A Note on Numerical Fluxes Conserving a Member of Harten's One-Parameter Family of Entropies for the Compressible Euler Equations

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Entropy-conserving numerical fluxes are a cornerstone of modern high-order entropy-dissipative discretizations of conservation laws. In addition to entropy conservation, other structural properties mimicking the continuous level such as pressure equilibrium and kinetic energy preservation are important. This note proves that there are no numerical fluxes conserving (one of) Harten's entropies for the compressible Euler equations that also preserve pressure equilibria and have a density flux independent of the pressure. This is in contrast to fluxes based on the physical entropy, where even kinetic energy preservation can be achieved in addition.

Key words. entropy stability, numerical fluxes, flux differencing, kinetic energy preservation, pressure equilibrium preservation, local linear stability

1 Introduction

Ever since the seminal work of Tadmor [22], researchers have been interested in entropy-dissipative numerical methods for conservation laws and related models. Usually, these methods have improved robustness properties, even for underresolved simulations. Nowadays, several means have been explored to ensure entropy stability. One of the most popular and successful approaches is based on entropy-conservative (EC) numerical fluxes. These can be used to construct high-order central-type methods using flux differencing [6] to which appropriate dissipation can be added. A recent alternative is the general algebraic approach of [1, 2].

There are several numerical fluxes for the compressible Euler equations conserving the physical (logarithmic) entropy [4, 10, 12]. Harten [8] studied another family of entropies for the compressible Euler equations. Although these entropies do not symmetrize the heat flux terms in the compressible Navier-Stokes equations [9], they are still of interest, for example to construct entropy splitting methods or related EC fluxes [21].

Entropy conservation alone is often insufficient to construct good numerical methods. Of course, dissipation and related issues such as shock capturing and positivity of the density and internal energy are also important but are not the focus of this contribution. Instead, preservation of the kinetic energy [11, 13, 14] and pressure equilibria [16, 20] is considered. Moreover, the numerical density flux should not depend on the pressure, in accordance with physical expectations. This is

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discussed further in [5, 12], where positivity failure could be identified for certain setups with large pressure jumps and constant densities and velocities, even in the presence of strong (numerical) dissipation.

The main contribution of this note is to prove that there are no numerical fluxes conserving Harten's entropies for the compressible Euler equations that also preserve pressure equilibria and have a density flux independent of the pressure (Section 2). Further discussion of this result is presented in Section 3.

2 Main result

It suffices to concentrate on the 1D compressible Euler equations

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{0}, \qquad \mathbf{u} = (\varrho, \varrho v, \varrho e)^T, \quad \mathbf{f}(\mathbf{u}) = (\varrho v, \varrho v^2 + p, (\varrho e + p)v)^T,$$
 (2.1)

where ϱ is the density, v the velocity, ϱe the total energy, and the pressure $p=(\gamma-1)(\varrho e-\varrho v^2/2)$ is given by the ideal gas law with ratio of specific heats $\gamma>1$. Harten [8] considered entropies of the form $U(\boldsymbol{u})=-\varrho h(s)$, where $s=\log(p/\varrho^\gamma)$ and h is a smooth function satisfying $h''(s)/h'(s)<1/\gamma$. A special one-parameter family of these entropies is given by

$$U(\mathbf{u}) = -\frac{\gamma + \alpha}{\gamma - 1} \varrho(p/\varrho^{\gamma})^{1/(\alpha + \gamma)}, \qquad \alpha > 0 \text{ or } \alpha < -\gamma,$$
(2.2)

where the restriction of the parameter α ensures convexity of U [21]. The members of this one-parameter family (2.2) of entropy functions for the compressible Euler equations are often referred to as Harten's entropies in the literature, e.g., in [21]. The associated entropy variables $\mathbf{w} = U'(\mathbf{u})$ and the flux potential ψ [22] are given by

$$\boldsymbol{w} = \frac{\varrho}{p} (p/\varrho^{\gamma})^{1/(\alpha+\gamma)} \left(-\frac{\alpha}{\gamma-1} \frac{p}{\varrho} - \frac{1}{2} v^2, v, -1 \right), \qquad \psi = \varrho (p/\varrho^{\gamma})^{1/(\alpha+\gamma)} v. \tag{2.3}$$

This note focuses on two-point numerical fluxes for the compressible Euler equations. Such a two-point numerical flux f^{num} is characterized as follows.

