OPTIMAL RANDOMIZED QUADRATURE FOR WEIGHTED SOBOLEV AND BESOV CLASSES WITH THE JACOBI WEIGHT ON THE BALL

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ABSTRACT. We consider the numerical integration

$$INT_d(f) = \int_{\mathbb{R}^d} f(x) w_{\mu}(x) dx$$

for the weighted Sobolev classes $BW^r_{p,\mu}$ and the weighted Besov classes $BB^r_{\tau}(L_{p,\mu})$ in the randomized case setting, where w_{μ} , $\mu \geq 0$, is the classical Jacobi weight on the ball \mathbb{B}^d , $1 \leq p \leq \infty$, $r > (d+2\mu)/p$, and $0 < \tau \leq \infty$. For the above two classes, we obtain the orders of the optimal quadrature errors in the randomized case setting are $n^{-r/d-1/2+(1/p-1/2)+}$. Compared to the orders $n^{-r/d}$ of the optimal quadrature errors in the deterministic case setting, randomness can effectively improve the order of convergence when p > 1.

1. Introduction

Let F_d be a class of continuous functions on D_d , where D_d is a compact subset of the Euclidean space \mathbb{R}^d with a probability measure ρ . The integral of a continuous function $f: F_d \to \mathbb{R}$ denotes by

(1.1)
$$\operatorname{INT}_{d}(f) = \int_{D_{d}} f(x)d\rho(x).$$

We want to approximate this integral $INT_d(f)$ by (deterministic) algorithms of the form

$$A_n(f) := \varphi_n(f(x_1), f(x_2), \dots, f(x_n)),$$

where $x_j \in D_d$ can be chosen adaptively and $\varphi_n : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary mapping. Adaption means that the selection of x_j may depend on the already computed values $f(x_1), f(x_2), \ldots, f(x_{j-1})$. We denoted by $\mathcal{A}_n^{\text{det}}$ the class of all algorithms of this form. If x_1, \ldots, x_n are fixed and φ_n is linear, i.e.,

$$A_n(f) = \sum_{j=1}^n \lambda_j f(x_j), \quad \lambda_j \in \mathbb{R}, \ j = 1, \dots, n,$$

then the algorithm A_n is called a linear algorithm. Such linear algorithm A_n is also called a quadrature formula. We say that a quadrature formula A_n is positive if $\lambda_j > 0, j = 1, \ldots, n$.

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The deterministic case error of A_n on F_d is given by

$$e^{\det}(F_d, A_n) := \sup_{f \in F_d} |\operatorname{INT}_d(f) - A_n(f)|,$$

and the minimal (optimal) deterministic case error on F_d given by

$$e_n^{\det}(F_d) := \inf_{A_n \in \mathcal{A}_n^{\det}} e^{\det}(F_d, A_n).$$

It was well known (see [3]) that if F_d is convex and balanced, then $e_n^{\text{det}}(F_d)$ can be achieved by linear algorithms. Hence $e_n^{\text{det}}(F_d)$ is also called the optimal quadrature error.

Randomized algorithms, called also Monte-Carlo algorithms, are understood as $\Sigma \otimes \mathcal{B}(F_d)$ measurable functions

$$(A^{\omega}) = (A^{\omega}(\cdot))_{\omega \in \Omega} : \Omega \times F_d \to \mathbb{R},$$

where $\mathcal{B}(F_d)$ denotes Borel σ -algebra of F_d , $(\Omega, \Sigma, \mathcal{P})$ is a suitable probability space, and for any fixed $\omega \in \Omega$, A^{ω} is a deterministic method with cardinality $n(f, \omega)$. The number $n(f, \omega)$ may be randomized and adaptively depend on the input, and the cardinality of (A^{ω}) is then defined by

$$\operatorname{Card}(A^{\omega}) := \sup_{f \in F_d} \mathbb{E}_{\omega} \, n(f, \omega) := \sup_{f \in F_d} \int_{\Omega} n(f, \omega) d\mathcal{P}(\omega).$$

We denote by $\mathcal{A}_n^{\text{ran}}$ the class of all randomized algorithms with cardinality not exceeding n.

The randomized case error of (A^{ω}) on F_d is defined by

$$e^{\operatorname{ran}}(F_d,(A^{\omega})) := \sup_{f \in F_d} \mathbb{E}_{\omega}|\operatorname{INT}_d(f) - A^{\omega}(f)|,$$

and the minimal (optimal) randomized case error on F_d is defined by

$$e_n^{\mathrm{ran}}(F_d) := \inf_{(A^{\omega}) \in \mathcal{A}_n^{\mathrm{ran}}} e^{\mathrm{ran}}(F_d, (A^{\omega})).$$

There are many papers devoted to investigating the integration problem (1.1) in the deterministic and randomized case settings. Compared to deterministic algorithms, randomized algorithms may speed up the order of convergence in many cases, especially for integration problem. We recall some known results.

Throughout the paper, the notation $a_n \approx b_n$ means $a_n \lesssim b_n$ and $a_n \gtrsim b_n$. Here, $a_n \lesssim b_n$ ($a_n \gtrsim b_n$) means that there exists a constant c > 0 independent of n such that $a_n \leq cb_n$ ($b_n \leq ca_n$).

(1) Consider the classical Sobolev class $BW_p^r([0,1]^d), 1 \leq p \leq \infty, r \in \mathbb{N}$, defined by

$$BW_p^r([0,1]^d) = \left\{ f \in L_p([0,1]^d) \, \middle| \, \sum_{|\alpha|_1 \le r} \|D^{\alpha} f\|_p \le 1 \right\},\,$$

and the Hölder class $C_d^{k,\gamma}$, $k \in \mathbb{N}_0$, $0 < \gamma \le 1$, defined by

$$C_d^{k,\gamma} := \big\{ f \in C([0,1]^d) \, \big| \, |D^{\alpha}f(x) - D^{\alpha}f(y)| \le \max_{1 \le i \le d} |x_i - y_i|^{\gamma}, \, |\alpha|_1 = k \big\},\,$$

where $\alpha \in \mathbb{N}_0^d$, $|\alpha|_1 := \sum_{i=1}^d \alpha_i$, and $D^{\alpha}f$ is the partial derivative of order α of f in the sense of distribution. Bakhvalov in [1] and [2] proved that

$$e_n^{\det}(C_d^{k,\gamma}) \asymp n^{-\frac{k+\gamma}{d}}$$
 and $e_n^{\det}(BW_{\infty}^r([0,1]^d)) \asymp n^{-\frac{r}{d}}$.

Novak extended the second equivalence result in [24] and [25], and proved that for $1 \le p < \infty$ and r > d/p,

$$e_n^{\det}(BW_n^r([0,1]^d)) \approx n^{-\frac{r}{d}}.$$

Meanwhile, Novak considered the randomized case errors of the above two classes in [24] and [25], and proved that

$$e_n^{\mathrm{ran}}(C_d^{k,\alpha}) \asymp n^{-\frac{k+\alpha}{d}-\frac{1}{2}},$$

and for $1 \le p \le \infty$ and r > d/p,

$$e_n^{\text{ran}}(BW_p^r([0,1]^d)) \approx n^{-\frac{r}{d} - \frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_+},$$

where $a_+ = \max(a, 0)$.

(2) Consider the anisotropic Sobolev class $BW_p^{\mathbf{r}}([0,1]^d)$, $1 \leq p \leq \infty$, $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{N}^d$, defined by

$$BW_p^{\mathbf{r}}([0,1]^d) = \big\{ f \in L_p([0,1]^d) \, \big| \, \sum_{j=1}^d \big\| \frac{\partial^{r_j} f}{\partial x_j^{r_j}} \big\|_p \le 1 \big\}.$$

Fang and Ye in [13] obtained for $g(\mathbf{r}) > d/p$,

$$e_n^{\det}(BW_p^{\mathbf{r}}([0,1]^d)) \simeq n^{-g(\mathbf{r})},$$

and for $g(\mathbf{r}) > 1/p$,

$$e_n^{\mathrm{ran}}(BW_p^{\mathbf{r}}([0,1]^d)) \simeq n^{-g(\mathbf{r}) - \frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_+},$$

where $g(\mathbf{r}) = \left(\sum_{j=1}^{d} r_j^{-1}\right)^{-1}$. For the anisotropic Hölder-Nikolskii classes, Fang and Ye obtained the similar results in [13].

