Numerical scheme for treatment of Uehling-Uhlenbeck equation for two-particle interactions in relativistic plasma

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Abstract

We present a new efficient method to compute Uehling-Uhlenbeck collision integral for all two-particle interactions in relativistic plasma with drastic improvement in computation time with respect to existing methods. Plasma is assumed isotropic in momentum space. The set of reactions consists of: Moeller and Bhabha scattering, Compton scattering, two-photon pair annihilation, and two-photon pair production, which are described by QED matrix elements. In our method exact energy and particle number conservation laws are fulfilled. Reaction rates are compared, where possible, with the corresponding analytical expressions and convergence of numerical rates is demonstrated.

Keywords: Uehling-Uhlenbeck equations, collision integral, binary interactions, relativistic plasma.

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1. Introduction

Relativistic plasma, for which $kT \ge mc^2$, where k is Boltzmann constant, c is speed of light, m is electron mass, T is temperature, is relevant in different branches of astrophysics. In the early universe ultrarelativistic electron-positron pairs contribute to the matter contents of the Universe [1]. X-ray and gamma-ray radiation from numerous astrophysical sources such as gamma-ray bursts [2, 3, 4], active galactic nuclei [5, 6], and X-ray binaries [7] points out to existence of relativistic electron-positron plasma in these objects. The upcoming high-energy laser facilities aiming at generation of femtosecond laser pulses with intensity more than $10^{21}W/cm^2$ aim generation of relativistic plasma by interacting laser pulses. At present relativistic electron-positron jets are generated by interaction of laser pulses with condensed matter [8, 9, 10, 11].

Solving the Boltzmann equations with collision integral containing a quantum cross-section represents the most general and complete method to describe a behavior of relativistic plasma [12, 13, 14]. The one-particle distribution function (DF) is defined on a seven dimensional space, three dimensions for the physical space and three dimensions for the momentum space, and one dimension for the time. Thus one has a multidimensional problem which is a real challenge from the computational point of view. Beside the dimensionality problem, there are other difficulties which are related to kinetic equations in general [15, 16]. Our main goal in this paper is to tackle the challenge associated with the calculation of the collision integral, dealing with two key issues. First, the computational cost related to the evaluation of the collision operator involving multidimensional integrals which should be solved in each point of the coordinate space. Second, the presence of multiple scales requires the development of adapted numerical schemes capable of solving stiff dynamics. Different deterministic approaches are used to tackle collision integral from a numerical point of view: finite volume, semi-Lagrangian and spectral schemes [16, 17, 18, 19, 20]. While the deterministic methods could normally reach high order of accuracy, the probabilistic ones, such as Monte-Carlo (MC) method, are often faster.

MC methods are traditionally used to model Coulomb interactions in nonrelativistic plasma[21, 22, 23, 24]. As a rule MC techniques are based on the random pairing of particles in close vicinity and the calculation of a scattering angle due to the interaction. Small-angle Coulomb collisions which allow small energy and momentum transfer are often described in diffusion approximation by the Fokker-Planck equation [25]. The principal feature of relativistic plasma is a presence of pair creation and pair annihilation processes, which are often included in MC based models [9, 10]. However Fokker-Planck approximation is no longer valid in relativistic plasma [26].

Classical Boltzmann equation does not take into account quantum statistics of particles. The generalization of classical Boltzmann equation including quantum corrections is Uehling-Uhlenbeck (U-U) equation, which contains additional Pauli blocking and Bose enhancement multipliers that give rise to equilibrium solution with Bose-Einstein and Fermi-Dirac distributions [27, 28]. The main problem of the MC methods in application to U-U equations is that total reaction rate is unknown as distribution function is unknown too. Compensation methods include smoothing of the delta-function distribution of MC-particles over cells in the phase space, but it suffers from a large number of simulation particles and cells needed to reproduce Bose-Einstein steady state distribution. Spectral methods based on the Fourier transformation of the velocity distribution function require very dense computational grid to reach high accuracy [29, 30, 31, 32, 33]. Process-oriented approach to the U-U collision integral presented in this work allows one to get high accuracy results with low computational cost.

In this paper we further develop the method first used in the work [34]. This method was successfully applied to follow the thermalization of relativistic plasma [35, 36, 26, 37] and to investigate thermalization timescales for an electron-positron plasma [38]. In section 2 we recall Boltzmann and UU equations and present usual scheme of their analytic treatment. Section 3 is devoted to the description of our numerical scheme, while section 4 shows comparison between our code results and known analytic formulae for non-degenerate case. Conclusion follows.

2. Formulation

The Boltzmann equation governs an evolution of one-particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$. We assume that plasma is homogeneous and isotropic in coordinate space and isotropic in momentum space, thus distribution function depends on absolute value of momentum (energy) and time. DF is normalized on particles concentration, so that $n = \int f(\mathbf{p}, t) d^3 p$.