Definition 2.1 (Entropy conservation [22]). The numerical flux f^{num} is EC if $[\![\boldsymbol{w}]\!] \cdot f^{\text{num}} - [\![\psi]\!] = 0$, where $[\![\boldsymbol{a}]\!] = a_+ - a_-$ is the common jump operator.

Definition 2.2 (Pressure equilibrium preservation [16]). A numerical flux $f^{\text{num}} = (f^{\text{num}}_{\varrho}, f^{\text{num}}_{\varrho e}, f^{\text{num}}_{$

As discussed in [7, 16, 20], pressure equilibria are important setups where the velocity and the pressure are constant. In this case, the compressible Euler equations are reduced to linear advection equations. Pressure equilibrium preserving schemes keep this property at the discrete level [16].

The main result of this note is

Theorem 2.3. There is no two-point numerical flux for the compressible Euler equations (2.1) that is entropy-conserving in the sense of Tadmor (Def. 2.1) for a member of Harten's one-parameter family of entropies (2.2), pressure equilibrium preserving (Def. 2.2), and has a density flux that does not depend on the pressure.

The proof of Theorem 2.3 is divided into the following steps.

Lemma 2.4. For $p \equiv \text{const}$ and $v \equiv \text{const}$, a numerical flux that is PEP and EC for a member of Harten's one-parameter family of entropies (2.2) has a density flux of the form

$$f_{\varrho}^{\text{num}} = -\frac{\gamma}{\alpha} \frac{\llbracket \varrho^{\alpha/(\alpha+\gamma)} \rrbracket}{\llbracket \varrho^{-\gamma/(\alpha+\gamma)} \rrbracket} v. \tag{2.4}$$

Proof. For constant pressure and velocity,

$$\begin{bmatrix} \boldsymbol{w} \end{bmatrix} \cdot \boldsymbol{f}^{\text{num}} - \begin{bmatrix} \boldsymbol{\psi} \end{bmatrix} = \left(-\frac{\alpha}{\gamma - 1} p^{1/(\alpha + \gamma)} \begin{bmatrix} \varrho^{-\gamma/(\alpha + \gamma)} \end{bmatrix} - \frac{1}{2} p^{(1 - \alpha - \gamma)/(\alpha + \gamma)} v^2 \begin{bmatrix} \varrho^{\alpha/(\alpha + \gamma)} \end{bmatrix} \right) \boldsymbol{f}^{\text{num}}_{\varrho} \\
+ p^{(1 - \alpha - \gamma)/(\alpha + \gamma)} v \begin{bmatrix} \varrho^{\alpha/(\alpha + \gamma)} \end{bmatrix} \boldsymbol{f}^{\text{num}}_{\varrho v} - p^{(1 - \alpha - \gamma)/(\alpha + \gamma)} \begin{bmatrix} \varrho^{\alpha/(\alpha + \gamma)} \end{bmatrix} \boldsymbol{f}^{\text{num}}_{\varrho e} \\
- p^{1/(\alpha + \gamma)} v \begin{bmatrix} \varrho^{\alpha/(\alpha + \gamma)} \end{bmatrix}.$$
(2.5)

A PEP flux is of the form $f_{\varrho v}^{\rm num} = v f_{\varrho}^{\rm num} + p$, $f_{\varrho e}^{\rm num} = \frac{1}{2} v^2 f_{\varrho}^{\rm num} + p v \gamma / (\gamma - 1)$. Thus,

Hence, the EC condition is equivalent to (2.4).

