# A new difference scheme for the time fractional diffusion equation 

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#### Abstract

In this paper we construct a new difference analog of the Caputo fractional derivative (called the $L 2-1_{\sigma}$ formula). The basic properties of this difference operator are investigated and on its basis some difference schemes generating approximations of the second and fourth order in space and the second order in time for the time fractional diffusion equation with variable coefficients are considered. Stability of the suggested schemes and also their convergence in the grid $L_{2}$ - norm with the rate equal to the order of the approximation error are proved. The obtained results are supported by the numerical calculations carried out for some test problems.


Keywords: fractional diffusion equation, finite difference method, stability, convergence

## 1. Introduction

Recently a noticeable growth of the attention of researches to the fractional differential equations has been observed. It is caused by numerous effective applications of fractional calculation to various areas of science and engineering [1, 2, 3, 4, 5, 6]. For example, mathematical language of fractional derivatives is irreplaceable for the description of the physical process of statistical transfer and, as it is known, leads to diffusion equations of fractional orders [7, 8].

Consider the time fractional diffusion equation with variable coefficients

$$
\begin{equation*}
\partial_{0 t}^{\alpha} u(x, t)=\mathcal{L} u(x, t)+f(x, t), 0<x<l, 0<t \leq T, \tag{1}
\end{equation*}
$$

[^0]\[

$$
\begin{equation*}
u(0, t)=0, \quad u(l, t)=0, \quad 0 \leq t \leq T, \quad u(x, 0)=u_{0}(x), \quad 0 \leq x \leq l \tag{2}
\end{equation*}
$$

\]

where

$$
\begin{equation*}
\partial_{0 t}^{\alpha} u(x, t)=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{\partial u(x, \eta)}{\partial \eta}(t-\eta)^{-\alpha} d \eta, \quad 0<\alpha<1 \tag{3}
\end{equation*}
$$

is the Caputo derivative of the order $\alpha$,

$$
\mathcal{L} u(x, t)=\frac{\partial}{\partial x}\left(k(x, t) \frac{\partial u}{\partial x}\right)-q(x, t) u
$$

$k(x, t) \geq c_{1}>0, q(x, t) \geq 0$ and $f(x, t)$ are sufficiently smooth functions.
The time fractional diffusion equation represents a linear integro - differential equation. Its solution not always can be found analytically; therefore it is necessary to use numerical methods. However, unlike the classical case, we require information about all the previous time layers, when numerically approximating a time fractional diffusion equation on a certain time layer. For that reason algorithms for solving the time fractional diffusion equations are rather time-consuming even in one - dimensional case. Upon transition to two - dimensional and three - dimensional problems their complexity considerably increases. In this regard constructing stable differential schemes of higher order approximation is a very important task.

A widespread difference approximation of fractional derivative (3) is the so-called $L 1$ method [2, 9] which is defined as follows

$$
\begin{equation*}
\partial_{0 t_{j+1}}^{\alpha} u(x, t)=\frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{j} \frac{u\left(x, t_{s+1}\right)-u\left(x, t_{s}\right)}{t_{s+1}-t_{s}} \int_{t_{s}}^{t_{s+1}} \frac{d \eta}{\left(t_{j+1}-\eta\right)^{\alpha}}+r^{j+1} \tag{4}
\end{equation*}
$$

where $0=t_{0}<t_{1}<\ldots<t_{j+1}$, and $r^{j+1}$ is the local truncation error. In the case of the uniform mesh, $\tau=t_{s+1}-t_{s}$, for all $s=0,1, \ldots, j+1$, it was proved that $r^{j+1}=\mathcal{O}\left(\tau^{2-\alpha}\right)$ [10, 11, 12]. The $L 1$ method has been widely used for solving the fractional differential equations with Caputo derivatives [10, 11, 12, 13, 14, 15, 16].

Difference schemes of the increased order of approximation such as the compact difference scheme [14, 17, 18, 19] and spectral method [11, 20, 21] were applied to improve the spatial accuracy of fractional diffusion equations. However, it is rather difficult to get a high-order time approximation due to the singularity of fractional derivatives.

A good approximation of the $L 1$ method is observed in case of a nonuniform mesh, when it is refined in a neighborhood of the point $t_{j+1}[9]$. Though the nonuniform mesh turns out to be more effective in comparison with the uniform one, it will not generate the second order of approximation in all points of the mesh.

In [22] a new difference analog of the Caputo fractional derivative with the order of approximation $\mathcal{O}\left(\tau^{3-\alpha}\right)$, called $L 1-2$ formula, is constructed. On the basis of this formula calculations of difference schemes for the timefractional sub-diffusion equations in bounded and unbounded spatial domains and the fractional ODEs are carried out. If the stability and convergence of difference schemes from [22] will be strictly proved, then this will undoubtedly be a significant progress in computing the time-fractional partial differential equations.

Using the energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the diffusion-wave equation with Caputo fractional derivative have been obtained in [15, 23].

In this paper a new difference analog of the fractional Caputo derivative with the order of approximation $\mathcal{O}\left(\tau^{3-\alpha}\right)$ for each $\alpha \in(0,1)$ is constructed. Properties of the obtained difference operator are studied. Difference schemes of the second and fourth order of approximation in space and the second order in time for the time fractional diffusion equation with variable coefficients are constructed. Using the method of energy inequalities, the stability and convergence of these schemes in the mesh $L_{2}$ - norm are proved. Numerical calculations of some test problems confirming reliability of the obtained results are carried out.

## 2. Family of difference schemes. Stability and convergence

In this section, families of difference schemes in a general form set on a non-uniform time mesh are investigated. A criterion of the stability of the difference schemes in the mesh $L_{2}$ - norm is obtained. The convergence of solutions of the difference schemes to the solution of the corresponding differential problem with the rate equal to the order of the approximation error is proved.

### 2.1. Family of difference schemes

In the rectangle $\bar{Q}_{T}=\{(x, t): 0 \leq x \leq l, 0 \leq t \leq T\}$ we introduce the mesh $\bar{\omega}_{h \tau}=\bar{\omega}_{h} \times \bar{\omega}_{\tau}$, where $\bar{\omega}_{h}=\left\{x_{i}=i h, i=0,1, \ldots, N ; h N=l\right\}$,
$\bar{\omega}_{\tau}=\left\{t_{j}: 0=t_{0}<t_{1}<t_{2}<\ldots<t_{M-1}<t_{M}=T\right\}$.
Basically the family of difference schemes, approximating problem (1)-(2) on the mesh $\bar{\omega}_{h \tau}$, has the form

$$
\begin{gather*}
{ }_{g} \Delta_{0 t_{j+1}}^{\alpha} y_{i}=\Lambda y_{i}^{\left(\sigma_{j+1}\right)}+\varphi_{i}^{j+1}, \quad i=1,2, \ldots, N-1, \quad j=0,1, \ldots, M-1,  \tag{5}\\
y(0, t)=0, \quad y(l, t)=0, \quad t \in \bar{\omega}_{\tau}, \quad y(x, 0)=u_{0}(x), \quad x \in \bar{\omega}_{h} \tag{6}
\end{gather*}
$$

where

$$
\begin{equation*}
{ }_{g} \Delta_{0 t_{j+1}}^{\alpha} y_{i}=\sum_{s=0}^{j}\left(y_{i}^{s+1}-y_{i}^{s}\right) g_{s}^{j+1}, \quad g_{s}^{j+1}>0 \tag{7}
\end{equation*}
$$

is a difference analog of the Caputo derivative of the order $\alpha(0<\alpha<1)$, $\Lambda$ is a difference operator approximating the continuous operator $\mathcal{L}$, such that the operator $-\Lambda$ preserves its positive definiteness $\left((-\Lambda y, y) \geq \varkappa\|y\|^{2}\right.$, $\varkappa>0$ ), for example

$$
\begin{equation*}
(\Lambda y)_{i}=\left(\left(a y_{\bar{x}}\right)_{x}-d y\right)_{i}=\frac{a_{i+1} y_{i+1}-\left(a_{i+1}+a_{i}\right) y_{i}+a_{i} y_{i-1}}{h^{2}}-d_{i} y_{i} \tag{8}
\end{equation*}
$$

$a, d$ and $\varphi$ are the mesh functions approximating $k, q$ and $f$, respectively, $y^{\left(\sigma_{j+1}\right)}=\sigma_{j+1} y^{j+1}+\left(1-\sigma_{j+1}\right) y^{j}, 0 \leq \sigma_{j+1} \leq 1$, at $j=0,1, \ldots, M-1$, $y_{\bar{x}, i}=\left(y_{i}-y_{i-1}\right) / h, y_{x, i}=\left(y_{i+1}-y_{i}\right) / h$.

### 2.2. Stability and convergence

Lemma 1. If $g_{j}^{j+1}>g_{j-1}^{j+1}>\ldots>g_{0}^{j+1}>0, j=0,1, \ldots, M-1$ then for any function $v(t)$ defined on the mesh $\bar{\omega}_{\tau}$ one has the inequalities

$$
\begin{gather*}
v^{j+1}\left({ }_{g} \Delta_{0 t}^{\alpha} v\right) \geq \frac{1}{2}{ }_{g} \Delta_{0 t}^{\alpha}\left(v^{2}\right)+\frac{1}{2 g_{j}^{j+1}}\left({ }_{g} \Delta_{0 t}^{\alpha} v\right)^{2},  \tag{9}\\
v^{j}{ }_{g} \Delta_{0 t}^{\alpha} v \geq \frac{1}{2}{ }_{g} \Delta_{0 t}^{\alpha}\left(v^{2}\right)-\frac{1}{2\left(g_{j}^{j+1}-g_{j-1}^{j+1}\right)}\left({ }_{g} \Delta_{0 t}^{\alpha} v\right)^{2}, \tag{10}
\end{gather*}
$$

where $g_{-1}^{1}=0$.
Proof. Let us consider the difference

$$
v^{j+1}{ }_{g} \Delta_{0 t}^{\alpha} v-\frac{1}{2}{ }_{g} \Delta_{0 t}^{\alpha}\left(v^{2}\right)
$$

$$
\begin{align*}
& =v^{j+1} \sum_{s=0}^{j} g_{s}^{j+1}\left(v^{s+1}-v^{s}\right)-\sum_{s=0}^{j} g_{s}^{j+1}\left(v^{s+1}-v^{s}\right)\left(\frac{v^{s+1}+v^{s}}{2}\right) \\
& =\sum_{s=0}^{j} g_{s}^{j+1}\left(v^{s+1}-v^{s}\right)\left(v^{j+1}-\frac{v^{s+1}+v^{s}}{2}\right) \\
& =\sum_{s=0}^{j} g_{s}^{j+1}\left(v^{s+1}-v^{s}\right)\left(\frac{1}{2}\left(v^{s+1}-v^{s}\right)+\sum_{k=s+1}^{j}\left(v^{k+1}-v^{k}\right)\right) \\
& =\frac{1}{2} \sum_{s=0}^{j} g_{s}^{j+1}\left(v^{s+1}-v^{s}\right)^{2}+\sum_{k=1}^{j}\left(v^{k+1}-v^{k}\right) \sum_{s=0}^{k-1} g_{s}^{j+1}\left(v^{s+1}-v^{s}\right) . \tag{11}
\end{align*}
$$

Here we consider the sums to be equal to zero if the upper summation index is less than the lower one.

