

EXPONENTIAL INTEGRATORS FOR STOCHASTIC MAXWELL'S EQUATIONS DRIVEN BY ITÔ NOISE*

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Abstract. This article presents explicit exponential integrators for stochastic Maxwell's equations driven by both multiplicative and additive noises. By utilizing the regularity estimate of the mild solution, we first prove that the strong order of the numerical approximation is $\frac{1}{2}$ for general multiplicative noise. Combing a proper decomposition with the stochastic Fubini's theorem, the strong order of the proposed scheme is shown to be 1 for additive noise. Moreover, for linear stochastic Maxwell's equation with additive noise, the proposed time integrator is shown to preserve exactly the symplectic structure, the evolution of the energy as well as the evolution of the divergence in the sense of expectation. Several numerical experiments are presented in order to verify our theoretical findings.

Key words. stochastic Maxwell's equation, exponential integrator, strong convergence, trace formula, average energy, average divergence.

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1. Introduction. In the context of electromagnetism, a common way to model precise microscopic origins of randomness (such as thermal motion of electrically charged micro-particles) is by means of stochastic Maxwell's equations [35]. Further applications of stochastic Maxwell's equations are: In [32], a stochastic model of Maxwell's field equations in 1 + 1 dimension is shown to be a simple modification of a random walk model due to Kac, which provides a basis for the telegraph equations. The work [27] studies the propagation of ultra-short solitons in a cubic nonlinear medium modeled by nonlinear Maxwell's equations with stochastic variations of media. To simulate a coplanar waveguide with uncertain material parameters, time-harmonic Maxwell's equations are considered in [4]. For linear stochastic Maxwell's equations driven by additive noise, the work [21] proves that the problem is a stochastic Hamiltonian partial differential equation whose phase flow preserves the multi-symplectic geometric structure. In addition, the averaged energy along the flow increases linearly with respect to time and the flow preserves the divergence in the sense of expectation, see [10]. Let us finally mention that linear stochastic Maxwell's equations are relevant in various physical applications, see e.g. [35, Chapter 3].

We now review the literature on the numerical discretisation of stochastic Maxwell's equations. The work [41] performs a numerical analysis of the finite element method and discontinuous Galerkin method for stochastic Maxwell's equations driven by colored noise. A stochastic multi-symplectic method for 3 dimensional problems with additive noise, based on stochastic variational principle, is studied in [21]. In partic-

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ular, it is shown that the implicit numerical scheme preserves a discrete stochastic multi-symplectic conservation law. The work [10] inspects geometric properties of the stochastic Maxwell’s equation with additive noise, namely the behavior of averaged energy and divergence, see below for further details. Especially, the authors of [10] investigate three novel stochastic multi-symplectic (implicit in time) methods preserving discrete versions of the averaged divergence. None of the proposed numerical schemes exactly preserve the behavior of the averaged energy. The work [22] proposes a stochastic multi-symplectic wavelet collocation method for the approximation of stochastic Maxwell’s equations with multiplicative noise (in the Stratonovich sense). For the same stochastic Maxwell’s equation as the one considered in this paper (see below for a precise definition), the recent reference [8] shows that the backward Euler–Maruyama method converges with mean-square convergence rate $\frac{1}{2}$. Finally, the preprint [9] studies implicit Runge–Kutta schemes for stochastic Maxwell’s equation with additive noise. In particular, a mean-square convergence of order 1 is obtained.

In the present paper, we construct and analyse an exponential integrator for stochastic Maxwell’s equations which is explicit (thus computationally more efficient than the above mentioned time integrators) and which enjoys excellent long-time behavior. Observe that exponential integrators are widely used for efficient time integrations of deterministic differential equations, see for instance [18, 7, 19, 12] and more specially [37, 31, 24, 39, 33] and references therein for Maxwell-type equations. In recent years, exponential integrators have been analysed in the context of stochastic (partial) differential equations (S(P)DEs). Without being too exhaustive, we mention analysis and applications of such numerical schemes for the following problems: stochastic differential equations [36, 25, 26]; stochastic parabolic equations [23, 29, 5, 15, 3]; stochastic Schrödinger equations [1, 11, 16]; stochastic wave equations [13, 40, 14, 2, 34] and references therein.

The main contributions of the present paper are:

- a strong convergence analysis of an explicit exponential integrator for stochastic Maxwell’s equations in \mathbb{R}^3 . By making use of regularity estimates of the exact and numerical solutions, the strong convergence order is shown to be $\frac{1}{2}$ for general multiplicative noise. Furthermore, by using a proper decomposition and stochastic Fubini’s theorem, we prove that the strong convergence order of the proposed scheme can achieve 1.
- an analysis of long-time conservation properties of an explicit exponential integrator for linear stochastic Maxwell’s equations driven by additive noise. Especially, we show that the proposed explicit time integrator is symplectic and satisfies a trace formula for the energy for all times, i. e. the linear drift of the averaged energy is preserved for all times. In addition, the numerical solution preserves the averaged divergence. This shows that the exponential integrator inherits the geometric structure and the dynamical behavior of the flow of the linear stochastic Maxwell’s equations. This is not the case for classical time integrators such as Euler–Maruyama type schemes.
- an efficient numerical implementation of two-dimensional models of stochastic Maxwell’s equations by explicit time integrators.

We would like to remark that the proofs of strong convergence for the exponential integrator use similar ideas present in various proofs of strong convergence from the literature. But, to the best of our knowledge, the present paper offers the first explicit time integrator for linear stochastic Maxwell’s equations that is of strong order 1, symplectic, exactly preserves the linear drift of the averaged energy, and preserves the

averaged divergence for all times. A weak convergence analysis of the proposed scheme for stochastic Maxwell's equations driven by multiplicative noise will be reported elsewhere.

An outline of the paper is as follows. Section 2 sets notations and introduces the stochastic Maxwell's equation. This section also presents assumptions to guarantee existence and uniqueness of the exact solution to the problem and shows its Hölder continuity. The exponential integrator for stochastic Maxwell's equation is introduced in Section 3, where we also prove its strong order of convergence for additive and multiplicative noise. In Section 4, we show that the proposed scheme has several interesting geometric properties: it preserves the evolution laws of the averaged energy, the evolution laws of the divergence, and the symplectic structure of the original linear stochastic Maxwell's equations with additive noise. We conclude the paper by presenting numerical experiments supporting our theoretical results in Section 5.

2. Well-posedness of stochastic Maxwell's equations. We consider the stochastic Maxwell's equation driven by multiplicative Itô noise

$$(1) \quad \begin{aligned} d\mathbf{U} &= A\mathbf{U} dt + \mathbb{F}(\mathbf{U}) dt + \mathbb{G}(\mathbf{U}) dW, \quad t \in (0, +\infty), \\ \mathbf{U}(0) &= (\mathbf{E}_0^\top, \mathbf{H}_0^\top)^\top \end{aligned}$$

supplemented with the boundary condition of a perfect conductor $\mathbf{n} \times \mathbf{E} = 0$ as in [21]. Here, $\mathbf{U} = (\mathbf{E}^\top, \mathbf{H}^\top)^\top$, is \mathbb{R}^6 -valued function whose domain \mathcal{O} is a bounded and simply connected domain in \mathbb{R}^3 with smooth boundary $\partial\mathcal{O}$. The unit outward normal vector to $\partial\mathcal{O}$ is denoted by \mathbf{n} . Moreover, dW stands for the formal time derivative of a Q -Wiener process W on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The Q -Wiener process can be written as $W(\mathbf{x}, t) = \sum_{k \in \mathbb{N}_+} Q^{\frac{1}{2}} e_k(\mathbf{x}) \beta_k(t)$, where $\{\beta_k\}_{k \in \mathbb{N}_+}$

is a sequence of mutually independent and identically distributed \mathbb{R} -valued standard Brownian motions; $\{e_k\}_{k \in \mathbb{N}_+}$ is an orthonormal basis of $U := \mathcal{L}^2(\mathcal{O}; \mathbb{R})$ consisting of eigenfunctions of a symmetric, nonnegative and of finite trace linear operator Q , i. e., $Qe_k = \eta_k e_k$, with $\eta_k \geq 0$ for $k \in \mathbb{N}_+$. Assumptions on \mathbb{F} and \mathbb{G} are provided below.

The Maxwell's operator A is defined by

$$(2) \quad A \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} := \begin{pmatrix} 0 & \epsilon^{-1} \nabla \times \\ -\mu^{-1} \nabla \times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \epsilon^{-1} \nabla \times \mathbf{H} \\ -\mu^{-1} \nabla \times \mathbf{E} \end{pmatrix}.$$

It has the domain $D(A) := H_0(\text{curl}, \mathcal{O}) \times H(\text{curl}, \mathcal{O})$, where

$$H(\text{curl}, \mathcal{O}) := \{\mathbf{U} \in (\mathcal{L}^2(\mathcal{O}))^3 : \nabla \times \mathbf{U} \in (\mathcal{L}^2(\mathcal{O}))^3\},$$

is termed by the curl-space and

$$H_0(\text{curl}, \mathcal{O}) := \{\mathbf{U} \in H(\text{curl}, \mathcal{O}) : \mathbf{n} \times \mathbf{U}|_{\partial\mathcal{O}} = \mathbf{0}\}$$

is the subspace of $H(\text{curl}, \mathcal{O})$ with zero tangential trace. In addition, ϵ and μ are bounded and uniformly positive definite functions:

$$\epsilon, \mu \in \mathcal{L}^\infty(\mathcal{O}), \quad \epsilon, \mu \geq \kappa > 0$$

with κ being a positive constant. These conditions on ϵ, μ ensure that the Hilbert space $V := (\mathcal{L}^2(\mathcal{O}))^3 \times (\mathcal{L}^2(\mathcal{O}))^3$ is equipped with the weighted scalar product

$$\left\langle \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{H}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{pmatrix} \right\rangle_V = \int_{\mathcal{O}} (\mu \langle \mathbf{H}_1, \mathbf{H}_2 \rangle + \epsilon \langle \mathbf{E}_1, \mathbf{E}_2 \rangle) d\mathbf{x},$$

where $\langle \cdot, \cdot \rangle$ stands for the standard Euclidean inner product. This weighted scalar product is equivalent to the standard inner product on $(\mathcal{L}^2(\mathcal{O}))^6$. Moreover, the corresponding norm, which stands for the electromagnetic energy of the physical system, induced by this inner product reads

$$\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_V^2 = \int_{\mathcal{O}} (\mu \|\mathbf{H}\|^2 + \epsilon \|\mathbf{E}\|^2) \, d\mathbf{x}$$

with $\|\cdot\|$ being the Euclidean norm. Based on the norm $\|\cdot\|_V$, the associated graph norm of A is defined by

$$\|\mathbb{V}\|_{D(A)}^2 := \|\mathbb{V}\|_V^2 + \|A\mathbb{V}\|_V^2.$$

It is well known that Maxwell's operator A is closed and that $D(A)$ equipped with the graph norm is a Banach space, see e.g. [30]. Moreover, A is skew-adjoint, in particular, for all $\mathbb{V}_1, \mathbb{V}_2 \in D(A)$,

$$\langle A\mathbb{V}_1, \mathbb{V}_2 \rangle_V = -\langle \mathbb{V}_1, A\mathbb{V}_2 \rangle_V.$$