Consider an interaction of two initial particles of type I and II which are in states 1 μ 2, correspondingly, and creation of two final particles of type III μ IV which are in states 3 μ 4, correspondingly. Let us image the process by the following scheme:

$$I_1 + II_2 \to III_3 + IV_4. \tag{1}$$

The corresponding inverse process is:

$$III_3 + IV_4 \to I_1 + II_2. \tag{2}$$

If every particle has momentum p_i , which lies in interval d^3p_i , then a number of interactions in unit time and unit space volume is:

$$w(p_1, p_2; p_3, p_4) f_I f_{II} d^3 p_1 d^3 p_2 d^3 p_3 d^3 p_4, (3)$$

. .

function w is called a transition rate for a given reaction. An effective cross-section is defined by the formula:

$$d\sigma = -\frac{w}{v}d^3p_3d^3p_4,\tag{4}$$

where $v = c\epsilon_1^{-1}\epsilon_2^{-1}\sqrt{(\epsilon_1\epsilon_2 - (\mathbf{p_1p_2})c^2)^2 - (m_1m_2c^4)^2}$ is a relative velocity of particles.

In quantum field theory an expression for interaction cross-section is:

$$d\sigma = \frac{\hbar^2 c^6}{(2\pi)^2} \frac{1}{v} \frac{|M_{if}|^2}{16\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4} \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) d^3 p_3 d^3 p_4,$$
(5)

where $|M_{if}|$ are a matrix elements calculated with a methods of quantum field theory.

Comparing two last formulas one can derive the following expression for transition rate:

$$w(p_3, p_4; p_1, p_2) = \frac{\hbar^2 c^6}{(2\pi)^2} \frac{|M_{if}|^2}{16\epsilon_1\epsilon_2\epsilon_3\epsilon_4} \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4),$$
(6)

Now let us write the Boltzmann equation for DF of particle I for a given process:

$$\dot{f}_{I} = \int d^{3}p_{2}d^{3}p_{3}d^{3}p_{4}[w(p_{3}, p_{4}; p_{1}, p_{2})f_{III}(\mathbf{p}_{3}, t)f_{IV}(\mathbf{p}_{4}, t) - w(p_{1}, p_{2}; p_{3}, p_{4})f_{I}(\mathbf{p}_{1}, t)f_{II}(\mathbf{p}_{2}, t)], \quad (7)$$

Process (q)	Ι	II	III	IV
Compton Scattering (CS)	e^{\pm}	γ	e^{\pm}	γ
Bhabha Scattering (BS)	e^{\pm}	e^{\mp}	e^{\pm}	e^{\mp}
Møller Scattering (MS)	e^{\pm}	e^{\pm}	e^{\pm}	e^{\pm}
Pair Annihilation (PA)	e^-	e^+	γ	γ
Pair Creation (PC)	γ	γ	e^-	e^+

Table 1: Two-particle processes in electron-positron-photon plasma

where a dot denotes time derivative. Equations for particle DFs of remaining types can be derived by the corresponding replacement of indices.

Specifically, for a scattering with I = III and II = IV the inverse process is the same as the direct one since pairs of indices (1, 2) and (3, 4) can be interchanged. The relation $w(p_3, p_4; p_1, p_2) = w(p_1, p_2; p_3, p_4)$ holds for all processes listed in Table 1. The right hand side of the Boltzmann equation is a collision integral denoted as $St f_I$. The first term in collision integral describes particle outcome and the second term describes particle income, we will denote it as St^-f and St^+f , respectively.

The generalization of Boltzmann equation for the case of particles obeying quantum statistics is U-U equation. For the particle I in the state 1 U-U equation has the following form:

$$\dot{f}_{I} = \int d^{3}\mathbf{p}_{2}d^{3}\mathbf{p}_{3}d^{3}\mathbf{p}_{4} \\
\times \left[w(p_{3}, p_{4}; p_{1}, p_{2})f_{III}(\mathbf{p}_{3}, t)f_{IV}(\mathbf{p}_{4}, t)\left(1 + \eta \frac{f_{I}(\mathbf{p}_{1}, t)}{2h^{-3}}\right)\left(1 + \eta \frac{f_{II}(\mathbf{p}_{2}, t)}{2h^{-3}}\right) \\
- w(p_{1}, p_{2}; p_{3}, p_{4})f_{I}(\mathbf{p}_{1}, t)f_{II}(\mathbf{p}_{2}, t)\left(1 + \eta \frac{f_{III}(\mathbf{p}_{3}, t)}{2h^{-3}}\right)\left(1 + \eta \frac{f_{IV}(\mathbf{p}_{4}, t)}{2h^{-3}}\right) \right],$$
(8)

where η is defined through

$$\eta = \begin{cases} +1, & \text{for Bose-Einstein statistics,} \\ -1, & \text{for Fermi-Dirac statistics,} \\ 0, & \text{for Maxwell-Boltzmann statistics.} \end{cases}$$
(9)

When incoming or outgoing particles coincide (I = II and/or III = IV) quantum indistinguishability gives the term $\frac{1}{2}$ in front of the corresponding outcome and income terms, see e.g. [39], [14].