(3) Consider the Sobolev class with bounded mixed derivative $BW_p^{r,\text{mix}}([0,1]^d)$, $r \in \mathbb{N}, 1 \le p \le \infty$, defined by

$$BW_p^{r,\min}([0,1]^d) = \{ f \in L_p([0,1]^d) \mid \sum_{|\alpha|_{\infty} \le r} ||D^{\alpha}f||_p \le 1 \},$$

where $|\alpha|_{\infty} := \max_{1 \leq i \leq d} \alpha_i$. The authors in [5, 14, 31, 32] obtained for r > 1/p and 1 ,

$$e_n^{\mathrm{det}}(BW_p^{r,\mathrm{mix}}([0,1]^d)) \asymp n^{-r}(\log n)^{\frac{d-1}{2}}.$$

It was shown in [19, 27, 35] that for $r > \max\{1/p, 1/2\}$ and 1 ,

$$e_n^{\mathrm{ran}}(BW_p^{r,\mathrm{mix}}([0,1]^d)) \asymp n^{-r-\frac{1}{2}+(\frac{1}{p}-\frac{1}{2})_+}.$$

(4) For the Sobolev class $BW_p^r(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, r > 0, on the sphere \mathbb{S}^{d-1} , it was proved in [4, 17, 37] that for r > (d-1)/p,

$$e_n^{\det}(BW_p^r(\mathbb{S}^{d-1})) \simeq n^{-\frac{r}{d-1}}.$$

Wang and Zhang in [39] obtained for r > (d-1)/p,

$$e_n^{\text{ran}}(BW_n^r(\mathbb{S}^{d-1})) \simeq n^{-\frac{r}{d-1} - \frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_+}$$

(5) For the generalized Besov class $BB_{p,\theta}^{\Omega}(\mathbb{S}^{d-1})$, $1 \leq p, \theta \leq \infty$, with the smoothness index Ω satisfying some conditions, Duan and Ye in [9] obtained

$$e_n^{\det}(BB_{n,\theta}^{\Omega}(\mathbb{S}^{d-1})) \simeq \Omega(n^{-\frac{1}{d-1}}),$$

and

$$e_n^{\text{ran}}(BB_{n,\theta}^{\Omega}(\mathbb{S}^{d-1})) \simeq \Omega(n^{-\frac{1}{d-1}})n^{-\frac{1}{2}+(\frac{1}{p}-\frac{1}{2})_+}.$$

We remark that if $\Omega(t) = t^r$, then $BB_{p,\theta}^{\Omega}(\mathbb{S}^{d-1})$ recedes to the usual Besov class $BB_{\theta}^{r}(L_{p}(\mathbb{S}^{d-1}))$.

(6) Dai and Wang in [6] investigated the weighted Besov class $BB_{\tau}^{r}(L_{p,w}(\mathbb{S}^{d-1}))$, $r > 0, 0 < \tau \leq \infty, 1 \leq p \leq \infty$, with an A_{∞} weight w on \mathbb{S}^{d-1} . They obtained for $r > s_{w}/p$,

$$e_n^{\det}(BB_{\tau}^r(L_{p,w}(\mathbb{S}^{d-1}))) \simeq n^{-\frac{r}{d-1}},$$

where s_w is a critical index for the A_{∞} weight w. This generalized the unweighted result of [18]. Meanwhile, they also obtained the corresponding results for the weighted Besov classes on the unit ball and on the standard simplex of the Euclidean space \mathbb{R}^d .

The above results indicate that randomized algorithms effectively improve the optimal rate of convergence in many cases. There is a vast literature of integration problems in the deterministic and randomized case settings, see for example, [6, 10, 11, 12, 13, 15, 25, 26, 34, 36, 43]. However, as far as we know, there are few results about integration problem on the unite ball \mathbb{B}^d in the randomized case setting.

Let $\mathbb{B}^d = \{x \in \mathbb{R}^d \mid |x| \leq 1\}$ be the unit ball of \mathbb{R}^d , where $x \cdot y$ is the usual inner product and $|x| = (x \cdot x)^{1/2}$ is the usual Euclidean norm. We denote by $L_{p,\mu} \equiv L_{p,\mu}(\mathbb{B}^d)$, 0 , the space of all measurable functions with finite quasi-norm

$$||f||_{p,\mu} := \left(\int_{\mathbb{R}^d} |f(x)|^p w_{\mu}(x) dx\right)^{1/p},$$

where $w_{\mu}(x) = b_d^{\mu}(1 - |x|^2)^{\mu - 1/2}$, $\mu \geq 0$ is the classical Jacobi weight on \mathbb{B}^d , normalized by $\int_{\mathbb{B}^d} w_{\mu}(x) dx = 1$. When $p = \infty$ we consider the space of continuous functions $C(\mathbb{B}^d)$ with the uniform norm. Let $BW_{p,\mu}^r$ and $BB_{\tau}^r(L_{p,\mu})$, $1 \leq p \leq \infty$, r > 0, $0 < \tau \leq \infty$, denote the weighted Sobolev class and the weighted Besov class on \mathbb{B}^d , respectively (see the precise definitions in Section 2). We remark that if $r > (d+2\mu)/p$, then the spaces $W_{p,\mu}^r$ and $B_{\tau}^r(L_{p,\mu})$ are compactly embedded into the space of continuous functions $C(\mathbb{B}^d)$.

This paper is concerned with numerical integration on \mathbb{B}^d

(1.2)
$$\operatorname{INT}_{d}(f) = \int_{\mathbb{R}^{d}} f(x)w_{\mu}(x)dx.$$

For the weighted Besov class $BB_{\tau}^{r}(L_{p,\mu})$, $1 \leq p \leq \infty$, $0 < \tau \leq \infty$, and $r > (d+2\mu)/p$, it follows from [6] that

(1.3)
$$e_n^{\det}(BB_{\tau}^r(L_{p,\mu})) \approx n^{-\frac{r}{d}}.$$

For the weighted Sobolev class $BW_{p,\mu}^r$, $1 \le p \le \infty$, $r > (d+2\mu)/p$, we obtain the similar result as follows.

Theorem 1.1. Let $1 \le p \le \infty$ and $r > (d + 2\mu)/p$. Then we have

(1.4)
$$e_n^{\det}(BW_{p,\mu}^r) \approx n^{-\frac{r}{d}}.$$

From (1.3) and (1.4), we know that the integration problems (1.2) for the weighted Besov class $BB_{\tau}^{r}(L_{p,\mu})$ and the weighted Sobolev class $BW_{p,\mu}^{r}$ is "intractable" in the deterministic setting if d is much larger than r. So it is natural to ask whether randomness improves the order of convergence. In this paper we

investigate randomized quadrature for $BW_{p,\mu}^r$ and $BB_{\tau}^r(L_{p,\mu})$. We obtain their sharp asymptotic orders of quadrature errors in the randomized case setting, and find that randomized algorithms provide a faster rate than that of deterministic ones for p > 1. Our main results can formulated as follows.

Theorem 1.2. Let $1 \le p \le \infty$, $0 < \tau \le \infty$, and $r > (d+2\mu)/p$. Then we have (1.5) $e_n^{\text{ran}}(BX_p^r) \approx n^{-\frac{r}{d} - \frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_+},$

where X_p^r denotes $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$.

Remark 1.3. We compare the results in the deterministic and randomized case settings. For p=1, the order of convergence is the same, which means that randomness does not help for p=1. Randomness does help for $1 . Indeed, randomness improve the order of convergence by a factor <math>n^{1-1/p}$ for $1 and <math>n^{1/2}$ for $2 \le p \le \infty$.

The organization of the paper is the following. Section 2 presents some facts about harmonic analysis on the ball. In Section 3 we use the filtered hyperinterpolation operators to approximate functions in $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$, and show that the filtered hyperinterpolation operators are asymptotically optimal algorithms in the sense of optimal recovery in some cases. We also give the proof of Theorem 1.1. Section 4 and Section 5 are devoted to proving the upper and lower estimates of the quantities $e_n^{\rm ran}(BX_p^r)$ as in Theorem 1.2, respectively.

2. Preliminaries

This section is devoted to give some basic knowledge about harmonic analysis on the unit ball \mathbb{B}^d .