In addition, the operator A generates a unitary C_0 -group $\mathbf{S}(t) := \exp(tA)$ via Stone's theorem, see for example [17]. According to the definition of unitary groups, one has

$$(3) \quad \|\mathbf{S}(t)\mathbb{V}\|_V = \|\mathbb{V}\|_V \quad \text{for all } \mathbb{V} \in V,$$

which means that the electromagnetic energy is preserved, for Maxwell's operator, see [20]. Besides, the unitary group $\mathbf{S}(t)$ satisfies the following properties which will be made use of in the next section.

LEMMA 2.1 (Theorem 3 with $\mathbf{q} = \mathbf{0}$ in [6]). *For the semigroup $\{\mathbf{S}(t); t \geq 0\}$ on V , it holds that*

$$(4) \quad \|\mathbf{S}(t) - Id\|_{L(D(A);V)} \leq Ct,$$

where the constant C does not depend on t . Here, $L(D(A);V)$ denotes the space of bounded linear operators from $D(A)$ to V .

Observe that, throughout the paper, C stands for a constant that may vary from line to line.

For two real-valued separable Hilbert spaces $(H_1, \langle \cdot, \cdot \rangle_{H_1}, \|\cdot\|_{H_1})$ and $(H_2, \langle \cdot, \cdot \rangle_{H_2}, \|\cdot\|_{H_2})$, we denote the set of Hilbert–Schmidt operators from H_1 to H_2 by $\mathcal{L}_2(H_1, H_2)$. It will be equipped with the norm

$$\|\Gamma\|_{\mathcal{L}_2(H_1, H_2)}^2 := \sum_{i=1}^{\infty} \|\Gamma\phi_i\|_{H_2}^2,$$

where $\{\phi_i\}_{i \in \mathbb{N}_+}$ is any orthonormal basis of H_1 . Furthermore, let $Q^{\frac{1}{2}}$ be the unique positive square root of the linear operator Q (defining the noise W). We also introduce the separable Hilbert space $U_0 := Q^{\frac{1}{2}}U$ endowed with the inner product $\langle u_1, u_2 \rangle_{U_0} := \langle Q^{-\frac{1}{2}}u_1, Q^{-\frac{1}{2}}u_2 \rangle_U$ for $u_1, u_2 \in U_0$, where we recall that $U = \mathcal{L}^2(\mathcal{O}; \mathbb{R})$.

LEMMA 2.2. *As a consequence of Lemma 2.1, for any $\Phi \in \mathcal{L}_2(U_0, D(A))$ and any $t \geq 0$, we have*

$$(5) \quad \|(\mathbf{S}(t) - Id)\Phi\|_{\mathcal{L}_2(U_0, V)} \leq Ct\|\Phi\|_{\mathcal{L}_2(U_0, D(A))}.$$

Proof Thanks to Lemma 2.1 and the definition of the Hilbert–Schmidt norm, we know that, for $\{e_k\}_{k \in \mathbb{N}_+}$ an orthonormal basis of U ,

$$\begin{aligned} \|(\mathbf{S}(t) - Id) \Phi\|_{\mathcal{L}_2(U_0, V)}^2 &= \sum_{k \in \mathbb{N}_+} \|(\mathbf{S}(t) - Id) \Phi Q^{\frac{1}{2}} e_k\|_V^2 \\ &\leq Ct^2 \sum_{k \in \mathbb{N}_+} \|\Phi Q^{\frac{1}{2}} e_k\|_{D(A)}^2 \leq Ct^2 \|\Phi\|_{\mathcal{L}_2(U_0, D(A))}^2, \end{aligned}$$

which proves the claim. \square

To guarantee existence and uniqueness of strong solutions to (1), we make the following assumptions:

ASSUMPTION 2.1 (Coefficients). *Assume that the coefficients of Maxwell’s operator (2) satisfy*

$$\epsilon, \mu \in \mathcal{L}^\infty(\mathcal{O}), \quad \epsilon, \mu \geq \kappa > 0$$

with some positive constant κ .

ASSUMPTION 2.2 (Initial value). *The initial value $\mathbb{U}(0)$ of the stochastic Maxwell’s equation (1) is a $D(A)$ -valued stochastic process with $\mathbb{E} \left[\|\mathbb{U}(0)\|_{D(A)}^p \right] < \infty$ for any $p \geq 1$.*

ASSUMPTION 2.3 (Nonlinearity). *We assume that the operator $\mathbb{F}: V \rightarrow V$ is continuous and that there exists constants $C_{\mathbb{F}}, C_{\mathbb{F}}^1 > 0$ such that*

$$\begin{aligned} \|\mathbb{F}(\mathbb{V}_1) - \mathbb{F}(\mathbb{V}_2)\|_V &\leq C_{\mathbb{F}} \|\mathbb{V}_1 - \mathbb{V}_2\|_V, \quad \mathbb{V}_1, \mathbb{V}_2 \in V, \\ \|\mathbb{F}(\mathbb{V}_1) - \mathbb{F}(\mathbb{V}_2)\|_{D(A)} &\leq C_{\mathbb{F}}^1 \|\mathbb{V}_1 - \mathbb{V}_2\|_{D(A)}, \quad \mathbb{V}_1, \mathbb{V}_2 \in D(A), \\ \|\mathbb{F}(\mathbb{V})\|_V &\leq C_{\mathbb{F}}(1 + \|\mathbb{V}\|_V), \quad \mathbb{V} \in V, \\ \|\mathbb{F}(\mathbb{V})\|_{D(A)} &\leq C_{\mathbb{F}}^1(1 + \|\mathbb{V}\|_{D(A)}), \quad \mathbb{V} \in D(A). \end{aligned}$$

ASSUMPTION 2.4 (Noise). *We assume that the operator $\mathbb{G}: V \rightarrow \mathcal{L}_2(U_0, V)$ satisfies*

$$(6) \quad \begin{aligned} \|\mathbb{G}(\mathbb{V}_1) - \mathbb{G}(\mathbb{V}_2)\|_{\mathcal{L}_2(U_0, V)} &\leq C_{\mathbb{G}} \|\mathbb{V}_1 - \mathbb{V}_2\|_V, \quad \mathbb{V}_1, \mathbb{V}_2 \in V, \\ \|\mathbb{G}(\mathbb{V}_1) - \mathbb{G}(\mathbb{V}_2)\|_{\mathcal{L}_2(U_0, D(A))} &\leq C_{\mathbb{G}}^1 \|\mathbb{V}_1 - \mathbb{V}_2\|_{D(A)}, \quad \mathbb{V}_1, \mathbb{V}_2 \in D(A), \\ \|\mathbb{G}(\mathbb{V})\|_{\mathcal{L}_2(U_0, V)} &\leq C_{\mathbb{G}}(1 + \|\mathbb{V}\|_V), \quad \mathbb{V} \in V, \\ \|\mathbb{G}(\mathbb{V})\|_{\mathcal{L}_2(U_0, D(A))} &\leq C_{\mathbb{G}}^1(1 + \|\mathbb{V}\|_{D(A)}), \quad \mathbb{V} \in D(A), \end{aligned}$$

where $C_{\mathbb{G}}, C_{\mathbb{G}}^1 > 0$ may depend on the operator Q . We recall that $\mathcal{L}_2(U_0, V)$ and $\mathcal{L}_2(U_0, D(A))$ denote the spaces of Hilbert–Schmidt operators from U_0 to V , resp. to $D(A)$.

We now present two examples of an operator \mathbb{G} verifying Assumption 2.4 (we only prove one of the inequality in (6), the others follow in a similar way).

For the first example (inspired by [21]), let $\mathcal{O} = [0, 1]^3$, $\epsilon = \mu = 1$ and consider $\mathbb{G} \equiv (\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2)^T$ for two real numbers λ_1 and λ_2 . The stochastic Maxwell’s equation (1) then becomes an SPDE driven by additive noise. In this case, one chooses the orthonormal basis of U to be $\sin(i\pi x_1) \sin(j\pi x_2) \sin(k\pi x_3)$, for $i, j, k \in \mathbb{N}_+$, and $x_1, x_2, x_3 \in [0, 1]$. Assuming for example that $\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2(U, \mathcal{H}_0^1)} < \infty$, where $\mathcal{H}_0^1 := \mathcal{H}_0^1(\mathcal{O}) = \{u \in \mathcal{H}^1(\mathcal{O}): u = 0 \text{ on } \partial\mathcal{O}\}$, one can get that $\mathbb{G}Q^{\frac{1}{2}}\mathbb{V} \in D(A)$ for all $\mathbb{V} \in D(A)$ and thus the last inequality in (6) holds.