For numerical evaluation phase space is divided into zones, in calculations we approximate continuous DF by its averaging over each zone (see Eq. (22)). For this purpose we add an integral over \mathbf{p}_1 in UU equation 8, the RHS of resulting equation will have the same form for each particle type differing only by sign and its integration limits:

$$\pm \int d^{3}\mathbf{p}_{1}d^{3}\mathbf{p}_{2}d^{3}\mathbf{p}_{3}d^{3}\mathbf{p}_{4} \\ \times \left[w(p_{3},p_{4};p_{1},p_{2})f_{III}(\mathbf{p}_{3},t)f_{IV}(\mathbf{p}_{4},t)\left(1+\eta\frac{f_{I}(\mathbf{p}_{1},t)}{2h^{-3}}\right)\left(1+\eta\frac{f_{II}(\mathbf{p}_{2},t)}{2h^{-3}}\right)\right. \\ \left.-w(p_{1},p_{2};p_{3},p_{4})f_{I}(\mathbf{p}_{1},t)f_{II}(\mathbf{p}_{2},t)\left(1+\eta\frac{f_{III}(\mathbf{p}_{3},t)}{2h^{-3}}\right)\left(1+\eta\frac{f_{IV}(\mathbf{p}_{4},t)}{2h^{-3}}\right)\right],$$
(10)

where the upper sign corresponds to particle type I and II and the lower sign corresponds to particle type III and IV.

Evaluating collision integral in the framework of reaction-oriented approach we use one expression (10) and distribute the result to each particle type according to integration limits in (10).

In this paper we deal with all two-particle QED processes in relativistic plasma, which are collected in Table 1. The exact QED matrix elements for these processes can be found in the standard textbooks, e.g. [40, 41].

Let us make a notice connected with a conservation laws for interacting particles. Energy and momentum conservations read

$$\hat{\varepsilon} = \varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4, \qquad \hat{\mathbf{p}} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4.$$
 (11)

There are 4 delta-functions in Eq. (6) representing conservation of energy and momentum (11). Three integrations over momentum of particle *III* can be performed immediately

$$\int d\mathbf{p}_3 \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \longrightarrow 1.$$
(12)

In the integration over energy ε_4 of particle IV it is necessary to take into account that ε_3 is now a function of energy and angles of particles I and II, as well as angles of particle IV, so we have

$$\int d\varepsilon_4 \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \longrightarrow \frac{1}{1 - (\beta_3/\beta_4) \mathbf{n}_3 \cdot \mathbf{n}_4},\tag{13}$$

where $\mathbf{n} = \mathbf{p}/p$ is the unit vector in the direction of particle momentum, $p = |\mathbf{p}| = \sqrt{(\varepsilon/c)^2 - m^2 c^2}$ is the absolute value of particle momentum, $\beta = pc/\varepsilon$, and a dot denotes scalar product of 3-vectors.

We use spherical coordinates in momentum space: $\{\varepsilon, \mu, \phi\}, \mu = \cos \vartheta$, where ε is the particle energy, and ϑ and ϕ are polar and azimuthal angles, respectively. Then energy and angles of particle *III* and energy of particle *IV* follow from energy and momentum conservation (11) and relativistic energy-momentum relation, namely

$$\varepsilon_4 = c\sqrt{p_4^2 + m_{IV}^2 c^2}, \qquad \mathbf{p}_4 = p_4 \mathbf{n}_4, \tag{14}$$

$$\varepsilon_3 = \hat{\varepsilon} - \varepsilon_4, \qquad \mathbf{p}_3 = \hat{\mathbf{p}} - \mathbf{p}_4,$$

$$\mathbf{n}_3 = \frac{\mathbf{p}_3}{p_3}, \qquad \mathbf{n}_4 = \frac{\mathbf{p}_4}{p_4},\tag{15}$$

$$\mathbf{n}_{3} = \left(\sqrt{1 - \mu_{3}^{2}}\cos\phi_{3}, \sqrt{1 - \mu_{3}^{2}}\sin\phi_{3}, \mu_{3}\right),\tag{16}$$

$$\mathbf{n}_4 = \left(\sqrt{1 - \mu_4^2} \cos \phi_4, \sqrt{1 - \mu_4^2} \sin \phi_4, \mu_4\right),\tag{17}$$

$$p_4 = \frac{AB \pm \sqrt{A^2 + 4m_{IV}^2 c^2 (B^2 - 1)}}{2(B^2 - 1)},$$
(18)

$$A = \frac{c}{\hat{\varepsilon}} [\hat{p}^2 + (m_{III}^2 - m_{IV}^2)c^2] - \frac{\hat{\varepsilon}}{c}, \qquad B = \frac{c}{\hat{\varepsilon}} \mathbf{n}_4 \cdot \hat{\mathbf{p}}.$$
$$\mathbf{n}_3 \cdot \mathbf{n}_4 = \mu_3 \mu_4 + \sqrt{(1 - \mu_3^2)(1 - \mu_4^2)} \cos(\phi_3 - \phi_4). \tag{19}$$