Let us introduce the following notation: $\sum_{s=0}^{k} g_{s}^{j+1}\left(v^{s+1}-v^{s}\right)=w^{k+1}$, then $v^{1}-v^{0}=\left(g_{0}^{j+1}\right)^{-1} w^{1}, v^{k+1}-v^{k}=\left(g_{k}^{j+1}\right)^{-1}\left(w^{k+1}-w^{k}\right), k=1,2, \ldots, j$ and rewrite the equality (11) as

$$
\begin{gathered}
\frac{1}{2}\left(g_{0}^{j+1}\right)^{-1}\left(w^{1}\right)^{2}+\frac{1}{2} \sum_{k=1}^{j}\left(g_{k}^{j+1}\right)^{-1}\left(w^{k+1}-w^{k}\right)^{2}+\sum_{k=1}^{j}\left(g_{k}^{j+1}\right)^{-1}\left(w^{k+1}-w^{k}\right) w^{k} \\
\quad=\frac{1}{2}\left(g_{0}^{j+1}\right)^{-1}\left(w^{1}\right)^{2}+\frac{1}{2} \sum_{k=1}^{j}\left(g_{k}^{j+1}\right)^{-1}\left(\left(w^{k+1}\right)^{2}-\left(w^{k}\right)^{2}\right) \\
\quad=\frac{1}{2}\left(g_{j}^{j+1}\right)^{-1}\left(w^{j+1}\right)^{2}+\frac{1}{2} \sum_{k=0}^{j-1} \frac{g_{k+1}^{j+1}-g_{k}^{j+1}}{g_{k+1}^{j+1} g_{k}^{j+1}}\left(w^{k+1}\right)^{2} \geq \frac{1}{2}\left(g_{j}^{j+1}\right)^{-1}\left(w^{j+1}\right)^{2}
\end{gathered}
$$

which is valid since $g_{k+1}^{j+1}-g_{k}^{j+1}>0, k=0,1, \ldots, j-1$.
Let us prove now the inequality (10). Since $v^{j}=v^{j+1}-\left(v^{j+1}-v^{j}\right)$, one obtains

$$
\begin{gathered}
v^{j} \Delta_{0 t}^{\alpha} v-\frac{1}{2} \Delta_{0 t}^{\alpha}\left(v^{2}\right)+\frac{1}{2\left(g_{j}^{j+1}-g_{j-1}^{j+1}\right)}\left(\Delta_{0 t}^{\alpha} v\right)^{2} \\
=v^{j+1} \Delta_{0 t}^{\alpha} v-\frac{1}{2} \Delta_{0 t}^{\alpha}\left(v^{2}\right)+\frac{1}{2\left(g_{j}^{j+1}-g_{j-1}^{j+1}\right)}\left(\Delta_{0 t}^{\alpha} v\right)^{2}-\left(v^{j+1}-v^{j}\right) \Delta_{0 t}^{\alpha} v \\
=\frac{1}{2}\left(g_{j}^{j+1}\right)^{-1}\left(w^{j+1}\right)^{2}+\frac{1}{2} \sum_{k=0}^{j-1} \frac{g_{k+1}^{j+1}-g_{k}^{j+1}}{g_{k+1}^{j+1} g_{k}^{j+1}}\left(w^{k+1}\right)^{2}
\end{gathered}
$$

$$
\begin{gathered}
+\frac{1}{2\left(g_{j}^{j+1}-g_{j-1}^{j+1}\right)}\left(w^{j+1}\right)^{2}-\left(g_{j}^{j+1}\right)^{-1}\left(w^{j+1}-w^{j}\right) w^{j+1} \\
=\frac{g_{j-1}^{j+1}}{2 g_{j}^{j+1}\left(g_{j}^{j+1}-g_{j-1}^{j+1}\right)}\left(w^{j+1}+\frac{g_{j}^{j+1}-g_{j-1}^{j+1}}{g_{j-1}^{j+1}} w^{j}\right)^{2}+\frac{1}{2} \sum_{k=0}^{j-2} \frac{g_{k+1}^{j+1}-g_{k}^{j+1}}{g_{k+1}^{j+1} g_{k}^{j+1}}\left(w^{k+1}\right)^{2} \geq 0 .
\end{gathered}
$$

The proof of the Lemma 1 is completed.
Corollary 1. If $g_{j}^{j+1}>g_{j-1}^{j+1}>\ldots>g_{0}^{j+1}>0$ and $\frac{g_{j}^{j+1}}{2 g_{j}^{j+1}-g_{j-1}^{j+1}} \leq \sigma_{j+1} \leq 1$, where $j=0,1, \ldots, M-1, g_{-1}^{1}=0$, then for any function $v(t)$ defined on the mesh $\bar{\omega}_{\tau}$ one has the inequality

$$
\begin{equation*}
\left(\sigma_{j+1} v^{j+1}+\left(1-\sigma_{j+1}\right) v^{j}\right)_{g} \Delta_{0 t}^{\alpha} v \geq \frac{1}{2}{ }^{g} \Delta_{0 t}^{\alpha}\left(v^{2}\right) . \tag{12}
\end{equation*}
$$

Theorem 1. If

$$
g_{j}^{j+1}>g_{j-1}^{j+1}>\ldots>g_{0}^{j+1} \geq c_{2}>0, \quad \frac{g_{j}^{j+1}}{2 g_{j}^{j+1}-g_{j-1}^{j+1}} \leq \sigma_{j+1} \leq 1
$$

where $j=0,1, \ldots, M-1, g_{-1}^{1}=0$, then the difference scheme (5) - (6) is unconditionally stable and its solution satisfies the following a priori estimate:

$$
\begin{equation*}
\left\|y^{j+1}\right\|_{0}^{2} \leq\left\|y^{0}\right\|_{0}^{2}+\frac{1}{2 \varkappa c_{2}} \max _{0 \leq j \leq M}\left\|\varphi^{j}\right\|_{0}^{2} \tag{13}
\end{equation*}
$$

where $(y, v)=\sum_{i=1}^{N-1} y_{i} v_{i} h,\|y\|_{0}^{2}=(y, y)$.
Proof. Taking the inner product of the equation (5) with $y^{\left(\sigma_{j+1}\right)}$, we have

$$
\begin{equation*}
\left(y^{\left(\sigma_{j+1}\right)},{ }_{g} \Delta_{0 t}^{\alpha} y\right)-\left(y^{\left(\sigma_{j+1}\right)}, \Lambda y^{\left(\sigma_{j+1}\right)}\right)=\left(y^{\left(\sigma_{j+1}\right)}, \varphi^{j+1}\right) \tag{14}
\end{equation*}
$$

Using inequality (12) and the positive definiteness of operator $A=-\Lambda$ from identity (14) one obtains

$$
\begin{equation*}
\frac{1}{2}{ }^{g} \Delta_{0 t}^{\alpha}\|y\|_{0}^{2}+\varkappa\left\|y^{\left(\sigma_{j+1}\right)}\right\|_{0}^{2} \leq \varepsilon\left\|y^{\left(\sigma_{j+1}\right)}\right\|_{0}^{2}+\frac{1}{4 \varepsilon}\left\|\varphi^{j+1}\right\|_{0}^{2}, \quad \varepsilon>0 . \tag{15}
\end{equation*}
$$

From (15), at $\varepsilon=\varkappa$ we get

$$
\begin{equation*}
{ }_{g} \Delta_{0 t}^{\alpha}\|y\|_{0}^{2} \leq \frac{1}{2 \varkappa}\left\|\varphi^{j+1}\right\|_{0}^{2} \tag{16}
\end{equation*}
$$

Let us rewrite inequality (16) in the form

$$
\begin{equation*}
g_{j}^{j+1}\left\|y^{j+1}\right\|_{0}^{2} \leq \sum_{s=1}^{j}\left(g_{s}^{j+1}-g_{s-1}^{j+1}\right)\left\|y^{s}\right\|_{0}^{2}+g_{0}^{j+1}\left\|y^{0}\right\|_{0}^{2}+\frac{1}{2 \varkappa}\left\|\varphi^{j+1}\right\|_{0}^{2} \tag{17}
\end{equation*}
$$

Noticing that $g_{0}^{j+1} \geq c_{2}>0$, we get

$$
\begin{equation*}
g_{j}^{j+1}\left\|y^{j+1}\right\|_{0}^{2} \leq \sum_{s=1}^{j}\left(g_{s}^{j+1}-g_{s-1}^{j+1}\right)\left\|y^{s}\right\|_{0}^{2}+g_{0}^{j+1}\left(\left\|y^{0}\right\|_{0}^{2}+\frac{1}{2 \varkappa c_{2}}\left\|\varphi^{j+1}\right\|_{0}^{2}\right) \tag{18}
\end{equation*}
$$

Denote

$$
E=\left\|y^{0}\right\|_{0}^{2}+\frac{1}{2 \varkappa c_{2}} \max _{0 \leq j \leq M}\left\|\varphi^{j}\right\|_{0}^{2}
$$

The inequality (18) is reduced to

$$
\begin{equation*}
g_{j}^{j+1}\left\|y^{j+1}\right\|_{0}^{2} \leq \sum_{s=1}^{j}\left(g_{s}^{j+1}-g_{s-1}^{j+1}\right)\left\|y^{s}\right\|_{0}^{2}+g_{0}^{j+1} E \tag{19}
\end{equation*}
$$

It is obvious that at $j=0$ the a priori estimate (13) follows from (19). Let us prove that (13) holds for $j=1,2, \ldots$ by using the mathematical induction method. For this purpose, let us assume that the a priori estimate (13) takes place for all $j=0,1, \ldots, k-1$ :

$$
\left\|y^{j+1}\right\|_{0}^{2} \leq E, \quad j=0,1, \ldots, k-1
$$

From (19) at $j=k$ one has

$$
\begin{align*}
& g_{k}^{k+1}\left\|y^{k+1}\right\|_{0}^{2} \leq \sum_{s=1}^{k}\left(g_{s}^{k+1}-g_{s-1}^{k+1}\right)\left\|y^{s}\right\|_{0}^{2}+g_{0}^{k+1} E \\
& \quad \leq \sum_{s=1}^{k}\left(g_{s}^{k+1}-g_{s-1}^{k+1}\right) E+g_{0}^{k+1} E=g_{k}^{k+1} E \tag{20}
\end{align*}
$$

The proof of Theorem 1 is completed.
A priori estimate (13) implies the stability of difference scheme (5) -(6).
Theorem 2. If the conditions of Theorem 1 are satisfied and difference scheme (5)-(6) has the approximation order $\mathcal{O}\left(N^{-r_{1}}+M^{-r_{2}}\right)$, where $r_{1}$ and $r_{2}$
are some known positive numbers, then the solution of difference scheme (5) (6) converges to the solution of differential problem (11)-(2) in the mesh $L_{2}$ norm with the rate equal to the order of the approximation error $\mathcal{O}\left(N^{-r_{1}}+\right.$ $M^{-r_{2}}$.

