# Central Limit Theorem for the volume of the zero set of Kostlan-Shub-Smale random polynomial systems. 

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#### Abstract

We establish the Central Limit Theorem, as the degree goes to infinity, for the normalized volume of the zero set of a rectangular Kostlan-Shub-Smale random polynomial system. This paper is a continuation of Central Limit Theorem for the number of real roots of Kostlan-Shub-Smale random polynomial systems by the same authors in which the case of square systems was considered. Our main tools are Kac-Rice formula and an expansion of the volume of the level set into the Itô-Wiener Chaos. Keywords: Kostlan-Shub-Smale random polynomial systems, co-area formula, Kac-Rice formula, central limit theorem. AMS sujet classification: Primary: 60F05, 30C15. Secondary: 60G60, 65H10


## 1 Introduction

The problem of studying the number of roots of random algebraic polynomials has attracted much interest for a long time. It is worth mentioning the seminal article of M. Kac on the subject [7] where a proof of the now famous Kac-Rice formula [5] was given. This formula establishes an analytical expression for computing the expectation of the number of zeros of a Gaussian random process. At the beginning the main interest was limited to compute the expected value of such a number. Later on there were also considered its variance [11] and the Central Limit Theorem (CLT) [12].

The algebraic systems of random polynomials of several variables were considered much later motivated by the inspiring work, due to Shub and Smale, for the understandig of the complexity of Bézout's theorem (see [16], [17], [18], [19] and [20]).

Kostlan [8] and Shub-Smale [17] studied random polynomials systems that are invariant under rotations. The properties of invariance of these polynomials allow considering them as functions over the multidimensional sphere. In Kostlan's paper an explicit expression for the expectation of the number of roots for a square system of such polynomials was given. For rectangular systems the study was directed to the behavior of the volume of their zero sets.

Wschebor [21], in a seminal work, gave for the first time a bound for a limit variance in the case when the degree of the system is controlled and the size of the system tends to infinity. A central limit theorem for this asymptotic scheme is still an open problem, even for the particular case of quadratic systems.

Another asymptotic regime naturally arises, namely: to fix the number of equations and variables and to let the degree grow to infinity. Under this scheme, the asymptotic variance for the number of roots of square systems was obtained in [2] and [10]. In the case of rectangular systems the asymptotic variance of the volume of the zero level set was given in [9]. Besides, in [3] a central limit theorem for the number of roots of the system in the square case was obtained.

The present paper extends the results listed in the last paragraph obtaining a central limit theorem for the volume of the zero set of a Kostlan-Shub-Smale random rectangular system as the (common) degree tends to infinity (see Section 2 for the precise definition). We also give an alternative (simpler) proof of the limit variance.

[^0]This paper is a continuation of $[2,3]$. The proof has the same structure, but new arguments are needed, see the comments and remarks.

The fundamental tools for the study of the zero sets are: on the one hand the Kac-Rice formulas for calculating the mean and variance of the functionals of these sets ([5]) and on the other hand the well-known theorem of the fourth moment to establish the CLT for nonlinear functionals of Gaussian processes or fields ([13]).

## 2 Main result

Consider a rectangular system $\mathbf{Y}_{d}$ of $r$ homogeneous polynomial equations in $m+1$ variables with common degree $d>1$. More precisely, let $\mathbf{Y}_{d}=\left(Y_{1}, \ldots, Y_{r}\right)$ with

$$
Y_{\ell}(t)=\sum_{|\boldsymbol{j}|=d} a_{j}^{(\ell)} t^{j} ; \quad \ell=1, \ldots, r,
$$

where

1. $\boldsymbol{j}=\left(j_{0}, \ldots, j_{m}\right) \in \mathbb{N}^{m+1}$ and $|\boldsymbol{j}|=\sum_{k=0}^{m} j_{k}$;
2. $a_{j}^{(\ell)} \in \mathbb{R}, \ell=1, \ldots, r,|\boldsymbol{j}|=d$;
3. $t=\left(t_{0}, \ldots, t_{m}\right) \in \mathbb{R}^{m+1}$ and $t^{j}=\prod_{k=0}^{m} t_{k}^{j_{k}}$.

The system $\mathbf{Y}_{d}$ has the Kostlan-Shub-Smale (KSS for short) distribution if the coefficients $a_{j}^{(\ell)}$ are independent centred normally distributed random variables with variances

$$
\operatorname{Var}\left(a_{j}^{(\ell)}\right)=\binom{d}{j}=\frac{d!}{j_{0}!j_{1}!\ldots j_{m}!}
$$

In the case $r<m$, we are interested in the zero level set of $\mathbf{Y}_{d}$. Note that, since $\mathbf{Y}_{d}$ is homogeneous, its roots consist of lines through the origin in $\mathbb{R}^{m+1}$. Then, each root ray of $\mathbf{Y}_{d}$ in $\mathbb{R}^{m+1}$ corresponds exactly to two (opposite) roots of $\mathbf{Y}_{d}$ on the unit sphere $S^{m}$ of $\mathbb{R}^{m+1}$. Hence, the unit sphere $S^{m}$ is a natural place where to consider the zero set.

By a Sard-type argument, the zero level set of $\mathbf{Y}_{d}$ on $S^{m}$ is, almost surely, a smooth sub-manifold of dimension $m-r$ (see for example Azaïs \& Wschebor [5, pp.177]). We denote by $\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})$ the ( $m-r$ )-volume of the zero level set (on the sphere).

Shub and Smale [17] and Kostlan [8] proved that $\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right]=2 d^{r / 2} c_{m, r}, r \leq m$, where $c_{m, r}$ is the geometric measure of the sphere $S^{m-r}$ as a sub-manifold of $S^{m}$. Letendre [9] and Letendre-Puchol [10], proved that there exists $0<V_{\infty}^{r}<\infty$ such that

$$
\begin{equation*}
\lim _{d \rightarrow \infty} \frac{\operatorname{Var}\left(\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right)}{d^{r-m / 2}}=V_{\infty}^{r} \tag{2.1}
\end{equation*}
$$

We include a different proof in Appendix 5.
We now establish the CLT for the rectangular case.
Theorem 1. Let $\mathbf{Y}_{d}$ be an $r \times(m+1) K S S$ homogeneous system. Then, if $r<m$, the standardized $(m-r)$-volume of the zero level set

$$
\overline{\mathcal{V}}_{d}=\frac{\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})-\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right]}{d^{\frac{r}{2}-\frac{m}{4}}}
$$

converges in distribution as $d \rightarrow \infty$ towards a centred normal random variable with finite positive variance.
Remark 2.1. Note that the variance of the volume of the zero level set exhibits a surprising behavior as $d \rightarrow \infty$. More precisely, as $d \rightarrow \infty$, if $r<m / 2$, then the variance tends to 0 , if $r=m / 2$ it tends to a constant and if $r>m / 2$ it tends to infinity. Thus, the normalization in the CLT either reduces or amplifies the oscillations of the volume of the zero level set.
Remark 2.2. This result can be extended to general functionals of the level sets using the same arguments.

Indeed, let us denote the zero level set of $\mathbf{Y}_{d}$ as

$$
\mathcal{C}_{\mathbf{Y}_{d}}(0)=\left\{t \in S^{m}: \mathbf{Y}_{d}(t)=0\right\} .
$$

If $g: S^{m} \rightarrow \mathbb{R}$ is an a.s. continuous function we define the level linear functional

$$
<g, \mathbf{1}_{\mathbf{Y}_{d}}>=\int_{\mathcal{C}_{\mathbf{Y}_{d}(0)}} g(t) d t,
$$

where $\mathbf{1}_{\mathbf{Y}_{d}}$ is the indicator function of $\mathcal{C}_{\mathbf{Y}_{d}}(0)$. Thus $\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})=<\mathbf{1}_{\mathbf{Y}_{d}}, \mathbf{1}_{\mathbf{Y}_{d}}>$. We point out that the study conducted in the present paper for $\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})$, could be made for those functionals. To explain a little how to proceed we need in first place a Kac-Rice formula for the first and second order for such a functional. This is considered in Chapter 6 of [5] (Theorm 6.10 p. 168). In the second place the asymptotic behavior of the variance and the Hermite expansion can be obtained in a similar form as we have made here. The reader can consult, for a near matter but for $m=1$, the following preprint [6].

## 3 Preliminaries

This paper is a continuation of [2] and [3]. For the ease of readability, we gather toghether here the notation and some basic results proved in $[2,3]$. We also include the celebrated Fourth moment Theorem [13, Th.11.8.1] which is used to get the asymptotic normality.

### 3.1 Notation and basic results

We denote the unit sphere in $\mathbb{R}^{m+1}$ by $S^{m}$ and its volume by $\kappa_{m}$. Concerning integration, the variables $s$ and $t$ denote points on $S^{m}$ and $d s$ and $d t$ denote the corresponding geometric measure. The variables $u$ and $v$ are in $\mathbb{R}^{m}$ and $d u$ and $d v$ are the associated Lebesgue measure. The variables $z$ and $\theta$ are reals and $d z$ and $d \theta$ are the associated differentials.

We use the Landau's big $O$ and small o notation. The set $\mathbb{N}$ of natural numbers contains 0 . Besides, Const will denote a universal constant that may change from a line to another.

Lemma 3.1 of [2] establishes that for an integrable $h:[-1,1] \rightarrow \mathbb{R}$ it holds that

$$
\begin{equation*}
\int_{S^{m} \times S^{m}} h(\langle s, t\rangle) d s d t=\kappa_{m} \kappa_{m-1} \int_{0}^{\pi} \sin ^{m-1}(\theta) h(\cos (\theta)) d \theta \tag{3.1}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ is the usual inner product in $\mathbb{R}^{m+1}$.
Concerning the distribution of $\mathbf{Y}_{d}$, we have

$$
\boldsymbol{\Gamma}_{d}(s, t):=\mathbb{E}\left[Y_{\ell}(s) Y_{\ell}(t)\right]=\langle s, t\rangle^{d} ; \quad s, t \in \mathbb{R}^{m+1}
$$

It follows that the distribution of the system $\mathbf{Y}_{d}$ is invariant under the action of the orthogonal group in $\mathbb{R}^{m+1}$. For $\ell=1, \ldots, r$, we denote by $Y_{\ell}^{\prime}(t)$ the derivative (along the sphere) of $Y_{\ell}(t)$ at the point $t \in S^{m}$ and by $Y_{\ell k}^{\prime}(t)$ its $k$-th component on a given basis of the tangent space of $S^{m}$ at the point $t$. We define the standardized derivative as

$$
\begin{equation*}
\bar{Y}_{\ell}^{\prime}(t):=\frac{Y_{\ell}^{\prime}(t)}{\sqrt{d}}, \quad \text { and } \quad \overline{\mathbf{Y}}_{d}^{\prime}(t):=\left(\bar{Y}_{1}^{\prime}(t), \ldots, \bar{Y}_{r}^{\prime}(t)\right) \tag{3.2}
\end{equation*}
$$

where $\bar{Y}_{\ell}^{\prime}(t)$ is a row vector. In [2] it is shown that $\left(\mathbf{Y}_{d}(t), \overline{\mathbf{Y}}_{d}^{\prime}(t)\right)$ is a vector random field, whose $r(1+m)$ entries are standard normal random variables with covariances depending upon the quantities

$$
\begin{align*}
\mathcal{A}(\theta) & =-\sqrt{d} \cos ^{d-1}(\theta) \sin (\theta)  \tag{3.3}\\
\mathcal{B}(\theta) & =\cos ^{d}(\theta)-(d-1) \cos ^{d-2}(\theta) \sin ^{2}(\theta) \\
\mathcal{C}(\theta) & =\cos ^{d}(\theta) \\
\mathcal{D}(\theta) & =\cos ^{d-1}(\theta)
\end{align*}
$$

where $\theta$ is the angle between $s$ and $t$ in $S^{m}$. More precisely, the variance-covariance matrix of the vector

$$
\left(Y_{\ell}(s), Y_{\ell}(t), \bar{Y}_{\ell}^{\prime}(s), \bar{Y}_{\ell}^{\prime}(t)\right)
$$

has the following form

$$
\left[\begin{array}{c|l|l}
A_{11} & A_{12} & A_{13}  \tag{3.4}\\
\hline A_{12}^{1} & I_{m} & A_{23} \\
\hline A_{13}^{\top} & A_{23}^{\top} & I_{m}
\end{array}\right],
$$

where $I_{m}$ is the $m \times m$ identity matrix,

$$
A_{11}=\left[\begin{array}{ll}
1 & \mathcal{C} \\
\mathcal{C} & 1
\end{array}\right], A_{12}=\left[\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
-\mathcal{A} & 0 & \cdots & 0
\end{array}\right], A_{13}=\left[\begin{array}{cccc}
\mathcal{A} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{array}\right],
$$

and $A_{23}=\operatorname{diag}(\mathcal{B}, \mathcal{D}, \ldots, \mathcal{D})_{m \times m}$.

