

Bipartite 3-Regular Counting Problems with Mixed Signs

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Abstract

We prove a complexity dichotomy for a class of counting problems expressible as bipartite 3-regular Holant problems. For every problem of the form $\text{Holant}(f \mid =_3)$, where f is any integer-valued ternary symmetric constraint function on Boolean variables, we prove that it is either P-time computable or #P-hard, depending on an explicit criterion of f . The constraint function can take both positive and negative values, allowing for cancellations. In addition, we discover a new phenomenon: there is a set \mathcal{F} with the property that for every $f \in \mathcal{F}$ the problem $\text{Holant}(f \mid =_3)$ is planar P-time computable but #P-hard in general, yet its planar tractability is by a *combination* of a holographic transformation by $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ to FKT *together* with an independent global argument.

1 Introduction

Holant problems encompass a broad class of counting problems [1, 2, 3, 9, 10, 11, 12, 17, 18, 20, 24, 25, 27]. For symmetric constraint functions this is also equivalent to edge-coloring models [21, 22]. These problems extend counting constraint satisfaction problems. Freedman, Lovász and Schrijver proved that some prototypical Holant problems, such as counting perfect matchings, cannot be expressed as vertex-coloring models known as graph homomorphisms [16, 19]. The classification program of counting problems is to classify as broad a class of these problems as possible into either #P-hard or P-time computable.

While much progress has been made for the classification of counting CSP [4, 6, 7, 14], and some progress for Holant problems [5], classifying Holant problems on regular bipartite graphs is particularly challenging. In a very recent paper [15] we initiated the study of Holant problems on the simplest setting of 3-regular bipartite graphs with *nonnegative* constraint functions. Admittedly, this is a severe restriction, because nonnegativity of the constraint functions rules out cancellation, which is a source of non-trivial P-time algorithms. Cancellation is in a sense the *raison d'être* for the Holant framework following Valiant's holographic algorithms [24, 25, 26]. The (potential) existence of P-time algorithms by cancellation is exciting, but at the same time creates obstacles if

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we want to classify every problem in the family into either P-time computable or #P-hard. At the same time, restricting to nonnegative constraint functions makes the classification theorem easier to prove. In this paper, we remove this nonnegativity restriction.

More formally, a Holant problem is defined on a graph where edges are variables and vertices are constraint functions. The aim of a Holant problem is to compute its partition function, which is a sum over all $\{0, 1\}$ -edge assignments of the product over all vertices of the constraint function evaluations. E.g., if every vertex has the EXACT-ONE function (which evaluates to 1 if exactly one incident edge is 1, and evaluates to 0 otherwise), then the partition function gives the number of perfect matchings. In this paper we consider Holant problems on 3-regular bipartite graphs $G = (U, V, E)$, where the Holant problem $\text{Holant}(f \mid =_3)$ computes the following partition function*

$$\text{Holant}(G) = \sum_{\sigma: E \rightarrow \{0,1\}} \prod_{u \in U} f(\sigma|_{E(u)}) \prod_{v \in V} (=_3)(\sigma|_{E(v)}),$$

where $f = [f_0, f_1, f_2, f_3]$ at each $u \in U$ is an integer-valued constraint function that evaluates to f_i if σ assigns exactly i among 3 incident edges $E(u)$ to 1, and $(=_3) = [1, 0, 0, 1]$ is the EQUALITY function on 3 variables (which is 1 iff all three are equal). E.g., if we take the EXACT-ONE function $f = [0, 1, 0, 0]$ then $\text{Holant}(f \mid =_3)$ counts the number of exact-3-covers; if f is the OR function $[0, 1, 1, 1]$ then $\text{Holant}(f \mid =_3)$ counts the number of all set covers.

The main theorem in this paper is a complexity dichotomy (Theorem 8.1): for any rational-valued function f of arity 3, the problem $\text{Holant}(f \mid =_3)$ is either #P-hard or P-time computable, depending on an explicit criterion on f . The main advance is to allow f to take both positive and negative values, thus cancellations in the sum $\sum_{\sigma: E \rightarrow \{0,1\}}$ can occur.

A major component of the classification program is to account for some algorithms, called holographic algorithms, that were initially discovered by Valiant [24]. These algorithms introduce quantum-like cancellations as the main tool. In the past 10 to 15 years we have gained a great deal of understanding of these mysteriously looking algorithms. In particular, it was proved in [12] that for all counting CSP with arbitrary constraint functions on Boolean variables, there is a precise 3-way division of problem types: (1) P-time computable in general, (2) P-time computable on planar structures but #P-hard in general, and (3) #P-hard even on planar structures. Moreover, every problem in type (2) is computable in P-time on planar structures by Valiant's holographic reduction to Fisher-Kasteleyn-Temperley algorithm (FKT) for planar perfect matchings. In [8] for (non-bipartite) Holant problems with symmetric constraint functions, the 3-way division above persists, but problems in (2) includes one more subtype unrelated to Valiant's holographic reduction. In this paper, we have a surprising discovery. We found a new set of functions \mathcal{F} which fits into type (2) problems above, but the planar P-time tractability is *neither* by Valiant's holographic reduction alone, *nor* entirely independent from it. Rather it is by a combination of a holographic reduction together with a global argument. An example of this set of problem is as follows: We say (X, \mathcal{S}) is a 3-regular k -uniform set system, if \mathcal{S} consists of a family of sets $S \subset X$ each of size $|S| = k$, and every $x \in X$ is in exactly 3 sets. If $k = 2$ this is just a 3-regular graph. We consider 3-regular 3-uniform set systems. We say \mathcal{S}' is a *leafless partial cover* if every $x \in \bigcup_{S \in \mathcal{S}'} S$ belongs to more than one set $S \in \mathcal{S}'$. We say x is *lightly covered* if $|\{S \in \mathcal{S}' : x \in S\}|$ is 2, and *heavily covered* if this number is 3.

*If we replace f by a set \mathcal{F} of constraint functions, each $u \in U$ is assigned some $f_u \in \mathcal{F}$, and replace $(=_3)$ by \mathcal{EQ} , the set of Equality of all arities, then $\text{Holant}(\mathcal{F} \mid \mathcal{EQ})$ can be taken as the definition of counting CSP.

Problem : `Weighted-Leafless-Partial-Cover`.

Input : A 3-regular 3-uniform set system (X, \mathcal{S}) .

Output : $\sum_{\mathcal{S}'} (-1)^l 2^h$, where the sum is over all leafless partial covers \mathcal{S}' , and l (resp. h) is the number of $x \in X$ that are lightly covered (resp. heavily covered).

One can show that this problem is just Holant ($f \mid =_3$), where $f = [1, 0, -1, 2]$. This problem is a special case of a set of problems of the form $f = [3a + b, a - b, -a + b, 3a - b]$. We show that all these problems belong to type (2) above, although they are not directly solvable by a holographic algorithm since they are provably not matchgates-transformable.

In this paper, we use MathematicaTM to perform symbolic computation. In particular, the procedure `CylindricalDecomposition` in MathematicaTM is an implementation (of a version) of Tarski's theorem on the decidability of the theory of real-closed fields. Some of our proof steps involve heavy symbolic computation. This stems from the bipartite structure. In order to preserve this structure, one has to connect each vertex from LHS to only vertices from RHS when constructing subgraph fragments called gadgets. In 3-regular bipartite graphs, it is easy to show that any gadget construction produces a constraint function that has the following restriction: the difference of the arities between the two sides is 0 mod 3. This severely limits the possible constructions within a moderate size, and a reasonable sized construction tends to produce gigantic polynomials. To “solve” some of these polynomials seems beyond direct manipulation by hand.

