1}|, j \in \mathbb{N}$. So we have

$$\mathbb{P}(\hat{U}_j = |\ln \lambda_{\sigma_j,k}| - |\ln \lambda_{\sigma_j,1}|) = \lambda_{\sigma_j,k}, \quad k \in \mathbb{N}, \quad j \in \mathbb{N},$$

i.e.

$$\mathbb{P}(\hat{U}_j = k | \ln \omega_j |) = (1 - \omega_j) \omega_j^k, \quad k \in \mathbb{N}_0, \quad j \in \mathbb{N}.$$

Here $\hat{U}_j \geqslant 0, j \in \mathbb{N}$. Next, we set

$$\hat{a}_d := a_d - \sum_{j=1}^d |\ln \lambda_{\sigma_j, 1}| = a_d - \sum_{j=1}^d |\ln(1 - \omega_j)|, \quad d \in \mathbb{N}.$$

Then

$$G_d(x) = \mathbb{P}\left(\frac{\sum_{j=1}^d \hat{U}_j - \hat{a}_d}{b_d} \leqslant x\right), \quad x \in \mathbb{R}, \quad d \in \mathbb{N},$$
(11)

For any $d \in \mathbb{N}$, $j \in \{1, ..., d\}$ and x > 0 we consider the following distribution tails:

$$\mathbb{P}(|\hat{U}_j| > xb_d) = \mathbb{P}(\hat{U}_j > xb_d) = \sum_{\substack{k \in \mathbb{N}_0: \\ k \mid \ln \omega_j \mid > xb_d}} (1 - \omega_j) \omega_j^k = \omega_j^{k_{j,d}(x)},$$

where $k_{j,d}(x) := \min\{k \in \mathbb{N}_0 : k|\ln \omega_j| > xb_d\}$. Since $k_{j,d}(x)|\ln \omega_j| > xb_d$, we have

$$\mathbb{P}(|\hat{U}_j| > xb_d) < e^{-xb_d}, \quad d \in \mathbb{N}, \quad j \in \{1, \dots, d\}, \quad x > 0.$$

Due to $b_d \to \infty$, $d \to \infty$, we obtain

$$\max_{j \in \{1, \dots, d\}} \mathbb{P}(|\hat{U}_j| > xb_d) \to 0, \quad d \to \infty.$$
(12)

This is the condition of uniform negligibility of \hat{U}_j/b_d in the sums $(\sum_{j=1}^n \hat{U}_j - \hat{a}_d)/b_d$. It is known, that under this condition and (11) the limit distribution in (9) is self-decomposable (see [17], p. 101, or [11], Theorem 10). Let L denote the Lévy spectral function of G. Due to the non-negativity of \hat{U}_j , we have L(x) = 0, x < 0 (see [7], p. 124, or [11], Theorem 11). \square

The next theorem provides a criterion for the asymptotics (8), where q is a quantile of a given self-decomposable distribution function. This is the main result of the paper.

Theorem 2 Let $(a_d)_{d\in\mathbb{N}}$ be a sequence, $(b_d)_{d\in\mathbb{N}}$ be a positive sequence such that $b_d \to +\infty$, $d \to \infty$. Let G be a self-decomposable distribution function with triplet (c, v, L) such that L(x) = 0, x < 0. Let a non-increasing function $q:(0,1)\to\mathbb{R}$ satisfy the equation $q(\varepsilon)=G^{-1}(1-\varepsilon^2)$ for all $\varepsilon\in(0,1)$. For the asymptotics

$$\forall \varepsilon \in (0,1) \quad \ln n^{Y_d}(\varepsilon) = a_d + q(\varepsilon)b_d + o(b_d), \quad d \to \infty, \tag{13}$$

the following ensemble of conditions is necessary and sufficient:

(A)
$$\lim_{d \to \infty} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| > \tau b_d}} \omega_j = -L(\tau), \quad \tau > 0;$$

(B)
$$\lim_{d \to \infty} \frac{1}{b_d} \left(\sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} - \hat{a}_d \right) = c + \gamma_\tau, \quad \tau > 0;$$

(C)
$$\lim_{\tau \to 0} \underline{\lim}_{d \to \infty} \frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = \lim_{\tau \to 0} \overline{\lim}_{d \to \infty} \frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = v;$$

where

$$\gamma_{\tau} := \int_{0}^{\tau} \frac{y^{3} dL(y)}{1 + y^{2}} - \int_{\tau}^{+\infty} \frac{y dL(y)}{1 + y^{2}}, \quad \tau > 0,$$
(14)

$$\hat{a}_d := a_d - \sum_{j=1}^d |\ln(1 - \omega_j)|, \quad d \in \mathbb{N}.$$
 (15)

The proof of this theorem is essentially based on the following lemma.

Lemma 1 For any x > 0 and $d \in \mathbb{N}$ the following identities hold:

$$\sum_{j=1}^{d} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid > x}} (1 - \omega_{j}) \omega_{j}^{k} = \sum_{\substack{j=1, \dots, d: \\ |\ln \omega_{j} \mid > x}} \omega_{j} + R_{0}(d, x),$$

$$\sum_{j=1}^{d} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leq x}} k \mid \ln \omega_{j} \mid (1 - \omega_{j}) \omega_{j}^{k} = \sum_{\substack{j=1, \dots, d: \\ |\ln \omega_{j} \mid \leq x}} \frac{|\ln \omega_{j}| \omega_{j}}{1 - \omega_{j}} - R_{1}(d, x),$$

$$\sum_{j=1}^{d} \left[\sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leq x}} k^{2} |\ln \omega_{j}|^{2} (1 - \omega_{j}) \omega_{j}^{k} - \left(\sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leq x}} k |\ln \omega_{j}| (1 - \omega_{j}) \omega_{j}^{k} \right)^{2} \right]$$

$$= \sum_{\substack{j=1, \dots, d: \\ |\ln \omega_{j} \mid \leq x}} \frac{|\ln \omega_{j}|^{2} \omega_{j}}{(1 - \omega_{j})^{2}} - R_{2}(d, x),$$

where

$$k_j(x) = \min\{k \in \mathbb{N} : k \geqslant 2, \, k|\ln \omega_j| > x\},\,$$

and

$$R_{0}(d,x) := \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant x}} \omega_{j}^{k_{j}(x)},$$

$$R_{1}(d,x) := \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant x}} \frac{|\ln \omega_{j}| \omega_{j}^{k_{j}(x)}}{1 - \omega_{j}} (k_{j}(x)(1 - \omega_{j}) + \omega_{j}),$$

$$R_{2}(d,x) := \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant x}} \frac{|\ln \omega_{j}|^{2} \omega_{j}^{k_{j}(x)}}{(1 - \omega_{j})^{2}} (k_{j}(x)^{2}(1 - \omega_{j})^{2}(1 + \omega_{j}^{k_{j}(x)})$$

$$+ 2k_{j}(x)(1 - \omega_{j})\omega_{j}^{k_{j}(x)+1} + (1 - \omega_{j})\omega_{j} + \omega_{j}^{k_{j}(x)+2}).$$

Proof of Lemma 1. We fix x > 0 and $d \in \mathbb{N}$.