Then we introduce these relations into collision integral (10). We also use spherical symmetry in momentum space to fix angles of the particle $I: \mu_1 =$ $1, \phi_1 = 0$, and to perform the integration over azimuthal angle of particle II: $\int d\phi_2 \longrightarrow 2\pi$, setting $\phi_2 = 0$ in the remaining integrals. Then final expression for collision integral is

$$\begin{aligned} \operatorname{St} f_{I} &= \frac{\hbar^{2}}{32\pi} \int d\varepsilon_{2} d\mu_{2} \ d\mu_{4} d\phi_{4} \frac{p_{2} p_{4} |M_{fi}|^{2}}{\varepsilon_{1} \varepsilon_{3} [1 - (\beta_{3}/\beta_{4}) \mathbf{n}_{3} \cdot \mathbf{n}_{4}]} \\ &\times \left[f_{III}(\varepsilon_{3}, t) f_{IV}(\varepsilon_{4}, t) \left(1 + \eta \frac{f_{I}(\varepsilon_{1}, t)}{2h^{-3}} \right) \left(1 + \eta \frac{f_{II}(\varepsilon_{2}, t)}{2h^{-3}} \right) \\ &- f_{I}(\varepsilon_{1}, t) f_{II}(\varepsilon_{2}, t) \left(1 + \eta \frac{f_{III}(\varepsilon_{3}, t)}{2h^{-3}} \right) \left(1 + \eta \frac{f_{IV}(\varepsilon_{4}, t)}{2h^{-3}} \right) \right]. \end{aligned}$$
(20)

For numerical integration, however, another expression is proved useful

$$\int d\varepsilon_{1} \operatorname{St} f_{I} = \frac{\hbar^{2}}{32\pi} \left[\int d\varepsilon_{3} \ d\varepsilon_{4} d\mu_{4} \ d\mu_{2} d\phi_{2} \frac{p_{2} p_{4} |M_{fi}|^{2}}{\varepsilon_{1} \varepsilon_{3} \left[1 - (\beta_{1}/\beta_{2}) \mathbf{n}_{1} \cdot \mathbf{n}_{2}\right]} \right] \\ \times f_{III}(\varepsilon_{3}, t) f_{IV}(\varepsilon_{4}, t) \left(1 + \eta \frac{f_{I}(\varepsilon_{1}, t)}{2h^{-3}}\right) \left(1 + \eta \frac{f_{II}(\varepsilon_{2}, t)}{2h^{-3}}\right) \\ - \int d\varepsilon_{1} \ d\varepsilon_{2} d\mu_{2} \ d\mu_{4} d\phi_{4} \frac{p_{2} p_{4} |M_{fi}|^{2}}{\varepsilon_{1} \varepsilon_{3} \left[1 - (\beta_{3}/\beta_{4}) \mathbf{n}_{3} \cdot \mathbf{n}_{4}\right]} \\ \times f_{I}(\varepsilon_{1}, t) f_{II}(\varepsilon_{2}, t) \left(1 + \eta \frac{f_{III}(\varepsilon_{3}, t)}{2h^{-3}}\right) \left(1 + \eta \frac{f_{IV}(\varepsilon_{4}, t)}{2h^{-3}}\right) \right], \quad (21)$$

where the first term is expressed in the form ready for replacement by the sum over incoming particles *III* and *IV*. In this term $\varepsilon_1, \mu_1, \phi_1, \varepsilon_2$ are given by relations (18) with indices exchange $1 \leftrightarrow 3, 2 \leftrightarrow 4, I \leftrightarrow III, II \leftrightarrow IV$.

This collision integral of any of two-particle processes is a four-dimensional integral in momentum space. In Sec. 3 we show how such integral is computed numerically on finite grid.

Here we note that in the case of homogeneous and isotropic pair plasma one has to satisfy only two conservation laws, namely of energy and particle number. Momentum conservation should be added for nonisotropic in momentum space DF, see e.g. [42]. In our method electric charge is conserved due to conservation of particles because we use between cell interpolation for the same kind of particles described in the next Section.