For the classical Jacobi weight

$$w_{\mu}(x) = b_d^{\mu} (1 - |x|^2)^{\mu - 1/2}, \ \mu \ge 0, \ b_d^{\mu} = \left(\int_{\mathbb{R}^d} (1 - |x|^2)^{\mu - 1/2} dx \right)^{-1},$$

on \mathbb{B}^d , denote by $L_{p,\mu} \equiv L_{p,\mu}(\mathbb{B}^d)$ (0 the space of all Lebesgue measurable functions <math>f on \mathbb{B}^d with the finite quasi-norm

$$||f||_{p,\mu} := \left(\int_{\mathbb{B}^d} |f(x)|^p w_{\mu}(x) dx\right)^{1/p}.$$

And when $p = \infty$ we consider the space of continuous functions $C(\mathbb{B}^d)$ with the uniform norm. In particular, $L_{2,\mu}$ is a Hilbert space with inner product

$$\langle f, g \rangle_{\mu} := \int_{\mathbb{R}^d} f(x)g(x)w_{\mu}(x)dx, \text{ for } f, g \in L_{2,\mu}.$$

Let Π_n^d be the space of all polynomials in d variables of total degree at most n. We denote by $\mathcal{V}_n^d(w_\mu)$ the space of all polynomials of degree n which are orthogonal to lower degree polynomials in $L_{2,\mu}$. It is well known (see [7, p.38 or p.229]) that the spaces $\mathcal{V}_n^d(w_\mu)$ are just the eigenspaces corresponding to the eigenvalues $-n(n+2\mu+d-1)$ of the second-order differential operator

$$D_{\mu} := \triangle - (x \cdot \nabla)^2 - (2\mu + d - 1)x \cdot \nabla,$$

where \triangle and ∇ are the Laplace operator and gradient operator, respectively. More precisely,

$$D_{\mu}P = -n(n+2\mu+d-1)P$$
, for all $P \in \mathcal{V}_n^d(w_{\mu})$.

It is easy to see that the spaces $\mathcal{V}_n^d(w_\mu)$ are mutually orthogonal in $L_{2,\mu}$. Let $\{\phi_{nk} \equiv \phi_{nk}^d : k = 1, 2, \dots, a_n^d\}$ be a fixed orthonormal basis for $\mathcal{V}_n^d(w_\mu)$, where $a_n^d := \dim \mathcal{V}_n^d(w_\mu)$. Then

$$\{\phi_{nk}: k=1,2,\ldots,a_n^d, n=0,1,2,\ldots\}$$

is an orthonormal basis for $L_{2,\mu}$. The orthogonal projector $\operatorname{Proj}_n: L_{2,\mu} \to \mathcal{V}_n^d(w_\mu)$ can be written as

$$(\operatorname{Proj}_n f)(x) = \sum_{k=1}^{a_n^d} \langle f, \phi_{nk} \rangle \phi_{nk}(x) = \langle f, P_n(w_\mu; x, \cdot) \rangle_{\mu},$$

where $P_n(w_\mu; x, y) = \sum_{k=1}^{a_n^u} \phi_{nk}(x)\phi_{nk}(y)$ is the reproducing kernel of $\mathcal{V}_n^d(w_\mu)$. See [41] for more details about $P_n(w_\mu; x, y)$.

For r > 0, we define the fractional power $(-D_{\mu})^{r/2}$ of the operator $-D_{\mu}$ on fby

$$(-D_{\mu})^{r/2}f := \sum_{k=0}^{\infty} (k(k+2\mu+d-1))^{r/2} \operatorname{Proj}_k f,$$

in the sense of distribution. By [42], we have for any $P \in \Pi_n^d$,

Given r > 0 and 1 , we define the weighted Sobolev space by

$$W_{p,\mu}^r \equiv W_{p,\mu}^r(\mathbb{B}^d) := \{ f \in L_{p,\mu} \mid ||f||_{W_{p,\mu}^r} := ||f||_{p,\mu} + ||(-D_\mu)^{r/2} f||_{p,\mu} < \infty \},$$

while the weighted Sobolev class $BW_{p,\mu}^r$ is defined to be the unit ball of the weighted Sobolev space $W_{p,\mu}^r$. We remark that if $r > (d+2\mu)/p$, then $W_{p,\mu}^r$ is compactly embedded into $C(\mathbb{B}^d)$.

Let $\eta \in C^{\infty}[0,+\infty)$ (a " C^{∞} -filter") satisfy

$$\chi_{[0,1]} \le \eta \le \chi_{[0,2]}$$
.

Here, χ_A denotes the characteristic function of A for $A \subset \mathbb{R}$. For $L \in \mathbb{N}$, we define the filtered polynomial operator by

(2.2)
$$V_L(f)(x) \equiv V_{L,\eta}(f)(x) := \sum_{k=0}^{\infty} \eta(\frac{k}{L}) \operatorname{Proj}_k(f)(x) = \langle f, K_{L,\eta}(x, \cdot) \rangle_{\mu},$$

where $f \in L_{1,\mu}$, and

(2.3)
$$K_{L,\eta}(x,y) = \sum_{k=0}^{\infty} \eta(\frac{k}{L}) P_k(w_{\mu}; x, y), \ x, y \in \mathbb{B}^d.$$

Then the following properties hold (see, for example, [28])):

- (a) $V_L(f) \in \Pi_{2L-1}^d$ for any $f \in L_{1,\mu}$;
- (b) $P = V_L(P)$ for any $P \in \Pi_L^d$; (c) $||V_L|| := ||V_L||_{(\infty,\infty)} = ||V_L||_{(1,1)} = \sup_{x \in \mathbb{B}^d} ||K_{L,\eta}(x,\cdot)||_{1,\mu} \lesssim 1$; (d) $||V_L||_{(p,p)} \le ||V_L|| \lesssim 1$ for $1 \le p \le \infty$; (e) $||f V_L(f)||_{p,\mu} \le (1 + ||V_L||_{(p,p)}) E_L(f)_{p,\mu} \lesssim E_L(f)_{p,\mu}$ for $1 \le p \le \infty$,
- where

$$||A||_{(p,p)} := \sup_{||f||_{p,\mu} \le 1} ||Af||_{p,\mu}$$

is the operator norm of a linear operator A on $L_{p,\mu}$, and $E_L(f)_{p,\mu}$ is the best approximation of $f \in L_{p,\mu}$ from Π_L^d defined by

$$E_L(f)_{p,\mu} := \inf_{P \in \Pi_r^d} ||f - P||_{p,\mu}.$$

We note that property (c) is essential.

Remark 2.1. Let η be a filter, i.e., η is a continuous function satisfying $\chi_{[0,1]} \leq \eta \leq \chi_{[0,2]}$. We may weaken the smoothness condition on η such that the operator norms $\|V_{L,\eta}\|$ are uniformly bounded. Wang and Sloan investigated the corresponding problem on the sphere, and gave the compact condition on η for which the operator norms of the filtered polynomial operators $V_{L,\eta}^{\mathbb{S}}$ on the sphere are uniformly bounded. Following the way in [38], Li obtained in [22] that the operator norms $\|V_{L,\eta}\|$ are uniformly bounded whenever $\eta \in W^{\lfloor \frac{d+2\mu+1}{2} \rfloor}BV$, where $W^rBV[a,b]$ denotes the set of all continuous functions η on [a,b] for which $\eta^{(r-1)}$ is absolutely continuous and $\eta_+^{(r)}$ and $\eta_-^{(r)}$ exist and are of bounded variation on [a,b] for $r \in \mathbb{N}$. Hence, if $\eta \in W^{\lfloor \frac{d+2\mu+1}{2} \rfloor}BV$, then properties (a)-(e) hold. Note that

$$C^{r+1}[a,b] \subset W^r BV[a,b] \subset C[a,b]$$
, for any $r \in \mathbb{N}$.

The condition $\eta \in W^{\lfloor \frac{d+2\mu+1}{2} \rfloor}BV$ is compact, since there exists an $\eta \in W^{\lfloor \frac{d+2\mu-1}{2} \rfloor}BV$ such that $\|V_{L,\eta}\|$ are not uniformly bounded.