Furthermore, the conditional distribution of $\left(\overline{\mathbf{Y}}_{d}^{\prime}(s), \overline{\mathbf{Y}}_{d}^{\prime}(t)\right)$ given that $\mathbf{Y}_{d}(s)=\mathbf{Y}_{d}(t)=0$ is centered normal with variance-covariance matrix

$$
\left[\begin{array}{c|c}
B_{11} & B_{12}  \tag{3.5}\\
\hline B_{12} & B_{22}
\end{array}\right],
$$

with $B_{11}=B_{22}=\operatorname{diag}\left(\sigma^{2}, 1, \ldots, 1\right)_{m \times m}$ and $B_{12}=\operatorname{diag}\left(\sigma^{2} \rho, \mathcal{D}, \ldots, \mathcal{D}\right)_{m \times m}$. Here,

$$
\sigma^{2}(\theta)=1-\frac{\mathcal{A}(\theta)^{2}}{1-\mathcal{C}(\theta)^{2}} ; \quad \rho(\theta)=\frac{\mathcal{B}(\theta)\left(1-\mathcal{C}(\theta)^{2}\right)-\mathcal{A}(\theta)^{2} \mathcal{C}(\theta)}{1-\mathcal{C}(\theta)^{2}-\mathcal{A}(\theta)^{2}}
$$

Finally, let us retrieve from [2] the asymptotics and bounds for these quantities after scaling $\theta=z / \sqrt{d}$.
Lemma 3.1 ([2]). There exists $0<\alpha<\frac{1}{2}$ such that for $\frac{z}{\sqrt{d}}<\frac{\pi}{2}$ it holds that:

$$
\begin{aligned}
|\mathcal{A}| & \leq z \exp \left(-\alpha z^{2}\right) ; & |\mathcal{B}| & \leq\left(1+z^{2}\right) \exp \left(-\alpha z^{2}\right) ; \\
|\mathcal{C}| & \leq|\mathcal{D}| \leq \exp \left(-\alpha z^{2}\right) ; & 1-\mathcal{C}^{2} & \geq \text { Const }\left(1-\exp \left(-2 \alpha z^{2}\right)\right) ; \\
0 \leq 1-\sigma^{2} & \leq \text { Const } \cdot \exp \left(-2 \alpha z^{2}\right) ; & |\rho| & \leq \text { Const } \cdot\left(1+z^{2}\right)^{2} \exp \left(-2 \alpha z^{2}\right)
\end{aligned}
$$

All the functions on the l.h.s. are evaluated at $\theta=z / \sqrt{d}$.
Lemma 3.2 ([2]). As $d \rightarrow+\infty$, it holds that

$$
\begin{aligned}
\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right) \rightarrow \exp \left(-z^{2}\right) ; \quad \mathcal{A} \rightarrow-z \exp \left(-z^{2} / 2\right) \\
\mathcal{B} \rightarrow\left(1-z^{2}\right) \exp \left(-z^{2} / 2\right) ; \quad \mathcal{C}, \mathcal{D} \rightarrow \exp \left(-z^{2} / 2\right) \\
\sigma^{2}\left(\frac{z}{\sqrt{d}}\right) \rightarrow \frac{1-\left(1+z^{2}\right) \exp \left(-z^{2}\right)}{1-\exp \left(-z^{2}\right)} \\
\rho\left(\frac{z}{\sqrt{d}}\right) \rightarrow \frac{\left(1-z^{2}-\exp \left(-z^{2}\right)\right) \exp \left(-z^{2} / 2\right)}{1-\left(1+z^{2}\right) \exp \left(-z^{2}\right)}
\end{aligned}
$$

### 3.2 Fourth Moment Theorem

We present here the well known Fourth Moment Theorem which helps us in the proof of Theorem 1.
Let $\mathbf{B}=\{B(\lambda): \lambda \geq 0\}$ be a standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is the $\sigma$-algebra generated by $\mathbf{B}$. The Wiener chaos is an orthogonal decomposition of $L^{2}(\mathbf{B})=L^{2}(\Omega, \mathcal{F}, \mathbb{P})$ :

$$
L^{2}(\mathbf{B})=\bigoplus_{q=0}^{\infty} \mathcal{C}_{q}
$$

where $\mathcal{C}_{0}=\mathbb{R}$ and for $q \geq 1, \mathcal{C}_{q}=\left\{I_{q}^{\mathrm{B}}\left(f_{q}\right): f_{q} \in L_{s}^{2}\left([0, \infty)^{q}\right)\right\}$ where $I_{q}^{\mathrm{B}}$ is the $q$-folded multiple integral w.r.t. B and $L_{s}^{2}\left([0, \infty)^{q}\right)$ the space of kernels $f_{q}:[0, \infty)^{q} \rightarrow \mathbb{R}$ which are square integrable and symmetric, that is, if $\pi$ is a permutation then $f_{q}\left(\lambda_{1}, \ldots, \lambda_{q}\right)=f_{q}\left(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(q)}\right)$. Equivalently, each square integrable functional $F$ of the Brownian motion $\mathbf{B}$ can be written as a sum of orthogonal random variables

$$
F=\mathbb{E}[F]+\sum_{q=1}^{\infty} I_{q}^{\mathbf{B}}\left(f_{q}\right),
$$

for some uniquely determined kernels $f_{q} \in L_{s}^{2}\left([0, \infty)^{q}\right)$.
Let $f_{q}, g_{q} \in L_{s}^{2}\left([0, \infty)^{q}\right)$, then for $n=0, \ldots, q$ we define the contraction by

$$
\begin{aligned}
& f_{q} \otimes_{n} g_{q}\left(\lambda_{1}, \ldots, \lambda_{2 q-2 n}\right)=\int_{[0, \infty)^{n}} f_{q}\left(z_{1}, \ldots, z_{n}, \lambda_{1}, \ldots, \lambda_{q-n}\right) \\
& \quad g_{q}\left(z_{1}, \ldots, z_{n}, \lambda_{q-n+1}, \ldots, \lambda_{2 q-2 n}\right) d z_{1} \ldots d z_{n}
\end{aligned}
$$

Now, we can state the generalization of the Fourth Moment Theorem.

Theorem 2 ([15] Theorem 11.8.3). Let $F_{d}$ be in $L_{s}^{2}(\mathbf{B})$ admit chaotic expansions

$$
F_{d}=\mathbb{E}\left[F_{d}\right]+\sum_{q=1}^{\infty} I_{q}\left(f_{d, q}\right)
$$

for some kernels $f_{d, q}$. Then, if $\mathbb{E}\left[F_{d}\right]=0$ and

1. for each fixed $q \geq 1, \operatorname{Var}\left(I_{q}\left(f_{d, q}\right)\right) \xrightarrow[d \rightarrow \infty]{ } V_{q}$;
2. $V:=\sum_{q=1}^{\infty} V_{q}<\infty$;
3. for each $q \geq 2$ and $n=1, \ldots, q-1$,

$$
\lim _{d \rightarrow \infty}\left\|f_{d, q} \otimes_{n} f_{d, q}\right\|_{L_{s}^{2}\left([0, \infty)^{2 q-2 n}\right)}=0
$$

4. $\lim _{Q \rightarrow \infty} \limsup _{d \rightarrow \infty} \sum_{q=Q+1}^{\infty} \operatorname{Var}\left(I_{q}\left(f_{d, q}\right)\right)=0$.

Then, $F_{d}$ converges in distribution towards the $N(0, V)$ distribution.

## 4 Proof of Theorem 1

Though the main lines of the proof are the same as in [3], there are some important technical differences since the volume of the level set is a more complicated object than the number of roots in the square case. We begin this section with an outline of the proof and a description of the main technical differences with [3].

- First, we obtain the Hermite or Wiener-chaos expansion of the standardized ( $m-r$ )-volume of the zero level set of $\mathbf{Y}_{d}$ on $S^{m}$.
In [3], the proof of the expansion profited of the fact that the number of roots of the system is locally constant and of Bézout's bound for it. In the present case, to obtain the chaotic expansion of the volume of the zero level set we need to change our arguments to deal with co-area and Rice formulas, see Lemma 4.1. In particular, we need to state the continuity of the volume of the level sets with respect to the level. this requires a careful use of Gaussian regression.
- In order to get the limit variance of the $q$-th chaotic component we obtain a domination depending on $q$ based on Mehler's formula and in Lemma 3.1 above. In [2] a global domination was obtained from the analysis of the limit variance of the number of roots of the system.
The asymptotic normality is obtained in the same way as in [3]. In particular, we deduce that the sufficient condition to get the asymptotic normality does not depend on the number of equations $r$ but on the covariances of the entries of $\mathbf{Y}_{d}$.
- We use a convenient partition of the sphere and the existence of a local limit process in order to prove the negligibility (of the variance) of the tail of the expansion.


### 4.1 Hermite expansion of the Volume

In this part we obtain the Hermite expansion of $\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})$; the $(m-r)$-volume of the zero level set.
Let $\delta_{0}(\mathbf{y})=\prod_{\ell=1}^{r} \delta_{0}\left(y_{\ell}\right)$ be the Dirac delta distribution for $\mathbf{y} \in \mathbb{R}^{r}$.
For $\varepsilon>0$ consider an approximation $\frac{1}{\varepsilon^{r}} \varphi\left(\frac{\mathbf{y}}{\varepsilon}\right)$ to Dirac's delta distribution, where we assume that $\varphi$ is a continuous density function with bounded support. Consider also the function $f$ defined as

$$
\begin{equation*}
f\left(\mathbf{y}^{\prime}\right)=\sqrt{\operatorname{det}\left(\mathbf{y}^{\prime}\left(\mathbf{y}^{\prime}\right)^{\top}\right)}, \tag{4.1}
\end{equation*}
$$

where $\mathbf{y}^{\prime}=\left(y_{1}^{\prime} ; \ldots ; y_{r}^{\prime}\right)=\left(y_{11}^{\prime}, y_{12}^{\prime}, \ldots, y_{1 m}^{\prime} ; y_{21}^{\prime}, \ldots, y_{2 m}^{\prime} ; \ldots, \ldots ; y_{r 1}^{\prime}, \ldots, y_{r m}^{\prime}\right)$ is an $m \times r$ matrix and $\left(\mathbf{y}^{\prime}\right)^{\top}$ its transpose. Sometimes, according to convenience, we understand $\mathbf{y}^{\prime}$ as a vector in $\mathbb{R}^{r \times m}$. Furthermore, for any $\gamma>0$ let

$$
\begin{equation*}
f_{\gamma}\left(\mathbf{y}^{\prime}\right)=f\left(\gamma y_{11}^{\prime}, y_{12}^{\prime}, \ldots, y_{1 m}^{\prime} ; \gamma y_{21}^{\prime}, \ldots, y_{2 m}^{\prime} ; \ldots, \ldots ; \gamma y_{r 1}^{\prime}, \ldots, y_{r m}^{\prime}\right) . \tag{4.2}
\end{equation*}
$$

Thus $f=f_{1}$.
Remark 4.1. The function $f_{\gamma}$ plays a key role in the alternative proof of the limit variance of the volume of the zero set of $\mathbf{Y}_{d}$ since it allows us to deal with the non-homogeneity of $f$ w.r.t. the first column in $\mathbf{y}$, see Appendix.

Similarly to the case of the number of roots [2], we can obtain the following convergence in $L^{2}(\mathbf{B})$. As said above, the expansion for the volume of the zero level set is more subtle than that of the number of roots in the square case.

