WORST CASE TRACTABILITY OF LINEAR PROBLEMS IN THE PRESENCE OF NOISE: LINEAR INFORMATION

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ABSTRACT. We study the worst case tractability of multivariate linear problems defined on separable Hilbert spaces. Information about a problem instance consists of noisy evaluations of arbitrary bounded linear functionals, where the noise is either deterministic or random. The cost of a single evaluation depends on its precision and is controlled by a cost function. We establish mutual interactions between tractability of a problem with noisy information, the cost function, and tractability of the same problem, but with exact information.

1. Introduction

Tractability of multivariate problems is nowadays one of the most active areas of information-based complexity; we mention only the three-volume monograph [3–5]. Tractability research concentrates on establishing both quantitative and qualitative properties of the interplay between the cost and accuracy of approximation, and the number of variables occurring in a multivariate computational problem. To the best of the authors' knowledge, all tractability research has hitherto concentrated on exact information, i.e., information consisting of exact evaluations of information functionals. The goal of this article is to extend tractability studies to include noisy information, where observations of functionals are contaminated by some noise.

We study tractability in the worst case setting, in the presence of deterministic (bounded) or random (Gaussian) noise. The model of noise and cost is adopted from [2,6]. That is, information is built out of a finite number of noisy evaluations of functionals, which are subject to our choice. Moreover, prior to their noisy evaluation it is also possible to set required precision σ , which is a bound on the absolute value of the noise in the deterministic case, and the standard deviation of a Gaussian variable in the case of random noise. The cost of a single evaluation with a given precision is controlled by a cost function \$, which is a part of the problem formulation. The higher the precision, the higher the cost.

The main theme of our work is a comparative study of exact and noisy information from the point of view of tractability of multivariate linear problems $S_d: F_d \to G_d$ acting between separable Hilbert spaces. We assume that noisy evaluations of any linear functionals with norm bounded by one are possible. The focus is on (strong) polynomial tractability, weak tractability, intractability, and the curse of dimensionality. We are interested in establishing mutual interactions between tractability of a multivariate problem with noisy information, the cost function, and tractability of the same problem, but with exact information. In particular, we seek for conditions guaranteeing equivalence of various tractability notions for both, the exact and noisy settings.

Such equivalence is established, for instance, for polynomial tractability provided the cost function grows polynomially. To give a flavor of our results, suppose that the problem with exact

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information is polynomially tractable, i.e., its (ε, d) -complexity is upper bounded by $Cd^q\varepsilon^{-p}$, where ε is the required error of approximation, and that the cost function grows polynomially, i.e., $\$(\sigma, d) \le 1 + Dd^t\sigma^{-2s}$. Then the same problem with noisy information is also polynomially tractable. Moreover, its complexity is essentially bounded as

$$\operatorname{comp}_{\$}(\varepsilon, d) \preccurlyeq d^{\overline{t} + q(\overline{s} + 1)} \varepsilon^{-\max(p(\overline{s} + 1), 2\overline{s})}$$

where $(\overline{s}, \overline{t}) = (s, t)$ for bounded noise, and $(\overline{s}, \overline{t}) = (s, t)/\max(1, s)$ for Gaussian noise, see Theorem 1 and Theorem 4. We stress that we do not know whether the exponents of polynomial tractability above are optimal. The point is that, unlike in the case of exact information, it is generally an open question how to optimally select functionals when their evaluations are corrupted by noise.

As for the technical part, it turns out that an important role in the analysis plays the complexity of a one-dimensional problem that relies on approximating an unknown real parameter from its noisy observations. This problem is trivial in the case of bounded noise, but far from that in the case of Gaussian noise, cf. [1, 2]. Some difficulty in showing lower bounds adds the fact that in the case of random noise one has to consider deterministic as well as randomized approximations. Indeed, although randomization is formally not allowed in the problem formulation, it can be mimicked with the help of adaption, cf. [7,8].

The paper is organized as follows. The scene is formally set in Section 2. The results for bounded noise are in Section 3, and those for Gaussian noise in Section 4. The Appendix contains some additional material concerning the optimal choice of information functionals in the case of bounded and Gaussian noise.

2. Preliminaries

We consider a multivariate problem $S = \{S_d\}_{d \geq 1}$ where

$$S_d: F_d \to G_d,$$

 F_d and G_d are separable Hilbert spaces, both over the reals, and S_d are nonzero continuous linear operators with norms

$$||S_d|| = \sup_{||f||_{F_d} \le 1} ||S_d(f)||_{G_d}.$$

2.1. **Information and approximation.** The values $S_d(f)$ for $f \in F_d$ are approximated based on information $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ about f, which consists of finitely many noisy values of some functionals at f. That is,

$$y_i = L_i(f) + e_i, \quad 1 \le i \le n,$$

where L_i are in a class $\Lambda_d \subset F_d^*$ of permissible functionals, and e_i is noise. A crucial assumption of the current paper is that arbitrary continuous functionals with norm at most one are allowed,

$$\Lambda_d = \{ L \in F_d^* : ||L|| \le 1 \},$$

where $||L|| = \sup_{\|f\|_{F_d} \le 1} |L(f)|$. The noise can be deterministic (bounded) or random (Gaussian),

$$|e_i| \le \sigma_i \quad \text{or} \quad e_i \stackrel{iid}{\sim} \mathcal{N}(0, \sigma_i^2),$$

where σ_i represents precision of the *i*th evaluation, and $\mathcal{N}(0,\sigma)$ is the standard zero-mean Gaussian distribution with variance σ^2 . Then an approximation to $S_d(f)$ is given as $\Phi(\mathbf{y})$, where

$$\Phi: Y \to G_d$$

called an algorithm, is an arbitrary mapping acting on the set Y of all possible values of information.

We now describe the information more formally. We first deal with *nonadaptive* (or parallel) information, in which case the functionals L_i and precisions σ_i are the same for all problem instances $f \in F_d$. In the case of bounded noise, nonadaptive information is a multi-valued operator, i.e., $N: F_d \to 2^Y$, where 2^Y is the power set of $Y = \mathbb{R}^n$, and

$$N(f) = \{ (L_1(f) + e_1, L_2(f) + e_2, \dots, L_n(f) + e_n) : |e_i| \le \sigma_i, 1 \le i \le n \}.$$

Then **y** is information about f iff $\mathbf{y} \in N(f)$.

In case of Gaussian noise, nonadaptive information \mathbf{y} about f is a realization of the random variable with n dimensional Gaussian distribution π_f whose mean element is $m_f = (L_1(f), \ldots, L_n(f))$ and correlation matrix $\Sigma = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2)$. Therefore nonadaptive information is now a mapping $N: F_d \to \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ is a set of probability distributions on the Borel sets of $Y = \mathbb{R}^n$, and

$$N(f) = \pi_f$$
 for $f \in F_d$.

Although we will mainly exploit nonadaptive information in this paper, in a generic approximation scheme we also allow a more general adaptive (or sequential) information, where the choice of the successive functionals L_i and precisions σ_i , as well as the number of them, depend on f and noise via the previously obtained values y_1, \ldots, y_{i-1} . The process of obtaining adaptive information $\mathbf{y} = (y_1, \ldots, y_n)$ about f can be schematically described as follows:

(1)
$$\begin{cases} y_1 &= L_1(f) + e_1, & \sigma_1, \\ y_2 &= L_2(f; y_1) + e_2, & \sigma_2(y_1), \\ y_3 &= L_3(f; y_1, y_2) + e_3, & \sigma_3(y_1, y_2), \\ \dots & \dots & \dots & \dots \\ y_n &= L_n(f; y_1, y_2, \dots, y_{n-1}) + e_n, & \sigma_n(y_1, y_2, \dots, y_{n-1}), \end{cases}$$

where $L_i(\cdot; y_1, \ldots, y_{i-1}) \in \Lambda_d$. The process terminates when $(y_1, y_2, \ldots, y_n) \in Y$, where the set Y of all values of information consists of finite sequences of (possibly) various lengths. For the termination criterion to be well defined we assume that for any infinite sequence $(y_1, y_2, y_3 \ldots)$ there is exactly one n such that $(y_1, \ldots, y_n) \in Y$. The corresponding operator N is for both, bounded and Gaussian noise, determined by the above construction. (In case of Gaussian noise appropriate measurability assumptions on $L(f; \cdot)$ and $\sigma_i(\cdot)$ have to be met.) For details, see [6, Sect. 2.7 & 3.7].

2.2. Cost function. We assume that we are free to choose the information functionals and precisions, but we have to pay more for more accurate evaluations. That is, the cost of a single noisy evaluation of L(f) for $f \in F_d$ with precision σ equals $\$(\sigma, d)$, where

$$\$: [0, +\infty) \times \{1, 2, 3, \ldots\} \to [1, +\infty]$$

is a cost function that is non-decreasing in both σ^{-1} and d. Note that $\$ \ge 1$, which corresponds to a natural assumption that one has to pay at least one unit even for 'slightest touch' of a functional. For instance,

$$\$(\sigma, d) = \begin{cases} +\infty, & 0 \le \sigma < \sigma_0, \\ 1, & \sigma_0 \le \sigma, \end{cases}$$

corresponds to the situation when one can only observe with precision σ_0 at cost 1. If, in addition, $\sigma_0 = 0$ then information is exact at the unit cost for all $\sigma \geq 0$ and $d \geq 1$. We distinguish several types of cost functions depending on how they grow as σ^{-1} and d increase. In particular, we have:

• polynomial growth in σ^{-1} and d iff

$$\$(\sigma, d) \le 1 + Dd^t \sigma^{-s}$$
 for all $d \ge 1$ and $\sigma \in (0, 1)$,

where D, t, s are some nonnegative numbers,

• sub-exponential growth in $\sigma^{-1} + d$ iff

$$\lim_{\sigma^{-1}+d\to\infty} \frac{\ln \$(\sigma,d)}{\sigma^{-1}+d} = 0,$$

• exponential growth in $\sigma^{-1} + d$ iff

$$\limsup_{\sigma^{-1}+d\to\infty} \frac{\ln \$(\sigma,d)}{\sigma^{-1}+d} > 0.$$

We will also consider corresponding growths in only one of the variables, σ^{-1} or d, with the other variable fixed. For instance, we have polynomial growth in σ^{-1} iff $\$(\sigma, d) \le D\psi(d)\sigma^{-s}$ for all $d \ge 1$ and $\sigma \in (0, 1)$, or we have sub-exponential growth in d iff $\lim_{d\to\infty} \ln \$(\sigma, d)/d = 0$ for all $\sigma \in (0, 1)$.