1. Let us prove the first identity. Observe that

$$\sum_{j=1}^{d} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_j \mid > x}} (1 - \omega_j) \omega_j^k = \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_j| > x}} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_j \mid > x}} (1 - \omega_j) \omega_j^k + \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_j| \leqslant x}} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_j \mid > x}} (1 - \omega_j) \omega_j^k.$$

Here we have

$$\sum_{\substack{j=1,\dots,d:\\|\ln\omega_j|>x}}\sum_{\substack{k\in\mathbb{N}:\\k|\ln\omega_j|>x}}(1-\omega_j)\omega_j^k=\sum_{\substack{j=1,\dots,d:\\|\ln\omega_j|>x}}\sum_{k\in\mathbb{N}}(1-\omega_j)\omega_j^k=\sum_{\substack{j=1,\dots,d:\\|\ln\omega_j|>x}}\omega_j,$$

and

$$\begin{split} \sum_{\substack{j=1,\dots,d:\\|\ln\omega_j|\leqslant x}} \sum_{\substack{k\in\mathbb{N}:\\k|\ln\omega_j|>x}} (1-\omega_j)\omega_j^k &= \sum_{\substack{j=1,\dots,d:\\|\ln\omega_j|\leqslant x}} \sum_{\substack{k\in\mathbb{N}:\\k\geqslant 2,\\k|\ln\omega_j|>x}} (1-\omega_j)\omega_j^k \\ &= \sum_{\substack{j=1,\dots,d:\\|\ln\omega_j|\leqslant x}} \sum_{\substack{k=k_j(x)}} (1-\omega_j)\omega_j^k = \sum_{\substack{j=1,\dots,d:\\|\ln\omega_j|\leqslant x}} \omega_j^{k_j(x)}. \end{split}$$

2. Let us prove the second identity. It is obvious that if $|\ln \omega_j| > x$ then there are no any $k \in \mathbb{N}$ such that $k |\ln \omega_j| \leq x$. Therefore

$$\sum_{j=1}^{d} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_j \mid \leqslant x}} k |\ln \omega_j| (1 - \omega_j) \omega_j^k = \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_j \mid \leqslant x}} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_j \mid \leqslant x}} k |\ln \omega_j| (1 - \omega_j) \omega_j^k$$

$$= \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_j \mid \leqslant x}} |\ln \omega_j| (1 - \omega_j) \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_j \mid \leqslant x}} k \omega_j^k.$$

Here

$$\sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_j \mid \leqslant x}} k \omega_j^k \ = \ \sum_{k=1}^{\infty} k \omega_j^k - \sum_{k=k_j(x)}^{\infty} k \omega_j^k.$$

It is well known that

$$\sum_{k=1}^{\infty} k\omega_j^k = \omega_j \sum_{k=1}^{\infty} k\omega_j^{k-1} = \frac{\omega_j}{(1-\omega_j)^2}.$$

Next, using this fact we get

$$\begin{split} \sum_{k=k_{j}(x)}^{\infty} k \omega_{j}^{k} &= \omega_{j}^{k_{j}(x)} \sum_{k=k_{j}(x)}^{\infty} k \omega_{j}^{k-k_{j}(x)} \\ &= \omega_{j}^{k_{j}(x)} \sum_{k=k_{j}(x)+1}^{\infty} (k-k_{j}(x)) \omega_{j}^{k-k_{j}(x)} + k_{j}(x) \omega_{j}^{k_{j}(x)} \sum_{k=k_{j}(x)}^{\infty} \omega_{j}^{k-k_{j}(x)} \\ &= \frac{\omega_{j}^{k_{j}(x)+1}}{(1-\omega_{j})^{2}} + \frac{k_{j}(x) \omega_{j}^{k_{j}(x)}}{1-\omega_{j}}. \end{split}$$

Then

$$\sum_{j=1,\dots,d:}^{d} \sum_{\substack{k \in \mathbb{N}: \\ |\ln \omega_{j}| \leqslant x}} k |\ln \omega_{j}| (1-\omega_{j}) \omega_{j}^{k} = \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_{j}| \leqslant x}} |\ln \omega_{j}| (1-\omega_{j}) \left(\frac{\omega_{j}}{(1-\omega_{j})^{2}} - \frac{\omega_{j}^{k_{j}(x)+1}}{(1-\omega_{j})^{2}} - \frac{k_{j}(x) \omega_{j}^{k_{j}(x)}}{1-\omega_{j}} \right) \\
= \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_{j}| \leqslant x}} \frac{|\ln \omega_{j}| \omega_{j}}{1-\omega_{j}} - \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_{j}| \leqslant x}} \left(\frac{|\ln \omega_{j}| \omega_{j}^{k_{j}(x)+1}}{1-\omega_{j}} + k_{j}(x) |\ln \omega_{j}| \omega_{j}^{k_{j}(x)} \right) \\
= \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_{j}| \leqslant x}} \frac{|\ln \omega_{j}| \omega_{j}}{1-\omega_{j}} - \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_{j}| \leqslant x}} \frac{|\ln \omega_{j}| \omega_{j}^{k_{j}(x)}}{1-\omega_{j}} \left(\omega_{j} + k_{j}(x) (1-\omega_{j}) \right).$$

3. We now prove the third identity. Let us consider the sum

$$S_{1} := \sum_{j=1}^{d} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leqslant x}} k^{2} |\ln \omega_{j}|^{2} (1 - \omega_{j}) \omega_{j}^{k} = \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_{j} \mid \leqslant x}} \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leqslant x}} k^{2} |\ln \omega_{j}|^{2} (1 - \omega_{j}) \omega_{j}^{k}$$

$$= \sum_{\substack{j=1,\dots,d: \\ |\ln \omega_{j} \mid \leqslant x}} |\ln \omega_{j}|^{2} (1 - \omega_{j}) \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leqslant x}} k^{2} \omega_{j}^{k}.$$

Here

$$\sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_j \mid \leqslant x}} k^2 \omega_j^k = \sum_{k=1}^{\infty} k^2 \omega_j^k - \sum_{k=k_j(x)}^{\infty} k^2 \omega_j^k.$$

It is easily seen that

$$\begin{split} \sum_{k=1}^{\infty} k^2 \omega_j^k &=& \sum_{k=1}^{\infty} k(k-1) \omega_j^k + \sum_{k=1}^{\infty} k \omega_j^k = \omega_j^2 \sum_{k=2}^{\infty} k(k-1) \omega_j^{k-2} + \sum_{k=1}^{\infty} k \omega_j^k \\ &= \frac{2 \omega_j^2}{(1 - \omega_j)^3} + \frac{\omega_j}{(1 - \omega_j)^2} = \frac{\omega_j + \omega_j^2}{(1 - \omega_j)^3}. \end{split}$$

Next, we write

$$\sum_{k=k_{j}(x)}^{\infty} k^{2} \omega_{j}^{k} = \sum_{k=k_{j}(x)}^{\infty} (k - k_{j}(x))^{2} \omega_{j}^{k} + 2k_{j}(x) \sum_{k=k_{j}(x)}^{\infty} k \omega_{j}^{k} - k_{j}(x)^{2} \sum_{k=k_{j}(x)}^{\infty} \omega_{j}^{k}$$

$$= \omega_{j}^{k_{j}(x)} \sum_{k=k_{j}(x)+1}^{\infty} (k - k_{j}(x))^{2} \omega_{j}^{k-k_{j}(x)} + 2k_{j}(x) \sum_{k=k_{j}(x)}^{\infty} k \omega_{j}^{k} - k_{j}(x)^{2} \sum_{k=k_{j}(x)}^{\infty} \omega_{j}^{k}.$$