Lemma 4.1. Almost surely and in the $L^{2}$ sense it holds that

$$
\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})=d^{\frac{r}{2}} \lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{r}} \int_{S^{m}} \varphi\left(\frac{\mathbf{Y}_{d}(t)}{\varepsilon}\right) f\left(\bar{Y}_{1}^{\prime}(t), \ldots, \bar{Y}_{r}^{\prime}(t)\right) d \mathbf{t}
$$

Proof. By the co-area formula we have

$$
\frac{1}{\varepsilon^{r}} \int_{\mathbb{R}^{r}} \varphi\left(\frac{\mathbf{u}}{\varepsilon}\right) \mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{u}) d \mathbf{u}=\frac{1}{\varepsilon^{r}} \int_{S^{m}} \varphi\left(\frac{\mathbf{Y}_{d}(t)}{\varepsilon}\right) f\left(\bar{Y}_{1}^{\prime}(t), \ldots, \bar{Y}_{r}^{\prime}(t)\right) d t
$$

where $\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{u})$ stands for the $(m-r)$-volume of the level set $\left\{t \in S^{m}: \mathbf{Y}_{d}(t)=\mathbf{u}\right\}$. Changing the variable in the left hand side integral we can write

$$
\mathcal{Q}_{\varepsilon}:=\int_{\mathbb{R}^{r}} \varphi(\mathbf{u}) \mathcal{V}_{\mathbf{Y}_{d}}(\varepsilon \mathbf{u}) d \mathbf{u}=\frac{1}{\varepsilon^{r}} \int_{S^{m}} \varphi\left(\frac{\mathbf{Y}_{d}(t)}{\varepsilon}\right) f\left(\bar{Y}_{1}^{\prime}(t), \ldots, \bar{Y}_{r}^{\prime}(t) d \mathbf{t},\right.
$$

we need to prove the convergence in $L^{2}(\mathbf{B})$ for this sequence. Let us evaluate

$$
\begin{equation*}
\mathbb{E}\left[\left(\mathcal{Q}_{\varepsilon}-\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right)^{2}\right]=\mathbb{E}\left[\mathcal{Q}_{\varepsilon}^{2}\right]-2 \mathbb{E}\left[\mathcal{Q}_{\varepsilon} \mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right]+\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}^{2}(\mathbf{0})\right] . \tag{4.3}
\end{equation*}
$$

But

$$
\mathbb{E}\left[\mathcal{Q}_{\varepsilon}^{2}\right]=\int_{\mathbb{R}^{r} \times \mathbb{R}^{r}} \varphi\left(\mathbf{u}_{1}\right) \varphi\left(\mathbf{u}_{2}\right) \mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}\left(\varepsilon \mathbf{u}_{1}\right) \mathcal{V}_{\mathbf{Y}_{d}}\left(\varepsilon \mathbf{u}_{2}\right)\right] d \mathbf{u}_{1} d \mathbf{u}_{2}
$$

and

$$
\mathbb{E}\left[\mathcal{Q}_{\varepsilon} \mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right]=\int_{\mathbb{R}^{r}} \varphi\left(\mathbf{u}_{1}\right) \mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}\left(\varepsilon \mathbf{u}_{1}\right) \mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right] d \mathbf{u}_{1}
$$

Using the Cauchy-Schwarz inequality we have

$$
\begin{equation*}
\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}\left(\varepsilon \mathbf{u}_{1}\right) \mathcal{V}_{\mathbf{Y}_{d}}\left(\varepsilon \mathbf{u}_{2}\right)\right] \leq\left(\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}\left(\varepsilon \mathbf{u}_{1}\right)^{2}\right] \mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}\left(\varepsilon \mathbf{u}_{2}\right)^{2}\right]\right)^{\frac{1}{2}} . \tag{4.4}
\end{equation*}
$$

Furthermore, below we show that the right hand side is a continuous function in the variable $\mathbf{u}$, obtaining

$$
\lim _{\varepsilon \rightarrow 0} \mathbb{E}\left[\mathcal{Q}_{\varepsilon}^{2}\right] \leq \mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}^{2}(\mathbf{0})\right] .
$$

Moreover, given that the process satisfies the hypothesis of Proposition 6.12 of Azaïs \& Wschebor book [5], it holds that

$$
\mathbb{P}\left\{\exists t: \operatorname{rank}\left(\mathbf{Y}_{d}^{\prime}(t)\right)<r, \mathbf{Y}_{d}(t)=\mathbf{u}\right\}=0
$$

Thus by using the implicit function theorem we have that the function $\mathcal{V}_{\mathbf{Y}_{d}}(\cdot)$ is a.s. continuous and by a classical result

$$
\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})=d^{\frac{r}{2}} \lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{r}} \int_{S^{m}} \varphi\left(\frac{\mathbf{Y}_{d}(t)}{\varepsilon}\right) f\left(\bar{Y}_{1}^{\prime}(t), \ldots, \bar{Y}_{r}^{\prime}(t) d t\right. \text { a.s. }
$$

In this form by Fatou's Lemma we get

$$
\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}^{2}(\mathbf{0})\right] \leq \lim _{\varepsilon \rightarrow 0} \mathbb{E}\left[\mathcal{Q}_{\varepsilon}^{2}\right] \leq \mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}^{2}(\mathbf{0})\right] .
$$

The same result can be obtained for the second addend of (4.3), in consequence the convergence in quadratic mean holds.

It remains to prove that the right hand side of (4.4) is a continuous function, we do that in the following. This issue is not present in [3]. By Kac-Rice formula

$$
\begin{equation*}
\mathbb{E}\left[\left(\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{u})\right)^{2}\right]=d^{r} \int_{S^{m} \times S^{m}} \mathbb{E}\left[f\left(\overline{\mathbf{Y}}_{d}^{\prime}(t)\right) f\left(\overline{\mathbf{Y}}_{d}^{\prime}(s)\right) \mid \mathbf{Y}_{d}(t)=\mathbf{Y}_{d}(s)=\mathbf{u}\right] p_{\mathbf{Y}(t), \mathbf{Y}(s)}(\mathbf{u}, \mathbf{u}) d t d s \tag{4.5}
\end{equation*}
$$

Clearly the density $p_{\mathbf{Y}(t), \mathbf{Y}(s)}(\mathbf{u}, \mathbf{u})$ is continuous as a function of $\mathbf{u}$. We deal now with the conditional expectation.

Recall the notation in (3.3). Let us define the vector $\mathbf{v}(\langle t, s\rangle)=(\mathcal{A}, 0, \ldots, 0)^{\top}$, a regression model gives that

$$
\begin{aligned}
\bar{Y}_{\ell}^{\prime}(t) & =\mathbf{v}(\langle t, s\rangle) \frac{\left\langle t, s s^{d}\right.}{1-\langle t, s\rangle^{2 d}} Y_{\ell}(t)-\mathbf{v}(\langle t, s\rangle) \frac{1}{1-\langle t, s\rangle^{2 d}} Y_{\ell}(s)+\xi_{\ell 1}(t, s), \\
\bar{Y}_{\ell}^{\prime}(s) & =-\mathbf{v}(\langle t, s\rangle) \frac{1}{1-\langle t, s\rangle^{2 d}} Y_{\ell}(t)+\mathbf{v}(\langle t, s\rangle) \frac{\langle t, s\rangle^{d}}{1-\langle t, s\rangle^{2 d}} Y_{\ell}(s)+\xi_{\ell 2}(t, s),
\end{aligned}
$$

with $\xi_{\ell 1}(t, s), \xi_{\ell 2}(t, s)$ centered Gaussian random variables independent from $Y_{\ell}(s)$ and $Y_{\ell}(t)$. In this form we get that the conditional distribution of $\left(\bar{Y}_{\ell}^{\prime}(t), \bar{Y}_{\ell}^{\prime}(s)\right)$ conditioned to $\mathbf{Y}(t)=\mathbf{Y}(s)=\mathbf{u}$ is normal with mean

$$
\binom{\mathbf{v}(\langle t, s\rangle)\left(\frac{\langle t, s\rangle^{d}-1}{1-\langle t, s\rangle^{2 d}}\right) u_{\ell}}{\mathbf{v}(\langle t, s\rangle)\left(\frac{\langle t, s\rangle^{d}-1}{1-\langle t, s\rangle^{2 d}}\right) u_{\ell}}=\binom{-\mathbf{v}(\langle t, s\rangle)\left(\frac{1}{1+\langle t, s\rangle^{d}}\right) u_{\ell}}{-\mathbf{v}(\langle t, s\rangle)\left(\frac{1}{1+\langle t, s\rangle^{d}}\right) u_{\ell}}
$$

and variance-covariance matrix (3.5). This result implies that the conditional expectation can be expressed through the following two vectors:

$$
\begin{aligned}
& \zeta_{1}:=\left(-\mathbf{v}(\langle t, s\rangle) \cdot \frac{1}{1+\langle t, s\rangle^{d}} \cdot \frac{u_{1}}{\sigma}+\bar{M}_{1}, \ldots,-\mathbf{v}(\langle t, s\rangle) \cdot \frac{1}{1+\langle t, s\rangle^{d}} \cdot \frac{u_{r}}{\sigma}+\bar{M}_{r}\right) \\
& \zeta_{2}:=\left(-\mathbf{v}(\langle t, s\rangle) \cdot \frac{1}{1+\langle t, s\rangle^{d}} \cdot \frac{u_{1}}{\sigma}+\bar{W}_{1}, \ldots,-\mathbf{v}(\langle t, s\rangle) \cdot \frac{1}{1+\langle t, s\rangle^{d}} \cdot \frac{u_{r}}{\sigma}+\bar{W}_{r}\right)
\end{aligned}
$$

where the $(r \times m)$-dimensional vectors

$$
\begin{aligned}
& \left(\bar{M}_{1}, \ldots, \bar{M}_{r}\right):=\left(M_{11}, \ldots, M_{1 m}, M_{21}, \ldots, M_{2 m}, \ldots, M_{r 1}, \ldots, M_{r m}\right), \\
& \left(\bar{W}_{1}, \ldots, \bar{W}_{r}\right):=\left(W_{11}, \ldots, W_{1 m}, W_{21}, \ldots, W_{2 m}, \ldots, W_{r 1}, \ldots, W_{r m}\right),
\end{aligned}
$$

are such that the $M_{l k}$ (resp. $W_{l k}$ ) are independent standard Gaussian random variables and

$$
\mathbb{E}\left[M_{l_{1} k_{1}} W_{l_{2} k_{2}}\right]=\rho \mathbf{1}_{\left\{l_{1}=l_{2}, k_{1}=k_{2}=1\right\}}+\mathcal{D} \mathbf{1}_{\left\{l_{1}=l_{2}, k_{1}=k_{2}>1\right\}} .
$$

In fact, we have

$$
\begin{equation*}
\mathbb{E}\left[f\left(\overline{\mathbf{Y}}_{d}^{\prime}(t)\right) f\left(\overline{\mathbf{Y}}_{d}^{\prime}(s)\right] \mid \mathbf{Y}_{d}(t)=\mathbf{Y}_{d}(s)=\mathbf{u}\right)=\mathbb{E}\left[f_{\sigma}\left(\zeta_{1}\right) f_{\sigma}\left(\zeta_{2}\right)\right] . \tag{4.6}
\end{equation*}
$$

Note that this is an example of the importance of the function $f_{\sigma}$. By the definition of the function $f_{\sigma}$ and the form of the vectors $\zeta_{1}$ and $\zeta_{2}$ this term is evidently a continuous function of the variable u.

Once we have the approximation in Lemma 4.1, the expansion follows as in [2].
To describe it, we introduce the Hermite polynomials $H_{n}(x), x \in \mathbb{R}$, by $H_{0}(x)=1, H_{1}(x)=x$ and $H_{n+1}(x)=x H_{n}(x)-n H_{n-1}(x)$ for $n \geq 1$. The tensorial versions are defined for multi-indices $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{r}\right) \in \mathbb{N}^{r}$ and $\boldsymbol{\beta}=\left(\beta_{11}, \ldots, \beta_{1 m}, \ldots, \beta_{r 1}, \ldots, \beta_{r m}\right)$, by

$$
\begin{gathered}
\mathbf{H}_{\alpha}(\mathbf{y})=H_{\alpha_{1}}\left(y_{1}\right) \ldots H_{\alpha_{r}}\left(y_{r}\right) \\
\widetilde{\mathbf{H}}_{\boldsymbol{\beta}}\left(\mathbf{y}^{\prime}\right)=H_{\beta_{11}}\left(y_{11}^{\prime}\right) \ldots H_{\beta_{1 m}}\left(y_{1 m}^{\prime}\right) \ldots H_{\beta_{r 1}}\left(y_{r 1}^{\prime}\right) \ldots H_{\beta_{r m}}\left(y_{r m}^{\prime}\right) .
\end{gathered}
$$

We denote the coefficients of Dirac's delta distribution in the Hermite basis of $L^{2}\left(\mathbb{R}^{r}, \phi_{r}(\mathbf{x}) d \mathbf{x}\right)$ by $b_{\alpha}$, where $\phi_{k}$ stands for the standard normal density function in $\mathbb{R}^{k}$. Readily we can show that $b_{\alpha}=0$ if at least one index $\alpha_{j}$ is odd, otherwise

$$
b_{\boldsymbol{\alpha}}=\frac{1}{\left[\frac{\alpha}{2}\right]!} \prod_{j=1}^{r} \frac{1}{\sqrt{2 \pi}}\left[-\frac{1}{2}\right]^{\left[\frac{\alpha_{j}}{2}\right]} .
$$

Since $f^{2}$ is a polynomial, $f \in L^{2}\left(\mathbb{R}^{r \times m}, \phi_{r \times m}\left(\mathbf{y}^{\prime}\right)\right) d \mathbf{y}^{\prime}$. For $f$ we have

$$
\begin{equation*}
f\left(\mathbf{y}^{\prime}\right)=\sum_{\beta} f_{\beta} \widetilde{\mathbf{H}}_{\beta}\left(\mathbf{y}^{\prime}\right) \tag{4.7}
\end{equation*}
$$

where $\boldsymbol{\beta}$ and $\widetilde{\mathbf{H}}_{\boldsymbol{\beta}}$ are as above and

$$
f_{\boldsymbol{\beta}}=\frac{1}{\boldsymbol{\beta}!} \int_{\mathbb{R}^{r \times m}} f\left(\mathbf{y}^{\prime}\right) \widetilde{\mathbf{H}}_{\boldsymbol{\beta}}\left(\mathbf{y}^{\prime}\right) \phi_{r \times m}\left(\mathbf{y}^{\prime}\right) d \mathbf{y}^{\prime} .
$$

Let us introduce the functions

$$
g_{q}\left(\mathbf{y}, \mathbf{y}^{\prime}\right)=\sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=q} b_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}} \mathbf{H}_{\alpha}(\mathbf{y}) \widetilde{\mathbf{H}}_{\boldsymbol{\beta}}\left(\mathbf{y}^{\prime}\right),
$$

where $\left(\mathbf{y}, \mathbf{y}^{\prime}\right) \in \mathbb{R}^{r} \times \mathbb{R}^{r \times m}$.
Thus similarly to [2] we can obtain the expansion.