We believe our dichotomy (Theorem 8.1) is valid even for (algebraic) real or complex-valued constraint functions. However, in this paper we can only prove it for rational-valued constraint functions. There are two difficulties of extending our proof beyond \mathbb{Q} . The first is that we use the idea of interpolating degenerate straddled functions, for which we need to ensure that the ratio of the eigenvalues of the interpolating gadget matrix is not a root of unity. With rational-valued constraint functions, the only roots of unity that can occur are in a degree 2 extension field. For general constraint functions, they can be arbitrary roots of unity. Another difficulty is that some MathematicaTM steps showing the nonexistence of some exceptional cases are only valid for \mathbb{Q} . We list the essential MathematicaTM procedures used in this proof in an appendix.

2 Preliminaries

We now introduce the concept of *gadget*. A gadget, such as those illustrated in Figure 1 to Figure 8, is a bipartite graph $G = (U, V, E_{\text{in}}, E_{\text{out}})$ with internal edges E_{in} and dangling edges E_{out} . There can be m dangling edges internally incident to vertices from U and n dangling edges internally incident to vertices from V . These $m + n$ dangling edges correspond to Boolean variables $x_1, \dots, x_m, y_1, \dots, y_n$ and the gadget defines a *signature*

$$f(x_1, \dots, x_m, y_1, \dots, y_n) = \sum_{\sigma: E_{\text{in}} \rightarrow \{0,1\}} \prod_{u \in U} f(\hat{\sigma}|_{E(u)}) \prod_{v \in V} (=_3)(\hat{\sigma}|_{E(v)}),$$

where $\hat{\sigma}$ denotes the extension of σ by the assignment on the dangling edges.

As indicated before, in the setting of 3-regular bipartite graph, we have limited number of symmetric gadgets with reasonable sizes. To preserve the bipartite structure, we must be careful in any gadget construction how each external wire is to be connected, i.e., as an input variable whether it is on the LHS (those of f which can be used to connect to $(=)_3$ on the RHS), or it is on the RHS (those of $(=)_3$ which can be used to connect to f on the LHS).



Figure 1: G_1

In each figure of gadgets presented later, we use a blue square to represent a signature from LHS, which under most of the cases will be $[1, a, b, c]$, a green circle to represent the ternary equality $[1, 0, 0, 1]$, and a black triangle to represent a unary signature whose values depend on the context.

A *signature grid* $\Omega = (G, \pi)$ over a signature set \mathcal{F} consists of a graph $G = (V, E)$ and a mapping π that assigns to each vertex $v \in V$ an $f_v \in \mathcal{F}$ and a linear order of the incident edges at v . For signature sets \mathcal{F} and \mathcal{G} , a bipartite signature grid over $(\mathcal{F} \mid \mathcal{G})$ is a signature grid $\Omega = (H, \pi)$ over $\mathcal{F} \cup \mathcal{G}$, where $H = (V, E)$ is a bipartite graph with bipartition $V = (V_1, V_2)$ such that $\pi(V_1) \subseteq \mathcal{F}$ and $\pi(V_2) \subseteq \mathcal{G}$. In this paper, we consider the bipartite Holant problem where $\mathcal{F} = \{f\}$ consists of a single rational-valued ternary symmetric Boolean function and $\mathcal{G} = \{[1, 0, 0, 1]\}$ consists of EQUALITY₃.

A *symmetric* signature is a function that is invariant under any permutation of its variables. The value of such a signature depends only on the Hamming weight of its input. We denote a ternary symmetric signature f by the notation $f = [f_0, f_1, f_2, f_3]$, where f_i is the value on inputs of Hamming weight i . The EQUALITY of arities 3 is $(=)_3 = [1, 0, 0, 1]$. A symmetric signature f is called (1) *degenerate* if it is the tensor power of a unary signature; (2) *Generalized Equality*, or Gen-Eq, if it is zero unless all inputs are equal. Affine signatures were discovered in the dichotomy for counting constraint satisfaction problems (#CSP) [5]. A (real valued) ternary symmetric signature is *affine* if it has the form $[1, 0, 0, \pm 1]$, $[1, 0, 1, 0]$, $[1, 0, -1, 0]$, $[1, 1, -1, -1]$ or $[1, -1, -1, 1]$, or by reversing the order of the entries, up to a constant factor. If f is degenerate, Gen-Eq, or affine, then the problem #CSP(f) and thus Holant($f \mid (=)_3$) is in P (for a more detailed exposition of this theory, see [5]). Our dichotomy asserts that, for all signatures f with $f_i \in \mathbb{Q}$, these three classes are the only tractable cases of the problem Holant($f \mid (=)_3$); all other signatures lead to #P-hardness.

By a slight abuse of notation, we say $[1, a, b, c]$ is #P-hard or in P depending on whether the problem Holant($[1, a, b, c] \mid (=)_3$) is #P-hard or in P, respectively. We shall invoke the following theorem [20] when proving our results:

Theorem 2.1. *Suppose $a, b \in \mathbb{C}$, and let $X = ab$, $Z = \left(\frac{a^3+b^3}{2}\right)^2$. Then Holant($[a, 1, b] \mid (=)_3$) is #P-hard except in the following cases, for which the problem is in P.*

1. $X = 1$;
2. $X = Z = 0$;
3. $X = -1$ and $Z = 0$;
4. $X = -1$ and $Z = -1$.

An important observation is that in the context of Holant($f \mid (=)_3$), every gadget construction produces a signature with $m \equiv n \pmod 3$, where m and n are the numbers of input variables (arities) from the LHS and RHS respectively. Thus, any construction that produces a signature purely on either the LHS or the RHS will have arity a multiple of 3. In order that our constructions are more manageable in size, we will make heavy use of *straddled gadgets* with $m = n = 1$ that do not belong to either side and yet can be easily iterated. The signatures of the iterated gadgets are represented by matrix powers.

Consider the binary straddled gadget G_1 in Figure 1. Its signature is $G_1 = \begin{bmatrix} 1 & b \\ a & c \end{bmatrix}$, where $G_1(i, j)$ (at row i column j) is the value of this gadget when the left dangling edge (from the “square”) and the right dangling edge (from the “circle” ($=_3$)) are assigned i and j respectively, for $i, j \in \{0, 1\}$. Iterating G_1 sequentially k times is represented by the matrix power G_1^k . It turns out that it is very useful either to produce directly or to obtain by interpolation a rank *deficient* straddled signature, which would in most cases allow us to obtain unary signatures on either side. With unary signatures we can connect to a ternary signature to produce binary signatures on one side and then apply Theorem 2.1. The proof idea of Lemma 2.2 is the same as in [15] for nonnegative signatures.