Proof. Let us introduce the error $z=y-u$ and substitute it into (5)-(6). Then we obtain the problem for the error

$$
\begin{gather*}
{ }_{g} \Delta_{0 t}^{\alpha} z_{i}=\Lambda z_{i}^{\left(\sigma_{j+1}\right)}+\psi_{i}^{j+1}, \quad i=1, \ldots, N-1, \quad j=0,1, \ldots, M-1  \tag{21}\\
z(0, t)=0, \quad z(l, t)=0, \quad t \in \bar{\omega}_{\tau}, \quad z(x, 0)=0, \quad x \in \bar{\omega}_{h} \tag{22}
\end{gather*}
$$

where $\psi_{i}^{j+1}=\Lambda u_{i}^{\left(\sigma_{j+1}\right)}-{ }_{g} \Delta_{0 t}^{\alpha} u_{i}+\varphi_{i}^{j+1}, \quad \psi_{i}^{j+1}=\mathcal{O}\left(N^{-r_{1}}+M^{-r_{2}}\right)$.
Since the conditions of Theorem 1 are fulfilled, then a priori estimate (13) holds true for the solution of problem (21)-(22) and, therefore, the following inequality takes place

$$
\|z\|_{0} \leq \frac{1}{\sqrt{2 \varkappa c_{2}}} \max _{0 \leq j \leq M}\left\|\psi^{j}\right\|_{0}=\mathcal{O}\left(N^{-r_{1}}+M^{-r_{2}}\right)
$$

which implies the convergence in the mesh $L_{2}$ - norm with the rate $\mathcal{O}\left(N^{-r_{1}}+\right.$ $M^{-r_{2}}$.

## 3. A new $\mathrm{L} 2-1_{\sigma}$ fractional numerical differentiation formula

In this section a difference analog of the Caputo fractional derivative with the approximation order $O\left(\tau^{3-\alpha}\right)$ is constructed and its basic properties are investigated.

Let us consider the uniform mesh $\bar{\omega}_{\tau}=\left\{t_{j}=j \tau, j=0,1, \ldots, M ; T=\right.$ $\tau M\}$. Let $\sigma=1-\frac{\alpha}{2}$, then for the Caputo fractional derivative of the order $\alpha, 0<\alpha<1$, of the function $u(t) \in \mathcal{C}^{3}[0, T]$ at the fixed point $t_{j+\sigma}, j \in$ $\{0,1, \ldots, M-1\}$ the following equalities hold

$$
\begin{gather*}
\partial_{0 t_{j+\sigma}}^{\alpha} u(\eta)=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t_{j+\sigma}} \frac{u^{\prime}(\eta) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}} \\
=\frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \frac{u^{\prime}(\eta) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}+\frac{1}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{u^{\prime}(\eta) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}} . \tag{23}
\end{gather*}
$$

As in [22], on each interval $\left[t_{s-1}, t_{s}\right](1 \leq s \leq j)$, denoting the quadratic interpolation $\Pi_{2, s} u(t)$ of $u(t)$ using three points $\left(t_{s-1}, u\left(t_{s-1}\right)\right),\left(t_{s}, u\left(t_{s}\right)\right)$ and $\left(t_{s+1}, u\left(t_{s+1}\right)\right)$, we get

$$
\begin{gather*}
\Pi_{2, s} u(t)=u\left(t_{s-1}\right) \frac{\left(t-t_{s}\right)\left(t-t_{s+1}\right)}{2 \tau^{2}} \\
-u\left(t_{s}\right) \frac{\left(t-t_{s-1}\right)\left(t-t_{s+1}\right)}{\tau^{2}}+u\left(t_{s+1}\right) \frac{\left(t-t_{s-1}\right)\left(t-t_{s}\right)}{2 \tau^{2}} \\
\left(\Pi_{2, s} u(t)\right)^{\prime}=u_{t, s}+u_{\bar{t}, s}\left(t-t_{s+1 / 2}\right)=u_{t, s-1}+u_{\bar{t}, s}\left(t-t_{s-1 / 2}\right), \tag{24}
\end{gather*}
$$

and

$$
\begin{equation*}
u(t)-\Pi_{2, s} u(t)=\frac{u^{\prime \prime \prime}\left(\bar{\xi}_{s}\right)}{6}\left(t-t_{s-1}\right)\left(t-t_{s}\right)\left(t-t_{s+1}\right), \tag{25}
\end{equation*}
$$

where $t \in\left[t_{s-1}, t_{s+1}\right], \bar{\xi}_{s} \in\left(t_{s-1}, t_{s+1}\right), u_{t, s}=\left(u\left(t_{s+1}\right)-u\left(t_{s}\right)\right) / \tau, u_{\bar{t}, s}=$ $\left(u\left(t_{s}\right)-u\left(t_{s-1}\right)\right) / \tau$.

In (23), we use $\Pi_{2, s} u(t)$ to approximate $u(t)$ on the interval $\left[t_{s-1}, t_{s}\right](1 \leq$ $s \leq j$ ). Taking into account the equality

$$
\begin{equation*}
\int_{t_{s-1}}^{t_{s}}\left(\eta-t_{s-1 / 2}\right)\left(t_{j+\sigma}-\eta\right)^{-\alpha} d \eta=\frac{\tau^{2-\alpha}}{1-\alpha} b_{j-s+1}^{(\alpha, \sigma)}, \quad 1 \leq s \leq j \tag{26}
\end{equation*}
$$

with
$b_{l}^{(\alpha, \sigma)}=\frac{1}{2-\alpha}\left[(l+\sigma)^{2-\alpha}-(l-1+\sigma)^{2-\alpha}\right]-\frac{1}{2}\left[(l+\sigma)^{1-\alpha}+(l-1+\sigma)^{1-\alpha}\right]$,
$l \geq 1$, from (23) and (24) we obtain the difference analog of the Caputo fractional derivative of the order $\alpha(0<\alpha<1)$ for the function $u(t)$ in the following form:

$$
\begin{gathered}
\partial_{0 t_{j+\sigma}}^{\alpha} u(\eta)=\frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \frac{u^{\prime}(\eta) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}+\frac{1}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{u^{\prime}(\eta) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}} \\
\quad \approx \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \frac{\left(\Pi_{2, s} u(\eta)\right)^{\prime} d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}+\frac{u_{t, j}}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}
\end{gathered}
$$

$$
\begin{gather*}
=\frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \frac{u_{t, s-1}+u_{\bar{t}, s}\left(\eta-t_{s-1 / 2}\right) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}+\frac{u_{t, j}}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}} \\
=\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{s=1}^{j}\left(a_{j-s+1}^{(\alpha, \sigma)} u_{t, s-1}+b_{j-s+1}^{(\alpha, \sigma)} u_{\bar{t}, s} \tau\right)+a_{0}^{(\alpha, \sigma)} u_{t, j}\right) \\
=\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{s=1}^{j}\left(a_{j-s+1}^{(\alpha, \sigma)} u_{t, s-1}+b_{j-s+1}^{(\alpha, \sigma)}\left(u_{t, s}-u_{t, s-1}\right)\right)+a_{0}^{(\alpha, \sigma)} u_{t, j}\right) \\
=\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j} c_{j-s}^{(\alpha, \sigma)} u_{t, s}=\Delta_{0 t_{j+\sigma}}^{\alpha} u \tag{27}
\end{gather*}
$$

where

$$
a_{0}^{(\alpha, \sigma)}=\sigma^{1-\alpha}, \quad a_{l}^{(\alpha, \sigma)}=(l+\sigma)^{1-\alpha}-(l-1+\sigma)^{1-\alpha}, \quad l \geq 1
$$

$c_{0}^{(\alpha, \sigma)}=a_{0}^{(\alpha, \sigma)}$, for $j=0 ;$ and for $j \geq 1$,

$$
c_{s}^{(\alpha, \sigma)}= \begin{cases}a_{0}^{(\alpha, \sigma)}+b_{1}^{(\alpha, \sigma)}, & s=0  \tag{28}\\ a_{s}^{(\alpha, \sigma)}+b_{s, 1}^{(\alpha, \sigma)}-b_{s}^{(\alpha, \sigma)}, & 1 \leq s \leq j-1 \\ a_{j}^{(\alpha, \sigma)}-b_{j}^{(\alpha, \sigma)}, & s=j\end{cases}
$$

We call the fractional numerical differentiation formula (27) for the Caputo fractional derivative of order $\alpha(0<\alpha<1)$ the $L 2-1_{\sigma}$ formula.