3. Numerical Scheme

The phase space is divided in zones. The zone $\Omega_{a,j,k}^{\alpha}$ for particle specie α corresponds to energy ε_a , cosine of polar angle μ_j and azimuthal angle ϕ_k , where indices run in the following ranges $1 \leq a \leq a_{\max}$, $1 \leq j \leq j_{\max}$, and $1 \leq k \leq k_{\max}$. The zone boundaries are $\varepsilon_{a\mp 1/2}$, $\mu_{j\mp 1/2}$, $\phi_{k\mp 1/2}$. The length of the *a*-th energy zone Ω_a^{α} is $\Delta \varepsilon_a \equiv \varepsilon_{a+1/2} - \varepsilon_{a-1/2}$. On finite grid f_{α} does not depend on μ and ϕ , and number density of particle α in zone *a* is

$$Y_{a}^{\alpha}(t) = 4\pi \int_{\varepsilon_{a-1/2}}^{\varepsilon_{a+1/2}} c^{-3} \varepsilon \sqrt{\varepsilon^{2} - m_{\alpha}^{2} c^{4}} f_{\alpha}(\varepsilon, t) d\varepsilon$$
$$= 4\pi c^{-3} \varepsilon_{a} \sqrt{\varepsilon_{a}^{2} - m_{\alpha}^{2} c^{4}} f_{\alpha}(\varepsilon_{a}, t) \Delta \varepsilon_{a}. \quad (22)$$

In this variables discretized U-U equation for particle I and energy zone a reads

$$\frac{dY_a^{\alpha}(t)}{dt} = \sum \left[\mathrm{St}^+ Y_a^I - \mathrm{St}^- Y_a^I \right], \qquad (23)$$

where the sum is taken over all processes involving particle I. Coefficients of particles income and outcome on the grid are obtained by integration of (21) for two-particle processes over the zone. The corresponding integrals are replaced by sums on the grid. For instance, coefficient of particle I outcome in two-particle process (1) is

$$\operatorname{St}^{-}Y_{a}^{I} = \frac{\hbar^{2}c^{4}}{8(4\pi)^{2}} \sum_{b,j,s,k} \Delta \mu_{j}^{II} \Delta \mu_{s}^{IV} \Delta \phi_{k}^{IV} |M_{fi}|^{2} \frac{p_{4}}{\varepsilon_{3}[1 - (\beta_{3}/\beta_{4})\mathbf{n}_{3} \cdot \mathbf{n}_{4}]} \times \\ \times \frac{Y_{a}^{I}(t)}{\varepsilon_{a}^{I}} \frac{Y_{b}^{II}(t)}{\varepsilon_{b}^{II}} \times \left[1 + \eta \frac{Y_{c}^{III}(t)}{\bar{Y}_{c}^{III}}\right] \left[1 + \eta \frac{Y_{d}^{IV}(t)}{\bar{Y}_{d}^{IV}}\right], \quad (24)$$

and coefficient of particle I income in process (2) from integration of (21) is

$$\operatorname{St}^{+}Y_{a}^{I} = \frac{\hbar^{2}c^{4}}{8(4\pi)^{2}} \sum_{c,d,j,s,k} C_{a}(\varepsilon_{1})\Delta\mu_{j}^{IV}\Delta\mu_{s}^{II}\Delta\phi_{k}^{II}|M_{fi}|^{2} \frac{p_{2}}{\varepsilon_{1}[1 - (\beta_{1}/\beta_{2})\mathbf{n}_{1}\cdot\mathbf{n}_{2}]} \times \frac{Y_{c}^{III}(t)}{\varepsilon_{c}^{III}} \frac{Y_{d}^{IV}(t)}{\varepsilon_{d}^{IV}} \times \left[1 + \eta \frac{Y_{a}^{I}(t)}{\bar{Y}_{a}^{I}}\right] \left[1 + \eta \frac{Y_{b}^{II}(t)}{\bar{Y}_{b}^{II}}\right], \quad (25)$$

where $\bar{Y}_a^{\alpha} = 4\pi \int_{\varepsilon_{a-1/2}}^{\varepsilon_{a+1/2}} c^{-3} \varepsilon \sqrt{\varepsilon^2 - m_{\alpha}^2 c^4} (2h^{-3}) d\varepsilon = 8\pi (hc)^{-3} \varepsilon_a \sqrt{\varepsilon_a^2 - m_{\alpha}^2 c^4} \Delta \varepsilon_a$, and

$$C_{a}(\varepsilon_{1}) = \begin{cases} \frac{\varepsilon_{a} - \varepsilon_{1}}{\varepsilon_{a} - \varepsilon_{a-1}}, & \varepsilon_{a-1} < \varepsilon_{1} < \varepsilon_{a}, \\ \frac{\varepsilon_{a+1} - \varepsilon_{1}}{\varepsilon_{a+1} - \varepsilon_{a}}, & \varepsilon_{a} < \varepsilon_{1} < \varepsilon_{a+1}, \\ 0, & \text{otherwise.} \end{cases}$$
(26)

In integration of (21) over the zone one can integrate out the δ -function $\int \delta(\varepsilon_1 - \varepsilon) d\varepsilon_1 \longrightarrow 1$. However, when energies of incoming particles are fixed on the grid,

the energies of outgoing particles are not on the grid. Hence an interpolation (26) is adopted, which enforces the exact number of particles and energy conservation in each two-particle process due to redistribution of outgoing particle α with energy ε over two energy zones $\Omega_n^{\alpha}, \Omega_{n+1}^{\alpha}$ with $\varepsilon_n < \varepsilon < \varepsilon_{n+1}$. Further we denote this technique as particle splitting.