For the second example (inspired by [8]), consider $\mathbb{G}(\mathbb{V}) = \mathbb{V}$ for $\mathbb{V} \in V$, the domain $\mathcal{O} = [0, 1]^3$ and $\epsilon = \mu = 1$. Taking the same orthonormal basis as above, and assuming in addition that $Q^{\frac{1}{2}} \in \mathcal{L}_2(U, \mathcal{H}^{1+\gamma}(\mathcal{O}))$ with $\gamma > \frac{3}{2}$, one gets for instance

$$(7) \quad \|\mathbb{G}(\mathbb{V})\|_{\mathcal{L}_2(U_0, D(A))} \leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2(U, \mathcal{H}^{1+\gamma})} (1 + \|\mathbb{V}\|_{D(A)}).$$

Using the definition of the graph norm one gets

$$\|\mathbb{G}(\mathbb{V})\|_{\mathcal{L}_2(U_0, D(A))}^2 = \sum_{k \in \mathbb{N}_+} \|\mathbb{V} Q^{\frac{1}{2}} e_k\|_V^2 + \sum_{k \in \mathbb{N}_+} \|A(\mathbb{V} Q^{\frac{1}{2}} e_k)\|_V^2.$$

Denoting $\mathbb{V} = (\mathbf{E}_{\mathbb{V}}^T, \mathbf{H}_{\mathbb{V}}^T)^T$ and using the definition of the operator A , one obtains

$$\begin{aligned} & \|\mathbb{G}(\mathbb{V})\|_{\mathcal{L}_2(U_0, D(A))}^2 \\ &= \sum_{k \in \mathbb{N}_+} \sum_{i=1,2,3} \|\mathbf{E}_{\mathbb{V}}^i Q^{\frac{1}{2}} e_k\|_U^2 + \sum_{k \in \mathbb{N}_+} \sum_{i=1,2,3} \|\mathbf{H}_{\mathbb{V}}^i Q^{\frac{1}{2}} e_k\|_U^2 \\ & \quad + \sum_{k \in \mathbb{N}_+} \left(\|\nabla \times (\mathbf{E}_{\mathbb{V}} Q^{\frac{1}{2}} e_k)\|_{U^3}^2 + \|\nabla \times (\mathbf{H}_{\mathbb{V}} Q^{\frac{1}{2}} e_k)\|_{U^3}^2 \right) \\ & \leq C \sum_{k \in \mathbb{N}_+} \|Q^{\frac{1}{2}} e_k\|_{L^\infty(\mathcal{O})}^2 \|\mathbb{V}\|_V^2 + \sum_{k \in \mathbb{N}_+} \left(\|\nabla \times (\mathbf{E}_{\mathbb{V}} Q^{\frac{1}{2}} e_k)\|_{U^3}^2 + \|\nabla \times (\mathbf{H}_{\mathbb{V}} Q^{\frac{1}{2}} e_k)\|_{U^3}^2 \right). \end{aligned}$$

We now illustrate how to estimate the term $\|\nabla \times (\mathbf{E}_{\mathbb{V}} Q^{\frac{1}{2}} e_k)\|_{U^3}^2$ as an example. Using the definition of the **curl** operator, one gets

$$\begin{aligned} \|\nabla \times (\mathbf{E}_{\mathbb{V}} Q^{\frac{1}{2}} e_k)\|_{U^3}^2 &= \left\| \frac{\partial}{\partial x_2} (\mathbf{E}_{\mathbb{V}}^3 Q^{\frac{1}{2}} e_k) - \frac{\partial}{\partial x_3} (\mathbf{E}_{\mathbb{V}}^2 Q^{\frac{1}{2}} e_k) \right\|_U^2 \\ & \quad + \left\| \frac{\partial}{\partial x_1} (\mathbf{E}_{\mathbb{V}}^3 Q^{\frac{1}{2}} e_k) - \frac{\partial}{\partial x_3} (\mathbf{E}_{\mathbb{V}}^1 Q^{\frac{1}{2}} e_k) \right\|_U^2 \\ & \quad + \left\| \frac{\partial}{\partial x_1} (\mathbf{E}_{\mathbb{V}}^2 Q^{\frac{1}{2}} e_k) - \frac{\partial}{\partial x_2} (\mathbf{E}_{\mathbb{V}}^1 Q^{\frac{1}{2}} e_k) \right\|_U^2 \\ & \leq C \|Q^{\frac{1}{2}} e_k\|_{L^\infty(\mathcal{O})}^2 \left(\left\| \frac{\partial}{\partial x_2} \mathbf{E}_{\mathbb{V}}^3 - \frac{\partial}{\partial x_3} \mathbf{E}_{\mathbb{V}}^2 \right\|_U^2 + \left\| \frac{\partial}{\partial x_1} \mathbf{E}_{\mathbb{V}}^3 - \nabla^3 \mathbf{E}_{\mathbb{V}}^1 \right\|_U^2 \right. \\ & \quad \left. + \left\| \frac{\partial}{\partial x_1} \mathbf{E}_{\mathbb{V}}^2 - \nabla^2 \mathbf{E}_{\mathbb{V}}^1 \right\|_U^2 \right) \\ & \quad + C \left(\left\| \frac{\partial}{\partial x_1} Q^{\frac{1}{2}} e_k \right\|_{L^\infty(\mathcal{O})}^2 + \left\| \frac{\partial}{\partial x_2} Q^{\frac{1}{2}} e_k \right\|_{L^\infty(\mathcal{O})}^2 + \left\| \frac{\partial}{\partial x_3} Q^{\frac{1}{2}} e_k \right\|_{L^\infty(\mathcal{O})}^2 \right) \|\mathbf{E}_{\mathbb{V}}\|_{U^3}^2 \\ & \leq C \|Q^{\frac{1}{2}} e_k\|_{L^\infty(\mathcal{O})}^2 \|\nabla \times \mathbf{E}_{\mathbb{V}}\|_{U^3}^2 + C \|\nabla Q^{\frac{1}{2}} e_k\|_{L^\infty(\mathcal{O})}^2 \|\mathbf{E}_{\mathbb{V}}\|_V^2. \quad \blacksquare \end{aligned}$$

Combing the above estimates, we obtain

$$\|\mathbb{G}(\mathbb{V})\|_{\mathcal{L}_2(U_0, D(A))}^2 \leq C \sum_{k \in \mathbb{N}_+} \|Q^{\frac{1}{2}} e_k\|_{L^\infty(\mathcal{O})}^2 (\|\mathbb{V}\|_V^2 + \|A\mathbb{V}\|_V^2) + C \sum_{k \in \mathbb{N}_+} \|\nabla Q^{\frac{1}{2}} e_k\|_{L^\infty(\mathcal{O})}^2 \|\mathbb{V}\|_V^2. \quad \blacksquare$$

Using the Sobolev embedding $\mathcal{H}^\gamma(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ for any $\gamma > \frac{3}{2}$, one finally obtains (7) and the linear growth property of \mathbb{G} .

The above assumptions suffice to establish well-posedness and regularity results of solutions to (1). This uses similar arguments as, for instance, [28, Theorem 9] (for a more general drift coefficient in (1)) and [8, Corollary 3.1].

LEMMA 2.3. *Let $T > 0$. Under the Assumptions 2.1-2.4, the stochastic Maxwell's equation (1) is strongly well posed and its solution \mathbb{U} satisfies*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathbb{U}(t)\|_{D(A)}^p \right] < C \left(1 + \mathbb{E} \left[\|\mathbb{U}(0)\|_{D(A)}^p \right] \right)$$

for any $p \geq 2$. Here, the constant C depends on p , T , Q , bounds for \mathbb{F} and \mathbb{G} , and $\mathbb{U}(0)$.

Subsequently we present a lemma on the Hölder regularity in time of solutions to (1). This result is important in analysing the approximation error of the proposed time integrator in Section 3.

LEMMA 2.4. *Let $T > 0$. Under the Assumptions 2.1-2.4, the solution \mathbb{U} of the stochastic Maxwell's equation (1) satisfies*

$$\mathbb{E} \left[\|\mathbb{U}(t) - \mathbb{U}(s)\|_V^{2p} \right] \leq C|t - s|^p,$$

for any $0 \leq s, t \leq T$, and $p \geq 1$. Here, the constant C depends on p , T , Q , bounds for \mathbb{F} and \mathbb{G} , and $\mathbb{U}(0)$.

The proof is very similar to the proof of [8, Proposition 3.2], we omit it for ease of presentation.

Based on the above regularity results for solutions to the stochastic Maxwell's equation (1), the work [8] shows mean-square convergence order $\frac{1}{2}$ of the backward Euler–Maruyama scheme (in temporal direction). In the next section, we design and analyse an explicit and effective numerical scheme, the exponential integrator, which has the rate of convergence 1 and preserves many inherent properties of the original problem (in the case of the stochastic Maxwell's equations with additive noise).

3. Exponential integrators for stochastic Maxwell's equations and error analysis. This section is concerned with a convergence analysis in strong sense of an exponential integrator for the stochastic Maxwell's equation (1). We first show an a priori estimate of the numerical solution. Then the strong convergence rate is studied in two cases, first when equation (1) is driven by additive noise and then for multiplicative noise.