Lemma 2. For any $\alpha \in(0,1)$ and $u(t) \in \mathcal{C}^{3}\left[0, t_{j+1}\right]$

$$
\begin{equation*}
\left|\partial_{0 t_{j+\sigma}}^{\alpha} u-\Delta_{0 t_{j+\sigma}}^{\alpha} u\right|=\mathcal{O}\left(\tau^{3-\alpha}\right) \tag{29}
\end{equation*}
$$

Proof. Let $\partial_{0 t_{j+\sigma}}^{\alpha} u-\Delta_{0 t_{j+\sigma}}^{\alpha} u=R_{1}^{j}+R_{j}^{j+\sigma}$, where

$$
\begin{gathered}
R_{1}^{j}=\frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \frac{u^{\prime}(\eta) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}-\frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \frac{\left(\Pi_{2, s} u(\eta)\right)^{\prime} d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}} \\
=\frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}}\left(u(\eta)-\Pi_{2, s} u(\eta)\right)^{\prime}\left(t_{j+\sigma}-\eta\right)^{-\alpha} d \eta
\end{gathered}
$$

$$
\begin{gathered}
=-\frac{\alpha}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}}\left(u(\eta)-\Pi_{2, s} u(\eta)\right)\left(t_{j+\sigma}-\eta\right)^{-\alpha-1} d \eta \\
=-\frac{\alpha}{6 \Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} u^{\prime \prime \prime}\left(\bar{\xi}_{s}\right)\left(\eta-t_{s-1}\right)\left(\eta-t_{s}\right)\left(\eta-t_{s+1}\right)\left(t_{j+\sigma}-\eta\right)^{-\alpha-1} d \eta \\
R_{j}^{j+\sigma}=\frac{1}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{u^{\prime}(\eta) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}-\frac{u_{t, j}}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}} \\
\frac{1}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{\left(u^{\prime}(\eta)-u_{t, j}\right) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}=\frac{1}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{\left(u^{\prime}\left(t_{j+1 / 2}\right)-u_{t, j}\right) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}+ \\
+\frac{u^{\prime \prime}\left(t_{j+1 / 2}\right)}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{\left(\eta-t_{j+1 / 2}\right) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}+\mathcal{O}\left(\tau^{3-\alpha}\right) \\
=\frac{u^{\prime \prime}\left(t_{j+1 / 2}\right)}{\Gamma(1-\alpha)} \int_{t_{j}}^{t_{j+\sigma}} \frac{\left(\eta-t_{j+1 / 2}\right) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}+\mathcal{O}\left(\tau^{3-\alpha}\right) .
\end{gathered}
$$

We estimate the error $R_{1}^{j}$ similarly to [22]:

$$
\begin{aligned}
& \left|R_{1}^{j}\right| \leq \frac{\alpha\left|u^{\prime \prime \prime}(\xi)\right|}{6 \Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}}\left(\eta-t_{s-1}\right)\left(t_{s}-\eta\right)\left(t_{s+1}-\eta\right)\left(t_{j+\sigma}-\eta\right)^{-\alpha-1} d \eta \\
\leq & \frac{\alpha\left|u^{\prime \prime \prime}(\xi)\right| \tau^{3}}{3 \Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}}\left(t_{j+\sigma}-\eta\right)^{-\alpha-1} d \eta=\frac{\alpha\left|u^{\prime \prime \prime}(\xi)\right| \tau^{3}}{3 \Gamma(1-\alpha)} \int_{0}^{t_{j}}\left(t_{j+\sigma}-\eta\right)^{-\alpha-1} d \eta \\
= & \frac{\left|u^{\prime \prime \prime}(\xi)\right| \tau^{3}}{3 \Gamma(1-\alpha)}\left(\frac{1}{\sigma^{\alpha} \tau^{\alpha}}-\frac{1}{(j+\sigma)^{\alpha} \tau^{\alpha}}\right) \leq \frac{\left|u^{\prime \prime \prime}(\xi)\right|}{3 \sigma^{\alpha} \Gamma(1-\alpha)} \tau^{3-\alpha}, \quad \xi \in\left(0, t_{j}\right) .
\end{aligned}
$$

Since

$$
\int_{t_{j}}^{t_{j+\sigma}} \frac{\left(\eta-t_{j+1 / 2}\right) d \eta}{\left(t_{j+\sigma}-\eta\right)^{\alpha}}=\frac{\tau t_{\sigma}^{1-\alpha}(2 \sigma+\alpha-2)}{2(1-\alpha)(2-\alpha)}=0
$$

the error $\left|R_{j}^{j+\sigma}\right|=\mathcal{O}\left(\tau^{3-\alpha}\right)$. Lemma 2 is proved.
3.1. Basic properties of the new $\mathbf{L} \mathbf{2}-\mathbf{1}_{\sigma}$ fractional numerical differentiation formula.
Lemma 3. For all $s=1,2, \ldots$ and $0<\alpha<1$ the following inequalities hold

$$
\frac{1}{2}<\varkappa_{s}<\frac{1}{2-\alpha}
$$

where

$$
\varkappa_{s}=\frac{(s+\sigma)^{2-\alpha}-(s-1+\sigma)^{2-\alpha}-(2-\alpha)(s-1+\sigma)^{1-\alpha}}{(2-\alpha)\left((s+\sigma)^{1-\alpha}-(s-1+\sigma)^{1-\alpha}\right)} .
$$

Proof. Let us consider two functions
$f_{\alpha}(x)=\frac{(x+1)^{2-\alpha}-x^{2-\alpha}-(2-\alpha) x^{1-\alpha}}{(2-\alpha)\left((x+1)^{1-\alpha}-x^{1-\alpha}\right)}=\int_{0}^{1} \frac{(z+x)^{1-\alpha}-x^{1-\alpha}}{(1+x)^{1-\alpha}-x^{1-\alpha}} d z, \quad x>0$
and

$$
g_{\alpha}(z, x)=\frac{(z+x)^{1-\alpha}-x^{1-\alpha}}{(1+x)^{1-\alpha}-x^{1-\alpha}}=\frac{z \int_{0}^{1} \frac{d \xi}{(x+z \xi)^{\alpha}}}{\int_{0}^{1} \frac{d \xi}{(x+\xi)^{\alpha}}}, \quad 0<z<1, \quad x>0 .
$$

For all $x>0$ and $0<z<1$ the following inequalities hold

$$
\int_{0}^{1} \frac{d \xi}{(x+\xi)^{\alpha}}<\int_{0}^{1} \frac{d \xi}{(x+z \xi)^{\alpha}}<\int_{0}^{1} \frac{d \xi}{(z x+z \xi)^{\alpha}}=z^{-\alpha} \int_{0}^{1} \frac{d \xi}{(x+\xi)^{\alpha}}
$$

Therefore, for the function $g_{\alpha}(z, x)$ for all $x>0$ and $0<z<1$ the inequalities

$$
\begin{equation*}
z<g_{\alpha}(z, x)<z^{1-\alpha} \tag{30}
\end{equation*}
$$

are valid.
Integrating (30) with respect to $z$ from 0 to 1 , we get the inequalities

$$
\frac{1}{2}<f_{\alpha}(x)<\frac{1}{2-\alpha}
$$

which hold for all $x>0$. Lemma 3 is proved.
Corollary 2. For any $\alpha(0<\alpha<1)$, it holds $b_{s}^{(\alpha, \sigma)}>0, s \geq 1$.

The latter follows from the equality

$$
b_{s}^{(\alpha, \sigma)}=\left[(s+\sigma)^{1-\alpha}-(s-1+\sigma)^{1-\alpha}\right]\left(\varkappa_{s}-\frac{1}{2}\right) .
$$

Lemma 4. For any $\alpha(0<\alpha<1)$ and $c_{s}^{(\alpha, \sigma)}(0 \leq s \leq j, j \geq 1)$ defined in (28), it holds

$$
\begin{gather*}
c_{j}^{(\alpha, \sigma)}>\frac{1-\alpha}{2}(j+\sigma)^{-\alpha},  \tag{31}\\
c_{0}^{(\alpha, \sigma)}>c_{1}^{(\alpha, \sigma)}>c_{2}^{(\alpha, \sigma)}>\ldots>c_{j-1}^{(\alpha, \sigma)}>c_{j}^{(\alpha, \sigma)},  \tag{32}\\
(2 \sigma-1) c_{0}^{(\alpha, \sigma)}-\sigma c_{1}^{(\alpha, \sigma)}>0, \tag{33}
\end{gather*}
$$

where $\sigma=1-\alpha / 2$.
Proof. For $j \geq 1$ we get

$$
\begin{aligned}
c_{j}^{(\alpha, \sigma)}= & a_{j}^{(\alpha, \sigma)}-b_{j}^{(\alpha, \sigma)}=\left((j+\sigma)^{1-\alpha}-(j-1+\sigma)^{1-\alpha}\right)\left(\frac{3}{2}-\varkappa_{j}\right) \\
& >\left((j+\sigma)^{1-\alpha}-(j-1+\sigma)^{1-\alpha}\right)\left(\frac{3}{2}-\frac{1}{2-\alpha}\right) \\
& >\frac{1-\alpha}{2} \int_{0}^{1} \frac{d \eta}{(j+\sigma-\eta)^{\alpha}}>\frac{1-\alpha}{2}(j+\sigma)^{-\alpha} .
\end{aligned}
$$

Inequality (31) is proved. Let us prove inequality (32).
For $1 \leq s \leq j-2(j \geq 3)$ we have

$$
\begin{gathered}
c_{s}^{(\alpha, \sigma)}-c_{s+1}^{(\alpha, \sigma)}=a_{s}^{(\alpha, \sigma)}-a_{s+1}^{(\alpha, \sigma)}+2 b_{s+1}^{(\alpha, \sigma)}-b_{s}^{(\alpha, \sigma)}-b_{s+2}^{(\alpha, \sigma)} \\
=\frac{1}{2}\left((s+2+\sigma)^{1-\alpha}-3(s+1+\sigma)^{1-\alpha}+3(s+\sigma)^{1-\alpha}-(s-1+\sigma)^{1-\alpha}\right) \\
+\frac{1}{2-\alpha}\left(-(s+2+\sigma)^{2-\alpha}+3(s+1+\sigma)^{2-\alpha}-3(s+\sigma)^{2-\alpha}+(s-1+\sigma)^{2-\alpha}\right) \\
=\frac{\alpha(1-\alpha)(1+\alpha)}{2} \int_{0}^{1} d z_{1} \int_{0}^{1} d z_{2} \int_{0}^{1} \frac{d z_{3}}{\left(s-1+\sigma+z_{1}+z_{2}+z_{3}\right)^{\alpha+2}} \\
+\alpha(1-\alpha) \int_{0}^{1} d z_{1} \int_{0}^{1} d z_{2} \int_{0}^{1} \frac{d z_{3}}{\left(s-1+\sigma+z_{1}+z_{2}+z_{3}\right)^{\alpha+1}}
\end{gathered}
$$

$$
>\frac{\alpha(1-\alpha)(1+\alpha)}{2}(s+2+\sigma)^{-\alpha-2}+\alpha(1-\alpha)(s+2+\sigma)^{-\alpha-1}>0 .
$$