The redistribution of final particles should also satisfy requirements of quantum statistics. Therefore if a process occurs, when fermionic final particle should be distributed over the quantum states which are fully occupied, such process should be forbidden. Thus we introduce the Bose enhancement/Pauli blocking coefficients in (24) and (25) as

$$\left[1+\eta \frac{Y_a^{\alpha}(t)}{\bar{Y}_a^{\alpha}}\right] = \min\left(1+\eta \frac{Y_n^{\alpha}(t)}{\bar{Y}_n^{\alpha}}, 1+\eta \frac{Y_{n+1}^{\alpha}(t)}{\bar{Y}_{n+1}^{\alpha}}\right).$$
(27)

The sum over angles μ_j, μ_s, ϕ_k can be found once and for all at the beginning of the calculations. We then store in the program for each set of the incoming and outgoing particles the corresponding terms and redistribution coefficients given by Eq. (26).

Representation of discretized collisional integral for particle I and energy zone a in processes (1), (2) is

$$\dot{Y}_{a}^{I} = \sum P_{abcd} \times Y_{c}^{III}(t) Y_{d}^{IV}(t) \times \left[1 + \eta \frac{Y_{a}^{I}(t)}{\bar{Y}_{a}^{I}}\right] \left[1 + \eta \frac{Y_{b}^{II}(t)}{\bar{Y}_{b}^{II}}\right] - \sum R_{abcd} \times Y_{a}^{I}(t) Y_{b}^{II}(t) \times \left[1 + \eta \frac{Y_{c}^{III}(t)}{\bar{Y}_{c}^{III}}\right] \left[1 + \eta \frac{Y_{d}^{IV}(t)}{\bar{Y}_{d}^{IV}}\right], \quad (28)$$

where constant coefficients P, R are obtained from the summation over angles in the sums (24), (25). In the nondegenerate case of Boltzmann equation the indices b in the first sum and c, d in the second sum become dummy, equation (28) can be partially summed and takes the following form:

$$\dot{Y}_a^I = \sum P_{acd} \times Y_c^{III}(t) Y_d^{IV}(t) - \sum R_{ab} \times Y_a^I(t) Y_b^{II}(t),$$
(29)

where $P_{acd} = \sum_{b} P_{abcd}$, $R_{ab} = \sum_{c,d} R_{abcd}$. The last quantity is essentially *reaction rate* usually used for description of binary processes and simply connected to the total cross section.

The full U-U equation (28) contains similar sums for all processes from Table 1. Each individual term in these sums appears in the system of discretized equations four times in emission and absorption coefficients for each particle entering a given process. Then each term can be computed only once and added to all corresponding sums, that is the essence of our *"reaction-oriented"* approach [43, 44].

We point out that unlike classical Boltzmann equation for binary interactions such as scattering, more general interactions are typically described by four collision integrals for each particle that appears both among incoming and outgoing particles.

4. Numerical results

The results of numerical calculations are presented below. As all known analytical expressions for reaction rates in relativistic plasma concern nondegenerate case, here we compare our results for collision integral to that of nondegenerate plasma. Notice that for Coulomb scattering we have implemented a cutoff scheme based on minimal scattering angle [26, 45].

We consider mildly relativistic plasma with

$$0.01 \lesssim e \lesssim 100,\tag{30}$$

where e is particle kinetic energy divided by electron rest energy, this range contains both relativistic and non-relativistic domains. The upper limit is chosen to avoid thermal production of other particles such as neutrinos and muons, while the lower limit is required to have sufficient pair density.

We introduce logarithmic energy grid with $a_{max} = 40$ nodes for all calculations and different homogeneous grids for angular variables, ϕ -grid is 2 time denser then μ -grid (typically μ -grid contains $j_{max} = 64$ nodes). To compare results with known analytical expressions we use definition of angle-averaged reaction rate per pair of particles

$$\overline{v\sigma}(e_1, e_2) = \int_{-1}^{+1} \frac{d\mu_2}{2} \int_{\vec{p}_3, \vec{p}_4} v d\sigma,$$
(31)

and angle-averaged emissivity per pair of particles

$$\overline{v\frac{d\sigma}{de_3}}(e_1, e_2, e_3) = \int_{-1}^{+1} \frac{d\mu_2}{2} \int_{\vec{p}_3, \vec{p}_4} v \frac{d\sigma}{de_3},$$
(32)

introduced by Svensson [46], where $d\sigma$ is given by standard definition (5) and we have used spherical symmetry as described before Eq. (20). We use Coppi & Blandford [47] analytical expressions (2.3), (3.2), (4.3) for $\overline{v\sigma}$, which corresponds to R_{ab} . Svensson [46] formula (55), Peer & Waxman [48] formulae (19, 28) are used for quantity $\overline{v \frac{d\sigma}{de}}$, which corresponds to $P_{abc}/\Delta e_a$. To compare numerical results with analytical ones, we introduce the follow-