Due to above remarks we get

$$\sum_{k=k_{j}(x)}^{\infty} k^{2} \omega_{j}^{k} = \omega_{j}^{k_{j}(x)} \left(\frac{2\omega_{j}^{2}}{(1-\omega_{j})^{3}} + \frac{\omega_{j}}{(1-\omega_{j})^{2}} \right) + 2k_{j}(x) \left(\frac{\omega_{j}^{k_{j}(x)+1}}{(1-\omega_{j})^{2}} + \frac{k_{j}(x)\omega_{j}^{k_{j}(x)}}{1-\omega_{j}} \right)$$

$$- k_{j}(x)^{2} \cdot \frac{\omega_{j}^{k_{j}(x)}}{1-\omega_{j}}$$

$$= \frac{2\omega_{j}^{k_{j}(x)+2}}{(1-\omega_{j})^{3}} + \frac{(2k_{j}(x)+1)\omega_{j}^{k_{j}(x)+1}}{(1-\omega_{j})^{2}} + \frac{k_{j}(x)^{2}\omega_{j}^{k_{j}(x)}}{1-\omega_{j}}.$$

Thus we obtain

$$S_{1} = \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant x}} |\ln \omega_{j}|^{2} (1-\omega_{j}) \left(\frac{\omega_{j}+\omega_{j}^{2}}{(1-\omega_{j})^{3}} - \frac{2\omega_{j}^{k_{j}(x)+2}}{(1-\omega_{j})^{3}} - \frac{(2k_{j}(x)+1)\omega_{j}^{k_{j}(x)+1}}{(1-\omega_{j})^{2}} - \frac{k_{j}(x)^{2}\omega_{j}^{k_{j}(x)}}{1-\omega_{j}} \right)$$

$$= \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant x}} |\ln \omega_{j}|^{2} \left(\frac{\omega_{j}+\omega_{j}^{2}}{(1-\omega_{j})^{2}} - \frac{2\omega_{j}^{k_{j}(x)+2}}{(1-\omega_{j})^{2}} - \frac{(2k_{j}(x)+1)\omega_{j}^{k_{j}(x)+1}}{1-\omega_{j}} - k_{j}(x)^{2}\omega_{j}^{k_{j}(x)} \right).$$

We now consider the sums

$$S_{2} := \sum_{j=1}^{d} \left(\sum_{\substack{k \in \mathbb{N}:\\ k \mid \ln \omega_{j} \mid \leqslant x}} k \mid \ln \omega_{j} \mid (1-\omega_{j})\omega_{j}^{k} \right)^{2}$$

$$= \sum_{\substack{j=1,\dots,d:\\ |\ln \omega_{j}| \leqslant x}} \left(\sum_{\substack{k \in \mathbb{N}:\\ k \mid \ln \omega_{j} \mid \leqslant x}} k \mid \ln \omega_{j} \mid (1-\omega_{j})\omega_{j}^{k} \right)^{2}$$

$$= \sum_{\substack{j=1,\dots,d:\\ |\ln \omega_{i}| \leqslant x}} |\ln \omega_{j}|^{2} (1-\omega_{j})^{2} \left(\sum_{k=1}^{\infty} k\omega_{j}^{k} - \sum_{k=k_{j}(x)}^{\infty} k\omega_{j}^{k} \right)^{2}.$$

By the above remarks, we have

$$\left(\sum_{k=1}^{\infty} k\omega_{j}^{k} - \sum_{k=k_{j}(x)}^{\infty} k\omega_{j}^{k}\right)^{2} = \left(\frac{\omega_{j}}{(1-\omega_{j})^{2}} - \frac{\omega_{j}^{k_{j}(x)+1}}{(1-\omega_{j})^{2}} - \frac{k_{j}(x)\omega_{j}^{k_{j}(x)}}{1-\omega_{j}}\right)^{2} \\
= \frac{\omega_{j}^{2}}{(1-\omega_{j})^{4}} + \frac{\omega_{j}^{2k_{j}(x)+2}}{(1-\omega_{j})^{4}} + \frac{k_{j}(x)^{2}\omega_{j}^{2k_{j}(x)}}{(1-\omega_{j})^{2}} - \frac{2\omega_{j}^{k_{j}(x)+2}}{(1-\omega_{j})^{4}} \\
- \frac{2k_{j}(x)\omega_{j}^{k_{j}(x)+1}}{(1-\omega_{j})^{3}} + \frac{2k_{j}(x)\omega_{j}^{2k_{j}(x)+1}}{(1-\omega_{j})^{3}}.$$

Hence we get

$$S_{2} = \sum_{\substack{j=1,\dots,d:\\|\ln\omega_{j}|\leq x}} |\ln\omega_{j}|^{2} \left(\frac{\omega_{j}^{2}}{(1-\omega_{j})^{2}} + \frac{\omega_{j}^{2k_{j}(x)+2}}{(1-\omega_{j})^{2}} + k_{j}(x)^{2} \omega_{j}^{2k_{j}(x)} - \frac{2\omega_{j}^{k_{j}(x)+2}}{(1-\omega_{j})^{2}} - \frac{2k_{j}(x)\omega_{j}^{k_{j}(x)+1}}{1-\omega_{j}} + \frac{2k_{j}(x)\omega_{j}^{2k_{j}(x)+1}}{1-\omega_{j}} \right).$$

Thus we have

$$S_{1} - S_{2} = \sum_{\substack{j=1,\dots,d:\\|\ln\omega_{j}| \leqslant x}} |\ln\omega_{j}|^{2} \left(\frac{\omega_{j}}{(1-\omega_{j})^{2}} - \frac{\omega_{j}^{k_{j}(x)+1}}{1-\omega_{j}} - k_{j}(x)^{2} \omega_{j}^{k_{j}(x)}\right)$$

$$- \frac{\omega_{j}^{2k_{j}(x)+2}}{(1-\omega_{j})^{2}} - k_{j}(x)^{2} \omega_{j}^{2k_{j}(x)} - \frac{2k_{j}(x)\omega_{j}^{2k_{j}(x)+1}}{1-\omega_{j}}\right)$$

$$= \sum_{\substack{j=1,\dots,d:\\|\ln\omega_{j}| \leqslant x}} \frac{|\ln\omega_{j}|^{2}\omega_{j}}{(1-\omega_{j})^{2}} - \sum_{\substack{j=1,\dots,d:\\|\ln\omega_{j}| \leqslant x}} \frac{|\ln\omega_{j}|^{2} \omega_{j}^{k_{j}(x)}}{(1-\omega_{j})^{2}} \left((1-\omega_{j})\omega_{j} + k_{j}(x)^{2}(1-\omega_{j})^{2} + \omega_{j}^{k_{j}(x)+1}\right)$$

$$= \sum_{\substack{j=1,\dots,d:\\|\ln\omega_{j}| \leqslant x}} \frac{|\ln\omega_{j}|^{2}\omega_{j}}{(1-\omega_{j})^{2}} - \sum_{\substack{j=1,\dots,d:\\|\ln\omega_{j}| \leqslant x}} \frac{|\ln\omega_{j}|^{2}\omega_{j}^{k_{j}(x)}}{(1-\omega_{j})^{2}} \left(k_{j}(x)^{2}(1-\omega_{j})^{2}(1+\omega_{j}^{k_{j}(x)}) + 2k_{j}(x)(1-\omega_{j})^{2}(1+\omega_{j}^{k_{j}(x)})\right)$$

$$+ 2k_{j}(x)(1-\omega_{j})\omega_{j}^{k_{j}(x)+1} + (1-\omega_{j})\omega_{j} + \omega_{j}^{k_{j}(x)+2}\right).$$

So we obtain the third identity. \Box

We now turn to proof of the theorem.