Lemma 2.5. For $v \equiv \text{const}$ and $p_- = p_+(\varrho_-/\varrho_+)^{\alpha/(\alpha+\gamma-1)}$, a numerical flux that is EC for a member of Harten's one-parameter family of entropies (2.2) has a density flux of the form

$$\boldsymbol{f}_{\varrho}^{\text{num}} = -\frac{\gamma - 1}{\alpha} \frac{\left[\left[\varrho^{\alpha/(\alpha + \gamma - 1)}\right]\right]}{\left[\left[\varrho^{(1 - \gamma)/(\alpha + \gamma - 1)}\right]\right]} v. \tag{2.7}$$

Proof. For this special choice of the pressure,

$$\frac{\varrho_{-}}{p_{-}}(p_{-}/\varrho_{-}^{\gamma})^{1/(\alpha+\gamma)} = \varrho_{-}^{\alpha/(\alpha+\gamma)}p_{-}^{(1-\alpha-\gamma)/(\alpha+\gamma)} = \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{-\alpha/(\alpha+\gamma)}p_{+}^{(1-\alpha-\gamma)/(\alpha+\gamma)}
= \varrho_{+}^{\alpha/(\alpha+\gamma)}p_{+}^{(1-\alpha-\gamma)/(\alpha+\gamma)} = \frac{\varrho_{+}}{p_{+}}(p_{+}/\varrho_{+}^{\gamma})^{1/(\alpha+\gamma)},$$
(2.8)

i.e., $[(\varrho/p)(p/\varrho^{\gamma})^{1/(\alpha+\gamma)}] = 0$. Thus, for $v \equiv \text{const}$, (2.3) yields

$$\llbracket \boldsymbol{w} \rrbracket \cdot \boldsymbol{f}^{\text{num}} - \llbracket \psi \rrbracket = -\frac{\alpha}{\gamma - 1} \llbracket (p/\varrho^{\gamma})^{1/(\alpha + \gamma)} \rrbracket \boldsymbol{f}_{\varrho}^{\text{num}} - \llbracket \varrho (p/\varrho^{\gamma})^{1/(\alpha + \gamma)} \rrbracket v. \tag{2.9}$$

Consequently, an EC flux must be of the form

$$\boldsymbol{f}_{\varrho}^{\text{num}} = -\frac{\gamma - 1}{\alpha} \frac{\llbracket \varrho(p/\varrho^{\gamma})^{1/(\alpha + \gamma)} \rrbracket}{\llbracket (p/\varrho^{\gamma})^{1/(\alpha + \gamma)} \rrbracket} v. \tag{2.10}$$

Inserting the pressure ratio, the fraction of the jump terms can be written as

$$\frac{\varrho_{+}^{\alpha/(\alpha+\gamma)}p_{+}^{1/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}p_{-}^{1/(\alpha+\gamma)}}{\varrho_{+}^{-\gamma/(\alpha+\gamma)}p_{+}^{1/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}p_{-}^{1/(\alpha+\gamma)}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/((\alpha+\gamma-1)(\alpha+\gamma))}}{\varrho_{+}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/((\alpha+\gamma-1)(\alpha+\gamma))}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/((\alpha+\gamma-1)(\alpha+\gamma))}}{\varrho_{+}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma-1)(\alpha+\gamma)}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/((\alpha+\gamma-1)(\alpha+\gamma))}}{\varrho_{+}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/((\alpha+\gamma-1)(\alpha+\gamma))}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/((\alpha+\gamma-1)(\alpha+\gamma))}}{\varrho_{+}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma-1)}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}}{\varrho_{+}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}}{\varrho_{+}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}}{\varrho_{-}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}}{\varrho_{-}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{-\gamma/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}}{\varrho_{-}^{-\gamma/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}} = \frac{\varrho_{+}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{-})^{\alpha/(\alpha+\gamma)}}{\varrho_{-}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{+})^{\alpha/(\alpha+\gamma)}} = \frac{\varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{-})^{\alpha/(\alpha+\gamma)}}{\varrho_{-}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{-})^{\alpha/(\alpha+\gamma)}} = \frac{\varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho_{-}/\varrho_{-}/\varrho_{-})^{\alpha/(\alpha+\gamma)}}{\varrho_{-}^{\alpha/(\alpha+\gamma)} - \varrho_{-}^{\alpha/(\alpha+\gamma)}} = \frac{\varrho_{-}^{\alpha/(\alpha+\gamma)}(\varrho$$

proving (2.7).