A total cost $\text{cost}_{\S}^{\text{sett}}(N)$ of given information N and error $e^{\text{sett}}(S_d, N, \Phi)$ of an algorithm Φ using it depend on a setting under consideration, and will be defined separately for each setting. The settings were distinguished by whether we have bounded or Gaussian noise.

2.3. Tractability notions. For a given setting, let

$$\operatorname{comp}_{\$}^{\operatorname{sett}}(\varepsilon, d) = \inf \left\{ \operatorname{cost}_{\$}^{\operatorname{sett}}(N) : N, \Phi \text{ such that } \operatorname{e}^{\operatorname{sett}}(S_d, N, \Phi) \le \varepsilon \|S_d\| \right\}$$

be the minimal cost of information sufficient to approximate S_d with (normalized) error ε . We call comp_{\$\settrum_{\substack}} (\varepsilon, d) the information (\varepsilon, d)-complexity, or simply (\varepsilon, d)-complexity of our problem. We consider the following tractability notions, cf. [3].}

• A multivariate problem $S = \{S_d\}_{d>1}$ is polynomially tractable iff

(2)
$$\operatorname{comp}_{\$}^{\operatorname{sett}}(\varepsilon, d) \leq C d^{q} \varepsilon^{-p} \quad \text{for all } d \geq 1 \text{ and } \varepsilon \in (0, 1),$$

where C, q, p are some nonnegative numbers. If, in addition, (2) holds with q = 0 then the problem is *strongly polynomially tractable*, and the infimum of p satisfying (2) with q = 0 is the *strong exponent*.

• A problem is weakly tractable iff

$$\lim_{\varepsilon^{-1}+d\to +\infty} \frac{\ln\left(\mathrm{comp}_\S^{\mathrm{sett}}(\varepsilon,d)\right)}{\varepsilon^{-1}+d} = 0.$$

- A problem is *intractable* iff it is not weakly tractable.
- A problem suffers from the curse of dimensionality iff there are $\varepsilon_0 > 0$, C > 0, and $\gamma > 0$, such that for infinitely many d we have

$$comp_{\$}^{sett}(\varepsilon_0, d) \ge C(1 + \gamma)^d.$$

Equivalently, we have the curse iff there is $\varepsilon_0 > 0$ such that

$$\limsup_{d\to\infty} \frac{\ln\left(\operatorname{comp}^{\operatorname{sett}}_{\$}(\varepsilon_0,d)\right)}{d} > 0.$$

We will later draw conclusions about tractability in the case of noisy information assuming we know tractability for exact information. As we already noticed, in the latter case we have $\$(\sigma,d)=1$, which means that we just count the number of functional evaluations. In the two settings considered in this paper the complexities in the case of exact information are the same and denoted by

$$n^{w}(\varepsilon, d)$$
,

where 'w' stands for 'worst'.

3. Worst case setting with bounded noise

In this section we assume that the noise is bounded. That is, information about f is given as $\mathbf{y} = (y_1, y_2, \dots, y_{n(\mathbf{y})})$ where

$$y_i = L_i(f; y_1, \dots, y_{i-1}) + e_i, \qquad |e_i| \le \sigma_i(y_1, \dots, y_{i-1}).$$

The (total) cost of information N is defined as

$$cost_{\$}^{ww}(N) = \sup_{\|f\|_{F_d} \le 1} \sup_{\mathbf{y} \in N(f)} \sum_{i=1}^{n(\mathbf{y})} \$(\sigma_i(y_1, \dots, y_{i-1})),$$

and the error of an algorithm Φ using information N as

$$e^{ww}(S_d, N, \Phi) = \sup_{\|f\|_{F_d} \le 1} \sup_{\mathbf{y} \in N(f)} \|S_d(f) - \Phi(\mathbf{y})\|_{G_d}.$$

We assume that S_d is a *compact* operator which, as well known, is necessary if we want to assure that $\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon,d) < +\infty$ for all $\varepsilon > 0$.

We now recall some auxiliary facts about the current setting that can be found, e.g., in [6].

Let $N: F_d \to 2^Y$ be arbitrary information and rad^{ww}(N) be its radius, i.e., the minimal error that can achieved using N. If N is nonadaptive and uses n functionals L_i with precisions σ_i then

(3)
$$\operatorname{rad}^{ww}(N) = \max \{ \|S_d(h)\|_{G_d} : \|h\|_{F_d} \le 1, |L_i(h)| \le \sigma_i, 1 \le i \le n \}.$$

Next we notice that we can restrict our considerations to algorithms using nonadaptive information. Indeed, for any adaptive information N^{ada} of the form (1) and with range Y^{ada} one can define nonadaptive information N^{non} with range $Y^{\text{non}} = \mathbb{R}^n$ where n is such that $(0, \ldots, 0) \in Y^{\text{ada}}$ and

$$(y_1,\ldots,y_n)\in N^{\mathrm{non}}(f)$$
 iff $y_i=L_i(f;\underbrace{0,\ldots,0}_{i-1})+e_i, |e_i|\leq \sigma_i(\underbrace{0,\ldots,0}_{i-1}), 1\leq i\leq n.$

Then $\operatorname{rad}^{\operatorname{ww}}(N^{\operatorname{non}}) \leq \operatorname{rad}^{\operatorname{ww}}(N^{\operatorname{ada}})$ and $\operatorname{cost}^{\operatorname{ww}}_{\$}(N^{\operatorname{non}}) \leq \operatorname{cost}^{\operatorname{ww}}_{\$}(N^{\operatorname{ada}})$, which means that adaption does not help. This and (3) imply that

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon, d) = \inf \left\{ \operatorname{cost}_{\$}^{\operatorname{ww}}(N) : N \text{-nonadaptive, } \operatorname{rad}^{\operatorname{ww}}(N) \le \varepsilon ||S_d|| \right\}.$$

To avoid notational difficulties, from now on we assume that $\dim(F_d) = +\infty$, which can obviously be done without loss of generality. Let $\{f_{d,j}^*\}_{j\geq 1}$ be the complete orthonormal system of eigenelements of $S_d^*S_d: F_d \to F_d$, and

$$\lambda_{d,1} \ge \lambda_{d,2} \ge \cdots \ge \lambda_{d,j} \ge \cdots$$

the corresponding eigenvalues. We have $||S_d|| = \sqrt{\lambda_{d,1}}$ and $\lim_{j\to\infty} \lambda_{d,j} = 0$. Furthermore, in the noiseless case, information

$$(4) N_n^d = \left(\langle \cdot, f_{d,1}^* \rangle_{F_d}, \dots, \langle \cdot, f_{d,n}^* \rangle_{F_d} \right)$$

is nth optimal, and its radius rad^w $(N_n^d) = \sqrt{\lambda_{d,n+1}}$, cf. [3]. Hence

(5)
$$n^{w}(\varepsilon, d) = \min \{ n : \sqrt{\lambda_{d,n+1}} \le \varepsilon \sqrt{\lambda_{d,1}} \}.$$

We first show a general though important result that will be used later.

Lemma 1. For all $\varepsilon \in (0,1)$ and $d \ge 1$ we have

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon, d) \ge \sum_{k=1}^{\operatorname{n}^{\operatorname{w}}(\varepsilon, d)} \$\left(\varepsilon \sqrt{\frac{\lambda_{d, 1}}{\lambda_{d, k}}}, d\right).$$

Hence, comp_{\$\frac{\text{w}}{\text{w}}(\varepsilon, d) \geq \max \{ \text{n}^{\text{w}}(\varepsilon, d) \\$ (1, d), \\$ (\varepsilon, d) \}.}

Proof. Let N be nonadaptive information using m functionals L_i with precisions σ_i , such that $\mathrm{rad}^{\mathrm{ww}}(N) \leq \varepsilon ||S_d||$. Assume without loss of generality that $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$. To prove the lemma, it suffices to show that

$$\sigma_k \le \varepsilon \sqrt{\frac{\lambda_{d,1}}{\lambda_{d,k}}} < 1, \qquad 1 \le k \le n^{w}(\varepsilon, d).$$

Let k be as above. The inequality '<' follows from (5). To show ' \leq ', define the linear subspace

$$V_{k-1} = \{ f \in F : L_i(f) = 0, 1 \le i \le k-1 \}$$
 (where $V_{k-1} = F$ if $k = 1$).