For $s=j-1(j \geq 2)$ we get

$$
\begin{gathered}
c_{s}^{(\alpha, \sigma)}-c_{s+1}^{(\alpha, \sigma)}=c_{j-1}^{(\alpha, \sigma)}-c_{j}^{(\alpha, \sigma)}=a_{j-1}^{(\alpha, \sigma)}-a_{j}^{(\alpha, \sigma)}+2 b_{j}^{(\alpha, \sigma)}-b_{j-1}^{(\alpha, \sigma)} \\
>a_{j-1}^{(\alpha, \sigma)}-a_{j}^{(\alpha, \sigma)}+2 b_{j}^{(\alpha, \sigma)}-b_{j-1}^{(\alpha, \sigma)}-b_{j+1}^{(\alpha, \sigma)} \\
>\frac{\alpha(1-\alpha)(1+\alpha)}{2}(j+1+\sigma)^{-\alpha-2}+\alpha(1-\alpha)(j+1+\sigma)^{-\alpha-1}>0 .
\end{gathered}
$$

For inequality (32) it remains to prove the case $s=0$, that is $c_{0}^{(\alpha, \sigma)}>c_{1}^{(\alpha, \sigma)}$ which obviously follows from (33). It is enough to prove inequality (33).

For $j=1$ we get

$$
\begin{aligned}
& (2 \sigma-1) c_{0}^{(\alpha, \sigma)}-\sigma c_{1}^{(\alpha, \sigma)}=(2 \sigma-1)\left(a_{0}^{(\alpha, \sigma)}+b_{1}^{(\alpha, \sigma)}\right)-\sigma\left(a_{1}^{(\alpha, \sigma)}-b_{1}^{(\alpha, \sigma)}\right) \\
& =\left(\frac{2 \sigma-1}{2 \sigma}-\frac{2 \sigma-1}{2}\right)(1+\sigma)^{1-\alpha}=\frac{(2 \sigma-1)(1-\sigma)}{2 \sigma}(1+\sigma)^{1-\alpha}>0 .
\end{aligned}
$$

For $j \geq 2$ we get

$$
\begin{gathered}
(2 \sigma-1) c_{0}^{(\alpha, \sigma)}-\sigma c_{1}^{(\alpha, \sigma)}=(2 \sigma-1)\left(a_{0}^{(\alpha, \sigma)}+b_{1}^{(\alpha, \sigma)}\right)-\sigma\left(a_{1}^{(\alpha, \sigma)}+b_{2}^{(\alpha, \sigma)}-b_{1}^{(\alpha, \sigma)}\right) \\
=\frac{4 \sigma-1}{2 \sigma}(1+\sigma)^{1-\alpha}-(2+\sigma)^{1-\alpha}=(1+\sigma)^{1-\alpha}\left(\frac{4 \sigma-1}{2 \sigma}-\left(1+\frac{1}{1+\sigma}\right)^{1-\alpha}\right) \\
\quad>(1+\sigma)^{1-\alpha}\left(\frac{4 \sigma-1}{2 \sigma}-1-\frac{1-\alpha}{1+\sigma}\right)=\frac{(2 \sigma-1)(1-\sigma)}{2 \sigma(1+\sigma)^{\alpha}}>0 .
\end{gathered}
$$

Here we used the inequality $(1+t)^{\gamma}<1+\gamma t$ which is valid for all $t>0$ and $0<\gamma<1$. Lemma 4 is proved.

### 3.2. Test example

In this subsection, the validity and numerical accuracy of the new presented $L 2-1_{\sigma}$ formula (27) are demonstrated by a test example.

Let us take a positive integer $M$, let $\tau=1 /(M-1+\sigma)$ and denote

$$
E_{L 2-1_{\sigma}}^{M}(\tau)=\left|\partial_{0 t_{M-1+\sigma}}^{\alpha} f(t)-\Delta_{0 t_{M-1+\sigma}}^{\alpha} f(t)\right| .
$$

Example. Let $f(t)=t^{4+\alpha}, 0<\alpha<1$. Compute the $\alpha$-order Caputo fractional derivative of $f(t)$ at $t=t_{M-1+\sigma}=1$ numerically.

The exact solution is given by

$$
\left.\partial_{0 t}^{\alpha} t^{4+\alpha}\right|_{t=1}=\frac{\Gamma(5+\alpha)}{24}
$$

Taking different temporal stepsizes $M=10, \quad 20, \quad 40, \quad 80, \quad 160$, 320, 640, 1280, $2560, \quad 5120$, we compute the example using $L 2-1_{\sigma}$ formula (27) and compare the results with those obtained with the help of the $L 1-2$ formula in [22]. Table 1 lists the computational errors and numerical convergence order $(C O)$ at $t_{M-1+\sigma}=1$ with different parameters $\alpha=0.9$, $0.5,0.1$.

Table 1.
Computational errors and convergence order with different temporal stepsizes

| $\alpha$ | $M$ | $E_{L 1-2}^{M}(\tau)[22]$ | $C O_{E_{L 1-2}}^{M}$ | $E_{L 2-1_{\sigma}}^{M}(\tau)$ | $C O_{E_{L 2-1 \sigma}^{M}}^{M}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 0.9 | 10 | $1.070471 e-1$ |  | $1.922978 e-2$ |  |
|  | 20 | $2.699702 e-2$ | 1.99 | $4.368964 e-3$ | 2.07 |
|  | 40 | $6.545547 e-3$ | 2.04 | $1.009364 e-3$ | 2.08 |
|  | 80 | $1.556707 e-3$ | 2.07 | $2.347614 e-4$ | 2.09 |
|  | 160 | $3.666902 e-4$ | 2.09 | $5.473732 e-5$ | 2.09 |
|  | 320 | $8.595963 e-5$ | 2.09 | $1.277246 e-5$ | 2.10 |
|  | 640 | $2.010152 e-5$ | 2.10 | $2.980723 e-6$ | 2.10 |
|  | 1280 | $4.694884 e-6$ | 2.10 | $6.955612 e-7$ | 2.10 |
|  | 2560 | $1.095840 e-6$ | 2.10 | $1.622925 e-7$ | 2.10 |
|  | 5120 | $2.556990 e-7$ | 2.10 | $3.786340 e-8$ | 2.10 |
|  |  |  |  |  |  |
| 0.5 | 10 | $1.350657 e-2$ |  | $3.756950 e-3$ |  |
|  | 20 | $2.612085 e-3$ | 2.37 | $7.231988 e-4$ | 2.33 |
|  | 40 | $4.861786 e-4$ | 2.43 | $1.367574 e-4$ | 2.38 |
|  | 80 | $8.864502 e-5$ | 2.46 | $2.544814 e-5$ | 2.42 |
|  | 160 | $1.597499 e-5$ | 2.47 | $4.673501 e-6$ | 2.44 |
|  | 320 | $2.859085 e-6$ | 2.48 | $8.495470 e-7$ | 2.46 |
|  | 640 | $5.095342 e-7$ | 2.49 | $1.532461 e-7$ | 2.47 |
|  | 1280 | $9.056389 e-8$ | 2.49 | $2.748687 e-8$ | 2.48 |
|  | 2560 | $1.606869 e-8$ | 2.49 | $4.909831 e-9$ | 2.48 |
|  | 5120 | $2.847764 e-9$ | 2.50 | $8.743961 e-10$ | 2.49 |
|  |  |  |  |  |  |
| 0.1 | 10 | $6.238229 e-4$ |  | $2.686107 e-4$ |  |
|  | 20 | $9.663202 e-5$ | 2.69 | $4.492624 e-5$ | 2.57 |
|  | 40 | $1.444281 e-5$ | 2.74 | $7.204745 e-6$ | 2.64 |
|  | 80 | $2.111896 e-6$ | 2.77 | $1.119177 e-6$ | 2.68 |
|  | 160 | $3.043133 e-7$ | 2.79 | $1.696376 e-7$ | 2.72 |
|  | 320 | $4.338827 e-8$ | 2.81 | $2.522442 e-8$ | 2.75 |
|  | 640 | $6.136347 e-9$ | 2.82 | $3.694254 e-9$ | 2.77 |
|  | 1280 | $8.622698 e-10$ | 2.83 | $5.344856 e-10$ | 2.79 |
|  | 2560 | $1.205229 e-10$ | 2.84 | $7.656497 e-11$ | 2.80 |
|  | 5120 | $1.676992 e-11$ | 2.85 | $1.087796 e-11$ | 2.82 |

## 4. A second order difference scheme for the time fractional diffusion equation

In this section for problem (1)-(2) a difference scheme with the approximation order $\mathcal{O}\left(h^{2}+\tau^{2}\right)$ is constructed. The stability of the constructed difference scheme as well as its convergence in the mesh $L_{2}$ - norm with the rate equal to the order of the approximation error is proved. The obtained results are supported with numerical calculations carried out for a test example.