To compare numerical results with analytical ones, we introduce the following quantity for each process

$$Q = \frac{1}{a_{\max}^2} \sum_{a,b} |R_{ab}/\overline{v}\overline{\sigma}(e_a, e_b) - 1|, \qquad (33)$$

expressing average relative deviation of numerical results from analytical ones for all energy grid nodes. Table 2 presents values of Q for selected number of angular grid nodes. It is evident that the relative error decreases with increasing of number of angular grid nodes reaching about 1 % with 128 nodes. This demonstrates convergence of numerical results to the corresponding analytical ones.

Below we present some representative plots for the reaction rates of all reactions together with analytical curves (where they are known). Energy is

Table 2: Values of Q for selected number of angular grid nodes $(a_{max} = 40, k_{max} = 2j_{max})$.

$Process/j_{max}$	16	32	64	128
CS	0.0855	0.0403	0.0207	0.0145
PA	0.0231	0.00693	0.00313	0.00138
PC	0.146	0.0657	0.0303	0.0116

Table 3: CPU time (in seconds) of each reaction initial angular integration for selected number of angular grid nodes ($a_{max} = 40, k_{max} = 2j_{max}$), and its exponent of computational cost $O(j_{max}^n)$.

$Process/j_{max}$	16	32	64	128	n
CS	2.215	14.48	113.2	590.1	2.7
PA	2.106	14.73	100.2	543.1	2.7
PC	0.531	3.619	28.82	223.2	2.9
MS	2.418	16.87	130.5	1030	2.9
BS	3.354	22.74	178.6	1113	2.8

measured in electron rest energy units. Presented results reproduce both nonrelativistic and relativistic energy cases. All computations were carried on Intel Core i3-7100 CPU @3.90 GHz processor using one processor core. The code is written in C and compiled in Windows 7 environment with Microsoft Visual Studio 2015 in fully optimized x64 mode. Computation time of initial angular integration of collision integrals for each reaction from Table 1 is shown in Table 3. It shows even lower than expected $O(j_{max}^3)$ behaviour due to kinematic cuts on the phase space of reactions.

Compton scattering presents well-known challenge for numerical treatment as all the analytical formulas for scattering rate behave badly numerically in different parameter areas, see e.g. [48, 49]. We easily bypass this difficulty as we numerically integrate well-behaved differential cross-section, as one can see for non-relativistic regime in Fig. 1 and for relativistic regime in Fig. 2. Figure 1 presents analytic photon spectrum for the reaction $\gamma + e^{\pm} \rightarrow \gamma' + e^{\pm'}$ as solid line and our numerical results shown by dots. Overall there is good agreement between numerical and analytical results. Small deviations in high-energy of the spectrum arise from leakage of the particles to kinematically forbidden area at the boundary of energy zones. Due to particle splitting (between cell interpolation) some final paricles would be placed on a grid node, that is kinematically forbidden, and it is indeed the case of Fig. 1. To show this effect we enlarge the plot range especially on this figure. On the other spectrum figures these points appear to be outside the presented plot range. There the maximum allowed photon energy is $e_{max} = 0.291$, but we have particles of energies from 0.275 up to e_{max} that are splitted between energy zones of 0.275 and 0.327 – the second is kinematically forbidden. Figure 2 shows the total reaction rate of the same process. Again there is good agreement between numerical and analytical re-



Figure 1: Compton scattered-photon distribution $v \frac{d\sigma}{de'}(e_{\gamma}, e_{\pm}, e'_{\gamma})$.



Figure 2: Compton scattering rate $\overline{v\sigma}(e_{\pm}, e_{\gamma})$.

sults. Small discrepancy arises from truncation of reactions where final particles get out of the grid to higher or lower energies. As a result numerical reaction rates are systematically lower than analytic ones.

Annihilation photon spectrum for reaction $e^+ + e^- \rightarrow \gamma + \gamma'$ is illustrated in Fig. 3 and total reaction rate in this process in Fig. 4. Figure 3 shows that the method is able to accurately reproduce the spectrum of annihilation photons in the range of more than two orders of magnitude. Reaction truncation errors, hardly seen at Fig. 4, are much lower for annihilation as low-energy photons are rare in this process.