Now we define weighted Besov spaces on the ball. Given $1 \le p \le \infty$, r > 0, and $0 < \tau \le \infty$, we define the weighted Besov space $B_{\tau}^{r}(L_{p,\mu})$ to be the space of all real functions f with quasi-norm

$$||f||_{B^r_{\tau}(L_{p,\mu})} := \begin{cases} ||f||_{p,\mu} + \Big(\sum_{j=0}^{\infty} 2^{jr\tau} E_{2^j}(f)_{p,\mu}^{\tau}\Big)^{1/\tau}, & 0 < \tau < \infty, \\ ||f||_{p,\mu} + \sup_{j \ge 0} 2^{jr} E_{2^j}(f)_{p,\mu}, & \tau = \infty, \end{cases}$$

while the weighted Besov class $BB_{\tau}^{r}(L_{p,\mu})$ is defined to be the unit ball of the weighted Besov space $B_{\tau}^{r}(L_{p,\mu})$.

There are other definitions of the weighted Besov spaces which are equivalent (see [20, Proposition 5.7]). We remark that if $1 \leq p, q \leq \infty$, $r > (d + 2\mu)/p$, then $B_{\tau}^{r}(L_{p,\mu})$ is compactly embedded into $C(\mathbb{B}^{d})$, and for $1 \leq p \leq \infty$, r > 0, $0 < \tau_{1} \leq \tau_{2} \leq \infty$,

$$B_{\tau_1}^r(L_{p,\mu}) \subset B_{\tau_2}^r(L_{p,\mu}) \subset B_{\infty}^r(L_{p,\mu}).$$

It follows from the Jackson inequality (see [42]) that for $f \in W_{p,\mu}^r$, $1 \le p \le \infty$, r > 0,

(2.4)
$$E_n(f)_{p,\mu} \lesssim n^{-r} ||f||_{W_{p,\mu}^r}.$$

This means that

$$W_{p,\mu}^r \subset B_{\infty}^r(L_{p,\mu}).$$

It can be seen that for $f \in B_{\tau}^{r}(L_{p,\mu}), 0 < \tau \leq \infty$,

(2.5)
$$E_n(f)_{p,\mu} \le 2^r n^{-r} ||f||_{B^r_{\tau}(L_{p,\mu})}.$$

We introduce a metric ρ on \mathbb{B}^d :

$$\rho(x,y) := \arccos\left((x,y) + \sqrt{1 - |x|^2}\sqrt{1 - |y|^2}\right).$$

For r > 0, $x \in \mathbb{B}^d$ and a positive integer n, we set

$$\mathbf{B}_{\rho}(x,r) := \{ y \in \mathbb{B}^d \mid \rho(x,y) \le r \}.$$

For $\varepsilon \in (0,1)$, we say that a finite subset $\Lambda \subset \mathbb{B}^d$ is maximal ε -separated if

$$\mathbb{B}^d \subset \bigcup_{\omega \in \Lambda} \mathbf{B}_{\rho}(\omega, \varepsilon) \text{ and } \min_{\omega \neq \omega'} \rho(\omega, \omega') \geq \varepsilon.$$

Note that such a maximal ε -separated set Λ exists and $\#\Lambda \simeq \varepsilon^{-d}$, where #A denotes the number of elements of a set A (see [28, Lemma 5.2]).

Finally, we give the Nikolskii inequalities on \mathbb{B}^d .

Lemma 2.2. ([20, Proposition 2.4]) Let $1 \le p, q \le \infty$ and $\mu \ge 0$. Then for any $P \in \Pi_n^d$ we have,

(2.6)
$$||P||_{q,\mu} \lesssim n^{(d+2\mu)(1/p-1/q)_+} ||P||_{p,\mu}.$$

3. FILTERED HYPERINTERPOLATION ON THE BALL

Let η be a filter such that properties (a)-(e) hold. We want to approximate the inner product integral (2.2) of $V_{L,\eta}(f)(x)$ by a positive quadrature rule of polynomial degree 3L. Following [30], we shall call the resulting operator "filtered hyperinterpolation".

For this purpose, we need positive quadrature rules on \mathbb{B}^d . For $L \in \mathbb{N}$, we assume that $Q_L(f) := \sum_{\omega \in \Lambda_L} \lambda_\omega f(\omega)$ is a positive quadrature rule on \mathbb{B}^d which is exact for $f \in \Pi^d_{3L}$, i.e., Λ_L is a finite subset of \mathbb{B}^d with $\#\Lambda_L \times L^d$, weights $\lambda_\omega > 0$, $\omega \in \Lambda_L$, satisfy, for all $P \in \Pi^d_{3L}$,

$$\int_{\mathbb{B}^d} P(x)w_{\mu}(x)dx = Q_L(P) = \sum_{\omega \in \Lambda_L} \lambda_{\omega} P(\omega).$$

Such positive quadrature rules exist. Indeed, for $L \in \mathbb{N}$, it follows from [7, Theorem 11.6.5] that for a given maximal δ/L -separated subset Λ_L of \mathbb{B}^d with $\delta \in (0, \delta_0)$ for some $\delta_0 > 0$, there exists a positive quadrature formula

$$\int_{\mathbb{B}^d} f(x) w_{\mu}(x) dx \approx Q_L(f) := \sum_{\omega \in \Lambda_L} \lambda_{\omega} f(\omega), \ \lambda_{\omega} > 0,$$

with $\lambda_{\omega} \simeq \int_{\mathbf{B}_{\sigma}(y,\delta/L)} w_{\mu}(x) dx$ and $\#\Lambda_{L} \simeq L^{d}$, which is exact for Π_{3L}^{d} .

For the above positive quadrature formula Q_L , we can define the discreted inner product $\langle \cdot, \cdot \rangle_{Q_L}$ on $C(\mathbb{B}^d)$ by

$$\langle f, g \rangle_{Q_L} := Q_L(fg) = \sum_{\omega \in \Lambda_L} \lambda_\omega f(\omega) g(\omega).$$

The filtered hyperinterpolation operator is defined by

(3.1)
$$G_L(f)(x) := \langle f, K_{L,\eta}(x,\cdot) \rangle_{Q_L} = \sum_{\omega \in \Lambda_L} \lambda_\omega f(\omega) K_{L,\eta}(x,\omega).$$

From [40] we know that

(3.2)
$$||G_L|| = \sup_{x \in \mathbb{B}^d} \sum_{\omega \in \Lambda_L} \lambda_\omega |K_{L,\eta}(x,\omega)| \lesssim 1.$$

The following theorem plays an important role in the proof of upper estimates.

Theorem 3.1. Let $1 \le p, q \le \infty$, $r > (d+2\mu)/p$, $0 < \tau \le \infty$, and G_L be given as in (3.1). Then for all $f \in X_p^r$, we have

(3.3)
$$||f - G_L(f)||_{q,\mu} \lesssim L^{-r + (d+2\mu)(\frac{1}{p} - \frac{1}{q})_+} ||f||_{X_p^r},$$

where X_p^r denotes $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$.

Remark 3.2. When $1 \le p = q \le \infty$ and $X_p^r = W_{p,\mu}^r$, (3.3) was obtained by Li in [22].

In order to prove Theorem 3.1, we need the following two lemmas.

Lemma 3.3. ([40, Theorem 3.1] and [21, Lemma 2.2]) Suppose that $n \in \mathbb{N}$, Ω_n is a finite subset of \mathbb{B}^d , and $\{\mu_\omega : \omega \in \Omega_n\}$ is a set of positive numbers. If there exists a $p_0 \in (0, \infty)$ such that for any $f \in \Pi_n^d$,

(3.4)
$$\sum_{\omega \in \Omega_n} \mu_{\omega} |f(\omega)|^{p_0} \lesssim \int_{\mathbb{B}^d} |f(x)|^{p_0} w_{\mu}(x) dx,$$

then the following regularity condition

(3.5)
$$\sum_{\omega \in \Omega_n \cap \mathbf{B}_{\rho}(y, \frac{1}{n})} \mu_{\omega} \lesssim \int_{\mathbf{B}_{\rho}(y, \frac{1}{n})} w_{\mu}(x) dx, \text{ for any } y \in \mathbb{B}^d,$$

holds.

Conversely, if the regularity condition (3.5) holds, then for any $1 \leq p < \infty$, $m \in \mathbb{N}$, $m \geq n$, $f \in \Pi_m^d$, we have

(3.6)
$$\sum_{\omega \in \Omega_n} \mu_{\omega} |f(\omega)|^p \lesssim \left(\frac{m}{n}\right)^{d+2\mu} \int_{\mathbb{B}^d} |f(x)|^p w_{\mu}(x) dx.$$

Lemma 3.4. Let $1 \leq p \leq \infty$, and $L \in \mathbb{N}$. Then for any $N \geq L$ and $P \in \Pi_N^d$, we have

(3.7)
$$||G_L(P)||_{p,\mu} \lesssim \left(\frac{N}{L}\right)^{\frac{d+2\mu}{p}} ||P||_{p,\mu}.$$

Proof. Our proof will be divided into three cases.