Lemma 2.2. *Given the binary straddled signature $G_1 = \begin{bmatrix} 1 & b \\ a & c \end{bmatrix}$, we can interpolate the degenerate binary straddled signature $\begin{bmatrix} y & xy \\ 1 & x \end{bmatrix}$, provided that $c \neq ab$, $a \neq 0$, $\Delta = \sqrt{(1-c)^2 + 4ab} \neq 0$ and $\frac{\Delta}{\mu}$ is not a root of unity, where $\lambda = \frac{-\Delta+(1+c)}{2}$, $\mu = \frac{\Delta+(1+c)}{2}$ are the two eigenvalues, and $x = \frac{\Delta-(1-c)}{2a}$ and $y = \frac{\Delta+(1-c)}{2a}$.*

Proof. We have $x+y = \Delta/a \neq 0$ and so $\begin{bmatrix} -x & y \\ 1 & 1 \end{bmatrix}^{-1}$ exists, and the matrix G_1 has the Jordan Normal Form

$$G_1 = \begin{pmatrix} 1 & b \\ a & c \end{pmatrix} = \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix}^{-1}.$$

Here the matrix G_1 is non-degenerate since $c \neq ab$, and so λ and μ are nonzero. Consider

$$D = \frac{1}{x+y} \begin{pmatrix} y & xy \\ 1 & x \end{pmatrix} = \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix}^{-1}.$$

Given any signature grid Ω where the binary degenerate straddled signature D appears n times, we form gadgets G_1^s where $0 \leq s \leq n$ by iterating the G_1 gadget s times and replacing each occurrence of D with G_1^s . (Here for $s = 0$ we simply replace each occurrence of D by an edge.) Denote the resulting signature grid as Ω_s . We stratify the assignments in the Holant sum for Ω according to assignments to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- $(0, 0)$ i times;
- $(1, 1)$ j times;

with $i+j = n$; all other assignments will contribute 0 in the Holant sum for Ω . The same statement is true for each Ω_s with the matrix $\begin{bmatrix} \lambda^s & 0 \\ 0 & \mu^s \end{bmatrix}$. Let $c_{i,j}$ be the sum, in Ω , over all such assignments of the products of evaluations of all other signatures other than that represented by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, including the contributions from $\begin{bmatrix} -x & y \\ 1 & 1 \end{bmatrix}$ and its inverse. The same quantities $c_{i,j}$ appear for each Ω_s , independent of s , with the substitution of the matrix $\begin{bmatrix} \lambda^s & 0 \\ 0 & \mu^s \end{bmatrix}$. Then, for $0 \leq s \leq n$, we have

$$\text{Holant}_{\Omega_s} = \sum_{i+j=n} (\lambda^i \mu^j)^s \cdot c_{i,j} \tag{2.1}$$

and $\text{Holant}_{\Omega} = c_{0,n}$.

Since λ/μ is not a root of unity, the quantities $\lambda^i \mu^{n-i}$ are pairwise distinct, thus (2.1) is a full ranked Vandermonde system. Thus we can compute Holant_{Ω} from Holant_{Ω_s} by solving the linear system in polynomial time. Thus we can interpolate D in polynomial time. \square

The next lemma allows us to get unary signatures.

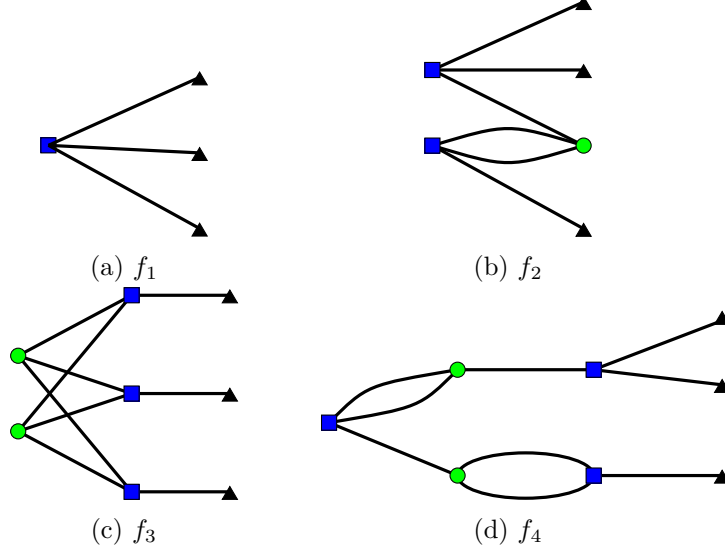


Figure 2: Four gadgets where each triangle represents the unary gadget $[1, x]$

Lemma 2.3. For $\text{Holant}([1, a, b, c] \mid =_3)$, $a, b, c \in \mathbb{Q}$, $a \neq 0$, with the availability of binary degenerate straddled signature $\begin{bmatrix} y & xy \\ 1 & x \end{bmatrix}$ (here $x, y \in \mathbb{C}$ can be arbitrary), in polynomial time

1. we can interpolate $[y, 1]$ on the LHS,

$$\text{Holant}(\{[1, a, b, c], [y, 1]\} \mid =_3) \leq_T \text{Holant}([1, a, b, c] \mid =_3);$$

2. we can interpolate $[1, x]$ on the RHS,

$$\text{Holant}([1, a, b, c] \mid \{ (=3), [1, x] \}) \leq_T \text{Holant}([1, a, b, c] \mid =_3),$$

except for two cases: $[1, a, a, 1]$, $[1, a, -1 - 2a, 2 + 3a]$.

Proof. For the problem $\text{Holant}(\{[1, a, b, c], [y, 1]\} \mid =_3)$, the number of occurrences of $[y, 1]$ on LHS is 0 mod 3, say $3n$, since the other signatures are both of arity 3. Now, for each occurrence of $[y, 1]$, we replace it with the binary straddled signature $\begin{bmatrix} y & xy \\ 1 & x \end{bmatrix}$, leaving $3n$ dangling edges on RHS yet to be connected to LHS, each of which represents a unary signature $[1, x]$. We build a gadget to connect every triple of such dangling edges. We claim that at least one of the connection gadgets in Figures 2a, 2b, 2c and 2d creates a nonzero global factor. The factors of these four gadgets are $f_1 = cx^3 + 3bx^2 + 3ax + 1$, $f_2 = (ab + c)x^3 + (3bc + 2a^2 + b)x^2 + (2b^2 + ac + 3a)x + ab + 1$, $f_3 = (a^3 + b^3 + c^3)x^3 + 3(a^2 + 2ab^2 + bc^2)x^2 + 3(a + 2a^2b + b^2c)x + 1 + 2a^3 + b^3$ and $f_4 = (ab + 2abc + c^3)x^3 + (2a^2 + b + 2a^2c + 3ab^2 + bc + 3b^2c)x^2 + (3a + 3a^2b + ac + 2b^2 + 2b^2c + ac^2)x + 1 + 2ab + abc$ respectively. By setting the four formulae to be 0 simultaneously, with $a \neq 0$, $a, b, c \in \mathbb{Q}$ and $x \in \mathbb{C}$, we found that there is no solution[†]. Thus, we can always “absorb” the left-over $[1, x]$ ’s at the cost of some easily computable nonzero global factor.

For the other claim on $[1, x]$ on RHS, i.e.,

$$\text{Holant}([1, a, b, c] \mid \{ (=3), [1, x] \}) \leq_T \text{Holant}([1, a, b, c] \mid =_3)$$

[†]We use Mathematica, where a complex x (or y) is written as $u + vi$, and the real and imaginary parts of f_i are both set to 0. The empty intersection of $f_1 = f_2 = f_3 = f_4 = 0$ is proved by cylindrical decomposition, an algorithm for Tarski’s theorem on real-closed fields.