Fix a time horizon $T > 0$ and an integer $N > 0$. Define a stepsize Δt such that $T = N\Delta t$. We then construct a uniform partition of the interval $[0, T]$

$$0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$$

with $t_n = n\Delta t$ for $n = 0, \dots, N$. Next, we consider the mild solution of the stochastic Maxwell's equation (1) on the small time interval $[t_k, t_{k+1}]$ (with $\mathbb{U}(t_k) = \mathbb{U}_k$):

$$\mathbb{U}(t_{k+1}) = \mathbf{S}(\Delta t)\mathbb{U}_k + \int_{t_k}^{t_{k+1}} \mathbf{S}(t_{k+1} - s)\mathbb{F}(\mathbb{U}(s)) ds + \int_{t_k}^{t_{k+1}} \mathbf{S}(t_{k+1} - s)\mathbb{G}(\mathbb{U}(s)) dW.$$

By approximating both integrals in the above mild solution at the left end point, one obtains the exponential integrator

$$(8) \quad \mathbb{U}_{k+1} = \mathbf{S}(\Delta t)\mathbb{U}_k + \mathbf{S}(\Delta t)\mathbb{F}(\mathbb{U}_k)\Delta t + \mathbf{S}(\Delta t)\mathbb{G}(\mathbb{U}_k)\Delta W_k,$$

where $\Delta W_k = \Delta W(t_{k+1}) - \Delta W(t_k)$ stands for Wiener increments. One readily sees that (8) is an explicit numerical approximation of the exact solution $\mathbb{U}(t_{k+1})$ of the stochastic Maxwell's equation (1).

In order to present a result on the strong error of the exponential integrator (8), we first show an a priori estimate of the numerical solution.

THEOREM 3.1. *Under the Assumptions 2.1-2.4, the numerical solution to the stochastic Maxwell's equation given by the exponential integrator (8) satisfies*

$$\mathbb{E} \left[\|\mathbb{U}_k\|_{D(A)}^{2p} \right] \leq C(\mathbb{U}_0, Q, T, p, \mathbb{F}, \mathbb{G})$$

for all $p \geq 1$ and $k = 0, 1, \dots, N$.

Proof. The numerical approximation given by the exponential integrator can be rewritten as

$$\mathbb{U}_k = \mathbf{S}(t_k)\mathbb{U}(0) + \Delta t \sum_{j=0}^{k-1} \mathbf{S}(t_k - t_j)\mathbb{F}(\mathbb{U}_j) + \sum_{j=0}^{k-1} \mathbf{S}(t_k - t_j)\mathbb{G}(\mathbb{U}_j)\Delta W_j.$$

Taking norm and expectation leads to, for $p \geq 1$,

$$\begin{aligned} \mathbb{E} \left[\|\mathbb{U}_k\|_{D(A)}^{2p} \right] &\leq C\mathbb{E} \left[\|\mathbf{S}(t_k)\mathbb{U}(0)\|_{D(A)}^{2p} \right] + C\mathbb{E} \left[\left\| \Delta t \sum_{j=0}^{k-1} \mathbf{S}(t_k - t_j)\mathbb{F}(\mathbb{U}_j) \right\|_{D(A)}^{2p} \right] \\ &\quad + C\mathbb{E} \left[\left\| \sum_{j=0}^{k-1} \mathbf{S}(t_k - t_j)\mathbb{G}(\mathbb{U}_j)\Delta W_j \right\|_{D(A)}^{2p} \right]. \end{aligned}$$

For the first term, using the definition of the graph norm and property (3), we obtain

$$\|\mathbf{S}(t_k)\mathbb{U}(0)\|_{D(A)}^{2p} = (\|\mathbf{S}(t_k)\mathbb{U}(0)\|_V + \|\mathbf{S}(t_k)A\mathbb{U}(0)\|_V)^{2p} = \|\mathbb{U}(0)\|_{D(A)}^{2p},$$

which leads to $\mathbb{E} \left[\|\mathbf{S}(t_k)\mathbb{U}(0)\|_{D(A)}^{2p} \right] = \mathbb{E} \left[\|\mathbb{U}(0)\|_{D(A)}^{2p} \right]$. Based on the linear growth property of \mathbb{F} and Hölder's inequality, the second term is estimated as follows

$$\begin{aligned} \left\| \Delta t \sum_{j=0}^{k-1} \mathbf{S}(t_k - t_j)\mathbb{F}(\mathbb{U}_j) \right\|_{D(A)}^{2p} &\leq C + C\Delta t^{2p} \left(\sum_{j=0}^{k-1} \|\mathbb{U}_j\|_{D(A)} \right)^{2p} \\ &\leq C + C\Delta t^{2p} k^{2p-1} \sum_{j=0}^{k-1} \|\mathbb{U}_j\|_{D(A)}^{2p}. \end{aligned}$$

One then obtains

$$\mathbb{E} \left[\left\| \Delta t \sum_{j=0}^{k-1} \mathbf{S}(t_k - t_j)\mathbb{F}(\mathbb{U}_j) \right\|_{D(A)}^{2p} \right] \leq C + C\Delta t \mathbb{E} \left[\sum_{j=0}^{k-1} \|\mathbb{U}_j\|_{D(A)}^{2p} \right].$$

The third term is equivalent to

$$\mathbb{E} \left[\left\| \sum_{j=0}^{k-1} \mathbf{S}(t_k - t_j)\mathbb{G}(\mathbb{U}_j)\Delta W_j \right\|_{D(A)}^{2p} \right] = \mathbb{E} \left[\left\| \int_0^{t_k} \mathbf{S} \left(t_k - \left[\frac{s}{\Delta t} \right] \Delta t \right) \mathbb{G}(\mathbb{U}_{[\frac{s}{\Delta t}]\Delta t}) dW(s) \right\|_{D(A)}^{2p} \right] \blacksquare$$

with $[\frac{s}{\Delta t}]$ being the integer part of $\frac{s}{\Delta t}$. The Burkholder–Davis–Gundy inequality for stochastic integrals and our assumption on \mathbb{G} give

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_0^{t_k} \mathbf{S} \left(t_k - [\frac{s}{\Delta t}] \Delta t \right) \mathbb{G}(\mathbb{U}_{[\frac{s}{\Delta t}] \Delta t}) dW(s) \right\|_{D(A)}^{2p} \right] \leq \\ & \leq C \mathbb{E} \left[\left(\int_0^{t_k} \left\| \mathbb{G}(\mathbb{U}_{[\frac{s}{\Delta t}] \Delta t}) \right\|_{\mathcal{L}_2(U_0, D(A))}^2 ds \right)^p \right] \\ & \leq C + C \mathbb{E} \left[\left(\int_0^{t_k} \left\| \mathbb{U}_{[\frac{s}{\Delta t}] \Delta t} \right\|_{D(A)}^2 ds \right)^p \right] = C + C \mathbb{E} \left[\left(\Delta t \sum_{j=0}^{k-1} \|\mathbb{U}_j\|_{D(A)}^2 \right)^p \right]. \end{aligned}$$

Using Hölder’s inequality, the last term in the above inequality becomes

$$\left(\Delta t \sum_{j=0}^{k-1} \|\mathbb{U}_j\|_{D(A)}^2 \right)^p \leq \Delta t^p k^{p-1} \sum_{j=0}^{k-1} \|\mathbb{U}_j\|_{D(A)}^{2p}.$$

Taking expectation, we then obtain

$$\mathbb{E} \left[\left\| \int_0^{t_k} \mathbf{S} \left(t_k - [\frac{s}{\Delta t}] \Delta t \right) \mathbb{G}(\mathbb{U}(s)) dW(s) \right\|_{D(A)}^{2p} \right] \leq C + C \Delta t \sum_{j=0}^{k-1} \mathbb{E} \left[\|\mathbb{U}_j\|_{D(A)}^{2p} \right].$$

Altogether, we get that

$$\mathbb{E} \left[\|\mathbb{U}_k\|_{D(A)}^{2p} \right] \leq C + C \Delta t \mathbb{E} \left[\sum_{j=0}^{k-1} \|\mathbb{U}_j\|_{D(A)}^{2p} \right].$$

A discrete Gronwall inequality concludes the proof. \square

Using the above theorem, we arrive at

COROLLARY 3.1. *Under the same assumptions as in Theorem 3.1, for all $p \geq 1$, there exists a constant $C := C(\mathbb{U}(0), Q, T, p, \mathbb{F}, \mathbb{G})$ such that*

$$(9) \quad \mathbb{E} \left[\sup_{0 \leq k \leq N} \|\mathbb{U}_k\|_{D(A)}^{2p} \right] \leq C.$$

Proof. The main idea to derive the estimate (9) is to properly estimate the stochastic integral

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq k \leq N} \left\| \sum_{j=0}^{k-1} \mathbf{S}(t_k - t_j) \mathbb{G}(\mathbb{U}_j) \Delta W_j \right\|_{D(A)}^{2p} \right] = \\ & = \mathbb{E} \left[\sup_{0 \leq k \leq N} \left\| \int_0^{t_k} \mathbf{S} \left(t_k - [\frac{s}{\Delta t}] \Delta t \right) \mathbb{G}(\mathbb{U}_{[\frac{s}{\Delta t}] \Delta t}) dW(s) \right\|_{D(A)}^{2p} \right]. \end{aligned}$$

Based on the unitarity of $S(\cdot)$, Burkholder–Davis–Gundy’s inequality, Hölder’s inequality, and our assumptions on \mathbb{G} , the right hand side (RHS) of the above equality

becomes

$$\begin{aligned} \text{RHS} &\leq C \mathbb{E} \left[\left(\int_0^T \left\| \mathbb{G}(\mathbb{U}_{[\frac{s}{\Delta t}]\Delta t} \right\|_{\mathcal{L}_2(U_0, D(A))}^2 ds \right)^p \right] \\ &\leq C + C \Delta t \sum_{j=0}^{N-1} \mathbb{E} \left[\|\mathbb{U}_j\|_{D(A)}^{2p} \right] \leq C, \end{aligned}$$

where we use the result of Theorem 3.1 in the last step. The estimations of the other terms in the numerical solution are done in a similar way as in the previous result. \square

We are now in position to show the error estimates of the exponential integrator for the stochastic Maxwell's equation (1) driven by additive noise.

THEOREM 3.2. *Let Assumptions 2.1-2.4 hold. Assume in addition that $\mathbb{F} \in C_b^2(V)$ and \mathbb{G} does not dependent on \mathbb{U} . The strong error of the exponential integrator (8) when applied to the stochastic Maxwell's equation (1) verifies, for all $p \geq 1$,*

$$\left(\mathbb{E} \left[\max_{k=0, \dots, N} \|\mathbb{U}(t_k) - \mathbb{U}_k\|_V^{2p} \right] \right)^{\frac{1}{2p}} \leq C \Delta t,$$

where the positive constant C depends on bounds for \mathbb{F} (and its derivatives) and \mathbb{G} , as well as on T , p and Q .