Proposition 4.1. With the same notation as above. We have, in the $L^{2}$ sense, that

$$
\overline{\mathcal{V}}_{d}=\frac{\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})-\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right]}{d^{\frac{\pi}{2}-\frac{m}{4}}}=d^{\frac{m}{4}} \sum_{q=1}^{\infty} \int_{S^{m}} g_{q}\left(\mathbf{Y}_{d}(t), \overline{\mathbf{Y}}^{\prime}(t)\right) d t .
$$

Remark 4.2. The same type of expansion can be obtained with minor modifications if instead of the volume of the zero set over the whole sphere we consider the volume of the set restricted to a Borel set $\mathcal{G} \subset S^{m}$.

The rest of the proof of Theorem 1 consists in the verification of the conditions of Theorem 2 with $F_{d}=\overline{\mathcal{V}}_{d}$.

### 4.2 Computing the variance of the $q$-th term

To compute the variance of the $q$-th term in the expansion we use Mehler's formula [5, L. 10.7]. We have

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_{S^{m}} g_{q}\left(\mathbf{Y}_{d}(t), \overline{\mathbf{Y}}_{d}^{\prime}(t)\right) d t\right)^{2}\right]=\sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=q\left|\boldsymbol{\alpha}^{\prime}\right|+\left|\boldsymbol{\beta}^{\prime}\right|=q} b_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}} b_{\boldsymbol{\alpha}^{\prime}} f_{\boldsymbol{\beta}^{\prime}} \\
& \times \int_{S^{m} \times S^{m}} \mathbb{E}\left[\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(t)) \mathbf{H}_{\boldsymbol{\alpha}^{\prime}}(\mathbf{Y}(s)) \widetilde{\mathbf{H}}_{\boldsymbol{\beta}}\left(\overline{\mathbf{Y}}^{\prime}(t)\right) \widetilde{\mathbf{H}}_{\boldsymbol{\beta}^{\prime}}\left(\overline{\mathbf{Y}}^{\prime}(s)\right)\right] d t d s
\end{aligned}
$$

Recall that the coefficients $b_{\alpha}$ are zero if one of the $\alpha_{j}$ is odd. Furthermore, the function $f(\mathbf{y})$ is even with respect to each column, thus its Hermite coefficients

$$
f_{\boldsymbol{\beta}}=f_{\boldsymbol{\beta}_{1}, \boldsymbol{\beta}_{2}, \ldots, \boldsymbol{\beta}_{r}}=\int_{\mathbb{R}^{r \times m}} \sqrt{\operatorname{det}\left(\mathbf{y}^{\prime}\left(\mathbf{y}^{\prime}\right)^{\top}\right)} \mathbf{H}_{\boldsymbol{\beta}_{1}}\left(y_{1}^{\prime}\right) \ldots \mathbf{H}_{\boldsymbol{\beta}_{r}}\left(y_{r}^{\prime}\right) \phi_{r \times m}\left(\mathbf{y}^{\prime}\right) d \mathbf{y}^{\prime}
$$

are zero if at least one of the $\boldsymbol{\beta}_{\ell}$ satisfies $\left|\boldsymbol{\beta}_{\ell}\right|=2 k+1$. In this form $\left|\beta_{\ell}\right|=\sum_{j=1}^{m} \beta_{\ell j}$ is necessarily even. Moreover, $q=|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|$ is also even.

By independence we have

$$
\begin{align*}
& \mathbb{E}\left[\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(t)) \widetilde{\mathbf{H}}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}(t)) \mathbf{H}_{\boldsymbol{\alpha}^{\prime}}(\mathbf{Y}(s)) \widetilde{\mathbf{H}}_{\boldsymbol{\beta}^{\prime}}\left(\overline{\mathbf{Y}}^{\prime}(s)\right)\right] \\
&= \prod_{\ell=1}^{r} \mathbb{E}\left[H_{\alpha_{\ell}}\left(Y_{\ell}(t)\right) \mathbf{H}_{\boldsymbol{\beta}_{\ell}}\left(\bar{Y}_{\ell}^{\prime}(t)\right) H_{\alpha_{\ell}^{\prime}}\left(Y_{\ell}(t)\right) \mathbf{H}_{\boldsymbol{\beta}_{\ell}^{\prime}}\left(\bar{Y}_{\ell}^{\prime}(t)\right)\right] \\
&= \prod_{\ell=1}^{r} \mathbb{E}\left[H_{\alpha_{\ell}}\left(Y_{\ell}(s)\right) H_{\alpha_{\ell}^{\prime}}\left(Y_{\ell}(t)\right) H_{\beta_{\ell 1}}\left(\bar{Y}_{\ell 1}^{\prime}(s)\right) H_{\beta_{\ell 1}^{\prime}}\left(\bar{Y}_{\ell 1}^{\prime}(t)\right)\right] \\
& \times \prod_{j=2}^{m} \mathbb{E}\left[H_{\beta_{\ell j}}\left(\bar{Y}_{\ell j}^{\prime}(s)\right) H_{\beta_{\ell j}^{\prime}}\left(\bar{Y}_{\ell j}^{\prime}(t)\right)\right] \tag{4.8}
\end{align*}
$$

In the second equality we used that the random vectors:

$$
\left(Y_{\ell}(s), Y_{\ell}(t), \bar{Y}_{\ell 1}^{\prime}(s), \bar{Y}_{\ell 1}^{\prime}(t)\right) ; \quad\left(\bar{Y}_{\ell j}^{\prime}(s), \bar{Y}_{\ell j}^{\prime}(t)\right) ; \quad j \geq 2
$$

are independent. Using Mehler's formula [5, L. 10.7], we get

$$
\mathbb{E}\left[H_{\beta_{\ell j}}\left(\bar{Y}_{\ell j}^{\prime}(s)\right) H_{\beta_{\ell j}^{\prime}}\left(\bar{Y}_{\ell j}^{\prime}(t)\right)\right]=\delta_{\beta_{\ell j} \beta_{\ell j}^{\prime}} \beta_{\ell j}!\left(\rho_{\ell j}^{\prime \prime}\right)^{\beta_{\ell j}}
$$

where $\rho_{\ell j}^{\prime \prime}=\rho_{\ell j}^{\prime \prime}(\langle s, t\rangle)=\mathbb{E}\left[\bar{Y}_{\ell j}^{\prime}(s) \bar{Y}_{\ell j}^{\prime}(t)\right]=\langle t, s\rangle^{d-1}$. Since $\sum_{j=1}^{m} \beta_{\ell j}$ is even, we have that either $\beta_{\ell 1}$ is even and then $\sum_{j=2}^{m} \beta_{\ell j}$ is even too or $\beta_{\ell 1}$ is odd and in this case $\sum_{j=2}^{m} \beta_{\ell j}$ is also odd.

For the first factor in the r.h.s. of (4.8), using again Mehler's formula we get

$$
\mathbb{E}\left[H_{\alpha_{\ell}}\left(Y_{\ell}(s)\right) H_{\alpha_{\ell}^{\prime}}\left(Y_{\ell}(t)\right) H_{\beta_{\ell 1}}\left(\bar{Y}_{\ell 1}^{\prime}(s)\right) H_{\beta_{\ell 1}^{\prime}}\left(\bar{Y}_{\ell 1}^{\prime}(t)\right)\right]=0
$$

if $\alpha_{\ell}+\beta_{\ell 1} \neq \alpha_{\ell}^{\prime}+\beta_{\ell 1}^{\prime}$. Otherwise, consider $\Lambda \subset \mathbb{N}^{4}$ defined by

$$
\Lambda=\left\{\left(d_{1}, d_{2}, d_{3}, d_{4}\right): d_{1}+d_{2}=\alpha_{\ell}, d_{3}+d_{4}=\beta_{\ell 1}, d_{1}+d_{3}=\alpha_{\ell}^{\prime}, d_{2}+d_{4}=\beta_{\ell 1}^{\prime}\right\}
$$

then

$$
\mathbb{E}\left[H_{\alpha_{\ell}}\left(Y_{\ell}(s)\right) H_{\alpha_{\ell}^{\prime}}\left(Y_{\ell}(t)\right) H_{\beta_{\ell 1}}\left(\bar{Y}_{\ell 1}^{\prime}(s)\right) H_{\beta_{\ell 1}^{\prime}}\left(\bar{Y}_{\ell 1}^{\prime}(t)\right)\right]=\sum_{\left(d_{i}\right) \in \Lambda} \frac{\alpha_{\ell}!\alpha_{\ell}^{\prime}!\beta_{\ell 1}!\beta_{\ell 1}^{\prime}!}{d_{1}!d_{2}!d_{3}!d_{4}!} \rho^{d_{1}}\left(\rho^{\prime}\right)^{d_{2}}\left(\rho^{\prime}\right)^{d_{3}}\left(\rho^{\prime \prime}\right)^{d_{4}}
$$

where $\rho=\rho(\langle s, t\rangle)=\mathbb{E}\left[Y_{\ell}(s) Y_{\ell}(t)\right], \rho^{\prime}=\mathbb{E}\left[Y_{\ell}(s) \bar{Y}_{\ell 1}^{\prime}(t)\right]=\mathbb{E}\left[\bar{Y}_{\ell 1}^{\prime}(s) Y_{\ell}(t)\right]$ and $\rho^{\prime \prime}=\mathbb{E}\left[\bar{Y}_{\ell 1}^{\prime}(s) \bar{Y}_{\ell 1}^{\prime}(t)\right]$. Note that the conditions defining the index set $\Lambda$ implies that the first factor in Equation (4.8) is

$$
\prod_{\ell=1}^{r} \sum_{\left(d_{i}\right) \in \Lambda} \frac{\alpha_{\ell}!\alpha_{\ell}^{\prime}!\beta_{\ell 1}!\beta_{\ell 1}^{\prime}!}{d_{1}!d_{2}!d_{3}!d_{4}!} \rho^{d_{1}}\left(\rho^{\prime}\right)^{d_{2}+d_{3}}\left(\rho^{\prime \prime}\right)^{d_{4}}
$$

Hence, if we change $\langle s, t\rangle$ by $-\langle s, t\rangle$, for each $\ell$ we have the factor

$$
\begin{array}{r}
(-1)^{d d_{1}} \cdot(-1)^{(d-1)\left(d_{2}+d_{3}\right)} \cdot(-1)^{d d_{4}}=(-1)^{d\left(d_{1}+d_{4}\right)+(d-1)\left(d_{2}+d_{3}\right)} \\
=(-1)^{d \alpha_{\ell}}(-1)^{d \beta_{\ell_{1}}^{\prime}}(-1)^{2\left(\alpha_{\ell}^{\prime}-d_{1}\right)}=(-1)^{d \beta_{\ell_{1}}^{\prime}}
\end{array}
$$

Remark 4.3. Changing $\langle t, s\rangle$ by $-\langle t, s\rangle$ in (4.8) and considering each term for $j=1, \ldots, r$ of the product, either $\beta_{\ell_{1}}^{\prime}$ and $\sum_{j=2}^{m} \beta_{\ell_{j}}^{\prime}$ are even and then the sign of this term does not change, or the two numbers are odd and then they have a minus in front and the sign neither change. Thus we get that the complete sign of (4.8) does not change.

Let us now define

$$
\begin{equation*}
\tilde{\mathcal{H}}_{q d}(\langle t, s\rangle)=\sum_{|(\boldsymbol{\alpha}, \boldsymbol{\beta})|=q\left|\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}^{\prime}\right)\right|=q} a_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}^{\prime}} \mathbb{E}\left[\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(t)) \widetilde{\mathbf{H}}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}(t)) \mathbf{H}_{\boldsymbol{\alpha}^{\prime}}(\mathbf{Y}(s)) \widetilde{\mathbf{H}}_{\boldsymbol{\beta}^{\prime}}\left(\overline{\mathbf{Y}}^{\prime}(s)\right)\right] . \tag{4.9}
\end{equation*}
$$

Set also, for $t \in S^{m}$,

$$
\begin{equation*}
\mathbf{Z}_{d}(t)=\left(Z_{1}(t), \ldots, Z_{r(1+m)}(t)\right)=\left(\mathbf{Y}_{d}(t), \overline{\mathbf{Y}}_{d}^{\prime}(t)\right) \tag{4.10}
\end{equation*}
$$

In this manner we can write

$$
\begin{aligned}
& d^{\frac{m}{2}} \mathbb{E}\left[\left(\int_{S^{m}} g_{q}\left(\mathbf{Z}_{d}(t)\right) d t\right)^{2}\right]=d^{\frac{m}{2}} \int_{S^{m} \times S^{m}} \tilde{\mathcal{H}}_{q d}(\langle t, s\rangle) d t d s \\
& \quad=\kappa_{m} \kappa_{m-1} d^{m / 2} \int_{0}^{\pi} \sin ^{m-1}(\theta) \tilde{\mathcal{H}}_{q d}(\cos (\theta)) d \theta \\
& \quad=2 \kappa_{m} \kappa_{m-1} \int_{0}^{\sqrt{d} \pi / 2} d^{(m-1) / 2} \sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right) \tilde{\mathcal{H}}_{q d}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right) d z
\end{aligned}
$$

For the second equality we used (3.1) about the integration on the sphere of an invariant by rotations function. In the third equality we use (deduced from Remark 4.3) the invariance of the function with respect to the change of variable $\varphi=\frac{\pi}{2}-\theta$ and finally we made $\theta=\frac{z}{\sqrt{d}}$.