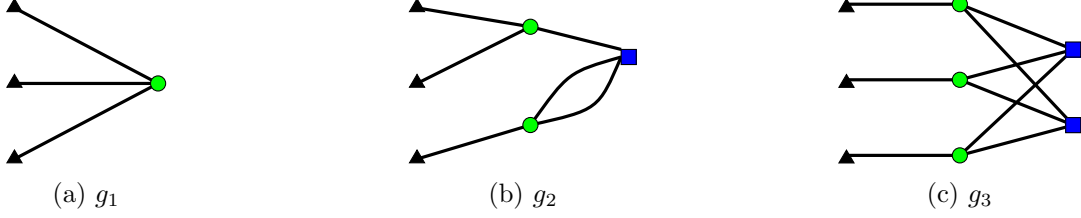


Figure 3: Three gadgets where each triangle represents the unary gadget $[y, 1]$

we use a similar strategy to “absorb” the left-over copies of $[y, 1]$ on the LHS by connecting them to $(=_3)$ in the gadgets in the Figures 3a, 3b or 3c. These gadgets produce factors $g_1 = y^3 + 1$, $g_2 = y^3 + by^2 + ay + c$ and $g_3 = y^3 + 3a^2y^2 + 3b^2y + c^2$ respectively. It can be directly checked that, for complex y , all these factors are 0 iff $y = -1$, and the signature has the form $[1, a, a, 1]$ or $[1, a, -2a - 1, 3a + 2]$. \square

A main thrust in our proof is we want to be assured that such degenerate binary straddled signature can be obtained, and the corresponding unary signatures in Lemma 2.3 can be produced. We now first consider the two exceptional cases $[1, a, a, 1]$ and $[1, a, -2a - 1, 3a + 2]$ where this is not possible.

Lemma 2.4. *The problem $[1, a, a, 1]$ is $\#P$ -hard unless $a \in \{0, \pm 1\}$ in which case it is in P.*

Proof. If $a = 0$ or $a = \pm 1$, then it is either Gen-Eq or degenerate or affine, and thus the problem $\text{Holant}([1, a, a, 1] \mid =_3)$ is in P. Now assume $a \neq 0$ and $a \neq \pm 1$.

Using the gadget G_1 , we have $\Delta = |2a|$ and $x = y = \Delta/2a = \pm 1$ depending on the sign of a . So we get the signature $[y, 1] = [\pm 1, 1]$ on LHS by Lemmas 2.2 and 2.3. Connecting two copies of $[y, 1]$ to $[1, 0, 0, 1]$ on RHS, we get $[1, 1]$ on RHS regardless of the sign. Connecting $[1, 1]$ to $[1, a, a, 1]$ on LHS, we get $[1 + a, 2a, 1 + a]$ on LHS. The problem $\text{Holant}([1 + a, 2a, 1 + a] \mid =_3)$ is $\#P$ -hard by Theorem 2.1 unless $a = 0, \pm 1$ or $-\frac{1}{3}$, thus we only need to consider the signature $[3, -1, -1, 3]$.

If $a = -\frac{1}{3}$, we apply holographic transformation with the Hadamard matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Note that $[3, -1, -1, 3] = 4((1, 0)^{\otimes 3} + (0, 1)^{\otimes 3}) - (1, 1)^{\otimes 3}$. Here each tensor power represents a truth-table of 8 entries, or a vector of dimension 8; the linear combination is the truth-table for the symmetric signature $f = [3, -1, -1, 3]$, which is in fact a short hand notation for the vector

$$(f_{000}, f_{001}, f_{010}, f_{011}, f_{100}, f_{101}, f_{110}, f_{111}) = (3, -1, -1, -1, -1, -1, -1, 3).$$

Also note that, $(1, 0)H = (1, 1)$, $(0, 1)H = (1, -1)$ and $(1, 1)H = (2, 0)$, thus $[3, -1, -1, 3]H^{\otimes 3} = 4((1, 1)^{\otimes 3} + (1, -1)^{\otimes 3}) - (2, 0)^{\otimes 3} = 4[2, 0, 2, 0] - [8, 0, 0, 0] = [0, 0, 8, 0]$, which is equivalent to $[0, 0, 1, 0]$ by a global factor. So, we get

$$\begin{aligned} \text{Holant}([3, -1, -1, 3] \mid (=_3)) &\equiv_T \text{Holant}([3, -1, -1, 3]H^{\otimes 3} \mid (H^{\otimes 3})^{-1}[1, 0, 0, 1]) \\ &\equiv_T \text{Holant}([0, 0, 1, 0] \mid [1, 0, 1, 0]) \\ &\equiv_T \text{Holant}([0, 0, 1, 0] \mid [0, 0, 1, 0]) \\ &\equiv_T \text{Holant}([0, 1, 0, 0] \mid [0, 1, 0, 0]) \end{aligned}$$

where the first reduction is by Valiant’s Holant theorem [25], the third reduction comes from the following observation: given a bipartite 3-regular graph $G = (V, U, E)$ where the vertices in V are

assigned the signature $[0, 0, 1, 0]$ and the vertices in U are assigned the signature $[1, 0, 1, 0]$, every nonzero term in the Holant sum must correspond to a mapping $\sigma : E \rightarrow \{0, 1\}$ where exactly two edges of any vertex are assigned 1. The fourth reduction is by simply flipping 0's and 1's. The problem $\text{Holant}([0, 1, 0, 0] \mid [0, 1, 0, 0])$ is the problem of counting perfect matchings in 3-regular bipartite graphs, which Dagum and Luby proved to be $\#P$ -complete (Theorem 6.2 in [13]). \square

Lemma 2.5. *The problem $[1, a, -2a - 1, 3a + 2]$ is $\#P$ -hard unless $a = -1$ in which case it is in P .*

Proof. Observe that the truth-table of the symmetric signature $[1, a, -2a - 1, 3a + 2]$ written as an 8-dimensional column vector is just

$$2(a+1) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes 3} \right) - \frac{a+1}{2} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{\otimes 3} \right) - a \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{\otimes 3}.$$

Here again, the tensor powers written as 8-dimensional vectors represent truth-tables, and the linear combination of these vectors “holographically” reconstitute a truth-table of the symmetric signature $[1, a, -2a - 1, 3a + 2]$.

We apply the holographic transformation with the Hadamard matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Note that $(1, 0)H = (1, 1)$, $(0, 1)H = (1, -1)$, $(1, 1)H = (2, 0)$ and $(1, -1)H = (0, 2)$, and we get

$$\begin{aligned} \text{Holant}([1, a, -2a - 1, 3a + 2] \mid (=3)) &\equiv_T \text{Holant}([1, a, -2a - 1, 3a + 2]H^{\otimes 3} \mid (H^{\otimes 3})^{-1}[1, 0, 0, 1]) \\ &\equiv_T \text{Holant}([0, 0, a + 1, -3a - 1] \mid [1, 0, 1, 0]) \\ &\equiv_T \text{Holant}([0, 0, a + 1, 0] \mid [0, 0, 1, 0]) \end{aligned} \tag{2.2}$$

where the last equivalence follows from the observation that for each nonzero term in the Holant sum, every vertex on the LHS has at least two of three edges assigned 1 (from $[0, 0, a + 1, -3a - 1]$), meanwhile every vertex on the RHS has at most two of three edges assigned 1 (from $[1, 0, 1, 0]$). The graph being bipartite and 3-regular, the number of vertices on both sides must equal, thus every vertex has exactly two incident edges assigned 1.