Proof of Theorem 2. We first recall that C(q) = (0,1), which follows from properties of self-decomposable distribution functions (see [11], Remarks 2 and 8). Thus, by Theorem 1 from [11], the condition (13) is equivalent to the convergence (9), where G is the given self-decomposable law with the triplet (c, v, L) and G_d , $d \in \mathbb{N}$, are defined by (10) for given a_d , b_d , and Y_d , $d \in \mathbb{N}$. According to (11), G_d , $d \in \mathbb{N}$, are distribution functions of centered and normalized sums of nonnegative independent random variables \hat{U}_j , $j \in \mathbb{N}$, satisfying (12). Hence for the convergence (9) the conditions (A_1) , (B), (C) of Theorem 11 from [11] (Appendix) are necessary and sufficient (with $Y_j := \hat{U}_j$, $j \in \mathbb{N}$, $A_d := \hat{a}_d$, $B_d := b_d$, $\gamma := c$, and $\sigma^2 := v$). Thus (13) is equivalent to the following ensemble of conditions:

$$\lim_{d \to \infty} \sum_{j=1}^{d} \mathbb{P}(\hat{U}_j > \tau b_d) = -L(\tau) \quad \text{for all} \quad \tau > 0;$$

$$\lim_{d\to\infty} \frac{1}{b_d} \left(\sum_{j=1}^d \mathbb{E} \left[\hat{U}_j \mathbb{1}(|\hat{U}_j| \leqslant \tau b_d) \right] - \hat{a}_d \right) = c + \gamma_\tau \quad \text{for all} \quad \tau > 0;$$

$$\lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{b_d^2} \sum_{j=1}^d \operatorname{Var} \left[\hat{U}_j \mathbb{1}(|\hat{U}_j| \leqslant \tau b_d) \right] = \lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{b_d^2} \sum_{j=1}^d \operatorname{Var} \left[\hat{U}_j \mathbb{1}(|\hat{U}_j| \leqslant \tau b_d) \right] = v.$$

Here γ_{τ} is defined by (14).

Let us write the sums in these conditions in terms of ω_i . First, note that

$$\mathbb{P}(\hat{U}_{j} > \tau b_{d}) = \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid > \tau b_{d}}} (1 - \omega_{j}) \omega_{j}^{k},$$

$$\mathbb{E}\left[\hat{U}_{j} \mathbb{1}(|\hat{U}_{j}| \leqslant \tau b_{d})\right] = \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leqslant \tau b_{d}}} k \mid \ln \omega_{j} \mid (1 - \omega_{j}) \omega_{j}^{k},$$

$$\operatorname{Var}\left[\hat{U}_{j} \mathbb{1}(|\hat{U}_{j}| \leqslant \tau b_{d})\right] = \mathbb{E}\left[\hat{U}_{j}^{2} \mathbb{1}(|\hat{U}_{j}| \leqslant \tau b_{d})\right] - \left(\mathbb{E}\left[\hat{U}_{j} \mathbb{1}(|\hat{U}_{j}| \leqslant \tau b_{d})\right]\right)^{2}$$

$$= \sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leqslant \tau b_{d}}} k^{2} |\ln \omega_{j}|^{2} (1 - \omega_{j}) \omega_{j}^{k} - \left(\sum_{\substack{k \in \mathbb{N}: \\ k \mid \ln \omega_{j} \mid \leqslant \tau b_{d}}} k |\ln \omega_{j} \mid (1 - \omega_{j}) \omega_{j}^{k}\right)^{2}.$$

Next, applying Lemma 1 we have

$$\sum_{j=1}^{d} \mathbb{P}(\hat{U}_{j} > \tau b_{d}) = \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| > \tau b_{d}}} \omega_{j} + R_{0}(d,\tau b_{d}),$$

$$\sum_{j=1}^{d} \mathbb{E}\left[\hat{U}_{j}\mathbb{1}(|\hat{U}_{j}| \leqslant \tau b_{d})\right] = \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant \tau b_{d}}} \frac{|\ln \omega_{j}| \omega_{j}}{1 - \omega_{j}} - R_{1}(d,\tau b_{d}),$$

$$\sum_{j=1}^{d} \operatorname{Var}\left[\hat{U}_{j}\mathbb{1}(|\hat{U}_{j}| \leqslant \tau b_{d})\right] = \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant \tau b_{d}}} \frac{|\ln \omega_{j}|^{2} \omega_{j}}{(1 - \omega_{j})^{2}} - R_{2}(d,\tau b_{d}),$$

where the functions R_0 , R_1 , and R_2 are defined as in Lemma 1. Therefore (13) is equivalent to the following three conditions

(A')
$$\lim_{d \to \infty} \left(\sum_{\substack{j=1,\dots,d:\\|\ln u_j| > \tau b}} \omega_j + R_0(d,\tau b_d) \right) = -L(\tau) \quad \text{for all} \quad \tau > 0;$$

(B')
$$\lim_{d \to \infty} \frac{1}{b_d} \left(\sum_{\substack{j=1,\dots,d:\\|\ln \omega_i| \leq \tau b_d}} \frac{|\ln \omega_j| \, \omega_j}{1 - \omega_j} - R_1(d,\tau b_d) - \hat{a}_d \right) = c + \gamma_\tau \quad \text{for all} \quad \tau > 0;$$

(C')
$$\lim_{\tau \to 0} \frac{\lim_{d \to \infty} \frac{1}{b_d^2} \left(\sum_{\substack{j=1,\dots,d:\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} - R_2(d,\tau b_d) \right) =$$

$$= \lim_{\tau \to 0} \overline{\lim_{d \to \infty} \frac{1}{b_d^2} \left(\sum_{\substack{j=1,\dots,d:\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} - R_2(d,\tau b_d) \right) = v^2.$$

We first show that (A), (B), (C) imply (A'), (B'), (C'). Due to the condition (C), there exist $\tau_0 > 0$ and $C_0 > 0$ such that for all $d \in \mathbb{N}$ we have

$$\frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d:\\|\ln \omega_j| \leqslant \tau_0 b_d}} \frac{|\ln \omega_j|^2 \,\omega_j}{(1-\omega_j)^2} \leqslant C_0. \tag{16}$$