Proof. Let us denote $\epsilon_k = \mathbb{U}(t_k) - \mathbb{U}_k$, for $k = 0, \dots, N$. We then have

$$\begin{aligned} \epsilon_{k+1} &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (\mathbf{S}(t_{k+1} - s) \mathbb{F}(\mathbb{U}(s)) - \mathbf{S}(t_{k+1} - t_j) \mathbb{F}(\mathbb{U}_j)) ds \\ &\quad + \sum_{j=0}^k \int_{t_j}^{t_{j+1}} ((\mathbf{S}(t_{k+1} - s) - \mathbf{S}(t_{k+1} - t_j)) \mathbb{G}) dW(s) \\ (10) \quad &=: Err_1^k + Err_2^k. \end{aligned}$$

We now rewrite the term Err_1^k as

$$\begin{aligned} Err_1^k &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (\mathbf{S}(t_{k+1} - s) (\mathbb{F}(\mathbb{U}(s)) - \mathbb{F}(\mathbb{U}(t_j)))) ds \\ &\quad + \sum_{j=0}^k \int_{t_j}^{t_{j+1}} ((\mathbf{S}(t_{k+1} - s) - \mathbf{S}(t_{k+1} - t_j)) \mathbb{F}(\mathbb{U}(t_j))) ds \\ &\quad + \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (\mathbf{S}(t_{k+1} - t_j) (\mathbb{F}(\mathbb{U}(t_j)) - \mathbb{F}(\mathbb{U}_j))) ds \\ &=: \mathbf{I}_1^k + \mathbf{I}_2^k + \mathbf{I}_3^k. \end{aligned}$$

We first estimate the term \mathbf{I}_1^k . Using a Taylor expansion, we obtain

$$\begin{aligned} \mathbb{F}(\mathbb{U}(s)) - \mathbb{F}(\mathbb{U}(t_j)) &= \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}(t_j))(\mathbb{U}(s) - \mathbb{U}(t_j)) \\ &\quad + \frac{1}{2} \frac{\partial^2 \mathbb{F}}{\partial u^2}(\Theta)(\mathbb{U}(s) - \mathbb{U}(t_j), \mathbb{U}(s) - \mathbb{U}(t_j)), \end{aligned}$$

where $\Theta := \theta\mathbb{U}(s) + (1 - \theta)\mathbb{U}(t_j)$, for some $\theta \in [0, 1]$, depends on $\mathbb{U}(s)$ and $\mathbb{U}(t_j)$. Combing this with the mild formulation of the exact solution on the interval $[t_j, s]$,

$$\mathbb{U}(s) = \mathbf{S}(s - t_j)\mathbb{U}(t_j) + \int_{t_j}^s \mathbf{S}(s - r)\mathbb{F}(\mathbb{U}(r)) dr + \int_{t_j}^s \mathbf{S}(s - r)\mathbb{G} dW(r),$$

we rewrite the term \mathbb{I}_1^k as

$$\mathbb{I}_1^k = \mathcal{A}_1^k + \mathcal{A}_2^k,$$

where we define

$$\begin{aligned} \mathcal{A}_1^k &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \mathbf{S}(t_{k+1} - s) \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}(t_j)) (\mathbf{S}(s - t_j) - Id) \mathbb{U}(t_j) ds \\ &\quad + \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \mathbf{S}(t_{k+1} - s) \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}(t_j)) \int_{t_j}^s \mathbf{S}(s - r) \mathbb{F}(\mathbb{U}(r)) dr ds \\ &\quad + \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \mathbf{S}(t_{k+1} - s) \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}(t_j)) \int_{t_j}^s \mathbf{S}(s - r) \mathbb{G} dW(r) ds \\ &=: \mathbb{II}_1^k + \mathbb{II}_2^k + \mathbb{II}_3^k, \end{aligned}$$

and

$$\mathcal{A}_2^k = \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \mathbf{S}(t_{k+1} - s) \frac{1}{2} \frac{\partial^2 \mathbb{F}}{\partial u^2}(\Theta) (\mathbb{U}(s) - \mathbb{U}(t_j), \mathbb{U}(s) - \mathbb{U}(t_j)) ds.$$

The assumption that $\mathbb{F} \in C_b^2(V)$ and the Hölder continuity of the exact solution \mathbb{U} in Lemma 2.4 provide us with the bound $\mathbb{E} [\|\mathcal{A}_2\|_V^{2p}] \leq C\Delta t^{2p}$. For the term \mathbb{II}_1 , we use property (3), the boundedness of the derivatives of \mathbb{F} and Lemma 2.1, combined with Hölder's inequality, to deduce that

$$\begin{aligned} \|\mathbb{II}_1^k\|_V &\leq \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \left\| \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}(t_j)) (\mathbf{S}(s - t_j) - Id) \mathbb{U}(t_j) \right\|_V ds \\ &\leq C \sum_{j=0}^k \int_{t_j}^{t_{j+1}} |s - t_j| \|\mathbb{U}(t_j)\|_{D(A)} ds \leq C(\Delta t)^2 \sum_{j=0}^k \|\mathbb{U}(t_j)\|_{D(A)} \\ &\leq C(\Delta t)^2 \left(\sum_{j=0}^k \|\mathbb{U}(t_j)\|_{D(A)}^{2p} \right)^{\frac{1}{2p}} \left(\frac{t_{k+1}}{\Delta t} \right)^{\frac{2p-1}{2p}} \\ &\leq C\Delta t \left(\sup_{0 \leq j \leq k} \|\mathbb{U}(t_j)\|_{D(A)}^{2p} \right)^{\frac{1}{2p}}. \end{aligned}$$

This leads to

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\mathbb{II}_1^k\|_V^{2p} \right] \leq C(\Delta t)^{2p} \mathbb{E} \left[\sup_{0 \leq j \leq N} \|\mathbb{U}(t_j)\|_{D(A)}^{2p} \right] \leq C(\Delta t)^{2p}$$

using Lemma 2.3. Next, we estimate the term Π_2^k . Using Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{aligned} \|\Pi_2^k\|_V &\leq C \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \int_{t_j}^s \|\mathbb{F}(\mathbb{U}(r))\|_V dr ds \\ &\leq C \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \int_{t_j}^s (1 + \|\mathbb{U}(r)\|_V) dr ds \\ &\leq C\Delta t + C \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (s - t_j)^{\frac{2p-1}{2p}} \left(\int_{t_j}^s \|\mathbb{U}(r)\|_V^{2p} dr \right)^{\frac{1}{2p}} ds \\ &\leq C\Delta t + C\Delta t \left(\sup_{0 \leq t \leq T} \|\mathbb{U}(t)\|_V^{2p} \right)^{\frac{1}{2p}}. \end{aligned}$$

From Lemma 2.3, It then follows that

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\Pi_2^k\|_V^{2p} \right] \leq C(\Delta t)^{2p} + C(\Delta t)^{2p} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathbb{U}(t)\|_V^{2p} \right] \leq C(\Delta t)^{2p}.$$

We now proceed to the estimation of the term Π_3^k . First notice that stochastic Fubini's theorem leads to

$$\begin{aligned} \Pi_3^k &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \mathbf{S}(t_{k+1} - s) \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}(t_j)) \int_{t_j}^s \mathbf{S}(s - r) \mathbb{G} dW(r) ds \\ &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \int_r^{t_{j+1}} \mathbf{S}(t_{k+1} - s) \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}(t_j)) \mathbf{S}(s - r) ds dW(r) \\ &= \int_0^{t_{k+1}} \int_r^{([\frac{r}{\Delta t}] + 1)\Delta t} \mathbf{S}(t_{k+1} - s) \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}([\frac{s}{\Delta t}]\Delta t)) \mathbf{S}(s - r) ds dW(r) \end{aligned}$$

and the integrand in the above equation is \mathcal{F}_r -adaptive. Then by the Burkholder–Davis–Gundy's inequality, we get

$$\begin{aligned} &\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\Pi_3^k\|_V^{2p} \right] \\ &\leq C \mathbb{E} \left[\left(\int_0^T \left\| \int_r^{([\frac{r}{\Delta t}] + 1)\Delta t} \mathbf{S}(t_{k+1} - s) \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}([\frac{s}{\Delta t}]\Delta t)) \mathbf{S}(s - r) ds \right\|_{\mathcal{L}_2(U_0, V)}^2 dr \right)^p \right]. \end{aligned}$$

Then, using the assumption that $\mathbb{F} \in C_b^2(V)$, we obtain

$$\begin{aligned} &\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\Pi_3^k\|_V^{2p} \right] \\ &\leq C \mathbb{E} \left[\left(\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left(\int_r^{t_{j+1}} \left\| \mathbf{S}(t_{k+1} - s) \frac{\partial \mathbb{F}}{\partial u}(\mathbb{U}(t_j)) \mathbf{S}(s - r) \right\|_{\mathcal{L}_2(U_0, V)} ds \right)^2 dr \right)^p \right] \\ &\leq C \mathbb{E} \left[\left(\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left(\int_r^{t_{j+1}} \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2(U, V)} ds \right)^2 dr \right)^p \right] \leq C(\Delta t)^{2p}. \end{aligned}$$

Thus, the above allows us to get the following estimate

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\mathcal{A}_1\|_V^{2p} \right] \leq C(\Delta t)^{2p},$$

which implies the estimate

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\mathbf{I}_1^k\|_V^{2p} \right] \leq C(\Delta t)^{2p}.$$

For the term \mathbf{I}_2^k , we use the unitary property of the semigroup (3) to get

$$\begin{aligned} \|\mathbf{I}_2^k\|_V &\leq \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \|(\mathbf{S}(t_{k+1} - s) - \mathbf{S}(t_{k+1} - t_j))\mathbb{F}(\mathbf{U}(t_j))\|_V \, ds \\ &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \|(\mathbf{S}(t_j - s) - Id)\mathbb{F}(\mathbf{U}(t_j))\|_V \, ds. \end{aligned}$$