Then by flipping 0's and 1's, $\text{Holant}([0, 0, a + 1, 0] \mid [0, 0, 1, 0]) \equiv_T \text{Holant}([0, a + 1, 0, 0] \mid [0, 1, 0, 0])$. For $a \neq -1$, this problem is equivalent to counting perfect matchings in bipartite 3-regular graphs, which is $\#P$ -complete by Theorem 6.2 in [13]. If $a = -1$, the signature $[1, -1, 1, -1] = [1, -1]^{\otimes 3}$ is degenerate, and thus in P . The holographic reduction also reveals that, not only the problem is in P , but the Holant sum is 0. \square

We can generalize Lemma 2.5 to get the following corollary.

Corollary 2.6. *The problem $\text{Holant}(f \mid (=3))$, where $f = [3a + b, -a - b, -a + b, 3a - b]$, is computable in polynomial time on planar graphs for all a, b , but is $\#P$ -hard on general graphs for all $a \neq 0$.*

Proof. The following equivalence is by a holographic transformation using $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$:

$$\begin{aligned} \text{Holant}(f \mid (=3)) &\equiv_T \text{Holant}(fH^{\otimes 3} \mid (H^{-1})^{\otimes 3}(=3)) \\ &\equiv_T \text{Holant}([0, 0, a, b] \mid [1, 0, 1, 0]) \\ &\equiv_T \text{Holant}([0, 0, a, 0] \mid [0, 0, 1, 0]) \\ &\equiv_T \text{Holant}([0, a, 0, 0] \mid [0, 1, 0, 0]) \end{aligned}$$

where the third reduction follows the same reasoning as in the proof of Lemma 2.5. When $a \neq 0$, $\text{Holant}([0, a, 0, 0] \mid [0, 1, 0, 0])$ is (up to a global nonzero factor) the perfect matching problem on 3-regular bipartite graphs. This problem is computable in polynomial time on planar graphs and the reductions are valid for planar graphs as well. It is $\#P$ -hard on general graphs (for $a \neq 0$). \square

Remark: The planar tractability of the problem $\text{Holant}(f \mid =_3)$, for $f = [3a+b, -a-b, -a+b, 3a-b]$, is a remarkable fact. It is neither accomplished by a holographic transformation to matchgates alone, nor entirely independent from it. One can prove that the signature f is not matchgates-transformable (for nonzero a, b ; see [5] for the theory of matchgates and the realizability of signatures by matchgates under holographic transformation). In previous complexity dichotomies, we have found that for the entire class of counting CSP problems over Boolean variables, all problems that are $\#P$ -hard in general, but P -time tractable on planar graphs, are tractable by the following universal algorithmic strategy—a holographic transformation to matchgates followed by the FKT algorithm [12]. On the other hand, for (non-bipartite) Holant problems with arbitrary symmetric signature sets, this category of problems (planar tractable but $\#P$ -hard in general) is completely characterized by two types [8]: (1) holographic transformations to matchgates, and (2) a separate kind that depends on the existence of “a wheel structure” (unrelated to holographic transformations and matchgates). Here in Corollary 2.6 we have found the first instance where a new type has emerged.

Proposition 2.7. *For $G_1 = \begin{bmatrix} 1 & b \\ a & c \end{bmatrix}$, with $a, b, c \in \mathbb{Q}$, if it is non-singular (i.e., $c \neq ab$), then it has two nonzero eigenvalues λ and μ . The ratio λ/μ is not a root of unity unless at least one of the following conditions holds:*

$$\begin{cases} c + 1 = 0 \\ ab + c^2 + c + 1 = 0 \\ 2ab + c^2 + 1 = 0 \\ 3ab + c^2 - c + 1 = 0 \\ 4ab + c^2 - 2c + 1 = 0 \end{cases} \quad (2.3)$$

Proof. We have $\lambda = \frac{-\Delta + (1+c)}{2}$ and $\mu = \frac{\Delta + (1+c)}{2}$, where $\Delta = \sqrt{(1-c)^2 + 4ab}$. Since $a, b, c \in \mathbb{Q}$, if $\frac{\lambda}{\mu}$ is a root of unity it belongs to an extension field of \mathbb{Q} of degree 2. Thus it can only be one of the following 8 values: $\pm 1, \pm i, \frac{\pm 1 \pm \sqrt{3}i}{2}$, where $i = \sqrt{-1}$. This gives us the cases listed in (2.3). \square

Now we introduce two more binary straddled signatures — G_2 and G_3 in Figure 4. The signature matrix of G_2 is $\begin{bmatrix} w & b' \\ a' & c' \end{bmatrix}$, where $w = 1 + 2a^3 + b^3$, $a' = a + 2a^2b + b^2c$, $b' = a^2 + 2ab^2 + bc^2$ and $c' = a^3 + 2b^3 + c^3$. Similar to G_1 , we have $\Delta' = \sqrt{(w - c')^2 + 4a'b'}$, two eigenvalues $\lambda' = \frac{-\Delta' + (w+c')}{2}$ and $\mu' = \frac{\Delta' + (w+c')}{2}$. If $a' \neq 0$, we have $x' = \frac{\Delta' - (w-c')}{2a'}$, $y' = \frac{\Delta' + (w-c')}{2a'}$ and if further $\Delta' \neq 0$ we can write its Jordan Normal Form as

$$G_2 = \begin{pmatrix} w' & b' \\ a' & c' \end{pmatrix} = \begin{pmatrix} -x' & y' \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda' & 0 \\ 0 & \mu' \end{pmatrix} \begin{pmatrix} -x' & y' \\ 1 & 1 \end{pmatrix}^{-1}. \quad (2.4)$$

The signature matrix of G_3 is $\begin{bmatrix} 1+ab & a^2+bc \\ a+b^2 & ab+c^2 \end{bmatrix}$. In this case we define $w = 1+ab$, $a' = a+b^2$, $b' = a^2+bc$ and $c' = ab+c^2$. Then the corresponding quantities $\Delta', \lambda', \mu', x', y'$ can be defined in the same way, and its Jordan Normal Form takes the same form as in (2.4).

Similar to Proposition 2.7, we have the following claim on G_2 and G_3 .

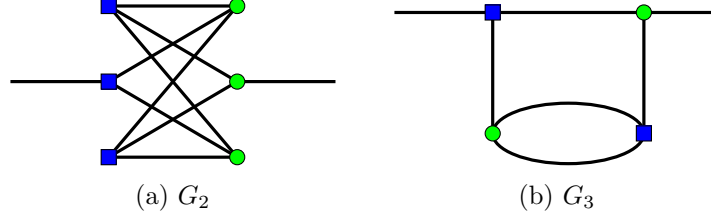


Figure 4: Two binary straddled gadgets

Proposition 2.8. *For each gadget G_2 and G_3 respectively, if the signature matrix is non-degenerate, then the ratio λ'/μ' of its eigenvalues is not a root of unity unless at least one of the following conditions holds, where $A = w + c'$, $B = (c' - w)^2 + 4a'b'$.*

$$\begin{cases} A = 0 \\ B = 0 \\ A^2 + B = 0 \\ A^2 + 3B = 0 \\ 3A^2 + B = 0 \end{cases} \quad (2.5)$$

Lemma 2.9. *Suppose $a, b, c \in \mathbb{Q}$, $a \neq 0$ and $c \neq ab$ and a, b, c do not satisfy any condition in (2.3). Let $x = \frac{\Delta - (1-c)}{2a}$, $y = \frac{\Delta + (1-c)}{2a}$ and $\Delta = \sqrt{(1-c)^2 + 4ab}$. Then for $\text{Holant}([1, a, b, c] \mid =_3)$,*

1. *we can interpolate $[y, 1]$ on LHS;*
2. *we can interpolate $[1, x]$ on RHS except for 2 cases: $[1, a, a, 1]$, $[1, a, -1 - 2a, 2 + 3a]$.*

Proof. This lemma follows from Lemma 2.2 and Lemma 2.3 using the binary straddled gadget G_1 with signature matrix $\begin{bmatrix} 1 & b \\ a & c \end{bmatrix}$. Note that $c \neq ab$ indicates that matrix G_1 is non-degenerate, and λ/μ not being a root of unity is equivalent to none of the equations in (2.3) holds. \square

We have similar statements corresponding to G_2 (resp. G_3). When the signature matrix is non-degenerate and does not satisfy any condition in (2.5), we can interpolate the corresponding $[y', 1]$ on LHS, and we can also interpolate the corresponding $[1, x']$ on RHS except when $y' = -1$.