Obviously codim $(V_{k-1}) \leq k-1$. Since for any h with $||h||_{F_d} \leq \sigma$ is $|L_i(h)| \leq \sigma$, we have

$$\operatorname{rad}^{\operatorname{ww}}(N_n) \geq \max\{\|S_d(h)\|_{F_d}: \|h\|_{F_d} \leq 1, h \in V_{k-1}, |L_i(h)| \leq \sigma_i, k \leq i \leq m\}$$

$$\geq \max\{\|S_d(h)\|_{F_d}: \|h\|_{F_d} \leq \sigma_k, h \in V_{k-1}\} \geq \sigma_k \sqrt{\lambda_{d,k}},$$

where we used the fact that the norm of S_d restricted to the subspace V_{k-1} is at least $\sqrt{\lambda_k}$. Hence $\sigma_k \leq \varepsilon \sqrt{\frac{\lambda_{d,1}}{\lambda_{d,k}}}$ since otherwise we would have $\mathrm{rad}^{\mathrm{ww}}(N) > \varepsilon \sqrt{\lambda_{d,1}} = \varepsilon \|S_d\|$.

To achieve upper bounds on tractability, we will use noisy version of the nonadaptive information N_n^d defined in (4). That is, for given d, n and σ_i we have $\mathbf{y} = (y_1, \dots, y_n) \in N_n^d(f)$ iff

(6)
$$y_i = \langle f, f_{d,i}^* \rangle_{F_d} + e_i, \quad \text{where} \quad |e_i| \le \sigma_i.$$

The radius of N_n^d can be estimated from above by the error of the approximation

$$\Phi_n^d(\mathbf{y}) = \sum_{i=1}^n y_i S_d(f_{d,i}^*).$$

Specifically, using $f = \sum_{i=1}^{\infty} \langle f, f_{d,i}^* \rangle_{F_d} f_{d,i}^*$ and orthogonality of $\{S_d(f_{d,i}^*)\}_{i \geq 1}$ in G_d we have

$$||S_{d}(f) - \Phi_{n}^{d}(\mathbf{y})||_{G_{d}}^{2} = \left| \left| -\sum_{i=1}^{n} e_{i} S_{d}(f_{d,i}^{*}) + \sum_{i=n+1}^{+\infty} \langle f, f_{d,i} \rangle_{F_{d}} S_{d}(f_{d,i}^{*}) \right| \right|_{G_{d}}^{2}$$
$$= \sum_{i=1}^{n} \lambda_{d,i} |e_{i}|^{2} + \sum_{i=n+1}^{+\infty} \lambda_{d,i} |\langle f, f_{d,i}^{*} \rangle_{F_{d}}|^{2}.$$

Taking the suprema with respect to $||f||_{F_d} \leq 1$ and $|e_i| \leq \sigma_i$ we obtain

(7)
$$e^{ww}(N_n^d, \Phi_n^d) = \sqrt{\sum_{i=1}^n \sigma_i^2 \lambda_{d,i} + \lambda_{d,n+1}}.$$

In particular, for exact information we restore the known result that $e^{ww}(S_d, N_n^d, \Phi_n^d) = \sqrt{\lambda_{d,n+1}}$, which is the minimal error when n exact functional evaluations are used. The cost of such approximation is obviously $\sum_{i=1}^{n} \$(\sigma_i, d)$.

3.1. **Polynomial tractability.** We use the following asymptotic notation. For two nonnegative functions of ε and d we write

$$\psi_1(\varepsilon, d) \preceq \psi_2(\varepsilon, d)$$
 iff $\psi_1(\varepsilon, d) \leq A \psi_2(\varepsilon, d)$,

for some $A < +\infty$ and all $\varepsilon \in (0,1)$ and $d \ge 1$.

Theorem 1. Consider a multivariate problem $S = \{S_d\}_{d>1}$.

- (i) The problem with noisy information is polynomially tractable if and only if
 - it is polynomially tractable for exact information, and
 - the cost function grows polynomially in σ^{-1} and d.
- (ii) The problem with noisy information is strongly polynomially tractable if and only if
 - it is strongly polynomially tractable for exact information, and
 - the cost function grows polynomially in σ^{-1} and is bounded in d for any $\sigma > 0$.
- (iii) Suppose that $n^{w}(\varepsilon, d) \leq Cd^{q}\varepsilon^{-p}$ and $\$(\sigma, d) \leq 1 + Dd^{t}\sigma^{-2s}$. If p = 0 and s = 0 then $\operatorname{comp}_{\$}^{ww}(\varepsilon, d) \leq d^{t+q}$; otherwise

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon,d) \, \preccurlyeq \, d^{t+q(s+1)} \left\{ \begin{array}{cc} \varepsilon^{-p(s+1)}, & p(s+1) > 2s, \\ \ln^{s+1}(1/\varepsilon) \, \varepsilon^{-2s}, & p(s+1) = 2s, \\ \varepsilon^{-2s}, & p(s+1) < 2s. \end{array} \right.$$

Proof. Suppose that the problem is polynomially tractable for noisy information, i.e.,

$$comp^{ww}(\varepsilon, d) \le Cd^q \varepsilon^{-p}.$$

Then we have by Lemma 1 that, on one hand,

$$n^{w}(\varepsilon, d) \le comp^{ww}(\varepsilon, d) \le Cd^{q}\varepsilon^{-p}$$

and, on the other hand,

$$\$(\sigma, d) \le \text{comp}^{ww}(\sigma, d) \le 1 + Cd^q \sigma^{-p}.$$

This proves the necessary conditions in (i) and (ii). The sufficient conditions follow from (iii).

In the proof of (iii) we distinguish several cases.

If s = 0 and $p \ge 0$ then exact observations are possible at cost $1 + Dd^t$, and therefore

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon,d) \leq (1+Dd^t) \operatorname{n}^{\operatorname{w}}(\varepsilon,d) \leq (1+D) C d^{t+q} \varepsilon^{-p} \preccurlyeq d^{t+q} \varepsilon^{-p}.$$

Assume s > 0. We first optimize the cost of obtaining an ε -approximation using information N_n^d with precisions $\sigma_i \leq 1$ together with the algorithm Φ_n^d . Let $e^{ww}(S_d, N_n^d, \Phi_n^d) \leq \varepsilon \sqrt{\lambda_1}$. The cost of N_n^d is upper bounded by

$$\psi_n(\sigma_1,\ldots,\sigma_n) = n + Dd^t \sum_{i=1}^n \sigma_i^{-2s}.$$

Minimizing ψ_n with respect to the condition $e^{ww}(S_d, N_n^d, \Phi_n^d)^2 = \sum_{i=1}^n \sigma_i^2 \lambda_{d,i} + \lambda_{d,n+1} \leq \lambda_{d,1} \varepsilon^2$ we obtain the optimal values

$$\hat{\sigma}_k^{-2} = \left(\frac{\lambda_{d,k}}{\lambda_{d,1}}\right)^{\frac{1}{s+1}} \sum_{i=1}^n \left(\frac{\lambda_{d,i}}{\lambda_{d,1}}\right)^{\frac{s}{s+1}} \left(\varepsilon^2 - \frac{\lambda_{d,n+1}}{\lambda_{d,1}}\right)^{-1}, \qquad 1 \le k \le n,$$

and

$$\psi_n(\hat{\sigma}_1, \dots, \hat{\sigma}_n) = n + Dd^t \left(\sum_{i=1}^n \left(\frac{\lambda_{d,i}}{\lambda_{d,1}} \right)^{\frac{s}{s+1}} \right)^{s+1} \left(\varepsilon^2 - \frac{\lambda_{d,n+1}}{\lambda_{d,1}} \right)^{-s}.$$

Now, taking $n = \max\left(2, n^{w}(\varepsilon/\sqrt{2}, d)\right)$ we have $\frac{\lambda_{d, n+1}}{\lambda_{d, 1}} \leq \frac{\varepsilon^{2}}{2} < \frac{\lambda_{d, n}}{\lambda_{d, 1}}$ and

$$\hat{\sigma}_k^{-2} \ge \hat{\sigma}_n^{-2} = \left(\frac{\lambda_{d,n}}{\lambda_{d,1}}\right)^{\frac{1}{s+1}} \sum_{i=1}^n \left(\frac{\lambda_{d,i}}{\lambda_{d,1}}\right)^{\frac{s}{s+1}} \left(\varepsilon^2 - \frac{\lambda_{d,n+1}}{\lambda_{d,1}}\right)^{-1} \ge n \left(\frac{\lambda_{d,n}}{\lambda_{d,1}}\right) \varepsilon^{-2} > \frac{n}{2} \ge 1,$$

i.e., $0 < \sigma_1 \le \cdots \le \sigma_n < 1$. Then an ε -approximation is obtained at cost

(8)
$$\operatorname{cost}_{\$}^{\operatorname{ww}}(N_n^d) \leq n + 2^s D d^t \left(\sum_{i=1}^n \left(\frac{\lambda_{d,i}}{\lambda_{d,1}} \right)^{\frac{s}{s+1}} \right)^{s+1} \varepsilon^{-2s}.$$

Assume now that we have polynomial tractability for exact information, i.e.,

$$n^{w}(\varepsilon, d) \le C d^{q} \varepsilon^{-p}$$
 for $d \ge 1$ and $\varepsilon \in (0, 1)$.