Since $\omega_j \in (0,1)$, the inequality $|\ln \omega_j| > 1 - \omega_j$ always holds. Hence we conclude that

$$\frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d:\\|\ln \omega_j| \leqslant \tau_0 b_d}} \omega_j \leqslant C_0, \quad d \in \mathbb{N}.$$

$$\tag{17}$$

We first show that $R_0(d, \tau b_d) \to 0$, $d \to \infty$, for every $\tau > 0$. We fix $\tau > 0$ and $\tau_* < \min\{\tau_0, \tau\}$. Observe that

$$R_0(d, \tau b_d) = \sum_{\substack{j=1,\dots,d:\\|\ln \omega_j| \leqslant \tau b_d}} \omega_j^{k_j(\tau b_d)} \leqslant \sum_{\substack{j=1,\dots,d:\\|\ln \omega_j| \leqslant \tau_* b_d}} \omega_j^{k_j(\tau b_d)} + \sum_{\substack{j=1,\dots,d:\\|\ln \omega_j| > \tau_* b_d}} \omega_j^{k_j(\tau b_d)}.$$

By the definition of $k_j(\cdot)$ (see Lemma 1), we have the inequalities $k_j(\tau b_d) \ge 2$ and $\omega_j^{k_j(\tau b_d)} \le e^{-\tau b_d}$, which give

$$R_0(d, \tau b_d) \leqslant e^{-\tau b_d} \sum_{\substack{j=1,\dots,d:\\ |\ln \omega_j| \leqslant \tau_* b_d}} 1 + \sum_{\substack{j=1,\dots,d:\\ |\ln \omega_j| > \tau_* b_d}} \omega_j^2.$$

Using conditions in the sums, we obtain

$$R_{0}(d, \tau b_{d}) \leqslant e^{-(\tau - \tau_{*})b_{d}} \sum_{\substack{j=1, \dots, d: \\ |\ln \omega_{j}| \leqslant \tau_{*}b_{d}}} \omega_{j} + e^{-\tau_{*}b_{d}} \sum_{\substack{j=1, \dots, d: \\ |\ln \omega_{j}| > \tau_{*}b_{d}}} \omega_{j}$$

$$\leqslant e^{-(\tau - \tau_{*})b_{d}} \sum_{\substack{j=1, \dots, d: \\ |\ln \omega_{j}| \leqslant \tau_{0}b_{d}}} \omega_{j} + e^{-\tau_{*}b_{d}} \sum_{\substack{j=1, \dots, d: \\ |\ln \omega_{j}| > \tau_{*}b_{d}}} \omega_{j}.$$

Here the first sum is less than $C_0b_d^2$ by (17) and the second sum is bounded by some constant C_1 due to (A). Thus

$$R_0(d, \tau b_d) \leqslant C_0 b_d^2 e^{-(\tau - \tau_*)b_d} + C_1 e^{-\tau_* b_d}.$$

Since $\tau > \tau_* > 0$ and $b_d \to \infty$, we obtain that $R_0(d, \tau b_d) \to 0$, $d \to \infty$. This, together with (A), yields (A').

We now consider

$$R_1(d, \tau b_d) = \sum_{\substack{j=1,\dots,d:\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j| \, \omega_j^{k_j(\tau b_d)}}{1 - \omega_j} \left(k_j(\tau b_d)(1 - \omega_j) + \omega_j \right).$$

According to the definition of $k_i(\cdot)$, we have

$$k_j(\tau b_d) \leqslant 2\left(k_j(\tau b_d) - 1\right) \leqslant \frac{2\tau b_d}{|\ln \omega_j|}.$$
(18)

Hence

$$R_1(d, \tau b_d) \leqslant 2\tau b_d \sum_{\substack{j=1,\dots,d:\\ |\ln \omega_j| \leqslant \tau b_d}} \omega_j^{k_j(\tau b_d)} + \sum_{\substack{j=1,\dots,d:\\ |\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} \cdot \omega_j^{k_j(\tau b_d)}.$$

Note that the function $\omega_j \mapsto \frac{|\ln \omega_j|\omega_j}{1-\omega_j}$ is bounded by some constant C_3 for all values $\omega_j \in (0,1)$. Therefore

$$R_1(d, \tau d) \leqslant (2\tau b_d + C_3) \sum_{\substack{j=1,\dots,d:\\ |\ln \omega_j| \leqslant \tau b_d}} \omega_j^{k_j(\tau b_d)} = (2\tau b_d + C_3) \cdot R_0(d, \tau b_d). \tag{19}$$

Since $R_0(d, \tau b_d) \to 0$, $d \to \infty$, we have $R_1(d, \tau d) = o(b_d)$, $d \to \infty$. Due to (B), this yields (B'). We now consider

$$R_{2}(d, \tau b_{d}) = \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leq \tau b_{d}}} \frac{|\ln \omega_{j}|^{2} \omega_{j}^{k_{j}(\tau b_{d})}}{(1-\omega_{j})^{2}} \left(k_{j}(\tau b_{d})^{2} (1-\omega_{j})^{2} (1+\omega_{j}^{k_{j}(\tau b_{d})})\right) + 2k_{j}(\tau b_{d})(1-\omega_{j})\omega_{j}^{k_{j}(\tau b_{d})+1} + \omega_{j}^{k_{j}(\tau b_{d})+2} + (1-\omega_{j})\omega_{j}\right).$$

Using the inequality (18),

$$R_{2}(d, \tau b_{d}) \leqslant (2\tau b_{d})^{2} \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}|\leqslant \tau b_{d}}} \omega_{j}^{k_{j}(\tau b_{d})} \left(1 + \omega_{j}^{k_{j}(\tau b_{d})}\right) + 4\tau b_{d} \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}|\leqslant \tau b_{d}}} \frac{|\ln \omega_{j}|}{1 - \omega_{j}} \omega_{j}^{2k_{j}(\tau b_{d}) + 1} + \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}|\leqslant \tau b_{d}}} \frac{|\ln \omega_{j}|^{2} \omega_{j}^{2k_{j}(\tau b_{d}) + 2}}{(1 - \omega_{j})^{2}} + \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}|\leqslant \tau b_{d}}} \frac{|\ln \omega_{j}|^{2} \omega_{j}^{k_{j}(\tau b_{d}) + 1}}{1 - \omega_{j}}.$$

The function $\omega_j \mapsto \frac{|\ln \omega_j|^2 \omega_j}{1-\omega_j}$ is bounded by some constant C_4 for all values $\omega_j \in (0,1)$. Using this and the same fact about $\omega_j \mapsto \frac{|\ln \omega_j| \omega_j}{1-\omega_j}$, we obtain

$$R_{2}(d, \tau b_{d}) \leqslant 4\tau^{2}b_{d}^{2} \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant \tau b_{d}}} \omega_{j}^{k_{j}(\tau b_{d})} \left(1 + \omega_{j}^{k_{j}(\tau b_{d})}\right) + 4\tau b_{d} \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant \tau b_{d}}} \omega_{j}^{2k_{j}(\tau b_{d})} + C_{4} \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant \tau b_{d}}} \omega_{j}^{2k_{j}(\tau b_{d})}.$$

Next, since $\omega_j^{k_j(\tau b_d)} \leqslant 1$, we have

$$R_{2}(d, \tau b_{d}) \leq (8\tau^{2}b_{d}^{2} + 4\tau b_{d} + C_{3}^{2} + C_{4}) \sum_{\substack{j=1,\dots,d:\\|\ln \omega_{j}| \leqslant \tau b_{d}}} \omega_{j}^{k_{j}(\tau b_{d})}$$

$$= (8\tau^{2}b_{d}^{2} + 4\tau b_{d} + C_{3}^{2} + C_{4})R_{0}(d, \tau b_{d}). \tag{20}$$

Here $R_0(d, \tau b_d) \to 0$, $d \to \infty$. Hence $R_2(d, \tau b_d) = o(b_d^2)$, $d \to \infty$. Due to (C), this yields (C').