According to Lemma 2.1 and the linear growth property of \mathbb{F} , the above term can be bounded by

$$\begin{aligned} \|\mathbf{I}_2^k\|_V &\leq C \sum_{j=0}^k \int_{t_j}^{t_{j+1}} |t_j - s| \|\mathbb{F}(\mathbf{U}(t_j))\|_{D(A)} \, ds \\ &\leq C(\Delta t)^2 \sum_{j=0}^k \|\mathbb{F}(\mathbf{U}(t_j))\|_{D(A)} \\ &\leq C\Delta t + C(\Delta t)^2 \sum_{j=0}^k \|\mathbf{U}(t_j)\|_{D(A)}. \end{aligned}$$

Taking the $2p$ -th power on both sides of the above inequality and then expectation, we obtain

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\mathbf{I}_2^k\|_V^{2p} \right] \leq C(\Delta t)^{2p} + C(\Delta t)^{2p} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathbf{U}(t)\|_{D(A)}^{2p} \right] \leq C(\Delta t)^{2p}$$

by Lemma 2.3 in Section 2. For the term \mathbf{I}_3^k , similarly as above, using properties of the semigroup and of \mathbb{F} , and Hölder's inequality, we obtain

$$\begin{aligned} \|\mathbf{I}_3^k\|_V &\leq \Delta t \sum_{j=0}^k \|\epsilon_j\|_V \leq \Delta t \left(\sum_{j=0}^k \|\epsilon_j\|_V^{2p} \right)^{\frac{1}{2p}} \left(\frac{t_{k+1}}{\Delta t} \right)^{\frac{2p-1}{2p}} \\ &\leq C\Delta t \left(\sum_{j=0}^k \|\epsilon_j\|_V^{2p} \right)^{\frac{1}{2p}} (\Delta t)^{\frac{1-2p}{2p}} = C(\Delta t)^{\frac{1}{2p}} \left(\sum_{j=0}^k \|\epsilon_j\|_V^{2p} \right)^{\frac{1}{2p}}. \end{aligned}$$

This gives us

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\mathbf{I}_3^k\|_V^{2p} \right] \leq C\Delta t \sum_{j=0}^{N-1} \mathbb{E} \left[\max_{l=0, \dots, j} \|\epsilon_l\|_V^{2p} \right].$$

The last term Err_2^k can be bounded as follows

$$\begin{aligned}
& \mathbb{E} \left[\max_{k=0, \dots, N-1} \|Err_2^k\|_V^{2p} \right] \\
&= \mathbb{E} \left[\max_{k=0, \dots, N-1} \left\| \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (\mathbf{S}(t_{k+1} - s) - \mathbf{S}(t_{k+1} - t_j)) \mathbb{G} dW(s) \right\|_V^{2p} \right] \\
&= \mathbb{E} \left[\max_{k=0, \dots, N-1} \left\| \int_0^{t_{k+1}} (\mathbf{S}(t_{k+1} - s) - \mathbf{S}(t_{k+1} - \lfloor \frac{s}{\Delta t} \rfloor \Delta t)) \mathbb{G} dW(s) \right\|_V^{2p} \right] \\
&\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left\| \int_0^t \mathbf{S}(t - \lfloor \frac{s}{\Delta t} \rfloor \Delta t) (\mathbf{S}(\lfloor \frac{s}{\Delta t} \rfloor \Delta t - s) - Id) \mathbb{G} dW(s) \right\|_V^{2p} \right].
\end{aligned}$$

Thanks to Burkholder–Davis–Gundy’s inequality and properties of the semigroup, we obtain

$$\begin{aligned}
\mathbb{E} \left[\max_{k=0, \dots, N-1} \|Err_2^k\|_V^{2p} \right] &\leq C \mathbb{E} \left[\left(\int_0^T \left\| (\mathbf{S}(\lfloor \frac{s}{\Delta t} \rfloor \Delta t - s) - Id) \mathbb{G} \right\|_{\mathcal{L}_2(U_0, V)}^2 ds \right)^p \right] \\
&= C \mathbb{E} \left[\left(\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left\| (\mathbf{S}(t_j - s) - Id) \mathbb{G} \right\|_{\mathcal{L}_2(U_0, V)}^2 ds \right)^p \right] \\
&\leq C \mathbb{E} \left[\left(\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} |t_j - s|^2 \left\| \mathbb{G} Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(U, D(A))}^2 ds \right)^p \right] \\
&\leq C(\Delta t)^{2p},
\end{aligned}$$

where we have used the linear growth property of \mathbb{G} in $\mathcal{L}_2(U_0, D(A))$ in the last step.

Collecting all the above estimates gives us the bound

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\epsilon_{k+1}\|_V^{2p} \right] \leq C(\Delta t)^{2p} + C\Delta t \sum_{j=0}^{N-1} \mathbb{E} \left[\max_{l=0, \dots, j} \|\epsilon_l\|_V^{2p} \right].$$

An application of Gronwall’s inequality yields

$$\left(\mathbb{E} \left[\max_{k=0, \dots, N} \|\epsilon_k\|_V^{2p} \right] \right)^{\frac{1}{2p}} \leq C\Delta t,$$

which means that the strong order of the exponential scheme is 1 if the noise is additive in the stochastic Maxwell’s equation (1). \square

Now we turn to the case where the stochastic Maxwell’s equation (1) is driven by a more general multiplicative noise.

THEOREM 3.3. *Let Assumptions 2.1–2.4 hold. The strong error of the exponential integrator (8) when applied to the stochastic Maxwell’s equation (1) verifies, for all $p \geq 1$,*

$$\left(\mathbb{E} \left[\max_{k=0, \dots, N} \|\mathbb{U}(t_k) - \mathbb{U}_k\|_V^{2p} \right] \right)^{\frac{1}{2p}} \leq C\Delta t^{\frac{1}{2}},$$

where the positive constant C depends on the Lipschitz coefficients of \mathbb{F} and \mathbb{G} , p , $\mathbb{U}(0)$, Q and T .

Proof. When the noise is multiplicative, the term Err_2^k in (10) becomes

$$Err_2^k = \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (\mathbf{S}(t_{k+1} - s)\mathbb{G}(\mathbb{U}(s)) - \mathbf{S}(t_{k+1} - t_j)\mathbb{G}(\mathbb{U}_j)) dW(s),$$

which can be rewritten as

$$\begin{aligned} Err_2^k &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \mathbf{S}(t_{k+1} - s)(\mathbb{G}(\mathbb{U}(s)) - \mathbb{G}(\mathbb{U}(t_j))) dW(s) \\ &\quad + \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (\mathbf{S}(t_{k+1} - s) - \mathbf{S}(t_{k+1} - t_j)) \mathbb{G}(\mathbb{U}(t_j)) dW(s) \\ &\quad + \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \mathbf{S}(t_{k+1} - t_j)(\mathbb{G}(\mathbb{U}(t_j)) - \mathbb{G}(\mathbb{U}_j)) dW(s) \\ &=: \text{III}_1 + \text{III}_2 + \text{III}_3. \end{aligned}$$

By Burkholder–Davis–Gundy’s inequality and the assumptions on \mathbb{G} , one obtains

$$\begin{aligned} &\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\text{III}_1\|_{\mathbb{V}}^{2p} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left\| \int_0^t \mathbf{S}(t-s)(\mathbb{G}(\mathbb{U}(s)) - \mathbb{G}(\mathbb{U}(\lfloor \frac{s}{\Delta t} \rfloor \Delta t))) dW(s) \right\|_{\mathbb{V}}^{2p} \right] \\ &\leq C \mathbb{E} \left[\left(\int_0^T \left\| \mathbb{G}(\mathbb{U}(s)) - \mathbb{G}(\mathbb{U}(\lfloor \frac{s}{\Delta t} \rfloor \Delta t)) \right\|_{\mathcal{L}_2(\mathbb{U}_0, \mathbb{V})}^2 ds \right)^p \right] \\ &\leq C \mathbb{E} \left[\left(\int_0^T \left\| \mathbb{U}(s) - \mathbb{U}(\lfloor \frac{s}{\Delta t} \rfloor \Delta t) \right\|_{\mathbb{V}}^2 ds \right)^p \right]. \end{aligned}$$

Based on Hölder’s inequality and the continuity of \mathbb{U} in Lemma 2.4, we have

$$\begin{aligned} &\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\text{III}_1\|_{\mathbb{V}}^{2p} \right] \\ &\leq C \mathbb{E} \left[\left(\left(\int_0^T \left\| \mathbb{U}(s) - \mathbb{U}(\lfloor \frac{s}{\Delta t} \rfloor \Delta t) \right\|_{\mathbb{V}}^{2p} ds \right)^{\frac{1}{p}} T^{\frac{p-1}{p}} \right)^p \right] \\ &\leq C \mathbb{E} \left[\int_0^T \left\| \mathbb{U}(s) - \mathbb{U}(\lfloor \frac{s}{\Delta t} \rfloor \Delta t) \right\|_{\mathbb{V}}^{2p} ds \right] \\ &\leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} |s - t_j|^p ds \leq C(\Delta t)^p. \end{aligned}$$