Proof of Theorem 2.3. For given α , γ , and ϱ_+ , such a flux needs to satisfy both (2.4) and (2.7), i.e.,

$$\gamma \frac{\llbracket \varrho^{\alpha/(\alpha+\gamma)} \rrbracket}{\llbracket \varrho^{-\gamma/(\alpha+\gamma)} \rrbracket} = (\gamma - 1) \frac{\llbracket \varrho^{\alpha/(\alpha+\gamma-1)} \rrbracket}{\llbracket \varrho^{(1-\gamma)/(\alpha+\gamma-1)} \rrbracket}.$$
 (2.12)

For fixed α and γ , it is easy to find $\varrho_{-} \neq \varrho_{+}$ such that this equation is not satisfied. Hence, a numerical flux with all properties listed in Theorem 2.3 cannot exist.

3 Discussion

There are no numerical fluxes conserving a member of Harten's one-parameter family of entropies (2.2) for the compressible Euler equations that also preserve pressure equilibria and have a density flux independent of the pressure (Theorem 2.3). This result is in contrast to fluxes conserving the physical (logarithmic) entropy, where kinetic energy preservation can be achieved in addition to all of these properties [13, 14], resulting in an essentially unique numerical flux [16].

Following the uniqueness proof of such a numerical flux based on the logarithmic entropy presented in [16], it is tempting to choose a density flux based on a simplified setting, e.g., for constant velocity and pressure as in Lemma 2.4. Then, kinetic energy preservation determines the momentum flux accordingly [13, 14] and one might expect to be able to use the EC criterion to find an energy flux. However, this does not work in general since the energy part of the numerical flux can be orthogonal to the jump of the entropy variables as in Lemma 2.5. In this case, such an approach will often lead to a blow-up of the derived component for the total energy flux. Hence, one cannot impose a form of the density flux in general.

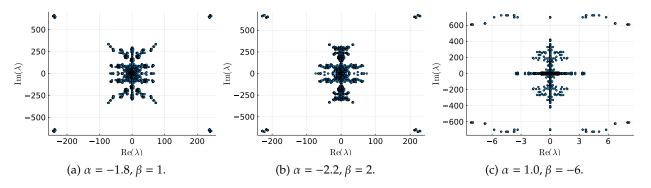


Figure 1: Spectra of discontinuous Galerkin semidiscretizations of the 2D compressible Euler equations conserving Harten's entropy (2.2) with parameter α using some values of $\beta = (\alpha + \gamma)/(1 - \gamma)$ recommended for entropy splitting discretizations [21] and $\alpha = 1$. The EC discretizations use the flux of [21] and tensor product Lobatto Legendre bases with polynomials of degree five on 4×4 elements. Using the central flux instead results in a spectrum with negligible imaginary part [7, 16].

Moreover, it is interesting to note that EC fluxes based on Harten's one-parameter family of entropies do not solve the local linear/energy stability issues discussed in [7, 16]. This is demonstrated by the spectra of discontinuous Galerkin semidiscretizations of the 2D compressible Euler equations shown in Figure 1. The setup uses the density wave example described in [16]. If the central flux $f^{\text{num}} = \{f\}$ is used instead of an EC flux, the spectrum of the resulting semidiscretization is essentially purely imaginary (ignoring floating point errors etc.), indicating (marginal) local linear/energy stability of the method [7, 16]. This property is desired and reasonable for this physical setup, since the density wave reduces the compressible Euler equations to linear advection equations, see also the extended discussion in [7, 16].

The semidiscretizations used to compute the spectra above are implemented using Trixi.jl [17, 19]. The Jacobian is computed using automatic differentiation [18] in Julia [3]. All source code required to reproduce the examples as well as additional material verifying the implementation and some calculations presented in this article are available online [15].

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