We now show that (A'), (B'), (C') imply (A), (B), (C). From (A') it follows that $\sup_{d\in\mathbb{N}} R_0(d, \tau b_d) < \infty$ for every $\tau > 0$. By this relation and (20), we have

$$\sup_{d\in\mathbb{N}} \frac{1}{b_d^2} R_2(d, \tau b_d) < \infty \quad \text{for every } \tau > 0.$$

This and (C') yield (16) for every $d \in \mathbb{N}$ and some $\tau_0 > 0$ and $C_0 > 0$. Hence, as we showed above, it follows that $R_0(d, \tau b_d) \to 0$, $d \to \infty$, for every $\tau > 0$. Here we use

$$\sup_{d \in \mathbb{N}} \sum_{\substack{j=1,\dots,d:\\ |\ln \omega_j| > \tau b_d}} \omega_j < \infty \quad \text{for every } \tau > 0,$$

which follows from (A'). Next, from (19) and (20) we correspondingly obtain that $R_1(d, \tau b_d) = o(b_d)$ and $R_2(d, \tau b_d) = o(b_d^2)$, $d \to \infty$, for every $\tau > 0$. These relations for $R_k(d, \tau b_d)$, k = 1, 2, 3, and conditions (A'), (B'), (C') give (A), (B), (C).

4 Examples

In this section we will study asymptotics for $n^{Y_d}(\varepsilon)$ under particular additional assumptions on the sequence of the length scale parameters σ_i , $j \in \mathbb{N}$.

We begin with the case, when the sequence $(\sigma_j)_{j\in\mathbb{N}}$ tends to to a non-negative constant σ . Observe that here approximation complexity $n^{Y_d}(\varepsilon)$ is unbounded due to Proposition 2. The following proposition specifies asymptotic (13) for this case.

Proposition 3 Suppose that $\sigma_j \to \sigma$, $0 \le \sigma < \infty$. Then

$$\forall \varepsilon \in (0,1) \quad \ln n^{Y_d}(\varepsilon) = a_d + \Phi^{-1}(1-\varepsilon^2)b_d + o(b_d), \quad d \to \infty, \tag{21}$$

where Φ is the distribution function of the standard Gaussian law,

$$a_d = \sum_{j=1}^d \left(\frac{|\ln \omega_j| \omega_j}{1 - \omega_j} + |\ln(1 - \omega_j)| \right), \qquad b_d = \rho \sqrt{d}, \quad d \in \mathbb{N},$$

$$\rho = \lim_{j \to \infty} \frac{|\ln \omega_j| \sqrt{\omega_j}}{1 - \omega_j} = \begin{cases} \frac{|\ln \omega| \sqrt{\omega}}{1 - \omega}, & \sigma > 0, \\ 1, & \sigma = 0, \end{cases} \qquad \omega = \lim_{j \to \infty} \omega_j = \left(1 + \frac{\sigma^2}{2} \left(1 + \sqrt{1 + \frac{4}{\sigma^2}}\right)\right)^{-1} for \quad \sigma > 0.$$

Remark 1 If $\sigma > 0$ then

$$a_d = \left(\frac{|\ln \omega|\omega}{1-\omega} + |\ln(1-\omega)|\right)d + o(d), \quad d \to \infty.$$

Indeed, since $\lim_{j\to\infty} \omega_j = \omega$ we have

$$a_d \sim d \cdot \lim_{d \to \infty} \left(\frac{|\ln \omega_d|\omega_d}{1 - \omega_d} + |\ln(1 - \omega_d)| \right) = d \cdot \left(\frac{|\ln \omega|\omega}{1 - \omega} + |\ln(1 - \omega)| \right), \quad d \to \infty.$$

Remark 2 If $\sigma = 0$ then a_d can increase arbitrarily fast.

It is seen from the inequality

$$a_d \geqslant |\ln(1 - \omega_d)|, \quad d \in \mathbb{N},$$

and from the relation $\lim_{d\to\infty}\omega_d=1$, which holds under the assumption $\sigma=0$.

More sharp asymptotics of a_d (up to $o(b_d)$) require additional assumptions on the rate of convergence of σ_j to σ as $j \to \infty$.

Proof of Proposition 3. It is well known that Φ is a self-decomposable distribution function with the triplet (0,1,L), where L(x)=0 for all $x \in \mathbb{R} \setminus \{0\}$. By Theorem 2 it is sufficient to check the following ensemble of conditions to obtain the asymptotics (21):

$$\lim_{d \to \infty} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| > \tau b_d}} \omega_j = 0, \quad \tau > 0; \tag{22}$$

$$\lim_{d \to \infty} \frac{1}{b_d} \left(\sum_{\substack{j=1,\dots,d\\|\ln \omega_i| \le \tau b_d}} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} - \hat{a}_d \right) = 0, \quad \tau > 0;$$
(23)

$$\lim_{\tau \to 0} \underline{\lim}_{d \to \infty} \frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = \lim_{\tau \to 0} \overline{\lim}_{d \to \infty} \frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = 1.$$
 (24)

Here \hat{a}_d is defined by (15).

Observe that $\lim_{j\to\infty}\omega_j\in(0,1]$, and, consequently, the sequence $(|\ln\omega_j|)_{j\in\mathbb{N}}$ is bounded. Since $b_d\to\infty$, $d\to\infty$, for every $\tau>0$ there exists $d_\tau\in\mathbb{N}$ such that $|\ln\omega_j|\leqslant\tau b_d$ holds for any $j\in\mathbb{N}$ and $d\geqslant d_\tau$. Therefore for every $\tau>0$ we have

$$\sum_{\substack{j=1,\dots,d\\|\ln\omega_j|>\tau b_d}} \omega_j = 0, \quad \text{and} \quad \sum_{\substack{j=1,\dots,d\\|\ln\omega_j|\leq\tau b_d}} \frac{|\ln\omega_j|\omega_j}{1-\omega_j} - \hat{a}_d = \sum_{j=1}^d \frac{|\ln\omega_j|\omega_j}{1-\omega_j} - \hat{a}_d = 0$$

for all sufficiently large d, i.e. we get (22) and (23). Next, we also have

$$\frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = \frac{1}{b_d^2} \sum_{j=1}^d \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = \frac{1}{\rho^2} \cdot \frac{1}{d} \sum_{j=1}^d \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2}$$

for all sufficiently large d. The arithmetic mean in the right-hand side tends to $\lim_{j\to\infty} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = \rho^2$. Thus for every $\tau > 0$ we get

$$\lim_{d \to \infty} \frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leq \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = 1,$$

that implies (24). \square

We now proceed to the case $\sigma_j \to \infty$ as $j \to \infty$. Here the asymptotics of $n^{Y_d}(\varepsilon)$ depends on the velocity of σ_j . In order to evidently illustrate the application of the general results from the previous section and to avoid routine unwieldy calculations, we will assume that

$$\sigma_j^2 \sim \beta j^{\alpha}, \quad j \to \infty,$$
 (25)

with some $\alpha > 0$ and $\beta > 0$.