Definition 2.10. *For $\text{Holant}([1, a, b, c] \mid =_3)$, with $a, b, c \in \mathbb{Q}$, $a \neq 0$, we say a binary straddled gadget G works if the signature matrix of G is non-degenerate and the ratio of its two eigenvalues λ/μ is not a root of unity.*

Remark: Explicitly, the condition that G_1 works is that $c \neq ab$ and a, b, c do not satisfy any condition in (2.3), which is just the assumptions in Lemma 2.9. G_1 works implies that it can be used to interpolate $[y, 1]$ on LHS, and to interpolate $[1, x]$ on RHS with two exceptions for which we already proved the dichotomy. The x, y are as stated in Lemma 2.9.

Similarly, when the binary straddled gadget G_2 (resp. G_3) works, for the corresponding values x' and y' , we can interpolate $[y', 1]$ on LHS, and we can interpolate $[1, x']$ on RHS except when $y' = -1$.

The ternary gadget G_4 in Figure 5 will be used in the proof here and later.

The unary signatures $\Delta_0 = [1, 0]$ and $\Delta_1 = [0, 1]$ are called the pinning signatures because they “pin” a variable to 0 or 1. One good use of having unary signatures is that we can use Lemma 2.12 to get the two pinning signatures. Pinning signatures are helpful as the following lemma shows.

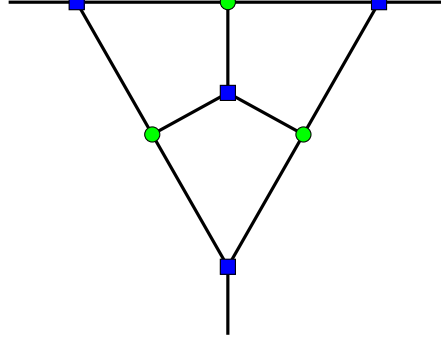


Figure 5: G_4

Lemma 2.11. *If Δ_0 and Δ_1 are available on the RHS in $\text{Holant}([1, a, b, c] \mid =_3)$, where $a, b, c \in \mathbb{Q}$, $ab \neq 0$, then the problem is $\#P$ -hard unless $[1, a, b, c]$ is affine or degenerate, in which cases it is in P .*

Proof. Connecting $[1, 0]$, $[0, 1]$ to $[1, a, b, c]$ on LHS respectively, we get binary signatures $[1, a, b]$ and $[a, b, c]$. Then we can apply Theorem 2.1, and the problem is $\#P$ -hard unless both $[1, a, b]$ and $[a, b, c]$ are in P . When $ab \neq 0$, both $[1, a, b]$ and $[a, b, c]$ are in P only when $[1, a, b, c] = [1, 1, 1 - 1]$ or $[1, -1, 1, 1]$ or $[1, -1, -1, -1]$ or $[1, 1, -1, 1]$ or $[1, 1, -1, -1]$ or $[1, -1, -1, 1]$, where the last two are affine and hence in P . Due to the symmetry by flipping 0 and 1 in the signature, it suffices to consider only $f = [1, 1, 1, -1]$ and $g = [1, -1, 1, 1]$; they are neither affine nor degenerate.

For both f and g we use the gadget G_4 to produce ternary signatures $f' = [1, 1, 3, 3]$ and $g' = [1, 1, -1, 3]$ respectively. Neither are among the exceptional cases above. So $\text{Holant}(f \mid =_3)$ and $\text{Holant}(g \mid =_3)$ are both $\#P$ -hard. \square

The following lemma lets us interpolate arbitrary unary signatures on RHS, in particular Δ_0 and Δ_1 , from a binary gadget with a straddled signature and a suitable unary signature s on RHS. Mathematically, the proof is essentially the same as in [23], but technically Lemma 2.12 applies to binary straddled signatures.

Lemma 2.12. *Let $M \in \mathbb{R}^{2 \times 2}$ be a non-singular signature matrix for a binary straddled gadget which is diagonalizable with distinct eigenvalues, and $s = [a, b]$ be a unary signature on RHS that is not a row eigenvector of M . Then $\{s \cdot M^j\}_{j \geq 0}$ can be used to interpolate any unary signature on RHS.*

3 Dichotomy when $ab \neq 0$ and G_1 works

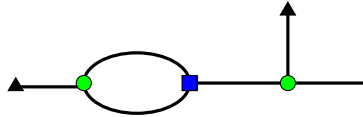


Figure 6: Non-linearity gadget, where a triangle represents the unary gadget $[y, 1]$

Let us introduce a *non-linearity* gadget in Figure 6 where the triangles represent $[y, 1]$. It is on the RHS with a unary signature $[y^2 + yb, ya + c]$. The following two lemmas will be used in proof of Theorem 3.3.

Lemma 3.1. *Let $a, b, c \in \mathbb{Q}$, $ab \neq 0$, and satisfy (con1) $a^3 - b^3 - ab(1 - c) = 0$ and (con2) $a^3 + ab + 2b^3 = 0$. Then $\text{Holant}([1, a, b, c] | =_3)$ is #P-hard unless it is $[1, -1, 1, -1] = [1, -1]^{\otimes 3}$, which is degenerate, in which case the problem is in P.*

Proof. If $a + b^2 = 0$, then $[1, a, b, c] = [1, -1, 1, -1]$ which is degenerate. Now we assume $a + b^2 \neq 0$. Here we use Gadget G_3 .

First assume G_3 works. Using $a + b^2 \neq 0$ together with (con1) and (con2), we can verify that $\Delta = \sqrt{4(a + b^2)(a^2 + bc) + (c^2 - 1)^2} \neq 0$, and we can write the Jordan Normal Form

$$G_3 = \begin{pmatrix} 1 + ab & a^2 + bc \\ a + b^2 & ab + c^2 \end{pmatrix} = \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix}^{-1},$$