Similarly, for the term III_2 , we obtain

$$\begin{aligned}
& \mathbb{E} \left[\max_{k=0, \dots, N-1} \|\text{III}_2\|_V^{2p} \right] \\
& \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left\| \int_0^t \mathbf{S}(t-s) - \mathbf{S}(t - [\frac{s}{\Delta t}] \Delta t) \mathbb{G}(\mathbb{U}([\frac{s}{\Delta t}] \Delta t)) dW(s) \right\|_V^{2p} \right] \\
& \leq C \mathbb{E} \left[\left(\int_0^T \left\| \left(\mathbf{S}(t-s) - \mathbf{S}(t - [\frac{s}{\Delta t}] \Delta t) \right) \mathbb{G}(\mathbb{U}([\frac{s}{\Delta t}] \Delta t)) \right\|_{\mathcal{L}_2(U_0, V)}^2 ds \right)^p \right] \\
& \leq CT^{p-1} \mathbb{E} \left[\int_0^T \left\| \left(\mathbf{S}(s - [\frac{s}{\Delta t}] \Delta t) - Id \right) \mathbb{G}(\mathbb{U}([\frac{s}{\Delta t}] \Delta t)) \right\|_{\mathcal{L}_2(U_0, V)}^{2p} ds \right] \\
& \leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} |s - [\frac{s}{\Delta t}] \Delta t|^{2p} \mathbb{E} \left[\left\| \mathbb{G}(\mathbb{U}([\frac{s}{\Delta t}] \Delta t)) \right\|_{\mathcal{L}_2(U_0, D(A))}^{2p} ds \right] \\
& \leq C(\Delta t)^{2p}.
\end{aligned}$$

For the last term III_3 , using Assumption 2.4, we get

$$\begin{aligned}
& \mathbb{E} \left[\max_{k=0, \dots, N-1} \|\text{III}_3\|_V^{2p} \right] \\
& \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left\| \int_0^t \mathbf{S}([\frac{s}{\Delta t}] \Delta t) (\mathbb{G}(\mathbb{U}([\frac{s}{\Delta t}] \Delta t)) - \mathbb{G}(\mathbb{U}([\frac{s}{\Delta t}] \Delta t))) dW(s) \right\|_V^{2p} \right] \\
& \leq C \mathbb{E} \left[\left(\int_0^T \left\| \mathbb{G}(\mathbb{U}([\frac{s}{\Delta t}] \Delta t)) - \mathbb{G}(\mathbb{U}([\frac{s}{\Delta t}] \Delta t)) \right\|_{\mathcal{L}_2(U_0, V)}^2 ds \right)^p \right] \\
& \leq C \mathbb{E} \left[\left(\int_0^T \left\| \mathbb{U}([\frac{s}{\Delta t}] \Delta t) - \mathbb{U}([\frac{s}{\Delta t}] \Delta t) \right\|_V^2 ds \right)^p \right] \\
& \leq C \Delta t \sum_{j=0}^{N-1} \mathbb{E} \left[\max_{l=0, \dots, j} \|\mathbb{U}(t_l) - \mathbb{U}_l\|_V^{2p} \right].
\end{aligned}$$

Altogether, we obtain

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\text{Err}_2^k\|_V^{2p} \right] \leq C(\Delta t)^p + C \Delta t \sum_{j=0}^{N-1} \mathbb{E} \left[\max_{l=0, \dots, j} \|\epsilon_l\|_V^{2p} \right],$$

where we recall the notation $\epsilon_l = \mathbb{U}(t_l) - \mathbb{U}_l$. Another difference with the proof for the additive noise case is estimating the term I_1^k . Using (3) and Assumption 2.3, we obtain

$$\begin{aligned}
\|\text{I}_1^k\|_V^{2p} & \leq \left(\sum_{j=0}^k \int_{t_j}^{t_{j+1}} \|\mathbf{S}(t_{k+1}-s)(\mathbb{F}(\mathbb{U}(s)) - \mathbb{F}(\mathbb{U}(t_j)))\|_V ds \right)^{2p} \\
& \leq C \left(\sum_{j=0}^k \int_{t_j}^{t_{j+1}} \|\mathbb{U}(s) - \mathbb{U}(t_j)\|_V ds \right)^{2p} \\
& \leq C \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \|\mathbb{U}(s) - \mathbb{U}(t_j)\|_V^{2p} ds.
\end{aligned}$$

Using Lemma 2.4, one gets

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\mathbf{I}_1^k\|_V^{2p} \right] \leq C \sum_{j=0}^k \int_{t_j}^{t_{j+1}} |s - t_j|^p ds \leq C(\Delta t)^p.$$

Putting all these estimates together yields

$$\mathbb{E} \left[\max_{k=0, \dots, N-1} \|\epsilon_{k+1}\|_V^{2p} \right] \leq C(\Delta t)^p + C\Delta t \sum_{j=0}^{N-1} \mathbb{E} \left[\max_{l=0, \dots, j} \|\epsilon_l\|_V^{2p} \right].$$

An application of Gronwall's inequality completes the proof, that is, one gets

$$\left(\mathbb{E} \left[\max_{k=0, \dots, N} \|\epsilon_k\|_V^{2p} \right] \right)^{\frac{1}{2p}} \leq C(\Delta t)^{\frac{1}{2}}.$$

□

4. Linear stochastic Maxwell's equations with additive noise. In this section, we study phenomena where the densities of the electric and magnetic currents are assumed to be linear. This is an important example of application of stochastic Maxwell's equations in physics, see e.g. [35, Chapter 3, pages 112-114]. We thus now inspect the long-time behavior of the exponential integrator applied to the linear stochastic Maxwell's equation with additive noise. We also briefly comment on the symplectic structure of the exact and numerical solutions. For simplicity of presentation, in this section we consider a similar setting as in [10]: we assume that $\epsilon = \mu = 1$, take $\mathbb{F} = 0$ and $\mathbb{G} = (\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2)^\top$ for two real numbers λ_1 and λ_2 . Then the stochastic Maxwell's equation (1) becomes the linear stochastic Maxwell's equation with additive noise:

$$(11) \quad \begin{aligned} d\mathbf{E} - \nabla \times \mathbf{H} dt &= \lambda_1 \mathbf{e} dW, \\ d\mathbf{H} + \nabla \times \mathbf{E} dt &= \lambda_2 \mathbf{e} dW, \end{aligned}$$

where $\mathbf{e} = (1, 1, 1)^\top$. In [10], it is shown that the averaged energy increases linearly with respect to the evolution of time and that the flow of the linear stochastic Maxwell's equation with additive noise preserves the divergence in the sense of expectation. We now recall these results and analyse the behavior of the exponential integrator with respect to the preservation of these geometric properties of the problem.

LEMMA 4.1 (Theorems 2.1 and 2.2 in [10], Theorem 3.1 in [9]). *Consider the linear stochastic Maxwell's equation (11) with a trace class noise. There exists a constant $K = 3(\lambda_1^2 + \lambda_2^2) \text{Tr}(Q)$ such that the averaged energy of the exact solution satisfies the trace formula*

$$\mathbb{E} [\Phi^{exact}(t)] = \mathbb{E} [\Phi^{exact}(0)] + Kt \quad \text{for all times } t,$$

where $\Phi^{exact}(t) := \int_{\mathcal{O}} (\|\mathbf{E}(t)\|^2 + \|\mathbf{H}(t)\|^2) dx$ denotes the energy of the problem.

Assume that $Q^{\frac{1}{2}} \in \mathcal{L}(\mathcal{L}^2(\mathcal{O}), \mathcal{H}^1(\mathcal{O}))$, then the solution to equation (11) preserves the averaged divergence

$$\mathbb{E} [\text{div}(\mathbf{E}(t))] = \mathbb{E} [\text{div}(\mathbf{E}(0))], \quad \mathbb{E} [\text{div}(\mathbf{H}(t))] = \mathbb{E} [\text{div}(\mathbf{H}(0))]$$

for all times t .

The solutions to Maxwell's equation (11) preserves the symplectic structure

$$\bar{\omega}(t) = \bar{\omega}(0) \quad \mathbb{P}\text{-a.s.},$$

where $\bar{\omega}(t) := \int_{\mathcal{O}} d\mathbf{E}(t, \mathbf{x}) \wedge d\mathbf{H}(t, \mathbf{x}) d\mathbf{x}$.

We now show that the proposed exponential integrator possesses the same long-time behavior as the exact solution to the linear stochastic Maxwell's equation. This is certainly not the case for traditional time integrators such as Euler–Maruyama's scheme, see the numerical experiments below. Recall, that under this setting, the exponential integrator applied to (11) reads

$$(12) \quad \mathbb{U}_{k+1} = \mathbf{S}(\Delta t)\mathbb{U}_k + \mathbf{S}(\Delta t)\mathbb{G}\Delta W_k.$$

We look at the trace formula for the energy first.

PROPOSITION 4.1. *The numerical scheme (12) satisfies the same trace formula for the energy as the exact solution to the linear stochastic Maxwell's equation*

$$\mathbb{E} [\Phi(t_k)] = \mathbb{E} [\Phi(0)] + Kt_k \quad \text{for all discrete times } t_k,$$

where we denote $\Phi(t_k) := \int_{\mathcal{O}} (\|\mathbf{E}_k\|^2 + \|\mathbf{H}_k\|^2) d\mathbf{x}$ the numerical energy, recall that $t_k = k\Delta t$ for $k = 1, 2, \dots$ and $K = 3(\lambda_1^2 + \lambda_2^2) \text{Tr}(Q)$ as in the above result.