The following assertion directly follows from Propositions 1 and 2.

Proposition 4 Let $(\sigma_j)_{j\in\mathbb{N}}$ satisfy (25) with some $\alpha > 0$ and $\beta > 0$. If $\alpha > 1$ then $\sup_{d\in\mathbb{N}} n^{Y_d}(\varepsilon) < \infty$ for every $\varepsilon \in (0,1)$. If $\alpha \leqslant 1$ then $n^{Y_d}(\varepsilon) \to \infty$ as $d \to \infty$ for every $\varepsilon \in (0,1)$.

We now propose the asymptotics of $n^{Y_d}(\varepsilon)$ for the case $\alpha \in (0,1]$.

Proposition 5 Let $(\sigma_i)_{i\in\mathbb{N}}$ satisfy (25) with some $\alpha\in(0,1]$ and $\beta>0$. Then

$$\forall \varepsilon \in (0,1) \quad \ln n^{Y_d}(\varepsilon) = a_d + \Phi^{-1}(1-\varepsilon^2)b_d + o(b_d), \quad d \to \infty,$$

where Φ is the distribution function of the standard Gaussian law,

$$a_d = \sum_{j=1}^d \left(\frac{|\ln \omega_j|\omega_j}{1 - \omega_j} + |\ln(1 - \omega_j)| \right), \quad and \quad b_d = \begin{cases} \frac{\alpha}{\sqrt{(1 - \alpha)\beta}} d^{\frac{1 - \alpha}{2}} \ln d, & \alpha \in (0, 1), \\ \frac{1}{\sqrt{3\beta}} (\ln d)^{3/2}, & \alpha = 1, \end{cases} \quad d \in \mathbb{N}.$$

Remark 3 The following asymptotic holds

$$a_d \sim \begin{cases} \frac{\alpha}{(1-\alpha)\beta} d^{1-\alpha} \ln d + \frac{1}{(1-\alpha)\beta} d^{1-\alpha}, & \alpha \in (0,1), \\ \frac{1}{2\beta} (\ln d)^2 + \frac{1}{\beta} \ln d, & \alpha = 1, \end{cases} \qquad d \to \infty.$$
 (26)

Indeed, since $\sigma_j \to \infty$ as $j \to \infty$, we have $\omega_j \sim \sigma_j^{-2} \sim (\beta j^{\alpha})^{-1}$, $j \to \infty$, due to (6) and (25). Then

$$a_d \sim \sum_{j=1}^d \left(\frac{\alpha \ln j}{\beta j^{\alpha}} + \frac{1}{\beta j^{\alpha}} \right) \sim \frac{\alpha}{\beta} \int_1^d \frac{\ln x}{x^{\alpha}} dx + \frac{1}{\beta} \int_1^d \frac{1}{x^{\alpha}} dx, \quad d \to \infty.$$

Using known asymptotic relations (see [1] and [14]), we obtain (26).

More sharp asymptotics of a_d (up to $o(b_d)$) require additional assumptions on the asymptotics of the difference of $\sigma_j^2 - \beta j^{\alpha}$ as $j \to \infty$.

Proof of Proposition 5. As in Proposition 3, according to Theorem 2, it is sufficient to check conditions (22)–(24) with given a_d , b_d , and with \hat{a}_d defined by (15), $d \in \mathbb{N}$.

Since $\omega_j \sim (\beta j^{\alpha})^{-1}$, $j \to \infty$, it is not difficult to see that for every $\tau > 0$

$$\frac{1}{\tau b_d} \max_{j \in \{1, \dots, d\}} |\ln \omega_j| \to 0, \quad d \to \infty.$$

Therefore for every $\tau > 0$ we have

$$\sum_{\substack{j=1,\dots,d\\|\ln\omega_j|>\tau b_d}} \omega_j = 0, \quad \text{and} \quad \sum_{\substack{j=1,\dots,d\\|\ln\omega_j|\leqslant\tau b_d}} \frac{|\ln\omega_j|\omega_j}{1-\omega_j} - \hat{a}_d = \sum_{j=1}^d \frac{|\ln\omega_j|\omega_j}{1-\omega_j} - \hat{a}_d = 0$$

for all sufficiently large d, i.e. we get (22) and (23). Next, we also have

$$\sum_{\substack{j=1,\dots,d\\|\ln\omega_j|\leqslant\tau b_d}} \frac{|\ln\omega_j|^2\omega_j}{(1-\omega_j)^2} = \sum_{j=1}^d \frac{|\ln\omega_j|^2\omega_j}{(1-\omega_j)^2}$$

for every $\tau > 0$ and for all sufficiently large d. Using the asimptotics $\omega_i \sim (\beta j^{\alpha})^{-1}$, $j \to \infty$, we get

$$\sum_{j=1}^{d} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = \sum_{j=1}^{d} \frac{\left(\alpha \ln j + \ln \beta + o(1)\right)^2}{\beta j^{\alpha} (1+o(1))} \sim \frac{\alpha^2}{\beta} \sum_{j=1}^{d} \frac{(\ln j)^2}{j^{\alpha}}, \quad d \to \infty.$$

By known asymptotic relations (see [1] and [14]), we obtain

$$\sum_{j=1}^{d} \frac{(\ln j)^2}{j^{\alpha}} \sim \int_{1}^{d} \frac{(\ln x)^2}{x^{\alpha}} dx \sim \begin{cases} \frac{1}{1-\alpha} d^{1-\alpha} (\ln d)^2, & \alpha \in (0,1), \\ \frac{1}{3} (\ln d)^3, & \alpha = 1, \end{cases} d \to \infty,$$

Hence for every $\tau > 0$

$$\sum_{\substack{j=1,\dots,d\\|\ln \omega_j|\leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} \sim \begin{cases} \frac{\alpha^2}{(1-\alpha)\beta} d^{1-\alpha} (\ln d)^2, & \alpha \in (0,1),\\ \frac{1}{3\beta} (\ln d)^3, & \alpha = 1, \end{cases} d \to \infty.$$

Thus

$$\lim_{d \to \infty} \frac{1}{b_d^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = 1, \quad \tau > 0,$$

that implies (24). \square

There exist particular cases, where the ε -component of asymptotics (13) is a quantile of a non-Gaussian self-decomposable distribution function. One such case is considered in the next proposition. There appear distribution functions D_{μ} , $\mu > 0$, of μ -convolution powers of the Dickman law (see [9] and [11]). It is known that D_{μ} has the spectral representation triplet $(\frac{\pi\mu}{4}, 0, L)$, where $L(x) = \mu \ln x \mathbb{1}(x \in (0, 1]), x \in \mathbb{R} \setminus \{0\}$.