### 4.1. Derivation of the difference scheme

Lemma 5. For any functions $k_{1}(x) \in \mathcal{C}_{x}^{3}$ and $v(x) \in \mathcal{C}_{x}^{4}$ the following equality is valid:

$$
\begin{gather*}
\left.\frac{d}{d x}\left(k_{1}(x) \frac{d}{d x} v(x)\right)\right|_{x=x_{i}} \\
=\frac{k_{1}\left(x_{i+1 / 2}\right) v\left(x_{i+1}\right)-\left(k_{1}\left(x_{i+1 / 2}\right)+k_{1}\left(x_{i-1 / 2}\right)\right) v\left(x_{i}\right)+k_{1}\left(x_{i-1 / 2}\right) v\left(x_{i-1}\right)}{h^{2}}+\mathcal{O}\left(h^{2}\right) . \tag{34}
\end{gather*}
$$

Let $u(x, t) \in \mathcal{C}_{x, t}^{4,3}$ be a solution of the problem (1)-(2). Let us consider equation (1) for $(x, t)=\left(x_{i}, t_{j+\sigma}\right) \in \bar{Q}_{T}, i=1,2, \ldots, N-1, j=0,1, \ldots, M-$ $1, \sigma=1-\alpha / 2$ :

$$
\begin{equation*}
\partial_{0 t_{j+\sigma}}^{\alpha} u=\left.\frac{\partial}{\partial x}\left(k(x, t) \frac{\partial u}{\partial x}\right)\right|_{\left(x_{i}, t_{j+\sigma}\right)}-q\left(x_{i}, t_{j+\sigma}\right) u\left(x_{i}, t_{j+\sigma}\right)+f\left(x_{i}, t_{j+\sigma}\right) \tag{35}
\end{equation*}
$$

Since

$$
\begin{gathered}
\left.\frac{\partial}{\partial x}\left(k(x, t) \frac{\partial u}{\partial x}\right)\right|_{\left(x_{i}, t_{j+\sigma}\right)}=k\left(x_{i}, t_{j+\sigma}\right) \frac{\partial^{2} u}{\partial x^{2}}\left(x_{i}, t_{j+\sigma}\right)+\frac{\partial k}{\partial x}\left(x_{i}, t_{j+\sigma}\right) \frac{\partial u}{\partial x}\left(x_{i}, t_{j+\sigma}\right) \\
=k\left(x_{i}, t_{j+\sigma}\right)\left(\sigma \frac{\partial^{2} u}{\partial x^{2}}\left(x_{i}, t_{j+1}\right)+(1-\sigma) \frac{\partial^{2} u}{\partial x^{2}}\left(x_{i}, t_{j}\right)\right) \\
+\frac{\partial k}{\partial x}\left(x_{i}, t_{j+\sigma}\right)\left(\sigma \frac{\partial u}{\partial x}\left(x_{i}, t_{j+1}\right)+(1-\sigma) \frac{\partial u}{\partial x}\left(x_{i}, t_{j}\right)\right)+\mathcal{O}\left(\tau^{2}\right) \\
=\sigma\left(k\left(x_{i}, t_{j+\sigma}\right) \frac{\partial^{2} u}{\partial x^{2}}\left(x_{i}, t_{j+1}\right)+\frac{\partial k}{\partial x}\left(x_{i}, t_{j+\sigma}\right) \frac{\partial u}{\partial x}\left(x_{i}, t_{j+1}\right)\right) \\
+(1-\sigma)\left(k\left(x_{i}, t_{j+\sigma}\right) \frac{\partial^{2} u}{\partial x^{2}}\left(x_{i}, t_{j}\right)+\frac{\partial k}{\partial x}\left(x_{i}, t_{j+\sigma}\right) \frac{\partial u}{\partial x}\left(x_{i}, t_{j}\right)\right)+\mathcal{O}\left(\tau^{2}\right)
\end{gathered}
$$

$$
\begin{gathered}
=\left.\sigma \frac{\partial}{\partial x}\left(k\left(x, t_{j+\sigma}\right) \frac{\partial}{\partial x} u\left(x, t_{j+1}\right)\right)\right|_{x=x_{i}} \\
+(1-\sigma) \\
\left.\frac{\partial}{\partial x}\left(k\left(x, t_{j+\sigma}\right) \frac{\partial}{\partial x} u\left(x, t_{j}\right)\right)\right|_{x=x_{i}}+\mathcal{O}\left(\tau^{2}\right) \\
q\left(x_{i}, t_{j+\sigma}\right) u\left(x_{i}, t_{j+\sigma}\right)=q\left(x_{i}, t_{j+\sigma}\right)\left(\sigma u\left(x_{i}, t_{j+1}\right)+(1-\sigma) u\left(x_{i}, t_{j}\right)\right)+\mathcal{O}\left(\tau^{2}\right),
\end{gathered}
$$

by virtue of Lemma 5 we have

$$
\left.\mathcal{L} u(x, t)\right|_{\left(x_{i}, t_{j+\sigma}\right)}=\sigma \Lambda u\left(x_{i}, t_{j+1}\right)+(1-\sigma) \Lambda u\left(x_{i}, t_{j}\right)+\mathcal{O}\left(h^{2}+\tau^{2}\right),
$$

where the difference operator $\Lambda$ is defined by formula (8) with the coefficients $a_{i}^{j+1}=k\left(x_{i-1 / 2}, t_{j+\sigma}\right), d_{i}^{j+1}=q\left(x_{i}, t_{j+\sigma}\right)$. Let $\varphi_{i}^{j+1}=f\left(x_{i}, t_{j+\sigma}\right)$, then with regard to Lemma 2 we get the difference scheme with the approximation order $\mathcal{O}\left(h^{2}+\tau^{2}\right)$ :

$$
\begin{gather*}
\Delta_{0 t_{j+\sigma}}^{\alpha} y_{i}=\Lambda y_{i}^{(\sigma)}+\varphi_{i}^{j+1}, \quad i=1,2, \ldots, N-1, \quad j=0,1, \ldots, M-1,  \tag{36}\\
y(0, t)=0, \quad y(l, t)=0, \quad t \in \bar{\omega}_{\tau}, \quad y(x, 0)=u_{0}(x), \quad x \in \bar{\omega}_{h} \tag{37}
\end{gather*}
$$

It is interesting to note that for $\alpha \rightarrow 1$ we obtain the Crank-Nicolson difference scheme.

### 4.2. Stability and convergence

Theorem 3. The difference scheme (36)-(37) is unconditionally stable and its solution satisfies the following a priori estimate:

$$
\begin{equation*}
\left\|y^{j+1}\right\|_{0}^{2} \leq\left\|y^{0}\right\|_{0}^{2}+\frac{l^{2} T^{\alpha} \Gamma(1-\alpha)}{4 c_{1}} \max _{0 \leq j \leq M}\left\|\varphi^{j}\right\|_{0}^{2} \tag{38}
\end{equation*}
$$

Proof. For the difference operator $\Lambda$ using Green's first difference formula and the embedding theorem [24] for the functions vanishing at $x=0$ and $x=l$, we get $(-\Lambda y, y) \geq \frac{4 c_{1}}{l^{2}}\|y\|_{0}^{2}$, that is for this operator it is possible to take $\varkappa=\frac{4 c_{1}}{l^{2}}$.

Since difference scheme (36)-(37) has the form (15)-(6), where $g_{s}^{j+1}=$ $\frac{c_{j-s}^{(\alpha, \beta)}}{\tau^{\alpha} \Gamma(2-\alpha)}$, then lemma 4 implies validity of the following inequalities:

$$
g_{0}^{j+1}=\frac{c_{j}^{(\alpha, \beta)}}{\tau^{\alpha} \Gamma(2-\alpha)}>\frac{1}{2 t_{j+\sigma}^{\alpha} \Gamma(1-\alpha)}>\frac{1}{2 T^{\alpha} \Gamma(1-\alpha)},
$$

$$
\begin{gathered}
g_{j}^{j+1}>g_{j-1}^{j+1}>\ldots>g_{0}^{j+1} \\
\frac{g_{j}^{j+1}}{2 g_{j}^{j+1}-g_{j-1}^{j+1}}<\sigma<1 .
\end{gathered}
$$

Therefore, validity of theorem 3 follows from theorem 1 . Theorem 3 is proved.
From theorem 2 it follows that if the solution and input data of problem (1) -(2) are sufficiently smooth, the solution of difference scheme (36)-(37) converges to the solution of the differential problem with the rate equal to the order of the approximation error $\mathcal{O}\left(h^{2}+\tau^{2}\right)$.

### 4.3. Numerical results

Numerical calculations are performed for a test problem when the function

$$
u(x, t)=\sin (\pi x)\left(t^{3}+3 t^{2}+1\right)
$$

is the exact solution of the problem (11)-(22) with the coefficients $k(x, t)=$ $2-\sin (x t), q(x, t)=1-\cos (x t)$ and $l=1, T=1$.

The errors $(z=y-u)$ and convergence order (CO) in the norms $\|\cdot\|_{0}$ and $\|\cdot\|_{\mathcal{C}\left(\bar{\omega}_{h \tau}\right)}$, where $\|y\|_{\mathcal{C}\left(\bar{\omega}_{h \tau}\right)}=\max _{\left(x_{i}, t_{j}\right) \in \bar{\omega}_{h \tau}}|y|$, are given in Table 2.

Table 2 shows that as the number of the spatial subintervals and time steps is increased keeping $h=\tau$, a reduction in the maximum error takes place, as expected and the convergence order of the approximate scheme is $\mathcal{O}\left(h^{2}\right)=\mathcal{O}\left(\tau^{2}\right)$, where the convergence order is given by the formula: $\mathrm{CO}=\log _{\frac{h_{1}}{h_{2}}} \frac{\left\|z_{1}\right\|}{\left\|z_{2}\right\|}\left(z_{i}\right.$ is the error corresponding to $\left.h_{i}\right)$.

Table 3 shows that if $h=1 / 1000$, then as the number of time steps of our approximate scheme is increased, a reduction in the maximum error takes place, as expected and the convergence order of time is $\mathcal{O}\left(\tau^{2}\right)$, where the convergence order is given by the following formula: $\mathrm{CO}=\log _{\frac{\tau_{1}}{\tau_{2}}} \frac{\left\|z_{1}\right\|}{\left\|z_{2}\right\|}$.