If p=0 then $\lambda_j=0$ for $j\geq |Cd^q|+1$ and we have from (8) that

$$\operatorname{cost}_{\$}^{\operatorname{ww}}(N_n^d) \preccurlyeq d^t |Cd^q|^{s+1} \varepsilon^{-2s} \preccurlyeq d^{t+q(s+1)} \varepsilon^{-2s}$$

Assume p > 0. We need to estimate the ratios $\lambda_{d,j}/\lambda_{d,1}$. For $1 \leq j \leq \lfloor Cd^q \rfloor + 1$ we have $\lambda_{d,j}/\lambda_{d,1} \leq 1$. Let $j \geq \lfloor Cd^q \rfloor + 2$. Let $\varepsilon_j \in (0,1)$ be such that $j = Cd^q \varepsilon_j^{-p} + 1$. Then $j-1 \geq n^w(\varepsilon_j,d)$, which implies

$$\sqrt{\frac{\lambda_{d,j}}{\lambda_{d,1}}} \le \varepsilon_j = \left(\frac{Cd^q}{j-1}\right)^{1/p}.$$

Hence for all $j \geq 1$

(9)
$$\frac{\lambda_{d,j}}{\lambda_{d,1}} \le \min\left(1, \left(\frac{Cd^q}{j-1}\right)^{2/p}\right).$$

Assuming $C \geq 2$ (which can be done without loss of generality) the estimates (8) and (9) give

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon,d) \leq \left\lfloor Cd^{q} \left(\frac{\varepsilon}{\sqrt{2}} \right)^{-p} \right\rfloor + 2^{s} Dd^{t} \left(\left\lfloor Cd^{q} \right\rfloor + 1 + \sum_{j=\left\lfloor Cd^{q} \right\rfloor + 2}^{\left\lfloor Cd^{q} \left(\varepsilon/\sqrt{2} \right)^{-p} \right\rfloor} \left(\frac{Cd^{q}}{j-1} \right)^{\frac{2s}{p(s+1)}} \right)^{s+1} \left(\frac{1}{\varepsilon} \right)^{2s}.$$

Using the formula

$$\sum_{i=k+1}^{n} j^{-\beta} \le \int_{k}^{n} x^{-\beta} dx = \begin{cases} \ln n - \ln k, & \beta = 1, \\ \frac{n^{1-\beta} - k^{1-\beta}}{1-\beta}, & \beta \ne 1, \end{cases}$$

with $2 \le k+1 \le n$ and $\beta = \frac{2s}{p(s+1)}$, we finally obtain the desired upper bounds.

Remark 1. The algorithm Φ_n^d is not optimal for N_n^d if information is contaminated by noise (6). Indeed, we have by (3) that

(10)
$$\operatorname{rad}^{\operatorname{ww}}(N_n^d) = \max \left\{ \left(\sum_{i=1}^{\infty} a_i^2 \lambda_{d,i} \right)^{1/2} : \sum_{i=1}^{n} a_i^2 \le 1, \ |a_i| \le \sigma_i, \ 1 \le i \le n \right\}$$

$$= \sqrt{\sum_{i=1}^{\ell} \sigma_i^2 \lambda_{d,i} + \left(1 - \sum_{i=1}^{\ell} \sigma_i^2 \right) \lambda_{d,\ell+1}} = \sqrt{\sum_{i=1}^{\ell} \sigma_i^2 (\lambda_{d,i} - \lambda_{d,\ell+1}) + \lambda_{d,\ell+1}},$$

where ℓ is the largest k satisfying $\sum_{i=1}^k \sigma_i^2 < 1$, cf. (7). Nevertheless, Φ_n^d gives optimal exponents of tractability when one relies only on information N_n^d .

To see this, let N_m^d be information (6) that uses precisions σ_i and whose radius is at most $\varepsilon \sqrt{\lambda_{d,1}}$. Then $m \geq n = \mathrm{n^w}(\varepsilon,d)$. A crucial observation is that, in view of (5) and (10), we then have $\sum_{i=1}^n \sigma_i^2 \leq 1$. Hence the cost of N_m^d can be lower bounded by minimization of $n + \sum_{i=1}^n \sigma_i^{-2s}$ (which does not exceed $\mathrm{cost^{ww}}(N_m^d)$) with respect to the condition $\sum_{i=1}^n \sigma_i^2 \lambda_{d,i} \leq \varepsilon^2 \lambda_{d,1}$ (which is weaker than $\mathrm{rad^{ww}}(N_m^d) \leq \varepsilon \sqrt{\lambda_{d,1}}$). In this way we obtain

$$cost_{\$}^{ww}(N_m^d) \ge n + Dd^t \left(\sum_{i=1}^n \left(\frac{\lambda_{d,i}}{\lambda_{d,1}} \right)^{\frac{s}{s+1}} \right)^{s+1} \varepsilon^{-2s}.$$

This bound differs from the upper bound in (8) at most by the factor of 2^s , which does not influence the exponents of polynomial tractability.

We believe that the tractability exponents in (iii) of Theorem 1 are best possible, but a formal justification of this fact is missing. The point is that these exponents are obtained using particular information N_n^d . On one hand, Proposition 1 of Appendix shows that this information is indeed optimal in some situations, even if we fix precisions σ_i . On the other hand, the following example shows that the cost can be sometimes significantly reduced by applying more sophisticated information.

Example 1. Suppose we approximate vectors $\vec{x} = (x_1, x_2)^T \in \mathbb{R}^2$ in the ℓ_2 norm. Consider noisy information consisting of 2n observations $\mathbf{y} = (\vec{y}_1^T, \dots, \vec{y}_n^T)$, where

(11)
$$\mathbb{R}^2 \ni \vec{y_i} = (R_i \vec{x})^T + \vec{e_i}, \qquad ||\vec{e_i}||_{\infty} \le \sigma < 1,$$

and

$$R_i = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix}, \qquad \theta_i = \frac{\pi(i-1)}{2n}, \qquad 1 \le i \le n,$$

is the clockwise rotation through the angle θ_i . Note that n=1 corresponds to $\mathbf{y}=\vec{x}+\vec{e}, \|\vec{e}\|_{\infty} \leq \sigma$, which is the information exploited in the proof of Theorem 1 for this particular problem.

Using geometrical arguments one can easily show that, given $n \geq 1$ and $\varepsilon \in (0,1)$, one has to use precision $\sigma_n(\varepsilon) = \varepsilon \cos\left(\frac{\pi}{4n}\right)$ to get an ε -approximation of \vec{x} , and then the cost of the ε -approximation equals $\cot(n,\varepsilon) = 2n\$(\sigma_n(\varepsilon))$, where \$\\$ is a cost function. Let \$\\$(\sigma) = 1 + \sigma^{-2s}\$ with s > 0. Then the cost is

(12)
$$\operatorname{cost}_{s}(n,\varepsilon) = 2n \left(1 + \left(\varepsilon \cos(\frac{\pi}{4n}) \right)^{-2s} \right).$$

Taking $n^* = \left\lceil \frac{\pi}{4} \sqrt{2s} \right\rceil$ we have the asymptotic equality

$$cost_s(n^*, \varepsilon) \approx \pi \sqrt{\frac{s}{2}} \left(1 + \sqrt{e} \varepsilon^{-2s} \right) \quad \text{as} \quad s \to \infty,$$

where we used the fact that $\lim_{x\to 0} (\cos x)^{-1/x^2} = \sqrt{e}$. Hence for $0 < \varepsilon < 1$ we have

$$\frac{\cos t_s(1,\varepsilon)}{\cos t_s(n^*,\varepsilon)} \approx \frac{2(1+2^s\varepsilon^{-2s})}{\pi\sqrt{s/2}(1+\sqrt{e}\varepsilon^{-2s})} \approx \frac{2^{s+1}}{\pi\sqrt{s\,e/2}}.$$

As we can see, for large s the 'rotated' information offers a serious improvement compared to the 'un-rotated' information consisting of only 2 observations as in the case of exact information.

3.2. Weak tractability and the curse of dimensionality.

Theorem 2. Consider a multivariate problem $S = \{S_d\}_{d\geq 1}$.

- (i) Suppose that the problem with noisy information is weakly tractable. Then
 - it is weakly tractable for exact information, and
 - the cost function grows sub-exponentially in $\sigma^{-1} + d$.
- (ii) Suppose that
 - the problem is weakly tractable for exact information, and
 - the cost function grows polynomially in σ^{-1} and sub-exponentially in d.

Then the same problem with noisy information is weakly tractable.

- (iii) Suppose that
 - the problem is strongly polynomially tractable for exact information with p < 2, and
 - the cost function grows sub-exponentially in $\sigma^{-1} + d$.

Then the same problem with noisy information is weakly tractable.

Proof. To show (i) we use Lemma 1. On one hand we have $n^w(\varepsilon, d) \leq comp_s^{ww}(\varepsilon, d)$, which implies

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln\left(\mathrm{n}^{\mathrm{w}}(\varepsilon,d)\right)}{\varepsilon^{-1}+d}\leq\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln\left(\mathrm{comp}^{\mathrm{ww}}_{\$}(\varepsilon,d)\right)-\ln\left(\$(1,d)\right)}{\varepsilon^{-1}+d}=0.$$

On the other hand $\$(\sigma, d) \le \text{comp}_{\$}^{\text{ww}}(\sigma, d)$, which implies

$$\lim_{\sigma^{-1}+d\to\infty}\frac{\ln\left(\$(\sigma,d)\right)}{\sigma^{-1}+d}\leq \lim_{\sigma^{-1}+d\to\infty}\frac{\ln\left(\operatorname{comp}_{\$}^{\operatorname{ww}}(\sigma,d)\right)}{\sigma^{-1}+d}=0.$$

Now we show (ii). Suppose that $\$(\sigma,d) \le 1 + D\sigma^{-2s}\kappa(d)$ where $\lim_{d\to\infty} \ln(\kappa(d))/d = 0$. Proceeding as in the proof of (iii) of Theorem 1 we get from (8) that

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon,d) \le n + 2^{s} D \,\kappa(d) \, n^{s+1} \varepsilon^{-2s} \quad \text{where} \quad n = \operatorname{n}^{\operatorname{w}} \left(\varepsilon / \sqrt{2}, d \right).$$

Hence, if the problem is weakly tractable for exact information, then

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\left(\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon,d)\right)}{\varepsilon^{-1}+d} = \lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\left(\kappa(d)\right)+(s+1)\ln n + 2s\ln(1/\varepsilon)}{\varepsilon^{-1}+d}$$
$$= (s+1)\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\left(\operatorname{n}^{\operatorname{w}}(\varepsilon/\sqrt{2},d)\right)}{\varepsilon^{-1}+d} = 0,$$

which means that the problem with noisy information is also weakly tractable.