The convergence follows by dominated convergence using for the covariances $\rho_{k, \ell}:=\mathbb{E}\left[Z_{k}(s) Z_{\ell}(t)\right]$, the bounds in Lemma 3.1 and the expression for the matrix (3.4). In this manner the integrand can be bounded by Const $\left(1+z^{2}\right)^{q} \exp \left(-q \alpha z^{2}\right)$. In conclusion we have

$$
V_{q}^{r}:=\lim _{d \rightarrow \infty} d^{\frac{m}{2}} \mathbb{E}\left[\left(\int_{S^{m}} g_{q}\left(\mathbf{Z}_{d}(t)\right) d t\right)^{2}\right]=2 \kappa_{m} \kappa_{m-1} \int_{0}^{\infty} z^{m-1} \tilde{\mathcal{H}}_{q}(z) d z \quad q \geq 1
$$

where

$$
\tilde{\mathcal{H}}_{q}(z):=\lim _{d \rightarrow \infty} \tilde{\mathcal{H}}_{q d}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right) .
$$

Implicit in $\tilde{\mathcal{H}}_{q}(z)$ we use the pointwise limits given in Lemma 3.2.
Remark 4.4. In the present case the domination is obtained via Mehler's formula separately for each fixed $q$. In [3] we used a less ad hoc argument profiting of the computation of the limit global variance (of the number of roots of the system) using Rice formula.

### 4.3 Point 2 of Theorem 2

Recall from (2.1) that

$$
V_{\infty}^{r}=\lim _{d \rightarrow \infty} \operatorname{Var}\left(\overline{\mathcal{V}}_{d}\right)=\lim _{d \rightarrow \infty} \sum_{q=0}^{\infty} d^{\frac{m}{2}} \mathbb{E}\left[\left(\int_{S^{m}} g_{q}\left(\mathbf{Z}_{d}(t)\right) d t\right)^{2}\right]
$$

The second equality follows from Parseval's identity. Thus, by Fatou's Lemma

$$
V^{r}:=\sum_{q=0}^{\infty} V_{q}^{r}=\sum_{q=0}^{\infty} \lim _{d \rightarrow \infty} d^{\frac{m}{2}} \mathbb{E}\left[\left(\int_{S^{m}} g_{q}\left(\mathbf{Z}_{d}(t)\right) d t\right)^{2}\right] \leq V_{\infty}^{r}<\infty
$$

Actually, equality holds as a consequence of Point 4 and the finiteness of $V_{\infty}^{r}$.

### 4.4 Normality of the $q$-th term

Lemma 4.3 below gives a sufficient condition on the covariances of the process $\mathbf{Z}_{d}$ in order to verify the convergence of the norm of the contractions (which in turn gives the asymptotic normality). Below we write the chaotic components

$$
I_{q, d}=d^{\frac{m}{4}} \int_{S^{m}} g_{q}\left(\mathbf{Z}_{d}(t)\right) d t
$$

in Proposition 4.1 as multiple stochastic integrals w.r.t. a standard Brownian motion $\mathbf{B}$ and use this fact in order to prove Lemma 4.3.

Let $\mathbf{B}=\{B(\lambda): \lambda \in[0, \infty)\}$ be a standard Brownian motion on $[0, \infty)$. By the isometric property of stochastic integrals there exist kernels $h_{t, \ell}$ such that the components of the vector $\mathbf{Z}_{d}$ defined in (4.10) can be written as:

$$
\begin{equation*}
Z_{\ell}(t)=\int_{0}^{\infty} h_{t, \ell}(\lambda) d B(\lambda), \ell=1, \ldots, r(m+1) \tag{4.11}
\end{equation*}
$$

The kernels $h_{t, \ell}$ can be computed explicitly from the definition of $Z_{\ell}$ writing the random coefficients as integrals w.r.t. the Brownian motion $\mathbf{B}$.

The two following lemmas are close to the equivalent lemmas in [2], their proofs are omitted. Note that the number of equations $r$ of the system $\mathbf{Y}_{d}$ does not play any role in the condition (4.12) below.
Lemma 4.2. With the same notation and assumptions as in Proposition 4.1, $I_{q, d}$ can be written as a multiple stochastic integral

$$
I_{q, d}=I_{q}^{\mathbf{B}}\left(g_{q, d}\right)=\int_{[0, \infty)^{q}} g_{q, d}(\boldsymbol{\lambda}) d B(\boldsymbol{\lambda})
$$

with

$$
g_{q, d}(\boldsymbol{\lambda})=d^{m / 4} \sum_{|\boldsymbol{\mu}|=q} a_{\boldsymbol{\mu}} \int_{S^{m}}\left(\otimes_{\ell=1}^{r(m+1)} h_{t, \ell}^{\otimes \gamma}\right)(\boldsymbol{\lambda}) d t
$$

where $h_{t, \ell}$ is defined in (4.11) and $I_{q}^{\mathbf{B}}$ is the $q$-folded multiple stochastic integral w.r.t. the Brownian motion B.

As $\Gamma_{d}(s, t)=\Gamma_{d}(\langle s, t\rangle)$, then $\Gamma_{d}$ can be seen as a function of one real variable.
Lemma 4.3. For $k=0,1,2$, let $\Gamma_{d}^{(k)}$ indicate the $k$-th derivative of $\Gamma_{d}:[-1,1] \rightarrow \mathbb{R}$. If

$$
\begin{equation*}
d^{m / 3} \int_{0}^{\pi / 2} \sin ^{m-1}(\theta)\left|\Gamma_{d}^{(k)}(\cos (\theta))\right| d \theta \underset{d \rightarrow+\infty}{\rightarrow} 0 \tag{4.12}
\end{equation*}
$$

then, for $n=1, \ldots, q-1$ and $g_{q, d}$ defined in Lemma 4.2:

$$
\left\|g_{q, d} \otimes_{n} g_{q, d}\right\|_{2} \rightarrow_{d \rightarrow \infty} 0
$$

Therefore, it suffices to verify (4.12). For $k=0,1,2$ we have

$$
\begin{aligned}
d^{m / 3} \int_{0}^{\pi / 2} \sin ^{m-1}(\theta)\left|\Gamma_{d}^{(k)}(\cos (\theta))\right| d \theta & =d^{m / 3} \int_{0}^{\sqrt{d} \pi / 2} \sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)\left|\Gamma_{d}^{(k)}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right)\right| \frac{d z}{\sqrt{d}} \\
& =\frac{1}{d^{m / 6}} \int_{0}^{\sqrt{d} \pi / 2} d^{\frac{m-1}{2}} \sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)\left|\Gamma_{d}^{(k)}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right)\right| d z
\end{aligned}
$$

Now $d^{\frac{m-1}{2}} \sin ^{m-1}(z / \sqrt{d}) \leq z^{m-1}$ and taking the worst case in Lemma 3.1 we have $\left|\Gamma_{d}^{(k)}(z / \sqrt{d})\right| \leq$ $\left(1+z^{2}\right) \exp \left(-\alpha z^{2}\right)$. Hence, the last integral is convergent and (4.12) follows.

### 4.5 Point 4 in Theorem 2

Let $\pi^{Q}$ be the projection on $\oplus_{q \geq Q} \mathcal{C}_{q}$. We need to bound the following quantity uniformly in $d$

$$
\begin{equation*}
\frac{d^{m / 2}}{4} \operatorname{Var}\left(\pi^{Q}\left(\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right)=\frac{1}{4} \sum_{q \geq Q} d^{m / 2} \int_{S^{m} \times S^{m}} \tilde{\mathcal{H}}_{q d}(\langle s, t\rangle) d s d t\right. \tag{4.13}
\end{equation*}
$$

where $\tilde{\mathcal{H}}_{q d}$ is defined in (4.9).
In order to bound this quantity we split the integral depending on the (geodesical) distance between $s, t \in S^{m}$

$$
\operatorname{dist}(s, t)=\arccos (\langle s, t\rangle),
$$

into the integrals over the regions $\{(s, t): \operatorname{dist}(s, t)<a / \sqrt{d}\}$ and its complement, $a$ will be chosen later. We bound each part in the following two subsections.

### 4.5.1 Off-diagonal term

In this subsection we consider the integral in the r.h.s. of (4.13) restricted to the off-diagonal region $\{(s, t): \operatorname{dist}(s, t) \geq a / \sqrt{d}\}$. This is the easier case since the covariances of $\mathbf{Z}_{d}$ are bounded away from 1.

We use Arcones' Lemma ([1], page 2245). Let $X$ be a standard Gaussian vector on $\mathbb{R}^{N}$ and $h: \mathbb{R}^{N} \rightarrow \mathbb{R}$ a measurable function such that $\mathbb{E}\left[h^{2}(X)\right]<\infty$ and let us consider its $L^{2}$ convergent Hermite's expansion

$$
h(x)=\sum_{q=0}^{\infty} \sum_{|\mathbf{k}|=q} h_{\mathbf{k}} \mathbf{H}_{\mathbf{k}}(x)
$$

The Hermite rank of $h$ is defined as

$$
\operatorname{rank}(h)=\inf \left\{\tau: \exists \mathbf{k},|\mathbf{k}|=\tau ; \mathbb{E}\left[(h(X)-\mathbb{E} h(X)) \mathbf{H}_{\mathbf{k}}(X)\right] \neq 0\right\}
$$

Then, we have
Lemma 4.4 ([1]). Let $W=\left(W_{1}, \ldots, W_{N}\right)$ and $Q=\left(Q_{1}, \ldots, Q_{N}\right)$ be two mean-zero Gaussian random vectors on $\mathbb{R}^{N}$. Assume that

$$
\mathbb{E}\left[W_{j} W_{k}\right]=\mathbb{E}\left[Q_{j} Q_{k}\right]=\delta_{j, k}
$$

for each $1 \leq j, k \leq N$. We define

$$
r^{(j, k)}=\mathbb{E}\left[W_{j} Q_{k}\right]
$$

Let $h$ be a function on $\mathbb{R}^{N}$ with finite second moment and Hermite rank $\tau, 1 \leq \tau<\infty$, define

$$
\psi:=\max \left\{\max _{1 \leq j \leq N} \sum_{k=1}^{N}\left|r^{(j, k)}\right|, \max _{1 \leq k \leq N} \sum_{j=1}^{N}\left|r^{(j, k)}\right|\right\}
$$

Then

$$
|\operatorname{Cov}(h(W), h(Q))| \leq \psi^{\tau} \mathbb{E}\left[h^{2}(W)\right]
$$

We apply this lemma for $N=r \times(1+m), W=\mathbf{Z}(s), Q=\mathbf{Z}(t)$ and to the function $h\left(\mathbf{y}, \mathbf{y}^{\prime}\right)=$ $g_{q}\left(\mathbf{y}, \mathbf{y}^{\prime}\right)$. Recalling that $\rho_{k, \ell}(s, t)=\rho_{k, \ell}(\langle s, t\rangle)=\mathbb{E}\left[Z_{k}(s) Z_{\ell}(t)\right]$, the Arcones' coefficient is now

$$
\psi(s, t)=\max \left\{\sum_{1 \leq k \leq m+m^{2}}\left|\rho_{k, \ell}(s, t)\right|, \sum_{1 \leq \ell \leq m+m^{2}}\left|\rho_{k, \ell}(s, t)\right|\right\}
$$