Case 1: $p = \infty$.

In this case, by (3.2) we have for $N \geq L$ and $P \in \Pi_N^d$,

$$||G_L(P)||_{\infty} \le ||G_L|| ||P||_{\infty} \le ||P||_{\infty}.$$

Case 2: p = 1.

In this case, since Q_L is a positive quadrature rule which is exact for Π_{3L}^d , then (3.4) is true for $\{\lambda_\omega\}_{\omega\in\Lambda_L}$ with $p_0=2$. This leads that the regular condition (3.5) holds. By property (c) and Lemma 3.3 we obtain for $N\geq L$ and $P\in\Pi_N^d$,

$$\begin{aligned} \|G_L(P)\|_{1,\mu} &= \|\sum_{\omega \in \Lambda_L} \lambda_\omega P(\omega) K_{L,\eta}(\cdot,\omega)\|_{1,\mu} \\ &\leq \sum_{\omega \in \Lambda_L} \lambda_\omega |P(\omega)| \|K_{L,\eta}(\cdot,\omega)\|_{1,\mu} \\ &\lesssim \sum_{\omega \in \Lambda_L} \lambda_\omega |P(\omega)| \lesssim (\frac{N}{L})^{d+2\mu} \|P\|_{1,\mu}. \end{aligned}$$

Case 3: 1 .

In this case, for $N \geq L$ and $P \in \Pi_N^d$, by the Hölder inequality, (3.2), property (c), and Lemma 3.3, we obtain

$$\begin{aligned} \|G_{L}(P)\|_{p,\mu}^{p} &= \int_{\mathbb{B}^{d}} \left| \sum_{\omega \in \Lambda_{L}} \lambda_{\omega} P(\omega) K_{L,\eta}(x,\omega) \right|^{p} w_{\mu}(x) dx \\ &\leq \int_{\mathbb{B}^{d}} \left(\sum_{\omega \in \Lambda_{L}} \lambda_{\omega} |P(\omega)| |K_{L,\eta}(x,\omega)| \right)^{p} w_{\mu}(x) dx \\ &\leq \int_{\mathbb{B}^{d}} \left(\sum_{\omega \in \Lambda_{L}} \lambda_{\omega} |P(\omega)|^{p} |K_{L,\eta}(x,\omega)| \right) \left(\sum_{\omega \in \Lambda_{L}} \lambda_{\omega} |K_{L,\eta}(x,\omega)| \right)^{p-1} w_{\mu}(x) dx \\ &\leq \left(\sum_{\omega \in \Lambda_{L}} \lambda_{\omega} |P(\omega)|^{p} \|K_{L,\eta}(\cdot,\omega)\|_{1,\mu} \right) \left(\sup_{x \in \mathbb{B}^{d}} \sum_{\omega \in \Lambda_{L}} \lambda_{\omega} |K_{L,\eta}(x,\omega)| \right)^{p-1} \\ &\lesssim \sum_{\omega \in \Lambda_{L}} \lambda_{\omega} |P(\omega)|^{p} \lesssim \left(\frac{N}{L} \right)^{d+2\mu} \|P\|_{p,\mu}^{p}, \end{aligned}$$

which proves (3.7).

Lemma 3.4 is proved.

Now we turn to prove Theorem 3.1.

Proof of Theorem 3.1.

Since X_p^r is continuously embedded into $B_{\infty}^r(L_{p,\mu})$, it suffices to show Theorem 3.1 for $B_{\infty}^r(L_{p,\mu})$.

Suppose that m is an integer satisfying $2^m \le L < 2^{m+1}$. Let $1 \le p, q \le \infty, r > (d+2\mu)/p$, and $f \in B^r_\infty(L_{p,\mu})$. Define $g_{2^k} \in \Pi^d_{2^k}$ by

$$E_{2^k}(f)_{p,\mu} = ||f - g_{2^k}||_{p,\mu},$$

and let $f_k = g_{2^k} - g_{2^{k-1}}$ for $k \ge 0$, where we set $g_{2^{-1}} = 0$. Note that since $\Pi_{2^k}^d$ are the finite dimensional linear spaces, the best approximant polynomials g_{2^k} always exist (see, for instance, [23, p. 17, Theorem 1]). It can be seen that $f_k \in \Pi_{2^k}^d$, and

the series $\sum_{k=j+1}^{\infty} f_k$ converges to $f - g_{2^j}$ in the uniform norm for each $j \ge -1$. We have

$$||f - G_L(f)||_{q,\mu} = ||\sum_{k=m+1}^{\infty} (f_k - G_L(f_k))||_{q,\mu}$$

$$\leq \sum_{k=m+1}^{\infty} ||f_k - G_L(f_k)||_{q,\mu}$$

$$\leq \sum_{k=m+1}^{\infty} ||f_k||_{q,\mu} + \sum_{k=m+1}^{\infty} ||G_L(f_k)||_{q,\mu}.$$
(3.8)

It follows from the Nikolskii inequality (2.6) and (2.5) we have

$$||f_{k}||_{q,\mu} \lesssim 2^{k(d+2\mu)(\frac{1}{p}-\frac{1}{q})_{+}} ||f_{k}||_{p,\mu}$$

$$\leq 2^{k(d+2\mu)(\frac{1}{p}-\frac{1}{q})_{+}} (||f-g_{2^{k}}||_{p,\mu} + ||f-g_{2^{k-1}}||_{p,\mu})$$

$$\lesssim 2^{k(d+2\mu)(\frac{1}{p}-\frac{1}{q})_{+}} E_{2^{k-1}}(f)_{p,\mu}$$

$$\lesssim 2^{-k(r-(d+2\mu)(\frac{1}{p}-\frac{1}{q})_{+})} ||f||_{B_{\infty}^{r}(L_{p,\mu})}.$$

$$(3.9)$$

This means that

$$\sum_{k=m+1}^{\infty} \|f_k\|_{q,\mu} \lesssim \sum_{k=m+1}^{\infty} 2^{-k(r-(d+2\mu)(\frac{1}{p}-\frac{1}{q})_+)} \|f\|_{B^r_{\infty}(L_{p,\mu})}$$

$$\approx 2^{-m(r-(d+2\mu)(\frac{1}{p}-\frac{1}{q})_+)} \|f\|_{B^r_{\infty}(L_{p,\mu})}$$

$$\approx L^{-r+(d+2\mu)(\frac{1}{p}-\frac{1}{q})_+} \|f\|_{B^r_{\infty}(L_{p,\mu})}.$$
(3.10)

Applying (3.9) and Lemma 3.4, we get for $k \geq m+1$,

$$\begin{split} \|G_L(f_k)\|_{q,\mu} &\lesssim L^{(d+2\mu)(\frac{1}{p}-\frac{1}{q})_+} \|G_L(f_k)\|_{p,\mu} \\ &\lesssim L^{(d+2\mu)(\frac{1}{p}-\frac{1}{q})_+} (\frac{2^k}{L})^{\frac{d+2\mu}{p}} \|f_k\|_{p,\mu} \\ &\lesssim L^{(d+2\mu)(\frac{1}{p}-\frac{1}{q})_+} (\frac{2^k}{L})^{\frac{d+2\mu}{p}} 2^{-kr} \|f\|_{B^r_{\infty}(L_{p,\mu})}. \end{split}$$

It follows that

$$\sum_{k=m+1}^{\infty} \|G_L(f_k)\|_{q,\mu} \lesssim \sum_{k=m+1}^{\infty} L^{(d+2\mu)(\frac{1}{p}-\frac{1}{q})} \left(\frac{2^k}{L}\right)^{\frac{d+2\mu}{p}} 2^{-kr} \|f\|_{B^r_{\infty}(L_{p,\mu})} \\
\lesssim L^{(d+2\mu)(\frac{1}{p}-\frac{1}{q})} \sum_{k=m+1}^{\infty} 2^{-k(r-\frac{d+2\mu}{p})} \|f\|_{B^r_{\infty}(L_{p,\mu})} \\
\lesssim L^{-r+(d+2\mu)(\frac{1}{p}-\frac{1}{q})} \|f\|_{B^r_{\infty}(L_{p,\mu})}.$$
(3.11)

Hence, for $f \in B_{\infty}^{r}(L_{p,\mu})$, by (3.8), (3.10), and (3.11) we have

$$||f - G_L(f)||_{q,\mu} \lesssim L^{-r + (d+2\mu)(\frac{1}{p} - \frac{1}{q})_+} ||f||_{B^r_{\infty}(L_{p,\mu})}.$$

This completes the proof of Theorem 3.1.