To show (iii), suppose that the problem with noisy information is strongly tractable for exact information, $n^{w}(\varepsilon, d) \leq C\varepsilon^{-p}$ where p < 2. Then, by (9),

$$A := \sum_{j=1}^{\infty} \frac{\lambda_{d,j}}{\lambda_{d,1}} \le 1 + C^{2/p} \sum_{j=1}^{\infty} j^{-2/p} \le 1 + \frac{p C^{2/p}}{2 - p} < +\infty.$$

Let

$$n = n^{w} \left(\frac{\varepsilon}{\sqrt{2}}, d \right) \le C \left(\frac{\sqrt{2}}{\varepsilon} \right)^{p}.$$

For the algorithm Φ_n^d using noisy information N_n^d with fixed precision $\sigma = \frac{\varepsilon}{\sqrt{2A}}$ we have by (7) that

$$e^{ww}(S_d, N_n^d, \Phi_n^d) = \sqrt{\sigma^2 \sum_{i=1}^n \lambda_{d,i} + \lambda_{d,n+1}} \le \sqrt{\lambda_{d,1}} \sqrt{\sigma^2 A + \frac{\lambda_{d,n+1}}{\lambda_{d,1}}} \le \sqrt{\lambda_{d,1}} \sqrt{\frac{1}{2}\varepsilon^2 + \frac{1}{2}\varepsilon^2} = \sqrt{\lambda_{d,1}} \varepsilon,$$

and

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon,d) \leq \operatorname{cost}^{\operatorname{ww}}(N_n^d) = n \, \$ \left(\frac{\varepsilon}{\sqrt{2A}}, d \right) \leq C \left(\frac{\sqrt{2}}{\varepsilon} \right)^p \$ \left(\frac{\varepsilon}{\sqrt{2A}}, d \right).$$

Hence

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\left(\operatorname{comp}^{\mathrm{ww}}_{\$}(\varepsilon,d)\right)}{\varepsilon^{-1}+d} \leq \lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln C + p\left(\frac{1}{2}\ln 2 + \ln\left(\frac{1}{\varepsilon}\right)\right) + \ln \$\left(\frac{\varepsilon}{\sqrt{2A}},d\right)}{\varepsilon^{-1}+d}$$
$$= \sqrt{2A} \lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln \$\left(\frac{\varepsilon}{\sqrt{2A}},d\right)}{\frac{\sqrt{2A}}{2}+d\sqrt{2A}} = 0,$$

where we used sub-exponential growth of the cost function.

Remark 2. The sufficient conditions in (iii) of Theorem 2 for weak tractability in the case of noisy information can be generalized as follows. Suppose that

$$n^{w}(\varepsilon, d) \leq C\varepsilon^{-p}\kappa(d),$$

where $\kappa(d)$ grows sub-exponentially in d. Then

$$A_n^d := \sum_{j=1}^n \frac{\lambda_{d,j}}{\lambda_{d,1}} \iff \left(\kappa(d)\right)^{\frac{2}{p}} \begin{cases} 1, & p < 2, \\ \ln n, & p = 2, \\ n^{1-2/p}, & p > 2. \end{cases}$$

Applying the information N_n^d and algorithm Φ_n^d with $n = n^w(\frac{\varepsilon}{\sqrt{2}}, d)$ and fixed precision $\sigma = \frac{\varepsilon}{\sqrt{A_n^d}}$, as in the proof of Theorem 2, we obtain that $e^{ww}(S_d, N_n^d, \Phi_n^d) \leq \sqrt{\lambda_{d,1}} \varepsilon$ and

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon, d) \le \operatorname{cost}^{\operatorname{ww}}(N_n^d) \preccurlyeq \varepsilon^{-p} \kappa(d) \, \$(\hat{\varepsilon}, d),$$

where

$$\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon, d) = \begin{cases} \varepsilon \left(\kappa(d)\right)^{-1/p}, & p < 2, \\ \varepsilon \left(\kappa(d) \ln(\kappa(d)\varepsilon^{-2}\right)^{-1/2}, & p = 2, \\ \varepsilon^{p/2} \left(\kappa(d)\right)^{-1/2}, & p > 2. \end{cases}$$

Hence the problem is weakly tractable for noisy information if the cost function satisfies

(13)
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln \$(\hat{\varepsilon},d)}{\varepsilon^{-1}+d} = 0.$$

Observe that (iii) of Theorem 2 is obtained by taking p < 2 and $\kappa(d) = 1$, in which case $\hat{\varepsilon} = \varepsilon$.

It is not clear whether the condition (13) is not only sufficient, but also necessary for weak tractability.

Since intractability is defined as the lack of weak tractability, necessary and sufficient conditions for a problem to be intractable follow immediately from Theorem 2. We move to the curse of dimensionality.

Theorem 3. Consider a multivariate problem $S = \{S_d\}_{d\geq 1}$.

- (i) Suppose that
 - the problem with exact information suffers from the curse of dimensionality, or
 - the cost function grows exponentially in d for some $\sigma_0 \geq 0$.

Then the same problem with noisy information also suffers from the curse.

- (ii) Suppose the problem with noisy information suffers from the curse of dimensionality. Then
 - the problem with exact information also suffers from the curse, or
 - the cost function grows faster than polynomially in σ^{-1} , or grows exponentially in d.
- (iii) Suppose the problem with noisy information suffers from the curse of dimensionality. Then
 - the problem is not strongly polynomially tractable for exact information, or
 - the cost function grows exponentially in d for some $\sigma_0 \geq 0$.

Proof. To show (i) it suffices to use again Lemma 1. If the curse is present for exact information then, owing to $\operatorname{comp_{\$}^{ww}}(\varepsilon_0, d) \ge \operatorname{n^w}(\varepsilon_0, d) \$(1, 1)$, it is also present for noisy information. If the cost function grows exponentially in d for some $\sigma_0 > 0$ then for $\varepsilon_0 = \sigma_0$ we have

$$\limsup_{d \to \infty} \frac{\ln \left(\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon_0, d) \right)}{d} \ge \limsup_{d \to \infty} \frac{\ln \left(\$(\sigma_0, d) \right)}{d} > 0.$$

To show (ii), assume that there is no curse for exact information, and $\$(\sigma,d) \le 1 + D\sigma^{-2s}\kappa(d)$, where $\lim_{d\to\infty} \ln(\kappa(d))/d = 0$. Then, applying the reasoning from the proof (ii) of Theorem 2 we get that $\operatorname{comp}_{\$}^{ww}(\varepsilon,d) \le n + 2^s D\kappa(d) n^{s+1} \varepsilon^{-2s}$ with $n = n^w(\varepsilon/\sqrt{2},d)$. Hence

$$\lim_{d \to \infty} \frac{\ln\left(\text{comp}_{\$}^{\text{ww}}(\varepsilon, d)\right)}{d} = (s+1) \lim_{d \to \infty} \frac{\ln\left(\text{n}^{\text{w}}(\varepsilon/\sqrt{2}, d)\right)}{d} = 0,$$

which means that the problem with noisy information does not suffer from the curse.

And finally, to show (iii) we assume that the problem is strongly polynomially tractably for exact information, i.e., $n^{w}(\varepsilon, d) \leq C\varepsilon^{-p}$, and that $\$(\sigma, d)$ grows sub-exponentially in d for all $\sigma > 0$. Using $\sum_{j=1}^{n} \lambda_{d,j}/\lambda_{d,1} \leq n$ and proceeding as in the proof of (iii) of Theorem 2 we obtain for $n(\varepsilon) = \left[C\left(\frac{\sqrt{2}}{\varepsilon}\right)^{p}\right]$ that

$$\lim_{d\to\infty}\frac{\ln\left(\operatorname{comp}^{\mathrm{ww}}_{\$}(\varepsilon,d)\right)}{d}\leq \lim_{d\to\infty}\frac{\ln\$\left(\frac{\varepsilon}{2n(\varepsilon)},\,d\right)}{d}=0,$$

which means that there is no curse for noisy information.

4. Worst case setting with Gaussian noise

In this section we assume that the noise is random. That is, information about f is given as $\mathbf{y} = (y_1, y_2, \dots, y_{n(\mathbf{y})})$ where

$$y_i = L_i(f; y_1, \dots, y_{i-1}) + e_i, \qquad e_i \sim \mathcal{N}(0, \sigma_i^2(y_1, \dots, y_{i-1})),$$

and the noise coming from different observations is independent. The (total) cost of information $N = {\pi_f}_{f \in F}$, where π_f is the probability distribution of information \mathbf{y} for given f, is defined as

$$cost_{\$}^{wa}(N) = \sup_{\|f\|_{F_d} \le 1} \int_{Y} \sum_{i=1}^{n(\mathbf{y})} \$(\sigma_i(y_1, \dots, y_{i-1})) \, \pi_f(\mathbf{d}\mathbf{y}),$$

and the error of an algorithm Φ using information N as

$$e^{\text{wa}}(S_d, N, \Phi) = \sup_{\|f\|_{F_d} \le 1} \left(\int_Y \|S_d(f) - \Phi(\mathbf{y})\|_{G_d}^2 \, \pi_f(d\mathbf{y}) \right)^{1/2}.$$

As before, S_d are compact operators.