Table 2.
$L_{2}$ - norm and maximum norm error behavior versus grid size reduction when $\tau=h$.

| $\alpha$ | $h$ | $\max _{0 \leq n \leq M}\left\\|z^{n}\right\\|_{0}$ | CO in $\\|\cdot\\|_{0}$ | $\\|z\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ | CO in $\\|\cdot\\| \\|_{C\left(\bar{\omega}_{h \tau}\right)}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 0.10 | $1 / 160$ | $1.0224 e-4$ |  | $1.4518 e-4$ |  |
|  | $1 / 320$ | $2.5558 e-5$ | 2.0001 | $3.6294 e-5$ | 2.0000 |
|  | $1 / 640$ | $6.3894 e-6$ | 2.0000 | $9.0733 e-6$ | 2.0000 |
| 0.50 | $1 / 160$ | $7.8417 e-5$ |  |  |  |
|  | $1 / 320$ | $1.9604 e-5$ | 2.0000 | $2.7882 e-5$ | 2.0000 |
|  | $1 / 640$ | $4.9009 e-6$ | 2.0000 | $6.9705 e-6$ | 2.0000 |
|  |  |  |  |  |  |
| 0.90 | $1 / 160$ | $6.6666 e-5$ |  | $9.4949 e-5$ |  |
|  | $1 / 320$ | $1.6669 e-5$ | 1.9998 | $2.3740 e-5$ | 1.9999 |
|  | $1 / 640$ | $4.1678 e-6$ | 1.9998 | $5.9360 e-6$ | 1.9998 |
|  |  |  |  |  |  |
| 0.99 | $1 / 160$ | $6.5660 e-5$ |  | $9.3532 e-5$ |  |
|  | $1 / 320$ | $1.6415 e-5$ | 2.0000 | $2.3384 e-5$ | 1.9999 |
|  | $1 / 640$ | $4.1039 e-6$ | 1.9999 | $5.8460 e-6$ | 2.0000 |

Table 3.
$L_{2}$ - norm and maximum norm error behavior versus $\tau$-grid size reduction when $h=1 / 1000$.

| $\alpha$ | $\tau$ | $\max _{0 \leq n \leq M}\left\\|z^{n}\right\\|_{0}$ | CO in $\\|\cdot\\|_{0}$ | $\\|z\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ | CO in $\\|\cdot\\| \\|_{C\left(\bar{\omega}_{h \tau}\right)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.10 | 1/10 | $1.9062 e-3$ |  | $2.6962 e-3$ |  |
|  | $1 / 20$ | $4.7789 e-4$ | 1.9959 | $6.7593 e-4$ | 1.9960 |
|  | 1/40 | $1.1779 e-4$ | 2.0205 | $1.6659 e-4$ | 2.0206 |
| 0.50 | 1/10 | $7.6326 e-3$ |  | $1.0795 e-2$ |  |
|  | $1 / 20$ | $1.9130 e-3$ | 1.9963 | $2.7058 e-3$ | 1.9962 |
|  | 1/40 | $4.7697 e-4$ | 2.0039 | $6.7461 e-4$ | 2.0039 |
| 0.90 | 1/10 | $1.0286 e-2$ |  | $1.4547 e-2$ |  |
|  | 1/20 | $2.5706 e-3$ | 2.0005 | $3.6357 e-3$ | 2.0004 |
|  | $1 / 40$ | $6.4066 e-4$ | 2.0045 | $9.0608 e-4$ | 2.0045 |
| 0.99 | 1/10 | $1.0449 e-2$ |  | $1.4777 e-2$ |  |
|  | 1/20 | $2.6102 e-3$ | 2.0011 | $3.6915 e-3$ | 2.0011 |
|  | 1/40 | $6.5050 e-4$ | 2.0045 | $9.1998 e-4$ | 2.0045 |

## 5. A higher order difference scheme for the time fractional diffusion equation

In this section for problem (1)-(2), we construct a difference scheme with the approximation order $\mathcal{O}\left(h^{4}+\tau^{2}\right)$ in the case when $k=k(t)$ and $q=q(t)$. The stability and convergence of the constructed difference scheme in the mesh $L_{2}$ - norm with the rate equal to the order of the approximation error are proved. The obtained results are supported by the numerical calculations carried out for a test example.

### 5.1. Derivation of the difference scheme

Let us assign a difference scheme to differential problem (1)-(2) in the case when $k=k(t)$ and $q=q(t)$ :

$$
\begin{align*}
\Delta_{0 t_{j+\sigma}}^{\alpha} \mathcal{H}_{h} y_{i} & =a^{j+1} y_{\bar{x} x, i}^{(\sigma)}-d^{j+1} \mathcal{H}_{h} y_{i}^{(\sigma)}+\mathcal{H}_{h} \varphi_{i}^{j+1}, i=1, \ldots, N-1, j=0,1, \ldots, M-1,  \tag{39}\\
y(0, t) & =0, \quad y(l, t)=0, \quad t \in \bar{\omega}_{\tau}, \quad y(x, 0)=u_{0}(x), \quad x \in \bar{\omega}_{h}, \tag{40}
\end{align*}
$$

where $\mathcal{H}_{h} v_{i}=v_{i}+h^{2} v_{\bar{x} x, i} / 12, i=1, \ldots, N-1, a^{j+1}=k\left(t_{j+\sigma}\right), d^{j+1}=q\left(t_{j+\sigma}\right)$, $\varphi_{i}^{j+1}=f\left(x_{i}, t_{j+\sigma}\right), \sigma=1-\alpha / 2$.

From [9] and Lemma 2 it follows that if $u \in \mathcal{C}_{x, t}^{6,3}$, then the difference scheme has the approximation order $\mathcal{O}\left(\tau^{2}+h^{4}\right)$.

### 5.2. Stability and convergence

The difference scheme (39)-(40) differs from (5)-(6) due to the presence of the operator $\mathcal{H}_{h}$. However, deriving an a priori estimate for the solution of difference scheme (39)-(40) does not differ significantly from proving Theorem 1.

Theorem 4. The difference scheme (39)-(40) is unconditionally stable and its solution satisfies the following a priori estimate:

$$
\begin{equation*}
\left\|\mathcal{H}_{h} y^{j+1}\right\|_{0}^{2} \leq\left\|\mathcal{H}_{h} y^{0}\right\|_{0}^{2}+\frac{l^{2} T^{\alpha} \Gamma(1-\alpha)}{c_{1}} \max _{0 \leq j \leq M}\left\|\mathcal{H}_{h} \varphi^{j}\right\|_{0}^{2} \tag{41}
\end{equation*}
$$

Proof. Taking the inner product of the equation (39) with $\mathcal{H}_{h} y^{(\sigma)}=$ $\left(\mathcal{H}_{h} y\right)^{(\sigma)}$, we have

$$
\begin{gather*}
\left(\mathcal{H}_{h} y^{(\sigma)}, \Delta_{0 t_{j+\sigma}}^{\alpha} \mathcal{H}_{h} y\right)-a^{j+1}\left(\mathcal{H}_{h} y^{(\sigma)}, y_{\bar{x} x}^{(\sigma)}\right) \\
+d^{j+1}\left(\mathcal{H}_{h} y^{(\sigma)}, \mathcal{H}_{h} y^{(\sigma)}\right)=\left(\mathcal{H}_{h} y^{(\sigma)}, \mathcal{H}_{h} \varphi^{j+1}\right), \tag{42}
\end{gather*}
$$

Let us transform the terms in identity (42) as

$$
\begin{gathered}
\left(\mathcal{H}_{h} y^{(\sigma)}, \Delta_{0 t_{j+\sigma}}^{\alpha} \mathcal{H}_{h} y\right) \geq \frac{1}{2} \Delta_{0 t_{j+\sigma}}^{\alpha}\left\|\mathcal{H}_{h} y\right\|_{0}^{2}, \\
\left.-\left(\mathcal{H}_{h} y^{(\sigma)}, y_{\bar{x} x}^{(\sigma)}\right)=-\left(y^{(\sigma)}, y_{\bar{x} x}^{(\sigma)}\right)-\frac{h^{2}}{12}\left\|y_{\bar{x} x}^{(\sigma)}\right\|_{0}^{2}=\| y_{\bar{x}}^{(\sigma)}\right]\left.\right|_{0} ^{2}-\frac{1}{12} \sum_{i=1}^{N-1}\left(y_{\bar{x}, i+1}^{(\sigma)}-y_{\bar{x}, i}^{(\sigma)}\right)^{2} h \\
\left.\left.\left.\left.\geq \| y_{\bar{x}}^{(\sigma)}\right]\left.\right|_{0} ^{2}-\frac{1}{3} \| y_{\bar{x}}^{(\sigma)}\right]\left.\right|_{0} ^{2}=\frac{2}{3} \| y_{\bar{x}}^{(\sigma)}\right]\left.\right|_{0} ^{2} \geq \frac{8}{3 l^{2}}\left\|y^{(\sigma)}\right\|_{0}^{2}, \quad \text { where } \quad \| y\right]\left.\right|_{0} ^{2}=\sum_{i=1}^{N} y_{i}^{2} h, \\
\left(\mathcal{H}_{h} y^{(\sigma)}, \mathcal{H}_{h} \varphi^{j+1}\right) \leq \varepsilon\left\|\mathcal{H}_{h} y^{(\sigma)}\right\|_{0}^{2}+\frac{1}{4 \varepsilon}\left\|\mathcal{H}_{h} \varphi^{j+1}\right\|_{0}^{2} \\
=\varepsilon \sum_{i=1}^{N-1}\left(\frac{y_{i-1}^{(\sigma)}+10 y_{i}^{(\sigma)}+y_{i+1}^{(\sigma)}}{12}\right)^{2} h+\frac{1}{4 \varepsilon}\left\|\mathcal{H}_{h} \varphi^{j+1}\right\|_{0}^{2} \leq \varepsilon\left\|y^{(\sigma)}\right\|_{0}^{2}+\frac{1}{4 \varepsilon}\left\|\mathcal{H}_{h} \varphi^{j+1}\right\|_{0}^{2} .
\end{gathered}
$$

Taking into account the above-performed transformations, from identity (42) at $\varepsilon=\frac{8 c_{1}}{3 l^{2}}$ one arrives at the inequality

$$
\Delta_{0 t_{j+\sigma}}^{\alpha}\left\|\mathcal{H}_{h} y\right\|_{0}^{2} \leq \frac{l^{2}}{8 c_{1}}\left\|\mathcal{H}_{h} \varphi^{j+1}\right\|_{0}^{2}
$$

The following process is similar to the proof of theorem 1, and it is omitted.
The norm $\left\|\mathcal{H}_{h} y\right\|_{0}$ is equivalent to the norm $\|y\|_{0}$, which follows from the inequalities

$$
\frac{5}{12}\|y\|_{0}^{2} \leq\left\|\mathcal{H}_{h} y\right\|_{0}^{2} \leq\|y\|_{0}^{2}
$$

Similarly to theorem 2 , we obtain the convergence result.
Theorem 5. Assume that $u(x, t) \in \mathcal{C}_{x, t}^{6,3}$ is the solution of the problem (11)-(2) in the case $k=k(t), q=q(t)$, and let $\left\{y_{i}^{j} \mid 0 \leq i \leq N, 1 \leq j \leq M\right\}$ be the solution of the difference scheme (39)-(40). Then it holds that

$$
\left\|u\left(\cdot, t_{j}\right)-y^{j}\right\|_{0} \leq C_{R}\left(\tau^{2}+h^{4}\right), \quad 1 \leq j \leq M
$$

where $C_{R}$ is a positive constant independent of $\tau$ and $h$.