Thus

$$
\left|\tilde{\mathcal{H}}_{q d}(\langle s, t\rangle)\right| \leq \psi(\langle s, t\rangle)^{q}\left\|g_{q}\right\|^{2}
$$

being $\left\|g_{q}\right\|^{2}=\mathbb{E}\left[g_{q}^{2}(\zeta)\right]$ for standard normal $\zeta$.
The following lemma is obtained as in [3].
Lemma 4.5. For $g_{q}$ it holds that $\left\|g_{q}\right\|^{2} \leq\|f\|_{2}^{2}$.
To bound the Arcones' coefficient $\psi(\langle s, t\rangle)$ we use the expressions in (3.1) thanks to the invariance of the distribution of $\mathbf{Y}_{d}\left(\right.$ and $\left.\mathbf{Z}_{d}\right)$ under isometries. It is not hard to see that the maximum in the definition of $\psi$ is $|\mathcal{C}|+|\mathcal{A}|$, see (3.3). From Lemma 3.1 it follows that $|\mathcal{C}|+|\mathcal{A}| \leq e^{-\alpha z^{2}}(1+z)$. For $z=2$ the bound takes the value $2 e^{-4 \alpha}$ which is less or equal to one if $\alpha \geq \frac{1}{4} \log 2$, this is always possible because the only restriction that we have is $\alpha<\frac{1}{2}$. Moreover, for $\delta$ small enough $e^{-\alpha z^{2}}(1+z) \geq 1$ if $z<\delta$. Thus, there exists an $a<2$ such that for all $z \geq a$ it holds $|\mathcal{C}|+|\mathcal{A}|<r_{0}<1$. Hence,

$$
\begin{aligned}
& \sup _{d} \sum_{q \geq Q} \frac{d^{m / 2}}{4} \int_{\left\{(s, t): \operatorname{dist}(s, t) \geq \frac{a}{\sqrt{d}}\right\}} \tilde{\mathcal{H}}_{q, d}(\langle s, t\rangle) d s d t \\
& \quad=\sup _{d} \frac{C_{m}}{4}\left|\sum_{q \geq Q} d^{\frac{m-1}{2}} \int_{a}^{\sqrt{d} \pi} \sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right) \tilde{\mathcal{H}}_{d}^{q}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right) d z\right| \\
& \quad \leq C_{m}\|f\|_{2}^{2} \sum_{q \geq Q} r_{0}^{q-1} \int_{a}^{\infty} z^{m-1}(1+z) e^{-\alpha z^{2}} d z \underset{Q \rightarrow \infty}{ } 0
\end{aligned}
$$

### 4.5.2 Diagonal term

It remains to prove that the integral in the r.h.s. of (4.13) restricted to the diagonal region $\{(s, t)$ : $\operatorname{dist}(s, t)<a / \sqrt{d}\}$ tends to 0 as $Q \rightarrow \infty$ uniformly in $d, a<2$ is fixed. This is the difficult part, we use an indirect argument.

Next proposition, whose proof is similar to that in [3], gives a convenient partition of the sphere based on the hyperspheric coordinates. For $\Theta=\left(\theta_{1}, \ldots, \theta_{m-1}, \theta_{m}\right) \in[0, \pi)^{m-1} \times[0,2 \pi)$ we write $x^{(m)}(\Theta)=\left(x_{1}^{(m)}(\Theta), \ldots, x_{m+1}^{(m)}(\Theta)\right) \in S^{m}$ in the following way

$$
x_{k}^{(m)}(\Theta)=\prod_{j=1}^{k-1} \sin \left(\theta_{j}\right) \cdot \cos \left(\theta_{k}\right), k \leq m \text { and } x_{m+1}^{(m)}(\Theta)=\prod_{j=1}^{m} \sin \left(\theta_{j}\right) ;
$$

with the convention that $\prod_{1}^{0}=1$.
Define the hyperspherical rectangle (HSR for short) with center $x^{(m)}(\tilde{\theta})$ with $\tilde{\theta}=\left(\tilde{\theta}_{1}, \ldots, \tilde{\theta}_{m}\right)$ and vector radius $\tilde{\eta}=\left(\tilde{\eta}_{1}, \ldots, \tilde{\eta}_{m}\right)$ as

$$
\operatorname{HSR}(\tilde{\theta}, \tilde{\eta})=\left\{x^{(m)}(\theta):\left|\theta_{i}-\tilde{\theta}_{i}\right|<\tilde{\eta}_{i}, i=1, \ldots, m\right\}
$$

Let $T_{t} S^{m}$ be the the tangent space to $S^{m}$ at $t$. This space can be identified with $t^{\perp} \subset \mathbb{R}^{m+1}$. Let $\phi_{t}: S^{m} \rightarrow t^{\perp}$ be the orthogonal projection over $t^{\perp}, \mathcal{C}_{\mathbf{Y}_{d}}(\mathbf{0})$ be the zero set of $\mathbf{Y}_{d}$ on $S^{m}$ and $\mathcal{V}$ its volume on $S^{m}$.
Proposition 4.2. For d large enough, there exists a partition of the unit sphere $S^{m}$ into HSRs $R_{j}: j=1, \ldots, k(m, d)=O\left(d^{m / 2}\right)$ and an extra set $E$ such that

1. $\operatorname{Var}\left(\mathcal{V}\left(\mathcal{C}_{\mathbf{Y}_{d}}(\mathbf{0}) \cap E\right)\right)=o\left(d^{r-\frac{m}{2}}\right)$.
2. The HSRs $R_{j}$ have diameter $O\left(\frac{1}{\sqrt{d}}\right)$ and if $R_{j}$ and $R_{\ell}$ do not share any border point (they are not neighbors), then $\operatorname{dist}\left(R_{j}, R_{\ell}\right) \geq \frac{1}{\sqrt{d}}$.
3. The projection of each of the sets $R_{j}$ on the tangent space at its center $c_{j}$, after normalizing by the multiplicative factor $\sqrt{d}$, converges to the rectangle $[-1 / 2,1 / 2]^{m}$ in the sense of Hausdorff distance. That is, the Hausdorff distance of

$$
\left[-\frac{1}{2}, \frac{1}{2}\right]^{m} \backslash \sqrt{d} \phi_{c_{j}}\left(R_{j}\right)
$$

tends to 0 as $d \rightarrow \infty$.
Arguing as in [3] it suffices to bound $\operatorname{Var}\left(\pi^{Q}\left(\mathcal{V}\left(\mathcal{C}_{\mathbf{Y}_{d}}(\mathbf{0}) \cap R_{j}\right)\right)\right)$, where $R_{0}$ is an HSR contained in the spherical cap

$$
C\left(e_{0}, \gamma / \sqrt{d}\right)=\left\{s: d\left(s, e_{0}\right)<\gamma / \sqrt{d}\right\} .
$$

for some $\gamma$ depending on $m$.
We use the local chart $\phi: C\left(e_{0}, \gamma / \sqrt{d}\right) \rightarrow B(0, \sin (\gamma / \sqrt{d})) \subset \mathbb{R}^{m}$ defined by

$$
\phi^{-1}(u)=\left(\sqrt{1-\|u\|^{2}}, u\right), \quad u \in B(0, \sin (\gamma / \sqrt{d}))
$$

to project this set over the tangent space. Define the random field $\mathcal{Y}_{d}: B(0, \gamma) \subset \mathbb{R}^{m} \rightarrow \mathbb{R}^{r}$, as

$$
\mathcal{Y}_{d}(u)=\mathbf{Y}_{d}\left(\phi^{-1}(u \sqrt{d})\right)
$$

Observe that the $\ell$ coordinates, $\mathcal{Y}_{d}^{(\ell)}$ say, of $\mathcal{Y}_{d}$ are independent. Clearly, the zero set of $\mathbf{Y}_{d}$ on $R \subset C\left(e_{0}, \gamma / \sqrt{d}\right)$ and the zero set of $\mathcal{Y}_{d}$ on $\phi(R \sqrt{d}) \subset B(0, \gamma)$ coincide. That is

$$
\mathcal{C}_{\mathbf{Y}_{d}}(\mathbf{0}) \cap R=\mathcal{C}_{\mathcal{Y}_{d}}(\mathbf{0}) \cap \phi(R \sqrt{d}) .
$$

Proposition 4.3. The sequence of processes $\mathcal{Y}_{d}^{(\ell)}(u)$ as well as its first and second order derivatives converge in the finite dimensional distribution sense towards the mean zero Gaussian processes $\mathcal{Y}_{\infty}$ with covariance function $\Gamma(u, v)=e^{-\frac{\|u-v\|^{2}}{2}}$ and its corresponding derivatives.

The proof of this proposition can be consulted in [14] and also in [3].
Remark 4.5. The local limit process $\mathcal{Y}_{\infty}$ has as coordinates $\left(\mathcal{Y}_{\infty}^{(1)}, \ldots, \mathcal{Y}_{\infty}^{(r)}\right)$ such that each one of them is an independent copy of the random field with covariance $\Gamma(u)=e^{-\frac{\|u\|^{2}}{2}}, u \in \mathbb{R}^{m}$. Then its covariance matrix writes

$$
\tilde{\Gamma}(u)=\operatorname{diag}(\Gamma(u), \ldots, \Gamma(u))
$$

The second derivative matrix $\tilde{\Gamma}^{\prime \prime}(u)$ can be written in a similar way, but here the blocks are equal to the matrix $\Gamma^{\prime \prime}(u)=\left(a_{i j}\right)$ where $a_{i j}=e^{-\frac{\|u\|^{2}}{2}} H_{1}\left(u_{i}\right) H_{1}\left(u_{j}\right)$ if $i \neq j$, and $a_{i i}=e^{-\frac{\|u\|^{2}}{2}} H_{2}\left(u_{i}\right)$. It follows from [4, Th.12] that the variance of $\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{\infty}}(\mathbf{0}) \cap K\right)$ is finite for any compact $K$.

The proof of the main theorem will be achieved as soon as we have proved the following proposition.
Proposition 4.4. For $\varepsilon>0$ there exist $Q_{0}$ and $d_{0}$ such that for $Q \geq Q_{0}$

$$
\sup _{d>d_{0}} \mathbb{E}\left[\left(\pi^{Q}\left(\mathcal{V}\left(\mathcal{C}_{\mathbf{Y}_{d}}(\mathbf{0}) \cap R_{j}\right)\right)\right)^{2}\right]<\varepsilon
$$

Proof. Let $R=R_{0} \subset C\left(e_{0}, \gamma / \sqrt{d}\right)$, By Remark 4.2, the Hermite expansion holds true also for the volume of the zero set of $\mathbf{Y}_{d}$ on any subset of $S^{m}$. Hence,

$$
\mathcal{V}\left(\mathcal{C}_{\mathbf{Y}_{d}}(\mathbf{0}) \cap R\right)=\sum_{q=0}^{\infty} d^{\frac{r}{2}} \int_{R} g_{q}\left(\mathbf{Z}_{d}(t)\right) d t .
$$

Let us define $\tilde{R}=\phi(R) \subset B\left(0, \sin \frac{a}{\sqrt{d}}\right) \subset \mathbb{R}^{m}$. It follows that

$$
\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{d}}(\mathbf{0}) \cap \tilde{R}\right)=\mathcal{V}\left(\mathcal{C}_{\mathbf{Y}_{d}}(\mathbf{0}) \cap R\right)=\sum_{q=0}^{\infty} d^{\frac{r}{2}} \int_{\tilde{R}} g_{q}\left(\mathcal{Y}_{d}(u), \mathcal{Y}_{d}^{\prime}(u)\right) J_{\phi}(u) d u
$$

where $J_{\phi}(u)=\left(1-\|u\|^{2}\right)^{-1 / 2}$ is the Jacobian. Rescaling $u=v / \sqrt{d}$ we get

$$
\frac{\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{d}}(\mathbf{0}) \cap \tilde{R}\right)}{d^{\frac{n}{2}-\frac{m}{4}}}=\sum_{q=0}^{\infty} \int_{\sqrt{d} \tilde{R}} g_{q}\left(\mathcal{Y}_{d}\left(\frac{v}{\sqrt{d}}\right), \mathcal{Y}_{d}^{\prime}\left(\frac{v}{\sqrt{d}}\right)\right) J_{\phi}\left(\frac{v}{\sqrt{d}}\right) d v .
$$

Besides, Kac-Rice formula, the domination for $\mathcal{H}_{q d}$ previously obtained, the convergence of $\mathcal{Y}_{d}$ to $\mathcal{Y}_{\infty}$ in Proposition 4.3 and the convergence, after normalization, of $\bar{R}$ to $[-1 / 2,1 / 2]^{m}$ in Proposition 4.2 yield

$$
\begin{equation*}
\operatorname{Var}\left(\frac{\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{d}}(\mathbf{0}) \cap \tilde{R}\right)}{d^{\frac{n}{2}-\frac{m}{4}}}\right) \underset{d \rightarrow \infty}{\rightarrow} \operatorname{Var}\left(\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{\infty}}(\mathbf{0}) \cap\left[-\frac{1}{2}, \frac{1}{2}\right]^{m}\right)\right) \tag{4.14}
\end{equation*}
$$