Proof. We first observe that $\Phi(t_k)$ stands for the norm $\|\mathbb{U}_k\|_V^2$ which we now compute

$$\begin{aligned} \|\mathbb{U}_k\|_V^2 &= \|\mathbf{S}(\Delta t)\mathbb{U}_{k-1}\|_V^2 + 2\langle \mathbf{S}(\Delta t)\mathbb{U}_{k-1}, \mathbf{S}(\Delta t)\mathbb{G}\Delta W_{k-1} \rangle_V \\ &\quad + \|\mathbf{S}(\Delta t)\mathbb{G}\Delta W_{k-1}\|_V^2 \\ &= \|\mathbb{U}_{k-1}\|_V^2 + 2\langle \mathbf{S}(\Delta t)\mathbb{U}_{k-1}, \mathbf{S}(\Delta t)\mathbb{G}\Delta W_{k-1} \rangle_V + \|\mathbb{G}\Delta W_{k-1}\|_V^2, \end{aligned}$$

which leads to

$$\mathbb{E} [\|\mathbb{U}_k\|_V^2] = \mathbb{E} [\|\mathbb{U}_{k-1}\|_V^2] + \mathbb{E} [\|\mathbb{G}\Delta W_{k-1}\|_V^2].$$

Moreover, using the definition of the $\|\cdot\|_V$ norm and Itô's isometry, one obtains

$$\begin{aligned} \mathbb{E} [\|\mathbb{G}\Delta W_{k-1}\|_V^2] &= 3(\lambda_1^2 + \lambda_2^2) \int_{\mathcal{O}} \mathbb{E} \left[\left\| \int_{t_{k-1}}^{t_k} dW(s) \right\|^2 \right] d\mathbf{x} \\ &= 3(\lambda_1^2 + \lambda_2^2) \Delta t \int_{\mathcal{O}} \left(\sum_{n \in \mathbb{N}_+} \eta_n e_n(x)^2 \right) d\mathbf{x} \\ &= 3(\lambda_1^2 + \lambda_2^2) \text{Tr}(Q) \Delta t = K \Delta t. \end{aligned}$$

A recursion concludes the proof. \square

The above proposition thus shows that the exact trace formula for the energy also holds for the numerical solution given by the exponential integrator (12). The following proposition shows that the exponential integrator (12) also preserves the discrete version of the averaged divergence exactly.

PROPOSITION 4.2. *The numerical approximation to the linear stochastic Maxwell's equation (11) given by the exponential integrator (12) exactly preserves the following discrete averaged divergence*

$$\mathbb{E} [\operatorname{div}(\mathbf{E}_k)] = \mathbb{E} [\operatorname{div}(\mathbf{E}_{k-1})], \quad \mathbb{E} [\operatorname{div}(\mathbf{H}_k)] = \mathbb{E} [\operatorname{div}(\mathbf{H}_{k-1})]$$

for all $k \in \mathbb{N}_+$.

Proof. Let us denote $(\operatorname{div}, \operatorname{div})(\mathbf{E}^T, \mathbf{H}^T)^T := (\operatorname{div}\mathbf{E}^T, \operatorname{div}\mathbf{H}^T)^T$. Taking now the divergence and expectation of both components of the numerical solution leads to

$$(13) \quad \mathbb{E} [(\operatorname{div}, \operatorname{div})\mathbb{U}_k] = \mathbb{E} [(\operatorname{div}, \operatorname{div})(\mathbf{S}(\Delta t)\mathbb{U}_{k-1})].$$

We next notice that $\mathbf{S}(\Delta t)\mathbb{U}_{k-1}$ is the solution of the deterministic Maxwell's equation at time $t = \Delta t$,

$$\begin{aligned} d\mathbf{E} - \nabla \times \mathbf{H} dt &= 0, \\ d\mathbf{H} + \nabla \times \mathbf{E} dt &= 0, \quad (\mathbf{E}^T, \mathbf{H}^T)^T(0) = \mathbb{U}_{k-1}. \end{aligned}$$

Using the property $\operatorname{div}(\nabla \times \cdot) = 0$ and a similar argument as in [10, Theorem 2.2], we obtain

$$(14) \quad (\operatorname{div}, \operatorname{div})(\mathbf{S}(\Delta t)\mathbb{U}_{k-1}) = (\operatorname{div}, \operatorname{div})(\mathbb{U}_{k-1}).$$

Finally, combing (13) and (14) yields the desired result. \square

Regarding the symplectic structure of the numerical solutions, we obtain the following result.

PROPOSITION 4.3. *The exponential integrator (12) has the discrete stochastic symplectic conservation law*

$$\bar{\omega}_1 = \int_{\mathcal{O}} d\mathbf{E}_1 \wedge d\mathbf{H}_1 dx = \int_{\mathcal{O}} d\mathbf{E}_0 \wedge d\mathbf{H}_0 dx = \bar{\omega}_0 \quad \mathbb{P}\text{-a.s.}$$

Proof. Taking the differential of the numerical solution (12) gives $d\mathbb{U}_{k+1} = d(\mathbf{S}(\Delta t)\mathbb{U}_k)$. Thus, showing symplecticity of the exponential integrator is equivalent to showing the symplecticity of the flow of the deterministic linear Maxwell's equation with initial value \mathbb{U}_k . This is a well know fact. \square

5. Numerical experiments. This section presents various numerical experiments in order to illustrate the main properties of the stochastic exponential integrator (8), denoted by SEXP below. We will compare this numerical scheme with the following classical ones:

- The Euler–Maruyama scheme (denoted by EM below)

$$(EM) \quad \mathbb{U}_{k+1} = \mathbb{U}_k + A\mathbb{U}_k\Delta t + \mathbb{F}(\mathbb{U}_k)\Delta t + \mathbb{G}(\mathbb{U}_k)\Delta W_k.$$

- The semi-implicit Euler–Maruyama scheme (denoted by SEM below)

$$(SEM) \quad \mathbb{U}_{k+1} = \mathbb{U}_k + A\mathbb{U}_{k+1}\Delta t + \mathbb{F}(\mathbb{U}_k)\Delta t + \mathbb{G}(\mathbb{U}_k)\Delta W_k.$$

Below, we consider the stochastic Maxwell's equation (1) with TM polarization on the domain $[0, 1] \times [0, 1]$. In this setting, the electric and magnetic fields are $\mathbf{E} = (0, 0, E_3)$, resp. $\mathbf{H} = (H_1, H_2, 0)$. The spatial discretisation is done by the staged uniform grid

from [38] with mesh sizes $\Delta x = \Delta y = 2^{-4}$. Unless stated otherwise, the initial condition reads

$$\begin{aligned} E_3(x, y, 0) &= 0.1 \exp(-50((x - 0.5)^2 + (y - 0.5)^2)) \\ H_1(x, y, 0) &= \text{rand}_y \\ H_2(x, y, 0) &= \text{rand}_x, \end{aligned}$$

where rand_x , resp. rand_y , are random initial values in one direction whereas the other direction is kept constant. This is done in order to have zero divergence. The eigenvalues of the linear operator Q are given by $3/(j^3 + k^3)$ for $j, k = 1, 2, \dots$

5.1. Strong convergence. We first illustrate the strong rates of convergence of the exponential integrator (8) stated in Theorems 3.2 and 3.3. To do this, we compute the errors $\mathbf{E} [\|\mathbf{U}^N - \mathbf{U}_{\text{ref}}(T)\|_V^2]$ at the final time $T = 0.5$ for time steps ranging from $\Delta t = 2^{-8}$ to $\Delta t_{\text{ref}} = 2^{-13}$ and report these errors in Figure 1. The reference solution is computed using the exponential integrator and the expected values are approximated by computing averages over $M_s = 500$ samples. We observed that using a larger number of samples ($M_s = 750$) does not significantly improve the behavior of the convergence plots. The theoretical rates of convergence of the exponential integrator stated in the above theorems are indeed observed in these plots.

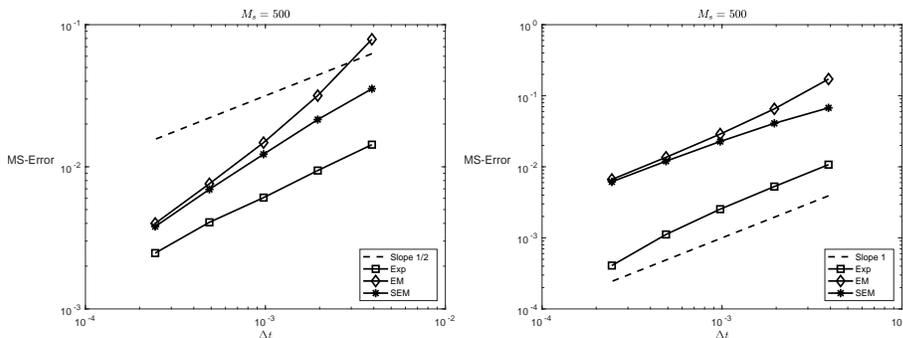


FIG. 1. *Strong rates of convergence for the stochastic Maxwell's equation with $\mathbb{F}(\mathbf{U}) = \mathbf{U} + \cos(\mathbf{U})$ and $\mathbb{G}(\mathbf{U}) = \sin(\mathbf{U})$ (left) and $\mathbb{F}(\mathbf{U}) = \mathbf{U}$ and $\mathbb{G}(\mathbf{U}) = \mathbf{1}^T$ (right).*

5.2. Averaged energy and divergence. We now illustrate the geometric properties of the exponential integrator stated in Section 4. We consider the problem (11) with $\lambda_1 = \lambda_2 = 0.5$, the time interval $[0, 5]$, a step size $\Delta t = 0.01$ and $M_s = 25000$ samples to approximate the expectations. The numerical averaged energies and divergences are displayed in Figure 2. The trace formula for the energy of the stochastic exponential integrator, as stated in Proposition 4.1, is observed in this figure (left and middle plots). This is in contrast with the wrong behavior of the SEM scheme and the EM scheme, where explosion in the energy is observed for the EM scheme (left plot). In this figure (right plot), one can also observe the preservation of the averaged divergence of the magnetic field along the numerical solution given by the exponential integrator. This confirms the result of Proposition 4.2.

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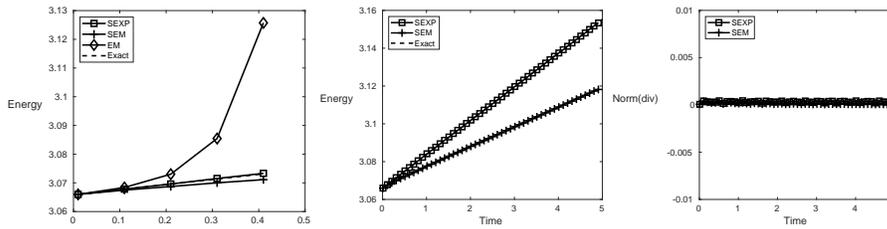


FIG. 2. Averaged energy on a short time (left) and on a longer time (middle), averaged divergence (right).

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