where $\lambda = \frac{1+2ab+c^2-\Delta}{2}$, $\mu = \frac{1+2ab+c^2+\Delta}{2}$, $x = \frac{\Delta+c^2-1}{2(a+b^2)}$, $y = \frac{\Delta-c^2+1}{2(a+b^2)}$. Because G_3 works, $[y, 1]$ on LHS is available. Use this $[y, 1]$ in the non-linearity gadget in Figure 6, we get the unary signature $[y^2 + yb, ya + c]$ on the RHS. By Lemma 2.12, we can interpolate any unary signature, in particular Δ_0 and Δ_1 on RHS and apply Lemma 2.11, unless $[y^2 + yb, ya + c]$ is proportional to a row eigenvector of G_3 , namely $[1, -y]$ and $[1, x]$. Thus the exceptions are $ya + c = x(y^2 + yb)$ and $ya + c = -y(y^2 + yb)$. Notice that now $xy = \frac{a^2+bc}{a+b^2}$. The first equation implies $c = ab$ or $a + b^2 = 0$ or $a^3 - b^3c + ab(-1 + c^2) = 0$. The second equation implies $a + b^2 = 0$ or $f_1 = 0$ where $f_1 = a^3 + 4a^6 + 3a^5b^2 + a^3b^3 - c - 4a^3c + 6a^4bc - 6a^2b^2c - b^3c - 3a^2b^5c - 3a^3c^2 - 3abc^2 - 4b^3c^2 - a^3b^3c^2 - 6ab^4c^2 - 4b^6c^2 + 3c^3 + 4a^3c^3 + 6a^2b^2c^3 + 3b^3c^3 + a^3c^4 + 3abc^4 + 4b^3c^4 - 3c^5 - b^3c^5 + c^7$. So there are four exceptional cases,

$$\begin{cases} c = ab \\ a + b^2 = 0 \\ a^3 - b^3c + ab(-1 + c^2) = 0 \\ f_1 = 0 \end{cases} \quad (3.6)$$

For each of them, together with (con1) and (con2), we get 3 equations and can solve them using Mathematica™. For rational a, b, c , when $ab \neq 0$, there are only two possible results — $[1, -1, 1, -1]$ and $[1, -\frac{1}{3}, -\frac{1}{3}, 1]$. The first one violates $a + b^2 \neq 0$, and the second has been proved to be #P-hard in Lemma 2.4. For all other cases when G_3 works, we have the pinning signatures Δ_0 and Δ_1 on the RHS and then the lemma is proved by Lemma 2.11.

Now suppose G_3 does not work. Then by Proposition 2.8, we get at least one more condition, either one in (2.5) or $(a + b^2)(a^2 + bc) = (1 + ab)(ab + c^2)$ which indicates that G_3 is degenerate. For each of the 6 conditions, together with (con1) and (con2), we can solve them using Mathematica™ for rational a, b, c . The only solution is $[1, -1, 1, -1]$ which violates $a + b^2 \neq 0$. The proof of the lemma is complete. \square

Lemma 3.2. *Let $a, b, c \in \mathbb{Q}$, $ab \neq 0$, and satisfy (con1) $a^3 - b^3 - ab(1 - c) = 0$ and (con2) $(a^4b + ab^4)^2 = (a^5 + b^4)(b^5 + a^4c)$. Then $\text{Holant}([1, a, b, c] | =_3)$ is #P-hard unless it is $[1, a, a^2, a^3]$, which is degenerate and thus in P.*

Proof. Eliminating c from (con1) and (con2) we get $a^{11} - a^9b + a^6b^4 + a^5b^6 - a^4b^5 - a^3b^7 + a^2b^9 - b^{10} = 0$, which, quite miraculously, can be factored as $(a^2 - b)(a^9 + a^4b^4 + a^3b^6 + b^9) = 0$. If $b = a^2$, then with (con1), we get $c = a^3$ thus the signature becomes $[1, a, a^2, a^3]$ which is degenerate. We assume $a^9 + a^4b^4 + a^3b^6 + b^9 = 0$, then the rest of the proof is essentially the same as Lemma 3.1. \square

Theorem 3.3. *For $a, b, c \in \mathbb{Q}$, $ab \neq 0$, if G_1 works, then the problem $\text{Holant}([1, a, b, c] \mid =_3)$ is $\#P$ -hard unless it is degenerate or Gen-Eq or affine, and thus in P .*

Proof. If $[1, a, b, c]$ has the form $[1, a, a, 1]$ or $[1, a, -1 - 2a, 2 + 3a]$ then the dichotomy has been proved in Lemmas 2.4 and 2.5 respectively. We now assume the signature is not of these two forms. By Lemma 2.9, when G_1 works, we can interpolate $[y, 1]$ on LHS and also $[1, x]$ on RHS.

Let us write down the Jordan Normal Form again:

$$G_1 = \begin{pmatrix} 1 & b \\ a & c \end{pmatrix} = \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix}^{-1},$$

and $\lambda = \frac{-\Delta + (1+c)}{2}$, $\mu = \frac{\Delta + (1+c)}{2}$, $x = \frac{\Delta - (1-c)}{2a}$, $y = \frac{\Delta + (1-c)}{2a}$, $\Delta = \sqrt{(1-c)^2 + 4ab}$.

Using $[y, 1]$ and the gadget in Figure 6, we get $[y^2 + yb, ya + c]$ on the RHS. We can interpolate Δ_0 and Δ_1 on RHS unless $[y^2 + yb, ya + c]$ is proportional to a row eigenvector of G_1 , namely $[1, -y]$ or $[1, x]$, according to Lemma 2.12. Thus the exceptions are $ya + c = (y^2 + yb)x$ or $ya + c = -y(y^2 + yb)$. The first equation implies $a^3 - b^3 - ab(1 - c) = 0$ or $c = ab$. The second equation implies $c = ab$ or $c = 1 + a - b$.

By assumption G_1 works, so $c \neq ab$. Thus, we consider two exceptional cases.

Case 1: $a^3 - b^3 - ab(1 - c) = 0$

In this case, we have $1 - c = \frac{a^3 - b^3}{ab}$ and thus $\Delta = \sqrt{(1 - c)^2 + 4ab} = \left| \frac{a^3 + b^3}{ab} \right|$. One condition $(4ab + c^2 - 2c + 1 = 0)$ in (2.3) is the same as $\Delta = 0$. Since G_1 works, we have $\Delta \neq 0$ and thus $a^3 + b^3 \neq 0$, which is equivalent to $a + b \neq 0$ when $a, b \in \mathbb{Q}$.

Subcase 1: $\frac{a^3 + b^3}{ab} > 0$. We have $[1, x] = [1, \frac{\Delta - (1-c)}{2a}] = [1, \frac{b^2}{a^2}]$ on RHS. Connect $[1, x]$ to $[1, a, b, c]$ on LHS, we get the binary signature $[1 + \frac{b^2}{a}, a + \frac{b^3}{a^2}, b + \frac{b^2c}{a^2}]$ on LHS. It is $\#P$ -hard (and thus the problem $[1, a, b, c]$ is $\#P$ -hard) unless one of the tractable conditions in Theorem 2.1 holds. It turns out that the only possibility is $X = 1$ in Theorem 2.1, which becomes $(a^2 - b)(a^3 + ab + 2b^3) = 0$. When $a^2 - b = 0$, together with $a^3 - b^3 = ab(1 - c)$, we have $c = a^3$, and thus $[1, a, b, c]$ is degenerate. When $a^3 + ab + 2b^3 = 0$, together with $a^3 - b^3 = ab(1 - c)$, by Lemma 3.1, $[1, a, b, c]$ is $\#P$ -hard (with $a + b \neq 0$ ruling out the exception).

Subcase 2: $\frac{a^3 + b^3}{ab} < 0$. We have $[y, 1] = [-\frac{b^2}{a^2}, 1]$ on LHS. Connecting two copies of $[y, 1]$ to $(=_3)$ we get $[y^2, 1] = [\frac{b^4}{a^4}, 1]$ on RHS. Connecting it back to LHS, we get a binary signature $[\frac{b^4}{a^4} + a, \frac{b^4}{a^3} + b, \frac{b^5}{a^4} + c]$ on LHS. It is $\#P$ -hard unless one of the tractable conditions in Theorem 2.1 holds. It turns out that the only possibility is $X = 1$ in Theorem 2.1, which becomes $(a^4b + ab^4)^2 = (a^5 + b^4)(b^5 + a^4c)$. Together with $a^3 - b^3 = ab(1 - c)$, by Lemma 3.2, $[1, a, b, c]$ is $\#P$ -hard unless it is degenerate.