Next we show that the filtered hyperinterpolation operators G_L are order optimal in sense of the optimal recovery in some cases. Let F_d be a class of continuous functions on D_d , and $(X, \|\cdot\|_X)$ be a normed linear space of functions on D_d , where D_d is a subset of \mathbb{R}^d . For $n \in \mathbb{N}$, the sampling numbers (or the optimal recovery) of F_d in X are defined by

$$g_n(F_d, X) := \inf_{\substack{\xi_1, \dots, \xi_n \in D_d \\ \varphi \colon \mathbb{R}^n \to X}} \sup_{f \in F_d} \|f - \varphi(f(\xi_1), \dots, f(\xi_n))\|_X,$$

where the infimum is taken over all n points ξ_1, \ldots, ξ_n in D_d and all mappings φ from \mathbb{R}^n to X. If in the above definition, the infimum is taken over all n points ξ_1, \ldots, ξ_n in D_d and all linear mappings φ from \mathbb{R}^n to X, we obtain the definition of the linear sampling numbers $g_n^{\text{lin}}(F_d, X)$.

It is well known (see [33]) that for a balanced convex set F_d ,

(3.12)
$$g_n^{\text{lin}}(F_d, X) \ge g_n(F_d, X) \ge \inf_{\substack{\xi_1, \dots, \xi_n \in D_d \\ f(\xi_1) = \dots = f(\xi_n) = 0}} \sup_{\substack{f \in F_d \\ f(\xi_1) = \dots = f(\xi_n) = 0}} ||f||_X.$$

Theorem 3.5. Let $1 \le q \le p \le \infty$, $0 < \tau \le \infty$, and $r > (d+2\mu)/p$. Then we have

(3.13)
$$g_n^{\text{lin}}(BX_p^r, L_{q,\mu}) \simeq g_n(BX_p^r, L_{q,\mu}) \simeq n^{-r/d},$$

where X_p^r denotes $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$.

In order to prove Theorem 3.5 we need the following lemma.

Lemma 3.6. ([6, Proposition 4.8]) Let X be a linear subspace of Π_N^d with dim $X \ge \varepsilon \dim \Pi_N^d$ for some $\varepsilon \in (0,1)$. Then there exists a function $f \in X$ such that $||f||_{p,\mu} \times 1$ for all 0 .

Proof of Theorem 3.5.

Without loss of generality we assume that n is sufficiently large. We choose $L \in \mathbb{N}$ such that

$$\#\Lambda_L \leq n \text{ and } \#\Lambda_L \approx n.$$

It follows from the definition of G_L and $g_n^{\text{lin}}(BX_p^r, L_{q,\mu})$ that

$$g_n^{\text{lin}}(BX_p^r, L_{q,\mu}) \le \sup_{f \in BX_n^r} \|f - G_L(f)\|_{q,\mu}.$$

By Theorem 3.1 and the above inequlity we have

(3.14)
$$g_n(BX_p^r, L_{q,\mu}) \le g_n^{\text{lin}}(BX_p^r, L_{q,\mu}) \lesssim L^{-r} \times n^{-r/d}$$

Now we show the lower bound. Let ξ_1, \ldots, ξ_n be any n distinct points on \mathbb{B}^d . Take a positive integer N such that $2n \leq \dim \Pi_N^d \leq Cn$, and denote

$$X_0 := \{ g \in \Pi_N^d \mid g(\xi_j) = 0 \text{ for all } j = 1, \dots, n \}.$$

Thus, X_0 is a linear subspace of Π_N^d with

$$\dim X_0 \ge \dim \Pi_N^d - n \ge \frac{1}{2} \dim \Pi_N^d.$$

It follows from Lemma 3.6 that there exists a function $g_0 \in X_0$ such that

$$||g_0||_{p_0,\mu} \approx 1$$
, for all $0 < p_0 \le \infty$.

Let $f_0(x) = N^{-r}(g_0(x))^2$ and $m \in \mathbb{N}$ such that $2^{m-1} \leq N < 2^m$. Then by the fact that $E_{2^j}(f_0)_{p,\mu} \leq \|f_0\|_{p,\mu}$ we have

$$||f_0||_{B_{\tau}^r(L_{p,\mu})} = ||f_0||_{p,\mu} + \Big(\sum_{j=0}^{m+1} 2^{jr\tau} E_{2^j}(f_0)_{p,\mu}^{\tau}\Big)^{1/\tau}$$

$$\lesssim \Big(\sum_{j=0}^{m+1} 2^{jr\tau}\Big)^{1/\tau} ||f_0||_{p,\mu} \lesssim 2^{mr} N^{-r} ||g_0||_{2p,\mu}^2 \lesssim 1.$$

By the fact that $f_0 \in \Pi_{2N}^d$ and (2.1) we have

$$(3.15) ||f_0||_{W^r_{p,\mu}} = ||f_0||_{p,\mu} + ||(-D_\mu)^{r/2} f_0||_{p,\mu} \lesssim N^r ||f_0||_{p,\mu} = ||g_0||_{2p,\mu}^2 \lesssim 1.$$

Hence, there exists a positive constant C such that $f_1 = Cf_0 \in BX_p^r$, and $f_1(\xi_1) = \cdots = f_1(\xi_n) = 0$. It follows from (3.12) that

$$g_n(BX_p^r, L_{q,\mu}) \ge \inf_{\xi_1, \dots, \xi_n \in D_d} \|f_1\|_{q,\mu} \gtrsim N^{-r} \inf_{\xi_1, \dots, \xi_n \in D_d} \|g_0\|_{2q,\mu}^2 \asymp N^{-r} \asymp n^{-r/d},$$

which combining with (3.14), gives (3.13).

The proof of Theorem 3.5 is finished.

Remark 3.7. It follows from the proof of Theorem 3.5 that for $1 \le q \le p \le \infty$, $0 < \tau \le \infty$, and $r > (d+2\mu)/p$,

$$g_n(BX_p^r, L_{q,\mu}) \simeq n^{-r/d} \simeq \sup_{f \in BX_n^r} \|f - G_L(f)\|_{q,\mu},$$

where X_p^r denotes $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$. This implies that the filtered hyperinterpolation operators are asymptotically optimal algorithms in the sense of optimal recovery for $1 \le q \le p \le \infty$.

Finally we prove Theorem 1.1.

Proof of Theorem 1.1.

Since $||f||_{B^r_{\infty}(L_{p,\mu})} \lesssim ||f||_{W^r_{p,\mu}}$ for $f \in W^r_{p,\mu}$, by (1.3) we obtain for $1 \leq p \leq \infty$ and $r > (d+2\mu)/p$,

$$e_n^{\det}(BW_{p,\mu}^r) \lesssim e_n^{\det}(BB_{\infty}^r(L_{p,\mu})) \lesssim n^{-r/d}.$$

Now we prove the lower bound. Let ξ_1, \ldots, ξ_n be any n distinct points on \mathbb{B}^d . Take a positive integer N such that $2n \leq \dim \Pi_N^d \leq Cn$. According to the proof of Theorem 3.5, there exists a function $f_1(x)$ such that

$$f_1 \in BW_{p,\mu}^r$$
, $f_1(\xi_1) = \dots = f_1(\xi_n) = 0$, $f_1(x) \ge 0$,

and

$$||f_1||_{1,\mu} = \int_{\mathbb{R}^d} f_1(x) w_{\mu}(x) dx \asymp N^{-r}.$$

It follows from [33] that for $BW^r_{p,\mu}$ which is a balanced convex set,

$$e_n^{\det}(BW_{p,\mu}^r) \ge \inf_{\substack{\xi_1,\dots,\xi_n \in \mathbb{B}^d \\ f(\xi_1)=\dots=f(\xi_n)=0}} \sup_{\substack{f \in F_d \\ f(\xi_1)=\dots=f(\xi_n)=0}} \left| \int_{\mathbb{B}^d} f(x)w_{\mu}(x)dx \right|$$
$$\ge \inf_{\substack{\xi_1,\dots,\xi_n \in \mathbb{B}^d \\ \xi_1,\dots,\xi_n \in \mathbb{B}^d}} \left| \int_{\mathbb{B}^d} f_1(x)w_{\mu}(x)dx \right| \gtrsim N^{-r} \times n^{-r/d}.$$

The proof of Theorem 1.1 is finished.