Define an auxiliary cost function $\widehat{\$}$ in such a way that $\widehat{\$}(\sigma,d)$ is the complexity of approximating a real parameter $f \in \mathbb{R}$ in the current setting using the cost function $\$(\cdot,d)$. (We stress that here f is not restricted to the interval [-1,1]. Possible approximations use noisy observations of f with adaptively chosen precisions σ_i .)

We clearly have $\widehat{\$} = \widehat{\$}$, and $\widehat{\$} \leq \$$ since the approximation $\widetilde{f} = f + e$, where $e \sim \mathcal{N}(0, \sigma^2)$, gives error σ at cost $\$(\sigma, d)$.

Lemma 2. For all $\varepsilon \in (0,1)$ and $d \ge 1$ we have

$$\operatorname{comp}_{\$}^{\operatorname{wa}}(\varepsilon, d) \ge \frac{\operatorname{n}^{\operatorname{w}}(2\varepsilon, d) + 1}{4}.$$

Also, there is $c \in (0,1)$ such that for all $\varepsilon \in (0,c)$ and $d \ge 1$ we have

$$\operatorname{comp}_{\$}^{\operatorname{wa}}(\varepsilon, d) \ge \frac{1}{2} \, \widehat{\$} \left(\frac{\varepsilon}{c\sqrt{2}}, d \right).$$

Proof. Let an algorithm Φ using information $N = \{\pi_f\}_{f \in F}$ be such that $e^{wa}(S_d, N, \Phi) \leq \varepsilon \sqrt{\lambda_1}$. Let

$$n = \sup_{\|f\|_{F_d} \le 1} \int_Y n(\mathbf{y}) \, \pi_f(\mathrm{d}\mathbf{y}).$$

Since $\$ \ge 1$, we have $n \le \operatorname{cost}^{\operatorname{wa}}_{\$}(N)$. Observe that any deterministic algorithm that uses noisy information can be interpreted as a randomized algorithm that uses exact information, where the noise is treated as a random parameter. Then, by [3, Theorem 4.42], there is deterministic algorithm using exact information of cardinality 4n-1 whose worst case error is at most 2ε . Hence

$$cost_{\$}^{wa}(N, \Phi) \ge n \ge \frac{n^{w}(2\varepsilon, d) + 1}{4}.$$

Taking the infimum with respect to all Φ and N we obtain the desired inequality.

To show the second inequality, we estimate the complexity of our problem from below by the complexity of the same problem, but with error taken over the interval $[-f_{d,1}^*, f_{d,1}^*]$ where, as before,

 $f_{d,1}^*$ is the eigenelement corresponding to the largest eigenvalue of $S_d^*S_d$. This is equivalent to the one-dimensional problem of approximating a prameter $f \in [-1, 1]$ from its noisy observations that is analyzed in Appendix of [2]. The worst case error of the latter can be lower bounded by the average error with respect to the two-point probability measure μ that assigns 1/2 to ± 1 . For any adaptive information such average error is not larger than $c \min(\sigma, 1)$ for some c > 0, where σ is such that

$$\sigma^{-2} = \int_{-1}^{1} \int_{Y} \sigma_{\mathbf{y}}^{-2} \, \pi_{f}(\mathrm{d}\mathbf{y}) \mu(\mathrm{d}f) = \int_{Y} \sigma_{\mathbf{y}}^{-2} \mu_{1}(\mathrm{d}\mathbf{y}), \qquad \sigma_{\mathbf{y}}^{-2} = \sum_{i=1}^{n(\mathbf{y})} \sigma_{i}^{-2} (y_{1}, \dots, y_{i-1}),$$

and μ_1 is the a priori distribution of information \mathbf{y} on Y, cf. [2, Lemma 3]. Another important property is that for any $\sigma_1, \ldots, \sigma_n$ and σ such that $\sigma^{-2} = \sum_{i=1}^n \sigma_i^{-2}$ we have

$$\sum_{i=1}^{n} \$(\sigma_i, d) \ge \widehat{\$}(\sigma, d),$$

which follows directly from the definition of $\hat{\$}$.

Let $A \subset Y$ be the set of all \mathbf{y} such that $\sigma_{\mathbf{y}}^{-2} \leq 2\sigma^{-2}$. Then $\mu_1(A) \geq 1/2$. Hence, if the error is at most $\varepsilon < c$ then $\sigma \leq \varepsilon/c$ and the cost is at least

$$\int_{A} \sum_{i=1}^{n(\mathbf{y})} \$ \left(\sigma_i(y_1, \dots, y_{i-1}), d \right) \mu_1(\mathrm{d}\mathbf{y}) \ge \int_{A} \sum_{i=1}^{n(\mathbf{y})} \widehat{\$} \left(\sigma_{\mathbf{y}}, d \right) \mu_1(\mathrm{d}\mathbf{y}) \ge \frac{1}{2} \widehat{\$} \left(\frac{\varepsilon}{c\sqrt{2}}, d \right),$$

as claimed. \Box

We now show some useful properties of $\hat{\$}$. For $d \geq 1$, define the two functions, h_1 and $h_{2,\lambda}$.

$$(0, +\infty) \ni x \mapsto h_1(x, d) = \$\left(\sqrt{\frac{1}{x}}, d\right),$$
 and

$$(0,\lambda) \ni x \mapsto h_{2,\lambda}(x,d) = \$\left(\sqrt{\frac{\lambda x}{\lambda - x}}, d\right).$$

Lemma 3. For any $d \ge 1$ we have the following.

(i) Suppose that $h_1(\cdot,d)$ is concave, and $h_{2,\lambda}(\cdot,d)$ is convex for all λ sufficiently large. Then

$$\widehat{\$}(\,\cdot\,,d) = \$(\,\cdot\,,d).$$

(ii) Suppose there is a line $\ell(x) = \alpha x$ supporting $h_1(\cdot, d)$ at some $x_0 > 0$, i.e., $\ell(x_0) = h_1(x_0, d)$ and $\ell(x) \le h(x, d)$ for all $x \ge 0$. Then for all $\sigma > 0$ we have

$$\frac{\alpha}{\sigma^2} \le \widehat{\$}(\sigma, d) \le \left\lceil \frac{\sigma_0^2}{\sigma^2} \right\rceil \frac{\alpha}{\sigma_0^2}, \quad where \quad \sigma_0^2 = 1/x_0.$$

(iii) We always have

$$\widehat{\$}(\sigma,d) \le \left\lceil \frac{\sigma_0^2}{\sigma^2} \right\rceil \$(\sigma_0,d), \quad \text{for any} \quad \sigma_0 > 0.$$

Proof. (i) We already noticed that $\hat{\$} \leq \$$. To bound $\hat{\$}$ from below we use the average case complexity of approximating $f \in \mathbb{R}$ from observations of f with Gaussian noise, where the average squared error and average cost are taken with respect to the one-dimensional Gaussian distribution μ_{λ} with mean zero and variance λ .

Consider first nonadaptive information consisting of n observations $y_i = f + e_i$ with precisions σ_i . Then the optimal algorithm is

$$\phi^*(\mathbf{y}) = \frac{\sigma^2 \lambda}{\sigma^2 + \lambda} \sum_{i=1}^n \frac{y_i}{\sigma_i^2}, \text{ where } \sigma^{-2} = \sum_{i=1}^n \sigma_i^{-2},$$

and its average squared error depends only on σ^2 and λ and equals

$$\int_{F} \int_{\mathbb{R}^{n}} |f - \phi^{*}(\mathbf{y})|^{2} \pi_{f}(\mathrm{d}\mathbf{y}) \mu_{\lambda}(\mathrm{d}f) = \frac{\sigma^{2} \lambda}{\sigma^{2} + \lambda},$$

cf. [6, Sect. 3.5]. By concavity of h_1 we have

$$h_{1}(\sigma^{-2}, d) = h_{1}\left(\sum_{i=1}^{n} \sigma_{i}^{-2}, d\right) \leq h_{1}(0, d) + h_{1}\left(\sum_{i=1}^{n} \sigma_{i}^{-2}, d\right) \leq h_{1}\left(\sum_{i=1}^{n-1} \sigma_{i}^{-2}, d\right) + h_{1}(\sigma_{n}^{-2}, d)$$

$$\leq h_{1}\left(\sum_{i=1}^{n-2} \sigma_{i}^{-2}, d\right) + h_{1}(\sigma_{n-1}^{-2}, d) + h_{1}(\sigma_{n}^{-2}, d) \leq \dots \leq \sum_{i=1}^{n} h_{1}(\sigma_{i}^{-2}, d),$$

so that $\$(\sigma,d) \le \sum_{i=1}^n \(σ_i,d) . This means that the cheapest way of obtaining an approximation with the average squared error $\sigma^2 < \lambda$ using nonadaptive information is to use just one observation y = f + e with variance $\sigma_1^2 = \sigma^2 \lambda / (\lambda - \sigma^2)$, for which the cost is

$$\psi_{\lambda}(\sigma) = \$\left(\sqrt{\frac{\sigma^2 \lambda}{\lambda - \sigma^2}}, d\right).$$

Since, by assumption, the function $\sigma \mapsto \psi_{\lambda}(\sqrt{\sigma})$ is convex for large λ , we can use [6, Lemma 3.9.2] to claim, that the cost $\psi_{\lambda}(\sigma)$ cannot be reduced using adaptive information. Hence $\widehat{\$}(\sigma) \geq \psi_{\lambda}(\sigma)$, and letting $\lambda \to \infty$ we obtain $\widehat{\$}(\sigma) \geq \(σ) , which forces $\widehat{\$}(\sigma) = \(σ) .