In fact, for the second moment we have
$\mathbb{E}\left[\left(\frac{\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{d}}(\mathbf{0}) \cap \tilde{R}\right)}{d^{\frac{r}{2}-\frac{m}{4}}}\right)^{2}\right]$

$$
\begin{gathered}
=d^{m} \int_{\tilde{R} \times \tilde{R}} \mathbb{E}\left[f\left(\mathcal{Y}_{d}^{\prime}(u)\right) f\left(\mathcal{Y}_{d}^{\prime}(v)\right) \mid \mathcal{Y}_{d}(u)=\mathcal{Y}_{d}(v)=0\right] p_{u, v}(0,0) J_{\phi}(u) J_{\phi}(v) d u d v \\
=\int_{\sqrt{d} \tilde{R} \times \sqrt{d} \tilde{R}} \mathbb{E}\left[\left.f\left(\mathcal{Y}_{d}^{\prime}\left(\frac{u}{\sqrt{d}}\right)\right) f\left(\mathcal{Y}_{d}^{\prime}\left(\frac{v}{\sqrt{d}}\right)\right) \right\rvert\, \mathcal{Y}_{d}\left(\frac{u}{\sqrt{d}}\right)=\mathcal{Y}_{d}\left(\frac{v}{\sqrt{d}}\right)=0\right] \\
\times p_{u, v}(0,0) J_{\phi}\left(\frac{u}{\sqrt{d}}\right) J_{\phi}\left(\frac{v}{\sqrt{d}}\right) d u d v \\
\underset{\substack{\rightarrow \infty}}{\rightarrow} \int_{\left(\left[\frac{1}{2}, \frac{1}{2}\right]^{m}\right)^{2}} \mathbb{E}\left[f\left(\mathcal{Y}_{\infty}^{\prime}(u)\right) f\left(\mathcal{Y}_{\infty}^{\prime}(v)\right) \mid \mathcal{Y}_{\infty}(u)=\mathcal{Y}_{\infty}(v)=0\right] p_{\mathcal{Y}_{\infty}(u), \mathcal{Y}_{\infty}(v)(0,0) d u d v} \\
=\mathbb{E}\left[\left(\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{\infty}}(\mathbf{0}) \cap\left[-\frac{1}{2}, \frac{1}{2}\right]^{m}\right)\right)^{2}\right]<\infty .
\end{gathered}
$$

The term of the square of the expectation is easier.
The same arguments show that for all $q$ we have

$$
V_{q, d}^{l o c}:=\operatorname{Var}\left(\pi_{q}\left(\frac{\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{d}}(\mathbf{0}) \cap \tilde{R}\right)}{d^{\frac{\tilde{n}}{2}-\frac{m}{4}}}\right)\right) \underset{d \rightarrow \infty}{\rightarrow} \operatorname{Var}\left(\pi_{q}\left(\mathcal{V}\left(\left(\mathcal{C}_{\mathcal{Y}_{\infty}}(\mathbf{0})\right) \cap\left[-\frac{1}{2}, \frac{1}{2}\right]^{m}\right)\right)\right)=: V_{q}^{l o c} .
$$

Thus, for all $Q$ it follows that $\sum_{q=0}^{Q} V_{q, d}^{l o c} \rightarrow_{d \rightarrow \infty} \sum_{q=0}^{Q} V_{q}^{\text {loc }}$. By Parseval's identity, (4.14) can be written as

$$
\sum_{q=0}^{\infty} V_{q, d}^{l o c} \underset{d \rightarrow \infty}{\rightarrow} \sum_{q=0}^{\infty} V_{q}^{l o c}
$$

Thus, by taking the difference we get

$$
\begin{equation*}
\sum_{q>Q} V_{q, d}^{l o c} \underset{d \rightarrow \infty}{\rightarrow} \sum_{q>Q} V_{q}^{l o c} . \tag{4.15}
\end{equation*}
$$

Given that the series $\sum_{q=0}^{\infty} V_{q}^{\text {loc }}$ is convergent, we can choose $Q_{0}$ such that for $Q \geq Q_{0}$ it holds $\sum_{q>Q}^{\infty} V_{q}^{\text {loc }} \leq \varepsilon / 2$. Hence, for this $Q_{0}$ and by using (4.15) we can choose $d_{0}$ such that for all $d>d_{0}$ and $Q \geq Q_{0}$

$$
\sum_{q>Q} V_{d, q}^{l o c} \leq \varepsilon .
$$

Namely, there exists $d_{0}$ such that for $Q \geq Q_{0}$

$$
\sup _{d>d_{0}} \mathbb{E}\left[\left(\pi^{Q}\left(\frac{\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_{d}}(\mathbf{0}) \cap \tilde{R}\right)}{d^{\frac{r}{2}-\frac{m}{4}}}\right)\right)^{2}\right]<\varepsilon .
$$

## 5 Appendix

Letendre [9] and Letendre \& Puchol [10] have studied the asymptotic variance of the volume of the zero set. We consider this problem using another method.

Write the variance as

$$
\operatorname{Var}\left(\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})\right)=\mathbb{E}\left[\left(\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})^{2}\right]-\left(\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0}]\right)^{2} .\right.\right.
$$

Let $f$ and $f_{\gamma}$ be defined as in (4.1) and (4.2) respectively.
We have already computed the second term, as in (4.5)-(4.6), and for the first one we apply the Rice formula for the second moment ([5, Ch.6]).

$$
\begin{align*}
\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})^{2}\right] & =\int_{S^{m} \times S^{m}} \mathbb{E}\left[f\left(\mathbf{Y}_{d}^{\prime}(t)\right) f\left(\mathbf{Y}_{d}^{\prime}(s)\right) \mid \mathbf{Y}(t)=\mathbf{Y}(s)=0\right] p_{\mathbf{Y}(t), \mathbf{Y}(s)}(0,0) d t d s \\
& =d^{r} \int_{\mathbb{S}^{m} \times \mathbb{S}^{m}} \mathbb{E}\left[f_{\sigma}\left(\zeta_{1}\right) f_{\sigma}\left(\zeta_{2}\right)\right] p_{\mathbf{Y}(t), \mathbf{Y}(s)}(0,0) d t d s, \tag{5.1}
\end{align*}
$$

where $p_{\mathbf{Y}(t), \mathbf{Y}(s)}(0,0)$ is the density of the vector $(\mathbf{Y}(t), \mathbf{Y}(s))$. By independence it holds

$$
p_{\mathbf{Y}(t), \mathbf{Y}(s)}(0,0)=\prod_{\ell=1}^{r} p_{Y_{\ell}(t), Y_{\ell}(s)}(0,0)=\frac{1}{(2 \pi)^{r}\left(1-\langle t, s\rangle^{2 d}\right)^{\frac{r}{2}}} .
$$

As in (4.7), we have

$$
f_{\gamma}\left(\mathbf{y}^{\prime}\right)=\sum_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}}(\gamma) \tilde{\mathbf{H}}_{\boldsymbol{\beta}}\left(\mathbf{y}^{\prime}\right),
$$

Remark 5.1. Let us point out that for $r=m$ it holds $f\left(\mathbf{y}^{\prime}\right)=\left|\operatorname{det} \mathbf{y}^{\prime}\right|$. Here $\mathbf{y}^{\prime}$ is the $m \times m$ matrix whose columns are the vectors $\left(y_{\ell 1}^{\prime}, \ldots, y_{\ell, m}^{\prime}\right)$ for $\ell=1, \ldots, m$. Furthermore, by the homogeneity of the determinant we have $f_{\gamma}\left(\mathbf{y}^{\prime}\right)=\gamma f\left(\mathbf{y}^{\prime}\right)$. The lack of this fact when $r<m$ implies that we need a different approach for obtaining the asymptotic variance.

Thus, the expansion (4.7) and the bi-dimensional Mehler's formula [5, Th.10.7] give

$$
\begin{align*}
d^{r} \mathbb{E}\left[f_{\sigma}\left(\zeta_{1}\right) f_{\sigma}\left(\zeta_{2}\right)\right] & =d^{r} \sum_{\boldsymbol{\beta}}\left[f_{\boldsymbol{\beta}}(\sigma(\langle t, s\rangle))\right]^{2} \boldsymbol{\beta}![\mathcal{D}(\langle t, s\rangle)]^{\left(|\boldsymbol{\beta}|-\sum_{\ell=1}^{r} \beta_{\ell 1}\right)}[\rho(\langle t, s\rangle)]^{\sum_{\ell=1}^{r} \boldsymbol{\beta}_{\ell 1}}  \tag{5.2}\\
& =d^{r} \mathcal{H}(\langle t, s\rangle) .
\end{align*}
$$

Using (3.1) we obtain

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})^{2}\right] & =\kappa_{m} \kappa_{m-1} \int_{0}^{\pi} \mathcal{H}(\cos (\theta)) \frac{d^{r}}{(2 \pi)^{r}} \frac{1}{\left(1-\cos ^{2 d}(\theta)\right)^{\frac{r}{2}}} \sin ^{m-1}(\theta) d \theta \\
& =\kappa_{m} \kappa_{m-1} \frac{d^{r-\frac{1}{2}}}{(2 \pi)^{r}} \int_{0}^{\sqrt{d} \pi} \mathcal{H}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right) \frac{\sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}} d z .
\end{aligned}
$$

By Parseval equality we have

$$
\sum_{\nu}\left|f_{\mathcal{\beta}}(\gamma)\right|^{2} \boldsymbol{\beta}!=\int_{\mathbb{R}^{r \times m}}\left|f_{\gamma}(\mathbf{x})\right|^{2} \varphi_{r \times m}(\mathbf{x}) d \mathbf{x} \leq m^{r}(\gamma \vee 1)^{2 r} \mathbb{E}\left[\sup _{1 \leq \ell \leq r, 1 \leq j \leq m}\left|\frac{Y_{\ell j}\left(\mathbf{e}_{0}\right)}{\sqrt{d}}\right|^{2 r}\right]
$$

Since $\sigma^{2}\left(\frac{z}{\sqrt{d}}\right) \leq 1($ Lemma 3.1), then

$$
\begin{equation*}
\sum_{\boldsymbol{\beta}}\left|f_{\mathcal{\beta}}\left(\sigma\left(\frac{z}{\sqrt{d}}\right)\right)\right|^{2} \boldsymbol{\beta}!<\mathbf{C} . \tag{5.3}
\end{equation*}
$$

Therefore, we can interchange the series with the integral obtaining

$$
\begin{equation*}
\mathbb{E}\left[\left(\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})^{2}\right]=\kappa_{m} \kappa_{m-1} \frac{d^{r-\frac{1}{2}}}{(2 \pi)^{r}} \sum_{\boldsymbol{\beta}} \int_{0}^{\sqrt{d} \pi} \mathcal{H}_{\boldsymbol{\beta}}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right) \frac{\sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}} d z,\right. \tag{5.4}
\end{equation*}
$$

where $\mathcal{H}$ is as in (5.2) and $\mathcal{H}_{\boldsymbol{\beta}}=\left[f_{\boldsymbol{\beta}}(\sigma(\langle t, s\rangle))\right]^{2} \boldsymbol{\beta}![\mathcal{D}(\langle t, s\rangle)]^{\left(|\boldsymbol{\beta}|-\sum_{\ell=1}^{r} \beta_{\ell 1}\right)}[\rho(\langle t, s\rangle)]^{\sum_{\ell=1}^{r} \boldsymbol{\beta}_{\ell 1}}$.
Thus, using the above notation and normalizing we have

$$
\begin{aligned}
& \mathbb{E}\left[\left(\frac{\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0}}{d^{\frac{\tau}{2}-\frac{m}{4}}}\right)^{2}\right]=\kappa_{m} \kappa_{m-1} \frac{d^{\frac{m-1}{2}}}{(2 \pi)^{r}} \int_{0}^{\sqrt{d} \pi} \sum_{|\boldsymbol{\beta}| \geq 1} \mathcal{H}_{\boldsymbol{\beta}}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right) \frac{\sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}} d z \\
&+\kappa_{m} \kappa_{m-1} \frac{d^{\frac{m-1}{2}}}{(2 \pi)^{r}} \int_{0}^{\sqrt{d} \pi}\left|f_{0}\left(\sigma\left(\frac{z}{\sqrt{d}}\right)\right)\right|^{2} \frac{\sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}} d z
\end{aligned}
$$

We start with the terms $|\boldsymbol{\beta}| \geq 1$. To apply the dominated convergence theorem we must look for a uniform bound. But let us begin with a remark.
Remark 5.2. The symmetrization argument used in step 3 of section 3.2 of [2] gives that the integral over $\left[\sqrt{d} \frac{\pi}{2}, \sqrt{d} \pi\right]$ of each term in the series (5.4) is equal to the integral of same term on $\left[0, \sqrt{d} \frac{\pi}{2}\right]$ except for a multiplication by $(-1)^{(d-1)|\boldsymbol{\beta}|}$. In this form the bound that is obtained for applying the dominated convergence theorem in the latter interval serves also for the former.