4. The upper estimates

This section is devoted to proving the upper estimates of the quantities $e_n^{\rm ran}(X_p^r)$ given as in (1.5). That is, for $1 \le p \le \infty$, $r > (d+2\mu)/p$, and $0 < \tau \le \infty$,

(4.1)
$$e_n^{\text{ran}}(BX_p^r) \lesssim n^{-\frac{r}{d} - \frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_+},$$

where X_p^r denotes $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$.

For this purpose, we will use the positive quadrature rule and the filtered hyperinterpolation operator to construct an randomized algorithm to attain the upper bounds. Due to Henrich [15], we need a concrete Monte Carlo method by virtue of the standard Monte Carlo algorithm. It is defined as follows: let $\{\xi_i\}_{i=1}^N$ be independent, \mathbb{B}^d -valued, distributed over \mathbb{B}^d with respect to the measure $w_{\mu}(x)dx$ random vectors on probability space (Ω, Σ, ν) . For any $h \in C(\mathbb{B}^d)$, we put

(4.2)
$$Q_N^{\omega}(h) = \frac{1}{N} \sum_{i=1}^N h(\xi_i(\omega)), \ \omega \in \Omega.$$

The following lemma can be drawn in a same way as in [15, Proposition 5.4]. Here we omit the proof.

Lemma 4.1. Let $1 \le p \le \infty$ and $\mu \ge 0$, Then for any $h \in C(\mathbb{B}^d)$, we have

(4.3)
$$\mathbb{E}_{\omega}|\text{INT}_{d}(h) - Q_{N}^{\omega}(h)| \lesssim N^{-\frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_{+}} ||h||_{p,\mu}.$$

Now we prove (4.1).

Proof of (4.1).

Let $n \in \mathbb{N}$. Without loss of generality we assume that n is sufficiently large. Then there exists a positive quadrature rule

$$\int_{\mathbb{B}^d} f(x) w_{\mu}(x) dx \approx Q_L(f) := \sum_{\xi \in \Lambda_L} \lambda_{\xi} f(\xi), \ \lambda_{\xi} > 0,$$

which is exact for Π_{3L}^d , where $\#\Lambda_L \leq n/2$ and $n \approx L^d$.

As in Section 3.1, we can construct the filtered hyperinterpolation operator by

$$G_L(f)(x) = \sum_{\xi \in \Lambda_L} \lambda_{\xi} f(\xi) K_{L,\eta}(x,\xi),$$

where $K_{L,\eta}$ is given as in (2.3). Hence, according to Theorem 3.1 we have for any $f \in X_p^r$, $1 \le p \le \infty$, $r > (d+2\mu)/p$,

$$(4.4) ||f - G_L(f)||_{p,\mu} \lesssim L^{-r} ||f||_{X_n^r}.$$

Let $N = \lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. We define the randomized algorithm (A_n^{ω}) by

$$A_n^{\omega}(f) = Q_N^{\omega}(f - G_L(f)) + INT_d(G_L(f)),$$

where $f \in C(\mathbb{B}^d)$, and Q_N^{ω} is the standard Monte Carlo algorithm given as in (4.2). We also note that

$$INT_d(G_L(f)) = \sum_{\xi \in \Lambda_{\varepsilon}} \lambda_{\xi} f(\xi).$$

Clearly, the algorithm (A_n^{ω}) use only at most $\#\Lambda_L + N \leq n$ function values of f. This means that $(A_n^{\omega}) \in \mathcal{A}_n^{\mathrm{ran}}$. Also the algorithm (A_n^{ω}) is a randomized linear algorithm. It is easy to check that

$$|INT_d(f) - A_n^{\omega}(f)| = |INT_d(g) - Q_N^{\omega}(g)|,$$

where $g = f - G_L(f)$. Combining with (4.3) and (4.4), we obtain for $f \in BX_p^r$, $r > (d + 2\mu)/p$,

$$\begin{split} \mathbb{E}_{\omega}|\text{INT}_{d}(f) - A_{n}^{\omega}(f)| &= \mathbb{E}_{\omega}|\text{INT}_{d}(g) - Q_{N}^{\omega}(g)| \\ &\lesssim N^{-\frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_{+}} \|f - G_{L}(f)\|_{p,\mu} \\ &\lesssim N^{-\frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_{+}} L^{-r} \approx n^{-\frac{r}{d} - \frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_{+}}, \end{split}$$

which leads to

$$e_n^{\text{ran}}(BX_p^r) \le e^{\text{ran}}(BX_p^r, (A_n^{\omega})) \lesssim n^{-\frac{r}{d} - \frac{1}{2} + (\frac{1}{p} - \frac{1}{2}) + \dots}$$

This completes the proof of (4.1).

5. Lower estimates

This section is devoted to proving the lower estimates of the quantities $e_n^{\rm ran}(X_p^r)$ given as in (1.5). That is, for $1 \le p \le \infty$, $r > (d+2\mu)/p$, and $0 < \tau \le \infty$,

(5.1)
$$e_n^{\operatorname{ran}}(BX_n^r) \gtrsim n^{-\frac{r}{d} - \frac{1}{2} + (\frac{1}{p} - \frac{1}{2})_+},$$

where X_p^r denotes $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$. Theorem 1.2 follows from (4.1) and (5.1) immediately.

The proof of (5.1) is based on the idea of Bakhvalov in [1] and Novak in [24, 25].

Lemma 5.1. (See [24, Lemma 3].)

- (a) Let $F \subset L_{1,\mu}$ and ψ_j , j = 1, ..., 4n, with the following conditions:
 - (i) the ψ_j have disjoint supports and satisfy

$$\operatorname{INT_d}(\psi_j) = \int_{\mathbb{R}^d} \psi_j(x) w_\mu(x) dx \ge \delta, \text{ for } j = 1, \dots, 4n.$$

(ii)
$$F_1 := \left\{ \sum_{i=1}^{4n} \alpha_i \psi_j \mid \alpha_j \in \{-1, 1\} \right\} \subset F.$$

Then

$$e_n^{\mathrm{ran}}(F) \ge \frac{1}{2} \delta n^{\frac{1}{2}}.$$

(b) We assume that instead of (ii) in statement (a) the property

(ii')
$$F_2 := \{ \pm \psi_j \mid j = 1, \dots, 4n \} \subset F.$$

Then

$$e_n^{\mathrm{ran}}(F) \ge \frac{1}{4}\delta.$$

By this lemma, we proceed to construct a sequence of functions $\{\psi_j\}_{j=1}^{4n}$ satisfying the conditions of Lemma 5.1 for $F = BX_p^r$, where X_p^r denotes $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$, $1 \le p \le \infty$, $r > (d+2\mu)/p$, $0 < \tau \le \infty$.

For a given $n \in \mathbb{N}$, choose a positive integer m satisfying $n \approx m^d$, $m \geq 6$, and $\{x_j\}_{j=1}^{4n} \subset \mathbf{B}(\mathbf{0}, \frac{2}{3})$ such that

$$\mathbf{B}(x_i, \frac{2}{m}) \bigcap \mathbf{B}(x_j, \frac{2}{m}) = \emptyset, \text{ if } i \neq j,$$

where $\mathbf{B}(\xi, r) = \{x \in \mathbb{B}^d \mid |x - \xi| \le r\}$ for $\xi \in \mathbb{B}^d$ and r > 0.

Let φ be a nonnegative C^{∞} -function on \mathbb{R}^d supported in \mathbb{B}^d and being equal to 1 on $\mathbf{B}(\mathbf{0}, \frac{1}{2})$. We define

$$\varphi_j(x) = \varphi(m(x - x_j)), \ j = 1, \dots, 4n.$$

Clearly,

supp
$$\varphi_j \subset \mathbf{B}(x_j, \frac{1}{m}) \subset \mathbf{B}(\mathbf{0}, \frac{5}{6}), \ j = 1, \dots, 4n,$$

and

(5.2)
$$\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_i = \emptyset, \text{ for } i \neq j.$$

It is easy to verify that

$$(5.3) \|\varphi_j\|_{p,\mu} \asymp \left(\int_{\mathbf{B}(x_j,\frac{1}{m})} |\varphi_j(x)|^p dx\right)^{1/p} = \left(\int_{\mathbf{B}(\mathbf{0},\frac{1}{m})} |\varphi(mx)|^p dx\right)^{1/p} \asymp m^{-d/p}.$$

We set

$$F_0 := \left\{ f_{\boldsymbol{\alpha}} := \sum_{j=1}^{4n} \alpha_j \varphi_j \mid \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_{4n}) \in \mathbb{R}^{4n} \right\}.$$

Then we have the following lemma.