(ii) If the cost function is $\$_1(\sigma,d) = \alpha\sigma^{-2}$ then we have from (i) that $\$_1 = \$_1$. (Note that here we violate the assumption that the cost is at least 1.) Since $\$(\cdot,d) \ge \$_1(\cdot,d)$ then $\$(\sigma,d) \ge \$_1(\sigma,d) = \alpha\sigma^{-2}$. On the other hand, we can approximate $f \in \mathbb{R}$ with error σ using n nonadaptive observations with the same precision σ_0/\sqrt{n} , where $n = \lceil \sigma_0^2/\sigma^2 \rceil$. Hence,

$$\widehat{\$}(\sigma, d) \le n \, \$(\sigma_0, d) = \left\lceil \frac{\sigma_0^2}{\sigma^2} \right\rceil \frac{\alpha}{\sigma_0^2}.$$

(iii) The bound can be easily obtained by repetitive observations of variance σ_0^2 , as in (ii). **Example 2.** Assume that the cost function grows polynomially,

$$\$(\sigma, d) = 1 + Dd^t \sigma^{-2s}.$$

For $s \leq 1$ the function $h_1(\cdot, d)$ is obviously concave, and

$$x \mapsto h_{2,\lambda}(x,d) = 1 + Dd^t \left(\frac{1}{x} - \frac{1}{\lambda}\right)^s$$

is convex for all $\lambda > 0$, and therefore $\hat{\$} = \$$.

For $s \ge 1$ the function $\$(\cdot, d)$ is supported at $x_0 = ((s-1)Dd^t)^{-1/s}$ by $\ell(x) = \alpha_d x$, where $\alpha_d = s(s-1)^{1/s-1}(Dd^t)^{1/s}$.

Hence $\widehat{\$}(\sigma, d)$ essentially equals $\alpha_d \sigma^{-2}$.

Remark 3. Lemma 2 is an analogue of Lemma 1. In the case of bounded noise the corresponding auxiliary cost function would always be $\hat{\$} = \$$.

Similarly to the case of bounded noise, for upper bounds on tractability we use

(14)
$$\Phi_n^d(\mathbf{y}) = \sum_{i=1}^n y_i S_d(f_{d,i}^*),$$

where y_i approximates $\langle f, f_{d,i}^* \rangle_{F_d}$ for all $f \in F$ with the expected squared error σ_i^2 , and with cost $\widehat{\$}(\sigma_i, d)$. Then, for the corresponding information we have

(15)
$$e^{\text{wa}}(S_d, N_n^d, \Phi_n^d) = \sqrt{\sum_{i=1}^n \sigma_i^2 \lambda_{d,i} + \lambda_{d,n+1}},$$

and $\operatorname{cost}^{\operatorname{wa}}_{\$}(N_n^d) = \sum_{i=1}^n \widehat{\$}(\sigma_i, d)$. Note that $\operatorname{e}^{\operatorname{wa}}(S_d, N_n^d, \Phi_n^d) = \operatorname{e}^{\operatorname{ww}}(S_d, N_n^d, \Phi_n^d)$, where in the deterministic case the noise of *i*th observation is bounded by σ_i , cf. (7). Hence we can adopt the proof technique from Section ?? with the cost function $\widehat{\$}$ to obtain complexity bounds in the case of random noise. In particular, Lemma 3(iii) gives the following general upper bound.

Corollary 1. Suppose that $n^w(\varepsilon, d) \leq C d^q \varepsilon^{-p}$. Then for any fixed $\sigma_0 > 0$ we have

$$\operatorname{comp}_{\$}^{\operatorname{wa}}(\varepsilon, d) \preceq \sigma_0^2 \$(\sigma_0, d) d^{2q} \begin{cases} \varepsilon^{-2p}, & p > 1, \\ \ln^2(1/\varepsilon) \varepsilon^{-2}, & p = 1, \\ \varepsilon^{-2}, & p < 1. \end{cases}$$

4.1. Polynomial tractability.

Theorem 4. Consider a multivariate problem $S = \{S_d\}_{d \geq 1}$.

- (i) The problem with noisy information is polynomially tractable if and only if
 - it is polynomially tractable for exact information, and
 - the auxiliary cost function grows polynomially in d for some $\sigma_0 > 0$.
- (ii) The problem with noisy information is strongly polynomially tractable if and only if
 - ullet it is strongly polynomially tractable for exact information, and
- the auxiliary cost function is bounded in d for some $\sigma_0 > 0$.
- (iii) Suppose that $n^{w}(\varepsilon, d) \leq Cd^{q}\varepsilon^{-p}$ and $\$(\sigma, d) \leq 1 + Dd^{t}\sigma^{-2s}$. If p = 0 and s = 0 then $\operatorname{comp}_{\$}^{ww}(\varepsilon, d) \leq d^{t+q}$; otherwise

$$\operatorname{comp}_{\$}^{\operatorname{ww}}(\varepsilon,d) \preccurlyeq d^{\overline{t}+q(\overline{s}+1)} \left\{ \begin{array}{ll} \varepsilon^{-p(\overline{s}+1)}, & p(\overline{s}+1) > 2\overline{s}, \\ \ln^{\overline{s}+1}(1/\varepsilon) \, \varepsilon^{-2\overline{s}}, & p(\overline{s}+1) = 2\overline{s}, \\ \varepsilon^{-2\overline{s}}, & p(\overline{s}+1) < 2s, \end{array} \right.$$

where $\overline{t} = \min(t, t/s)$ and $\overline{s} = \min(s, 1)$.

Proof. Suppose the problem with noise is polynomially tractable, i.e., $\operatorname{comp}^{\operatorname{wa}}(\varepsilon, d) \leq C d^q \varepsilon^{-p}$ for $\varepsilon \in (0, 1)$ and $d \geq 1$. Then we have by Lemma 2 that, on one hand,

$$n^{w}(\varepsilon, d) \le 4 \operatorname{comp}^{wa}(\frac{\varepsilon}{2}, d) - 1 \le 4 C d^{q}(\frac{\varepsilon}{2})^{-p} \le d^{q} \varepsilon^{-p},$$

and, on the other hand,

$$\widehat{\$}(\sigma_0, d) \le 2 \operatorname{comp}^{\operatorname{wa}}(c\sqrt{2}\,\sigma_0, d) \le 2 + 2 \, C d^q (c\sqrt{2}\,\sigma_0)^{-p}.$$

This proves the necessary conditions in (i) and (ii). The sufficient conditions follow from Corollary 1. To show (iii) it suffices to note that $\widehat{\$}(\sigma,d) \preceq d^{\overline{t}}\sigma^{\overline{s}}$, as in Example 2, and use Theorem 1.

Remark 4. As in the case of bounded noise, see Remark 1, for Gaussian noise the algorithm Φ_n^d is not optimal for information N_n^d ; however, one can show that the exponents of tractability in (iii) of Theorem 4 cannot be improved when one relies only on information N_n^d . If arbitrary information is allowed then the optimal exponents are unknown. The point is that we do not know in general how to optimally choose the information. For a particular case, see Proposition 2 of Appendix.

4.2. Weak tractability and the curse of dimensionality.

Theorem 5. Consider a multivariate problem $S = \{S_d\}_{d \geq 1}$.

- (i) The problem with noisy information is weakly tractable if and only if the same problem with exact information is weakly tractable.
- (ii) The problem with noisy information suffers from the curse of dimensionality if and only if the same problem with exact information suffers from the curse.

Proof. (i) The necessary condition follows from the first inequality in Lemma 2. For sufficiency it is enough to consider the information N_n^d and algorithm Φ_n^d with $n = n^w(\varepsilon/\sqrt{2}, d)$ and $\sigma_i^2 = \varepsilon^2/(2n)$ for all i. Then the error is ε and the cost is $n \, \widehat{\$}(\varepsilon/\sqrt{n}, d) \le \beta n^2/\varepsilon^2$ for some $\beta > 0$. This implies

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln\left(\operatorname{comp}^{\mathrm{wa}}_{\$}(\varepsilon,d)\right)}{\varepsilon^{-1}+d}\leq\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln\beta+2\ln n+2\ln(1/\varepsilon)}{\varepsilon^{-1}+d}=0.$$

(ii) The sufficiency follows again from the first inequality in Lemma 2. For necessity it is enough to consider the same approximation as in (i) to get that if the problem with exact information does not suffer from the curse then for all ε we have

$$\lim_{d \to \infty} \frac{\ln\left(\text{comp}_{\$}^{\text{wa}}(\varepsilon, d)\right)}{d} \le \lim_{d \to \infty} \frac{\ln\beta + 2\ln n + 2\ln(1/\varepsilon)}{d} = 0,$$

i.e., the problem with noise as well does not suffer from the curse.

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APPENDIX

Let F and G be unitary spaces with dim $F = m < +\infty$, and $S : F \to G$ a nonsingular linear operator. Let $\{x_i^*\}_{i=1}^m$ be an orthonormal basis in F of eigenelements of the operator S^*S and $\{\lambda_i\}_{i=1}^m$ the corresponding eigenvalues, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$.