### 5.3. Numerical results

In this subsection we present a test example for a numerical investigation of difference scheme (39)-(40).

Consider the following problem:

$$
\begin{align*}
\partial_{0 t}^{\alpha} u(x, t) & =k(t) \frac{\partial^{2} u}{\partial x^{2}}(x, t)-q(t) u(x, t)+f(x, t), 0<x<1,0<t \leq 1  \tag{43}\\
u(0, t) & =0, \quad u(1, t)=0, \quad 0 \leq t \leq 1, \quad u(x, 0)=0, \quad 0 \leq x \leq 1 \tag{44}
\end{align*}
$$

where $k(t)=e^{t}, \quad q(t)=1-\sin (2 t)$,

$$
f(x)=\left[\pi^{2} t^{2} e^{t}+t^{2}(1-\sin (2 t))+\frac{2 t^{2-\alpha}}{\Gamma(3-\alpha)}\right] \sin (\pi x)
$$

whose exact analytical solution reads $u(x, t)=t^{2} \sin (\pi x)$.
Table 4 presents the $L_{2}$ - norm, the maximum norm errors and the temporal convergence order for $\alpha=0.75,0.85,0.95$. Here we can see that the order of convergence in time is two.

Table 5 shows that if $\tau=1 / 20000$ is kept fixed, while $h$ varies, then one obtains the expected fourth-order spatial accuracy.

Table 6 shows that as the number of spatial subintervals and time steps is increased keeping $h^{2}=\tau$, a reduction in the maximum error takes place, as expected and the convergence order of the approximate of the scheme is $\mathcal{O}\left(h^{4}\right)$.

In Table 7 for the case $N=\lceil\sqrt{M}\rceil$ the maximum error, the convergence order and CPU time (seconds) are given. For this case we obtain the expected rate of convergence $\mathcal{O}\left(\tau^{2}\right)$.

## Table 4.

$L_{2}$ - norm and maximum norm error behavior versus $\tau$-grid size reduction when $h=1 / 100$.

| $\alpha$ | $\tau$ | $\max _{0 \leq n \leq M}\left\\|z^{n}\right\\|_{0}$ | CO in $\\|\cdot\\|_{0}$ | $\\|z\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ | CO in $\\|\cdot\\| \\|_{C\left(\bar{\omega}_{h \tau}\right)}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 0.75 | $1 / 10$ | $1.6336 e-3$ |  | $2.3103 e-3$ |  |
|  | $1 / 20$ | $4.0889 e-4$ | 1.9983 | $5.7826 e-4$ | 1.9983 |
|  | $1 / 40$ | $1.0229 e-4$ | 1.9990 | $1.4466 e-4$ | 1.9990 |
|  | $1 / 80$ | $2.5581 e-5$ | 1.9995 | $3.6177 e-5$ | 1.9995 |
| 0.85 | $1 / 10$ | $1.7130 e-3$ |  |  |  |
|  | $1 / 20$ | $4.2856 e-4$ | 1.9989 | $6.4225 e-3$ |  |
|  | $1 / 40$ | $1.0718 e-4$ | 1.9994 | $1.5158 e-4$ | 1.9989 |
|  | $1 / 80$ | $2.6801 e-5$ | 1.9997 | $3.7902 e-5$ | 1.9997 |
|  |  |  |  |  |  |
| 0.95 | $1 / 10$ | $1.7582 e-3$ |  | $2.4865 e-3$ |  |
|  | $1 / 20$ | $4.3967 e-4$ | 1.9996 | $6.2179 e-4$ | 1.9996 |
|  | $1 / 40$ | $1.0993 e-4$ | 1.9998 | $1.5547 e-4$ | 1.9998 |
|  | $1 / 80$ | $2.7484 e-5$ | 1.9999 | $3.8868 e-5$ | 1.9999 |

Table 5.
$L_{2}$ - norm and maximum norm error behavior versus $h$-grid size reduction when $\tau=1 / 20000$.

| $\alpha$ | $h$ | $\max _{0 \leq n \leq M}\left\\|z^{n}\right\\|_{0}$ | CO in $\\|\cdot\\|_{0}$ | $\\|z\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ | CO in $\\|\cdot\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ |
| :--- | :--- | :--- | :---: | :---: | :---: |
| 0.10 | $1 / 4$ | $1.1004 e-3$ |  | $1.5562 e-3$ |  |
|  | $1 / 8$ | $6.7512 e-5$ | 4.0267 | $9.5476 e-5$ | 4.0267 |
|  | $1 / 16$ | $4.2000 e-6$ | 4.0067 | $5.9397 e-6$ | 4.0067 |
|  | $1 / 32$ | $2.6213 e-7$ | 4.0021 | $3.7070 e-7$ | 4.0021 |
| 0.50 |  |  |  |  |  |
|  | $1 / 4$ | $1.0836 e-3$ |  | $1.5325 e-3$ |  |
|  | $1 / 8$ | $6.6485 e-5$ | 4.0267 | $9.4024 e-5$ | 4.0267 |
|  | $1 / 16$ | $4.1360 e-6$ | 4.0067 | $5.8491 e-6$ | 4.0067 |
|  | $1 / 32$ | $2.5790 e-7$ | 4.0034 | $3.6472 e-7$ | 4.0034 |
|  |  |  |  |  |  |
| 0.90 | $1 / 4$ | $1.0654 e-3$ |  | $1.5067 e-3$ |  |
|  | $1 / 8$ | $6.5371 e-5$ | 4.0266 | $9.2449 e-5$ | 4.0266 |
|  | $1 / 16$ | $4.0665 e-6$ | 4.0068 | $5.7510 e-6$ | 4.0068 |
|  | $1 / 32$ | $2.5346 e-7$ | 4.0040 | $3.5844 e-7$ | 4.0040 |

Table 6.
$L_{2}$ - norm and maximum norm error behavior versus grid size reduction when $h^{2}=\tau$.

| $\alpha$ | $h$ | $\max _{0 \leq n \leq M}\left\\|z^{n}\right\\|_{0}$ | CO in $\\|\cdot\\|_{0}$ | $\\|z\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ | CO in $\\|\cdot\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 0.10 | $1 / 10$ | $2.4349 e-5$ |  | $3.4434 e-5$ |  |
|  | $1 / 20$ | $1.5166 e-6$ | 4.0049 | $2.1448 e-6$ | 4.0049 |
|  | $1 / 40$ | $9.4708 e-8$ | 4.0012 | $1.3394 e-7$ | 4.0012 |
|  | $1 / 80$ | $5.9180 e-9$ | 4.0003 | $8.3693 e-9$ | 4.0003 |
| 0.50 | $1 / 10$ | $1.4211 e-5$ |  |  |  |
|  | $1 / 20$ | $8.8285 e-7$ | 4.0087 | $1.0097 e-5$ |  |
|  | $1 / 40$ | $5.5094 e-8$ | 4.0022 | $7.7914 e-8$ |  |
|  | $1 / 80$ | $3.4420 e-9$ | 4.0006 | $4.8677 e-9$ | 4.0087 |
|  |  |  |  |  | 4.0022 |
| 0.90 | $1 / 10$ | $1.5119 e-5$ |  | $2.1381 e-5$ |  |
|  | $1 / 20$ | $9.5080 e-7$ | 3.9910 | $1.3446 e-6$ | 3.9911 |
|  | $1 / 40$ | $5.9571 e-8$ | 3.9965 | $8.4247 e-8$ | 3.9964 |
|  | $1 / 80$ | $3.7274 e-9$ | 3.9984 | $5.2714 e-9$ | 3.9984 |

Table 7.
Maximum norm error behavior versus grid size reduction
when $N=\lceil\sqrt{M}\rceil$ and CPU time (seconds).

| $\alpha$ | $M$ | $\\|z\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ | CO in $\\|\cdot\\|_{C\left(\bar{\omega}_{h \tau}\right)}$ | CPU(s) |
| :--- | :--- | :---: | :---: | :---: |
| 0.70 | 10 | $2.0986 e-3$ |  | 0.0156 |
|  | 30 | $2.1085 e-4$ | 2.0916 | 0.0468 |
|  | 90 | $2.3672 e-5$ | 1.9905 | 0.1404 |
|  | 270 | $2.6359 e-6$ | 1.9980 | 0.5460 |
|  | 810 | $2.9428 e-7$ | 1.9956 | 3.0108 |
|  | 2430 | $3.2802 e-8$ | 1.9971 | 22.2925 |
|  |  |  |  |  |
| 0.80 | 10 | $2.1403 e-3$ |  | 0.0156 |
|  | 30 | $2.2690 e-4$ | 2.0427 | 0.0468 |
|  | 90 | $2.5342 e-5$ | 1.9953 | 0.1716 |
|  | 270 | $2.8146 e-6$ | 2.0004 | 0.5616 |
|  | 810 | $3.1383 e-7$ | 1.9968 | 3.2604 |
|  | 2430 | $3.4962 e-8$ | 1.9976 | 23.3065 |
|  |  |  |  |  |
| 0.90 | 10 | $2.2549 e-3$ |  | 0.0156 |
|  | 30 | $2.4088 e-4$ | 2.0358 | 0.0468 |
|  | 90 | $2.6745 e-5$ | 2.0007 | 0.1404 |
|  | 270 | $2.9607 e-6$ | 2.0033 | 0.5460 |
|  | 810 | $3.2949 e-7$ | 1.9986 | 3.6670 |
|  | 2430 | $3.6670 e-8$ | 1.9985 | 22.7605 |

## 6. Conclusion

In this paper, the stability and convergence of a family of difference schemes approximating the time fractional diffusion equation of a general form is studied. Sufficient conditions for the unconditional stability of such difference schemes are obtained. For proving the stability of a wide class of difference schemes approximating the time fractional diffusion equation, it is simple enough to check the stability conditions obtained in this paper. A new difference approximation of the Caputo fractional derivative with the approximation order $\mathcal{O}\left(\tau^{3-\alpha}\right)$ is constructed. The basic properties of this difference operator are investigated. New difference schemes of the second and fourth approximation order in space and the second approximation order in time for the time fractional diffusion equation with variable coefficients
are constructed as well. The stability and convergence of these schemes in the mesh $L_{2}$ - norm with the rate equal to the order of the approximation error are proved. The method can be easily extended to other time fractional partial differential equations with other boundary conditions.

Numerical tests completely confirming the obtained theoretical results are carried out. In all the calculations MATLAB is used.

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