Balance between pair creation and annihilation represent an independent test for the numerical scheme, as it is not automatically satisfied due to different numerical treatment of incoming and outgoing particles in the reactions. Pair creation spectra for reaction $\gamma_1 + \gamma_2 \rightarrow e^+ + e^-$ are reproduced well, see Fig. 5, as well as total reaction rates, see Fig. 6. Numerical balance can be checked by the form of particle distributions in numerical equilibrium, that was verified to be within 5 % of corresponding Boltzmann distributions.

For completeness we present also the results for Møller and Bhabha scattering, they show that these processes are indeed dominant for electrons and



Figure 3: Distribution for pair annihilation $\overline{v \frac{d\sigma}{de_{\gamma}}}(e_{\gamma}, e_{+}, e_{-}).$



Figure 4: Pair annihilation rate $\overline{v\sigma}(e_+,e_-).$



Figure 5: Distribution for pair production $\overline{v \frac{d\sigma}{de_{\pm}}}(e_{\gamma_1}, e_{\gamma_2}, e_{\pm}).$



Figure 6: Pair production rate $\overline{v\sigma}(e_{\gamma_1}, e_{\gamma_2})$.



Figure 7: Bhabha scattering rate $\overline{v\sigma}(e_+, e_-)$.

positrons in relativistic plasma, compare Figs. 7, 8 with Figs. 2, 4, 6.

Finally, we present a time evolution of energy density and concentration to demonstrate the difference between the classical Boltzmann and U-U equations. Both systems (28) and (29) were solved numerically with the same initial conditions under $a_{max} = 60$, $j_{max} = 64$, $k_{max} = 2j_{max}$ and using Gear's method for resulting stiff ODE system [50].

The energy spectrum $d\rho/d\varepsilon$ is shown instead of the distribution function f, that are related by $d\rho/d\varepsilon = 4\pi |\mathbf{p}|\varepsilon^2 c^{-2} f$. We chose an initial state without electrons and positrons but with photons only, initial spectrum has a power-law shape $d\rho/d\varepsilon = a(\varepsilon/\varepsilon_0)^b$, with $a = 3.63 \times 10^{28} \text{ cm}^{-3}$ and b = -0.438, $\varepsilon_0 = 1 \text{ erg}$, between e = 0.157 and e = 157. The initial spectrum corresponds to a total energy density $\rho = 4.10 \times 10^{26} \text{ erg cm}^{-3}$ and a total number density of particles $n = 8.15 \times 10^{31} \text{ cm}^{-3}$. In general, initial spectrum can have an arbitrary shape and thermalization process transforms it to an equilibrium form. Fig. 11 represents energy spectra at final equilibrium state. They attain corresponding shapes of Boltzmann and Bose-Einstein/Fermi-Dirac with some deviations in high-energy tails that are attributed to reaction truncation errors described before. We note that total energy and number densities do not change in time due to particle splitting applied, this feature does not depend on a form of a system of equations or a type of numerical ODE solver.



Figure 8: Møller scattering rate $\overline{v\sigma}(e_{\pm}, e_{\pm})$.



Figure 9: Time evolution of energy density in components of electron-positron-photon plasma: photon energy density (blue), electron/positron energy density (orange), total energy density (green). Solid lines correspond to Boltzmann case, dashed lines correspond to Uehling-Uhlenbeck case.



Figure 10: Time evolution of number density in components of electron-positron-photon plasma: photon concentration (blue), electron/positron concentration (orange), total concentration (green). Solid lines correspond to Boltzmann case, dashed lines correspond to Uehling-Uhlenbeck case. Note the difference in the final pair and photon density due to rest mass of elecron/positron (in both cases) and difference in statistics (for U-U case).



Figure 11: Final energy spectra at $t_{final} = 10^{-14}$ s. Solid lines are equilibrium Boltzmann and Bose-Einstein/Fermi-Dirac fits of numerical results: photon Boltzmann energy spectrum (blue), photon Bose-Einstein energy spectrum (cyan), pairs Boltzmann energy spectrum (orange), pairs Fermi-Dirac energy spectrum (red).

5. Conclusions

In this paper, we propose a new numerical method to accurately calculate Uehling–Uhlenbeck collision integral for two-particle interactions in relativistic plasma. Exact energy and particle number conservation laws are achieved by using interpolation scheme (26). After calculation of collision integral discretized Uehling–Uhlenbeck equations transforms into system of ODEs, which can be treated by various methods suitable to solve stiff ODEs. The method admits parallelization on GPU/CPU. Improvement in computation time with respect to previous work is achieved. Our reaction-oriented approach can be easily applied to any other types of particles and any other binary interactions, for instanse, weak interactions of neutrinos or electromagnetic ones of protons. Generalization of the proposed method for triple interactions is straightforward.

Our results show that reaction rates in relativistic plasma are well reproduced with moderate number of grid nodes in energy and angles (see Figures and Table 1) both for non-relativistic and relativistic particle energies. This allows development of an efficient method of solution for relativistic Uehling–Uhlenbeck equation.

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