Lemma 5.2. If $f_{\alpha} \in F_0$, then for r > 0, $1 \le p \le \infty$, and $0 < \tau \le \infty$,

where X_p^r denotes $W_{p,\mu}^r$ or $B_{\tau}^r(L_{p,\mu})$, and

$$\|\boldsymbol{\alpha}\|_{\ell_p^{4n}} := \begin{cases} \left(\sum_{j=1}^{4n} |\alpha_j|^p\right)^{1/p}, & 1 \le p < \infty, \\ \max_{1 \le j \le 4n} |\alpha_j|, & p = \infty. \end{cases}$$

Proof. Indeed, for any $f_{\alpha} \in F_0$, it follows from (5.2) and (5.3) that

(5.5)
$$||f_{\alpha}||_{p,\mu} \asymp m^{-d/p} ||\alpha||_{\ell_p^{4n}}.$$

For a positive integer v > r, by the definition of $-D_{\mu}$ and (5.2), it is easy to verify that

$$\operatorname{supp}(-D_{\mu})^{v}\varphi_{i} \bigcap \operatorname{supp}(-D_{\mu})^{v}\varphi_{j} = \emptyset, \text{ for } i \neq j,$$

and

$$\|(-D_{\mu})^v \varphi_j\|_{p,\mu} \lesssim m^{2v-d/p},$$

which leads to

(5.6)
$$\|(-D_{\mu})^{v} f_{\alpha}\|_{p,\mu} \lesssim m^{2v-d/p} \|\alpha\|_{\ell_{p}^{4n}}.$$

It follows from the Kolmogorov type inequality (see [8, Theorem 8.1]) that

$$(5.7) \|(-D_{\mu})^{r/2} f_{\alpha}\|_{p,\mu} \lesssim \|(-D_{\mu})^{v} f_{\alpha}\|_{p,\mu}^{\frac{r}{2v}} \|f_{\alpha}\|_{p,\mu}^{\frac{2v-r}{2v}} \lesssim m^{r-d/p} \|\alpha\|_{\ell_{n}^{4n}},$$

which combining with (5.5), we obtain (5.4) for $W_{p,\mu}^r$.

By the fact that

$$||f_{\alpha}||_{B_{\infty}^{r}(L_{n,\mu})} \lesssim ||f_{\alpha}||_{B_{\infty}^{r}(L_{n,\mu})}, \ 0 < \tau < \infty,$$

it suffices to show (5.4) for $B_{\tau}^{r}(L_{p,\mu})$, $0 < \tau < \infty$. Since $E_{2^{j}}(f_{\alpha})_{p,\mu} \leq ||f_{\alpha}||_{p,\mu}$ for any $j \geq 0$, by (5.4) we have

(5.8)
$$\sum_{2^{j} < m} \left(2^{jr} E_{2^{j}}(f_{\boldsymbol{\alpha}})_{p,\mu} \right)^{\tau} \leq \|f_{\boldsymbol{\alpha}}\|_{p,\mu}^{\tau} \sum_{2^{j} < m} 2^{jr\tau}$$

$$\approx m^{r\tau} \|f_{\boldsymbol{\alpha}}\|_{p,\mu}^{\tau} \approx m^{(r-d/p)\tau} \|\boldsymbol{\alpha}\|_{\ell_{p}^{4n}}^{\tau}.$$

Choose a positive number v > r, by (2.4) we obtain for any $j \ge 0$,

$$E_{2^j}(f_{\boldsymbol{\alpha}})_{p,\mu} \lesssim 2^{-j\upsilon} \|f_{\boldsymbol{\alpha}}\|_{W_{p,\mu}^{\upsilon}},$$

which combining with (5.4) for $W_{p,\mu}^r$, we get

$$\sum_{2^{j} \geq m} \left(2^{jr} E_{2^{j}} (f_{\boldsymbol{\alpha}})_{p,\mu} \right)^{\tau} \leq \|f_{\boldsymbol{\alpha}}\|_{W_{p,\mu}^{\upsilon}}^{\tau} \sum_{2^{j} \geq m} 2^{j(r-\upsilon)\tau} \\
\approx m^{(r-\upsilon)\tau} \|f_{\boldsymbol{\alpha}}\|_{W_{p,\mu}^{\upsilon}}^{\tau} \lesssim m^{(r-d/p)\tau} \|\boldsymbol{\alpha}\|_{\ell_{p}^{4n}}^{\tau}.$$
(5.9)

It follows from (5.5), (5.8), and (5.9) that

$$||f||_{B_{\tau}^{r}(L_{p,\mu})} := ||f||_{p,\mu} + \left(\sum_{j=0}^{\infty} \left(2^{jr} E_{2^{j}}(f_{\alpha})_{p,\mu}\right)^{\tau}\right)^{1/\tau} \lesssim m^{r-d/p} ||\alpha||_{\ell_{p}^{4n}}.$$

This completes the proof of Lemma 5.2.

Finally we turn to prove (5.1).

Proof of (5.1).

First we consider the case $2 \le p \le \infty$. By the fact that

$$X_{\infty}^r \subset X_p^r, \ 2 \le p \le \infty,$$

it suffices to consider the case $p = \infty$. It follows from (5.4) that when $\alpha_j \in \{-1,1\}, j = 1, \ldots, 4n$,

$$||f_{\boldsymbol{\alpha}}||_{X_{\infty}^r} \lesssim m^r ||\boldsymbol{\alpha}||_{\ell_{\infty}^{4n}} \lesssim m^r.$$

Hence, there exists a positive constant C_1 such that $C_1m^{-r}f_{\alpha} \in BX_{\infty}^r$. Set

$$\psi_j(x) := C_1 m^{-r} \varphi_j(x), \ j = 1, \dots, 4n.$$

We have

$$F_1 := \left\{ \sum_{j=1}^{4n} \alpha_j \psi_j \mid \alpha_j \in \{-1, 1\}, \ j = 1, \dots, 4n \right\} \subset BX_{\infty}^r.$$

It follows from (5.3) that

INT_d(
$$\psi_j$$
) = $\int_{\mathbb{R}^d} \psi_j(x) w_\mu(x) dx = C_1 m^{-r} ||\varphi_j||_{1,\mu} \asymp m^{-r-d}$,

Applying Lemma 5.1 (a) we obtain for $2 \le p \le \infty$,

(5.10)
$$e_n^{\text{ran}}(BX_n^r) \ge e_n^{\text{ran}}(BX_\infty^r) \gtrsim m^{-r-d} n^{1/2} \asymp n^{-\frac{r}{d} - \frac{1}{2}}.$$

Next we consider the case $1 \le p < 2$. It follows from (5.4) that

$$\|\pm\varphi_j\|_{X_p^r}\lesssim m^{r-d/p}.$$

Hence, there exists a positive constant C_2 such that

$$\psi_j(x) := C_2 m^{-r+d/p} \varphi_j(x) \in BX_p^r, \ j = 1, \dots, 4n.$$

We have

$$F_2 := \{ \pm \psi_j : j = 1, \dots, 4n \} \subset BX_p^r.$$

It follows from (5.3) that

$$INT_{d}(\psi_{j}) = \int_{\mathbb{R}^{d}} \psi_{j}(x) w_{\mu}(x) dx = C_{2} m^{-r+d/p} \|\varphi_{j}\|_{1,\mu} \approx m^{-r+d/p-d}.$$

Applying Lemma 5.1 (b), we obtain for $1 \le p < 2$,

$$e_n^{\operatorname{ran}}(BX_p^r) \gtrsim m^{-r+d/p-d} \approx n^{-\frac{r}{d}+\frac{1}{p}-1},$$

which combining with (5.10), gives the lower bounds of $e_n^{\text{ran}}(BX_p^r)$ for $1 \leq p \leq \infty$. This completes the proof of (5.1).

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