Using Lemma 3.1 it holds that there exists a $d_{0}$ such that for $\frac{z}{\sqrt{d}}<\frac{\pi}{2}$ and $d>d_{0}$ it holds

$$
|\rho| \leq \mathbf{C}\left(1+z^{2}\right)^{2} \exp \left(-2 \alpha z^{2}\right) \text { and } \mathcal{D} \leq \exp \left(-2 \alpha z^{2}\right)
$$

By the Remark 5.2 it is enough to study only the interval $\left[0, \sqrt{d} \frac{\pi}{2}\right]$. In this form we get

$$
\begin{aligned}
\left|\sum_{|\boldsymbol{\beta}| \geq 1}\left[f_{\boldsymbol{\beta}}\left(\sigma\left(\frac{z}{\sqrt{d}}\right)\right)\right]^{2} \boldsymbol{\beta}!\left[\mathcal{D}\left(\frac{z}{\sqrt{d}}\right)\right]^{|\boldsymbol{\beta}|-\sum_{\ell=1}^{r} \beta_{\ell 1}}\left[\rho\left(\frac{z}{\sqrt{d}}\right)\right]^{\sum_{\ell=1}^{r} \beta_{\ell 1}}\right| \\
\leq \mathbf{C} \sum_{|\boldsymbol{\beta}| \geq 1}\left[f_{\boldsymbol{\beta}}\left(\sigma\left(\frac{z}{\sqrt{d}}\right)\right)\right]^{2} \boldsymbol{\beta}!\left(1+z^{2}\right)^{2} \exp \left(-2 \alpha z^{2}\right) \\
\leq \mathbf{C}\left(1+z^{2}\right)^{2} \exp \left(-2 \alpha z^{2}\right)
\end{aligned}
$$

Above we have used (5.3).
It remains to consider the integral over the interval $\left[0, z_{0}\right]$. But now the integrand can be bounded by

$$
\begin{equation*}
\mathbf{C} \frac{1}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}} d^{\frac{m-1}{2}} \sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right) \leq \mathbf{C} \frac{1}{\left(1-\exp \left(-2 \alpha z^{2}\right)\right)^{\frac{r}{2}}} z^{m-1} \tag{5.5}
\end{equation*}
$$

and the function on the right hand side is integrable whenever $r<m$.
In this manner, applying the dominated convergence theorem we get

$$
\begin{aligned}
& \lim _{d \rightarrow \infty} \kappa_{m} \kappa_{m-1} \frac{d^{\frac{m-1}{2}}}{(2 \pi)^{r}} \int_{0}^{\sqrt{d} \pi} \sum_{|\boldsymbol{\beta}| \geq 1} \mathcal{H}_{\boldsymbol{\beta}}\left(\cos \left(\frac{z}{\sqrt{d}}\right)\right) \frac{\sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}} d z \\
&=\kappa_{m} \kappa_{m-1} \frac{1}{(2 \pi)^{r}} \int_{0}^{\infty} \sum_{|\boldsymbol{\beta}| \geq 1} \mathcal{H}_{\boldsymbol{\beta}}(z) \frac{z^{m-1}}{\left(1-\exp \left(-z^{2}\right)\right)^{\frac{r}{2}}} d z
\end{aligned}
$$

The exact expression for the function $\mathcal{H}_{\boldsymbol{\beta}}(z)$ is obtained by using the results of the step 2 of [2]. In fact

$$
\begin{aligned}
\mathcal{H}_{\boldsymbol{\beta}}(z)=\left|f_{\mathcal{\beta}}\left(\left(\frac{1-\left(1+z^{2}\right) \exp \left(-z^{2}\right)}{1-\exp \left(-z^{2}\right)}\right)^{1 / 2}\right)\right|^{2} \boldsymbol{\beta !} & \left(\exp \left(-\frac{z^{2}}{2}\right)\right)^{|\boldsymbol{\beta}|-\sum_{\ell=1}^{r} \beta_{\ell 1}} \\
& \times\left(\frac{\left(1-z^{2}-\exp \left(-z^{2}\right)\right) \exp \left(-z^{2} / 2\right)}{1-\left(1+z^{2}\right) \exp \left(-z^{2}\right)}\right)^{\sum_{\ell=1}^{r} \beta_{\ell 1}} .
\end{aligned}
$$

To end our proof it only remains to consider the zero term in the expansion. Consider first

$$
\mathcal{I}_{d}=\kappa_{m} \kappa_{m-1} \frac{d^{\frac{m-1}{2}}}{(2 \pi)^{r}} \int_{0}^{\sqrt{d} \pi}\left[\left|f_{0}\left(\sigma\left(\frac{z}{\sqrt{d}}\right)\right)\right|^{2}-\left|f_{0}(1)\right|^{2}\right] \frac{\sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}} d z .
$$

But

$$
\begin{aligned}
\left|\left|f_{0}\left(\sigma\left(\frac{z}{\sqrt{d}}\right)\right)\right|^{2}-\left|f_{0}(1)\right|^{2}\right| & =\mathbf{C}\left|\int_{\mathbb{R}^{r \times m}} \int_{0}^{1} \sum_{\ell=1}^{r} \frac{\partial f_{\xi \sigma+(1-\xi)}(\mathbf{x})}{\partial x_{\ell 1}} d \xi\left(1-\sigma\left(\frac{z}{\sqrt{d}}\right)\right) \varphi_{r \times m}(\mathbf{x}) d \mathbf{x}\right| \\
& \leq \mathbf{C} \mathbf{1}_{\left\{z \leq z_{0}\right\}}+\mathbf{C} \exp \left(-2 \alpha z^{2}\right) \mathbf{1}_{\left\{z>z_{0}\right\}} .
\end{aligned}
$$

Yielding by the dominated convergence theorem that

$$
\lim _{d \rightarrow \infty} \mathcal{I}_{d}=\frac{\kappa_{m} \kappa_{m-1}}{(2 \pi)^{r}} \int_{0}^{\infty}\left[\left|f_{0}\left(\left(\frac{1-\left(1+z^{2}\right) \exp \left(-z^{2}\right)}{1-\exp \left(-z^{2}\right)}\right)^{1 / 2}\right)\right|^{2}-\left|f_{0}(1)\right|^{2}\right] \frac{z^{m-1}}{\left(1-\exp \left(-z^{2}\right)\right)^{\frac{r}{2}}} d z
$$

Finally, we will consider the remaining term. In first place let us point out that

$$
\kappa_{m} \kappa_{m-1}\left|f_{0}(1)\right|^{2} \frac{d^{\frac{m-1}{2}}}{(2 \pi)^{r}} \int_{0}^{\sqrt{d} \pi} \sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right) d z=\frac{1}{d^{r-\frac{m}{2}}}\left(\mathbb{E}\left[\mathcal{V}\left(\mathcal{C}_{\mathbf{Y}_{d}}(\mathbf{0})\right)\right]\right)^{2} .
$$

Then, substracting this term we get

$$
\mathcal{J}_{d}=\kappa_{m} \kappa_{m-1} \frac{d^{\frac{m-1}{2}}}{(2 \pi)^{r}} \int_{0}^{\sqrt{d} \pi}\left|f_{0}(1)\right|^{2}\left[\frac{1}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}}-1\right] \sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right) d z
$$

The convergence at 0 follows from

$$
\frac{1}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}}-1 \leq \frac{1}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}},
$$

using (5.5).
On the large interval we use the lower bound for $1-\mathcal{C}^{2}$ and the upper bound for $\mathcal{C}$ in Lemma 3.1 to obtain

$$
\left|d^{\frac{m-1}{2}}\left[\frac{1}{\left(1-\cos ^{2 d}\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{r}{2}}}-1\right] \sin ^{m-1}\left(\frac{z}{\sqrt{d}}\right)\right| \leq \mathbf{C} \exp \left(-2 \alpha z^{2}\right) \frac{z^{m-1}}{\left(1-\exp \left(-2 \alpha z^{2}\right)\right)^{\frac{3 r}{2}}}
$$

Since these two bounds allow applying the dominated convergence theorem it holds

$$
\lim _{d \rightarrow \infty} \mathcal{J}_{d}=\kappa_{m} \kappa_{m-1} \frac{\left|f_{0}(1)\right|^{2}}{(2 \pi)^{r}} \int_{0}^{\infty}\left[\frac{1}{\left(1-\exp \left(-z^{2}\right)\right)^{\frac{n}{2}}}-1\right] z^{m-1} d z
$$

Hence, it is possible to write a closed formula for the limit variance. Indeed, we obtain

$$
\begin{aligned}
\lim _{d \rightarrow 0} \operatorname{Var}\left(\frac{\mathcal{V}_{\mathbf{Y}_{d}}(\mathbf{0})}{d^{\frac{r}{2}}-\frac{m-1}{4}}\right) & =\kappa_{m} \kappa_{m-1}\left\{\frac{1}{(2 \pi)^{r}} \int_{0}^{\infty}\left(\sum_{|\boldsymbol{\beta}| \geq 1} \mathcal{H}_{\boldsymbol{\beta}}(z)\right) \frac{1}{\left(1-\exp \left(-z^{2}\right)\right)^{\frac{r}{2}}} z^{m-1} d z\right. \\
& +\frac{1}{(2 \pi)^{r}} \int_{0}^{\infty}\left[\left|f_{0}\left(\left(\frac{1-\left(1+z^{2}\right) \exp \left(-z^{2}\right)}{1-\exp \left(-z^{2}\right)}\right)^{1 / 2}\right)\right|^{2}-\left|f_{0}^{r}(1)\right|^{2}\right] \frac{z^{m-1}}{\left(1-\exp \left(-z^{2}\right)\right)^{\frac{r}{2}}} d z \\
& \left.+\frac{\left|f_{0}(1)\right|^{2}}{(2 \pi)^{r}} \int_{0}^{\infty}\left[\frac{1}{\left(1-\exp \left(-z^{2}\right)\right)^{\frac{r}{2}}}-1\right] z^{m-1} d z\right\}
\end{aligned}
$$

The result follows.

## References

[1] M. Arcones. Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. Ann. Probab., 22, no. 4, 2242-2274 (1994).
[2] D. Armentano, J-M. Azaïs, F. Dalmao and J.R. León. On the asymptotic variance of the number of real roots of random polynomial systems. Proc. Amer. Math. Soc., 147, no. 1, 205-214 (2019).
[3] D. Armentano, J-M. Azaïs, F. Dalmao and J.R. León. Central Limit Theorem for the number of real roots of Kostlan-Shub-Smale random polynomial systems. To appear in American Journal of Mathematics.
[4] J.M. Azaïs, J.R. León. Necessary and sufficient conditions for the finiteness of the second moment of the measure of level sets. Electron. J. Probab., 25 (2020), Paper No. 107, 15 pp.
[5] J-M Azaïs and M. Wschebor. Level sets and extrema of random processes and fields. John Wiley \& Sons Inc., Hoboken, NJ, ISBN: 978-0-470-40933-6 (2009).
[6] C. Berzin. Estimation of Local Anisotropy Based on Level Sets. arXiv:1801.03760
[7] M. Kac. On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc., 49, 314-320 (1943).
[8] E. Kostlan. On the distribution of roots of random polynomials. In The work of Smale in differential topology, From Topology to Computation: Proceedings of the Smalefest, (Berkeley, CA, 1990), 419-431, Springer, New York, (1993).
[9] T. Letendre. Variance of the volume of random real algebraic submanifolds. Trans. Amer. Math. Soc., 371, no. 6, 4129-4192 (2019).
[10] T. Letendre T, M. Puchol. Variance of the volume of random real algebraic submanifolds II, Indiana University Mathematics Journal, 68 (6), pp.1649-1720 (2019).
[11] N. B. Maslova. The variance of the number of real roots of random polynomials. Teor. Verojatnost. i Primenen, 19, 36-51 (1974).
[12] N.B. Maslova. The distribution of the number of real roots of random polynomials. Teor. Verojatnost. i Primenen, 19, 488-500 (1974).
[13] I. Nourdin and G. Peccati. Normal approximations with Malliavin calculus. From Stein's method to universality. Cambridge Tracts in Mathematics, 192. Cambridge University Press, Cambridge, xiv+239 pp. ISBN: 978-1-107-01777-1 (2012).
[14] F. Nazarov, M. Sodin. Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. Journal of Mathematical Physics, Analysis, Geometry, V12, No. 3 pp 205-278 (2016).
[15] G. Peccati and M. Taqqu. Wiener chaos: moments, cumulants and diagrams. A survey with computer implementation. Supplementary material available online. Bocconi \& Springer Series, 1. Springer, Milan; Bocconi University Press, Milan, xiv+274 pp. ISBN: 978-88-470-1678-1 (2011).
[16] M. Shub and S. Smale. Complexity of Bézout's Theorem I: geometric aspects. Journal of AMS, 6:459-501, (1993).
[17] M. Shub and S. Smale. Complexity of Bézout's theorem. II. Volumes and Computational algebraic geometry (Nice, 1992), 267-285, Progr. Math., 109, Birkhäuser Boston, Boston, MA, (1993).
[18] M. Shub and S. Smale. Complexity of Bézout's Theorem III: condition number and packing. Journal of Complexity, 9:4-14, (1993).
[19] M. Shub and S. Smale. Complexity of Bézout's Theorem IV: probability of success; extensions. SIAM J. of Numer. Anal., 33:128-148, (1996).
[20] M. Shub and S. Smale. Complexity of Bézout's Theorem V: polynomial time. Theoretical Computer Science, 133:141-164, (1994).
[21] M. Wschebor On the Kostlan-Shub-Smale model for random polynomial systems. Variance of the number of roots. J. Complexity, 21 No. 6, 773-789 (2005).


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