Proposition 6 Suppose that $\sigma_j^2 \sim \beta j \ln j$, $j \to \infty$, with some $\beta > 0$. Then

$$\forall \varepsilon \in (0,1) \quad \ln n^{Y_d}(\varepsilon) = D_{1/\beta}^{-1}(1-\varepsilon^2) \ln d + o(\ln d), \quad d \to \infty.$$

Proof of Proposition 6. We use Theorem 2 with $a_d := 0$, $b_d := \max\{\ln d, 1\}$, $d \in \mathbb{N}$, $G := D_{1/\beta}$, $c = \frac{\pi}{4\beta}$, v = 0, and $L(x) = \frac{1}{\beta} \ln x \mathbb{1}(x \in (0, 1])$, $x \in \mathbb{R} \setminus \{0\}$. So it is sufficient to check the following ensemble of conditions:

$$\lim_{d \to \infty} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| > \tau \ln d}} \omega_j = -\frac{1}{\beta} \ln \tau \mathbb{1}(\tau \in (0,1]), \quad \tau > 0, \tag{27}$$

$$\lim_{d \to \infty} \frac{1}{\ln d} \left(\sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \le \tau \ln d}} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} + \sum_{j=1}^d |\ln(1 - \omega_j)| \right) = \frac{\pi}{4\beta} + \gamma_\tau, \quad \tau > 0, \tag{28}$$

$$\lim_{\tau \to 0} \frac{\lim}{d \to \infty} \frac{1}{(\ln d)^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau \ln d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = \lim_{\tau \to 0} \frac{\overline{\lim}}{d \to \infty} \frac{1}{(\ln d)^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau \ln d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = 0.$$
 (29)

Here

$$\gamma_{\tau} = \frac{1}{\beta} \int_{0}^{\tau} \frac{x^{2} \mathbb{1}(x \in (0, 1))}{1 + x^{2}} dx - \frac{1}{\beta} \int_{\tau}^{+\infty} \frac{\mathbb{1}(x \in (0, 1))}{1 + x^{2}} dx, \quad \tau > 0.$$

Due to the assumption, we have

$$\omega_j \sim \sigma_j^{-2} \sim \frac{1}{\beta j \ln j}, \quad j \to \infty.$$
 (30)

We first check (27). It is easily seen that for $\tau > 1$ we have $|\ln \omega_j| \leq \tau \ln d$, $j = 1, \ldots, d$, for all sufficiently large d. Therefore (27) obviously holds in this case. For $\tau \in (0, 1]$ we set

$$j_{d,\tau} = \min\{j \in \mathbb{N} : |\ln \omega_j| > \tau \ln d\}, \quad d \in \mathbb{N}.$$
(31)

Due to (30), it is not difficult to see that $j_{d,\tau} \leq d$ for all sufficiently large d and $\ln j_{d,\tau} \sim \tau |\ln d|$, $d \to \infty$. Next, observe that

$$\sum_{\substack{j=1,\dots,d\\|\ln\omega_j|>\tau b_d}}\omega_j=\sum_{j=j_{d,\tau}}^d\omega_j\sim\sum_{j=j_{d,\tau}}^d\frac{1}{\beta j\ln j},\quad d\to\infty.$$

Using the known asymptotics (see [14], 2.13, p. 21)

$$\sum_{k=1}^{n} \frac{1}{k \ln k} = \ln \ln n + C + o(1), \quad n \to \infty,$$
 (32)

with a constant C, we obtain

$$\sum_{j=j_{d,\tau}}^{d} \frac{1}{\beta j \ln j} = \frac{1}{\beta} (\ln \ln d - \ln \ln j_{d,\tau}) + o(1)$$
$$= \frac{1}{\beta} (\ln \ln d - \ln \ln d^{\tau}) + o(1)$$
$$= -\frac{1}{\beta} \ln \tau + o(1), \quad d \to \infty.$$

Thus (27) holds.

We now check (28). Observe that for every $\tau > 0$

$$\frac{\pi}{4\beta} + \gamma_{\tau} = \frac{\pi}{4\beta} + \frac{1}{\beta} \int_{0}^{\min\{\tau,1\}} \frac{x^{2}}{1+x^{2}} dx - \frac{1}{\beta} \int_{\min\{\tau,1\}}^{1} \frac{1}{1+x^{2}} dx$$

$$= \frac{\pi}{4\beta} + \frac{1}{\beta} \int_{0}^{\min\{\tau,1\}} dx - \frac{1}{\beta} \int_{0}^{1} \frac{1}{1+x^{2}} dx$$

$$= \frac{1}{\beta} \min\{\tau,1\}. \tag{33}$$

Next, due to (30) and (32), we have

$$\sum_{j=1}^{d} |\ln(1 - \omega_j)| \sim \sum_{j=1}^{d} \omega_j \sim \sum_{j=1}^{d} \frac{1}{\beta j \ln j} \sim \frac{1}{\beta} \ln \ln d = o(\ln d), \quad d \to \infty.$$
 (34)

Next, if $\tau > 1$ then $|\ln \omega_j| \leq \tau \ln d$, $j = 1, \ldots, d$, for all sufficiently large d. Therefore

$$\sum_{\substack{j=1,\dots,d\\|\ln\omega_i|\leq\tau b_d}} \frac{|\ln\omega_j|\omega_j}{1-\omega_j} = \sum_{j=1}^d \frac{|\ln\omega_j|\omega_j}{1-\omega_j} \sim \sum_{j=1}^d \frac{1}{\beta j} \sim \frac{1}{\beta} \ln d, \quad d\to\infty.$$

If $\tau \in (0, 1]$ then, according to (31), we write

$$\sum_{\substack{j=1,\dots,d\\ |\ln \omega_i| \leq \tau b_d}} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} = \sum_{j=1}^{j_{d,\tau}-1} \frac{|\ln \omega_j| \omega_j}{1 - \omega_j} \sim \sum_{j=1}^{j_{d,\tau}-1} \frac{1}{\beta j} \sim \frac{1}{\beta} \ln j_{d,\tau} \sim \frac{\tau}{\beta} \ln d, \quad d \to \infty.$$

Therefore we have for every $\tau > 0$

$$\sum_{\substack{j=1,\dots,d\\|\ln\omega_j|\leqslant\tau b_d}} \frac{|\ln\omega_j|\omega_j}{1-\omega_j} \sim \frac{1}{\beta} \min\{\tau,1\} \ln d, \quad d\to\infty.$$
 (35)

Thus (33), (34), and (35) together yield (28).

Next, we check the condition (29). Let us fix $\tau \in (0,1)$ and consider

$$\sum_{\substack{j=1,\dots,d\\\ln\omega_j|\leqslant\tau b_d}}\frac{|\ln\omega_j|^2\omega_j}{(1-\omega_j)^2}=\sum_{j=1}^{j_{d,\tau}-1}\frac{|\ln\omega_j|^2\omega_j}{(1-\omega_j)^2}\sim\sum_{j=1}^{j_{d,\tau}-1}\frac{\ln j}{\beta j}\sim\frac{(\ln j_{d,\tau})^2}{2\beta}\sim\frac{\tau^2(\ln d)^2}{2\beta},\quad d\to\infty.$$

Therefore

$$\lim_{d \to \infty} \frac{1}{(\ln d)^2} \sum_{\substack{j=1,\dots,d\\|\ln \omega_j| \leqslant \tau b_d}} \frac{|\ln \omega_j|^2 \omega_j}{(1-\omega_j)^2} = \frac{\tau^2}{2\beta}.$$

From this we obtain (29). \square

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