We fix the numbers

$$0 = \sigma_1 = \dots = \sigma_{n_0} < \sigma_{n_0+1} \le \dots \le \sigma_n$$

(where $n_0 = 0$ if all σ_i s are positive) and for any functionals L_i define

$$R^{ww}(L_1, L_2, \dots, L_n) = \max \{ ||Sx||_Y : x \in X, |L_i x| \le \sigma_i, 1 \le i \le n \}.$$

This is obviously the radius of information $y_i = L_i x + e_i$, $1 \le i \le n$, in the worst case setting with bounded noise, $|e_i| \le \sigma_i$, where the worst case error is taken with respect to the whole space F (instead of the unit ball of F), cf. (3).

Proposition 1. Let n = m. For any functionals L_i with $||L_i|| \le 1$ for $1 \le i \le n$, we have

(16)
$$R^{ww}(L_1, L_2, \dots, L_n) \ge \left(\sum_{i=m+1}^n \sigma_i^2 \lambda_i\right)^{1/2}.$$

The equality above holds for $L_i^* = \langle \cdot, x_i^* \rangle_F$.

Proof. We can assume without loss of generality that the functionals L_i are linearly independent since otherwise $R^{ww}(L_1, L_2, \ldots, L_n)$ is infinite.

Let $\{x_j\}_{j=1}^n$ be the basis in F that is adjoint to $\{L_i\}_{i=1}^n$, i.e., $L_i x_j = \delta_{i,j}$ for $1 \leq i, j \leq n$. Let

$$V_i = \operatorname{span}(x_j : 1 \le j \le n, j \ne i).$$

Observe that

$$||L_i|| = \max_{x \neq 0} \frac{|L_i x|}{||x||_F} = \max_{v \in V_i} \frac{1}{||x_i + v||_F} = \frac{1}{\operatorname{dist}(x_i, V_i)}.$$

Hence the condition $||L_i|| \le 1$ is equivalent to

$$\operatorname{dist}(x_i, V_i) \ge 1.$$

Since for any $x = \sum_{j=1}^{n} c_j x_j \in F$ is $L_i x = c_i$, inequality (16) can be equivalently rewritten as

(17)
$$\max\left\{\left\|\sum_{i=1}^{n} c_{i} S x_{i}\right\|_{G}^{2} : \left|c_{i}\right| \leq \sigma_{i}, 1 \leq i \leq n\right\} \geq \sum_{i=1}^{n} \sigma_{i}^{2} \lambda_{i}.$$

To show (17) we use induction on n. For n = 1 we have

$$\max_{|c_1| \le \sigma_1} \|c_1 S x_1\|_G^2 = \max_{|c_1| \le \sigma_1} c_1^2 \lambda_1 = \sigma_1^2 \lambda_1.$$

Suppose $n \geq 2$. Let

$$W = \operatorname{span}(Sx_1, \dots, Sx_{n-1})$$

and Z be the subspace of im A that is perpendicular to W. We obviously have dim W = n - 1 and dim Z = 1. Let us decompose S as

$$S = P_W S + P_Z S,$$

where P_W and P_Z are the orthogonal projections onto W and Z, respectively, and denote by

$$S_W: V_n \to G$$
 and $S_Z: V_n^{\perp} \to G$

the operators P_WS and P_ZS restricted, respectively, to V_n and V_n^{\perp} . Let

$$\widehat{\lambda}_1 \ge \dots \ge \widehat{\lambda}_{n-1} > 0$$

be the eigenvalues of $S_W^*S_W: V_n \to V_n$, and $\widehat{\lambda}_n > 0$ be the only eigenvalue of $S_Z^*S_Z: V_n^{\perp} \to V_n^{\perp}$. Observe that for $1 \le i \le n-1$ the distance of x_i from the subspace span $(x_j: j \in \{1, \ldots, n\} \setminus \{i, n\})$ is at least 1. Then, by induction hypothesis, there are c_i such that $|c_i| \le \sigma_i$, $1 \le i \le n-1$, and

$$\left\| \sum_{i=1}^{n-1} c_i S x_i \right\|_G^2 \ge \sum_{i=1}^{n-1} \sigma_i^2 \widehat{\lambda}_i.$$

Letting v be the orthogonal projection of x_n onto V^{\perp} we also have that

$$||Sx_n||_G^2 \ge ||P_Z Sx_n||_G^2 = ||P_Z Sv||_G^2 = \sigma_n^2 \widehat{\lambda}_n.$$

Setting $c_n = \sigma_n$ if $\left\langle \sum_{i=1}^{n-1} c_i Sx_i, Sx_n \right\rangle_G \geq 0$, and $c_n = -\sigma_n$ otherwise, we obtain

$$\left\| \sum_{i=1}^{n-1} c_i Sx_i + c_n Sx_n \right\|_G^2 \ge \left\| \sum_{i=1}^{n-1} c_i Sx_i \right\|_G^2 + \left\| Sx_n \right\|_G^2 \ge \sum_{i=1}^n \sigma_i^2 \widehat{\lambda}_i.$$

To complete the proof of (17) it suffices to use the fact that

$$\sum_{i=1}^{k} \widehat{\lambda}_i \le \sum_{i=1}^{k} \lambda_i \quad \text{for} \quad 1 \le k \le n-1, \quad \text{and} \quad \sum_{i=1}^{n} \widehat{\lambda}_i = \sum_{i=1}^{n} \lambda_i,$$

which implies $\sum_{i=1}^{n} \sigma_i^2 \hat{\lambda}_i \ge \sum_{i=1}^{n} \sigma_i^2 \lambda_i$.

The remaining part of the proposition is obvious, since orthogonality of $\{Sx_i^*\}_{i=1}^n$ implies that for any c_i with $|c_i| \leq \sigma_i$ we have

$$\left\| \sum_{i=1}^{n} c_{i} S x_{i}^{*} \right\|_{G}^{2} = \sum_{i=1}^{n} c_{i}^{2} \| S x_{i}^{*} \|_{G}^{2} = \sum_{i=1}^{n} c_{i}^{2} \lambda_{i} \leq \sum_{i=1}^{n} \sigma_{i}^{2} \lambda_{i}.$$

Proposition 1 should be confronted with the related result in the case of Gaussian noise. Let

$$R^{\text{wa}}(L_1, L_2, \dots, L_n) = \inf_{\Phi} \sup_{x \in F} \left(\int_{\mathbb{R}^n} ||Sx - \Phi(\mathbf{y})||_G^2 \pi_x(d\mathbf{y}) \right)^{1/2},$$

where π_f is the *n*-dimensional Gaussian measure with mean (L_1x, \ldots, L_nx) and covariance matrix $\operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2)$, be the minimal error in the worst case setting with Gaussian noise, where again the worst case error is taken with respect to the whole space F. In this case the optimal choice of the functionals L_i is known for all $n \geq m$, but their construction is much more complicated.

Proposition 2. Let $n \geq m$. For any functionals L_i with $||L_i|| \leq 1$ for $1 \leq i \leq n$, we have

(18)
$$R^{\text{wa}}(L_1, L_2, \dots, L_n) \ge \left(\sum_{i=n_0+1}^m \widehat{\sigma}_i^2 \lambda_i\right)^{1/2},$$

where $\widehat{\sigma}_{n_0+1} \leq \cdots \leq \widehat{\sigma}_m$ minimize the sum $\sum_{i=n_0+1}^m \eta_i^2 \lambda_i$ with respect to all $\eta_{n_0+1} \leq \cdots \leq \eta_m$ satisfying

$$\sum_{i=k}^{m} \eta_i^{-2} \le \sum_{i=k}^{n} \sigma_i^{-2} \quad for \quad k = n_0 + 1 \le k \le n,$$

and $\sum_{i=n_0+1}^{m} \eta_i^{-2} = \sum_{i=i_0+1}^{n} \sigma_i^{-2}$. The equality in (18) holds for $L_i^* = \langle \cdot, x_i^* \rangle_F$, $1 \le i \le n_0$, and

$$L_{n_0+i}^* = \sigma_{n_0+i} \sum_{j=1}^{m-n_0} w_{i,j} \langle \cdot, x_{n_0+j}^* \rangle_F, \qquad 1 \le i \le n - n_0,$$

where the matrix $W = \{w_{i,j}\} \in \mathbb{R}^{(n-n_0)\times(m-n_0)}$ satisfies

$$W^T W = \operatorname{diag}(\widehat{\sigma}_{n_0+1}^{-2}, \dots, \widehat{\sigma}_m^{-2}) \quad and \quad ||W^T \vec{e}_i||_2^2 = \sigma_{n_0+i}^{-2}, \quad 1 \le i \le n - n_0.$$

A proof of this proposition as well as construction of the matrix W can be derived from Theorems 3.8.1 and 3.8.2 and Lemma 2.8.1 in [6]. One considers the average case error with respect to the Gaussian measure on \mathbb{R}^m with mean zero and covariance matrix λI_m (where I_m is the $m \times m$ identity matrx). The worst case error is obtained as the limiting case when λ increases to infinity.

We add that for n=m we have $L_i^*=\langle \,\cdot\,,x_i^*\rangle_F$ for all $1\leq i\leq n$ if and only if

$$\sigma_{n_0+i}^{-2} = \frac{\lambda_{n_0+i}^{1/2}}{\sum_{j=n_0+1}^n \lambda_j^{1/2}} \sum_{j=n_0+1}^n \sigma_j^{-2} \quad \text{for} \quad 1 \le i \le n - n_0,$$

and then $R(\langle \cdot, x_1^* \rangle_F, \dots, \langle \cdot, x_n^* \rangle_F) = \left(\sum_{i=n_0+1}^n \sigma_i^2 \lambda_i\right)^{1/2}$.

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