Case 2: $1 + a - b - c = 0$

In this case, $\Delta = |a + b|$, and since G_1 works, one condition is $4ab + c^2 - 2c + 1 = 0$ in (2.3) which says $\Delta \neq 0$, and thus $a + b \neq 0$.

If $a + b > 0$, then $x = \frac{a+b-(1-c)}{2a} = 1$. Then we can interpolate $[1, x] = [1, 1]$ on RHS (as $y = \frac{a+b+(1-c)}{2a} = \frac{b}{a} \neq -1$). Else, $a + b < 0$, $y = \frac{-a-b+(1-c)}{2a} = \frac{-a-b+(b-a)}{2a} = -1$. We can get $[1, 1]$

on RHS by connecting two copies of $[y, 1] = [-1, 1]$ to $[1, 0, 0, 1]$. Then connecting $[1, 1]$ to $[1, a, b, c]$ on LHS we get a binary signature $[a + 1, a + b, a + 1]$ on LHS. Again we can apply Theorem 2.1 to it, and conclude that it is $\#P$ -hard. It turns out that the only feasible case of tractability is $X = 1$ in Theorem 2.1, which leads to $[1, a, -1 - 2a, 2 + 3a]$, but we assumed $[1, a, b, c]$ is not of this form. This proves the $\#P$ -hardness of $\text{Holant}([1, a, b, c] \mid =_3)$. \square

4 Dichotomy for $[1, a, b, 0]$

Theorem 4.1. *The problem $[1, a, b, 0]$ for $a, b \in \mathbb{Q}$ is $\#P$ -hard unless it is degenerate or affine, and thus in P .*

Proof. If $ab \neq 0$ and G_1 works, then this is proved in Theorem 3.3 (in this case, it cannot be a Gen-Eq). If $a = b = 0$, it is degenerate and in P . We divide the rest into three cases:

1. $ab \neq 0$ and G_1 does not work;
 2. $f = [1, a, 0, 0]$ with $a \neq 0$;
 3. $f = [1, 0, b, 0]$ with $b \neq 0$.
- Case 1: $ab \neq 0$ in $f = [1, a, b, 0]$ and G_1 does not work. Since $c = 0 \neq ab$, this implies that at least one equation in (2.3) holds. After a simple derivation, we have the following family of signatures to consider: $[1, a, -\frac{1}{ka}, 0]$, for $k = 1, 2, 3, 4$.

We use G_4 to produce another symmetric ternary signature in each case. If the new signature is $\#P$ -hard, then so is the given signature. We will describe the case $[1, a, -\frac{1}{a}, 0]$ in more detail; the other three types ($k = 2, 3, 4$) are similar.

For $k = 1$, the gadget G_4 produces $g = [3a^3 + 4, a^4 - a - \frac{2}{a^2}, -a^2 + \frac{1}{a} + \frac{1}{a^4}, a^3 + 3]$. For $a = -1$, this is $[1, 0, -1, 2]$, which has the form $[1, a', -1 - 2a', 2 + 3a']$ and is $\#P$ -hard by Lemma 2.5. Below we assume $a \neq -1$. Then all entries of g are nonzero.

We claim that the gadget G_1 works using g . Since $a \in \mathbb{Q}$, it can be checked that g is non-degenerate since $(a^4 - a - \frac{2}{a^2})(-a^2 + \frac{1}{a} + \frac{1}{a^4}) = (3a^3 + 4)(a^3 + 3)$ has no solution, and that no equation in (2.3) has a solution applied to g . Hence, G_1 works using g and we may apply Theorem 3.3 to g . Using the fact that $a \in \mathbb{Q}$, one can show that g cannot be a Gen-Eq because it has no zero entry, nor can it be affine or degenerate. Thus $[1, a, -\frac{1}{a}, 0]$ is $\#P$ -hard.

- Case 2: $f = [1, a, 0, 0]$ with $a \neq 0$. The gadget G_4 produces $g' = [3a^3 + 1, a^4 + a, a^2, a^3]$. Since $a \in \mathbb{Q}$, $3a^3 + 1 \neq 0$. If $a = -1$, $g' = [-2, 0, 1, -1]$ and it suffices to consider $[1, -1, 0, 2]$, in which case G_3 works where the matrix $G_3 = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$. We can interpolate $[1, x] = [1, -\frac{3+\sqrt{5}}{2}]$ on RHS. Connect it back to $[1, -1, 0, 2]$ and get a binary signature $[\frac{5+\sqrt{5}}{2}, -1, -(3+\sqrt{5})]$ on LHS, which, by Theorem 2.1, is $\#P$ -hard. Thus, $[1, -1, 0, 2]$ is $\#P$ -hard and so is $[1, a, 0, 0]$.

Else, $a \neq -1$. We claim that the gadget G_1 works using g' . The signature g' is non-degenerate since $a \in \mathbb{Q}$ is nonzero and thus $(a^4 + a)a^2 \neq (3a^3 + 1)a^3$. Also no equation in (2.3) has a solution applied to g' . Hence, G_1 works using g' and we may apply Theorem 3.3 to g' . Using the fact that $a \in \mathbb{Q}$, one can show that g' cannot be a Gen-Eq because it has no zero entry, nor can it be affine or degenerate. Thus $[1, a, 0, 0]$ is $\#P$ -hard.

- Case 3: $f = [1, 0, b, 0]$ with $b \neq 0$. The gadget G_1 produces a binary straddled signature $G_1 = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \\ 0 & \end{bmatrix} \cdot \begin{bmatrix} 1 & b \end{bmatrix}$ which decomposes into a unary signature $[1, b]$ on RHS and a unary signature $[1, 0]$ on LHS. This gives us a reduction $\text{Holant}(f \mid \{ (=3), [1, b] \}) \leq_T \text{Holant}(f \mid =_3)$. To see that, notice that in any signature grid for the problem $\text{Holant}(f \mid \{ (=3), [1, b] \})$, the number of

occurrences of $[1, b]$ is $0 \bmod 3$, say $3n$. We can replace each occurrence $[1, b]$ by G_1 , leaving $3n$ extra copies of $[1, 0]$ on the LHS. These can all be absorbed by connecting to $(=3)$.

Now, if we connect $[1, b]$ to $[1, 0, b, 0]$ and get a binary signature $[1, b^2, b]$ on LHS. Thus, $\text{Holant}([1, b^2, b] \mid =_3) \leq_T \text{Holant}(f \mid \{(=3), [1, b]\})$. The problem $\text{Holant}([1, b^2, b] \mid =_3)$ is $\#P$ -hard except $b = \pm 1$, by Theorem 2.1, which implies that $\text{Holant}(f \mid =_3)$ is also $\#P$ -hard. If $b = \pm 1$, then f is affine, and $\text{Holant}(f \mid =_3)$ is in P . \square

5 Dichotomy for $[1, a, 0, c]$

Theorem 5.1. *The problem $[1, a, 0, c]$ with $a, c \in \mathbb{Q}$ is $\#P$ -hard unless $a = 0$, in which case it is Gen-Eq and thus in P .*

Proof. When $a = 0$, it is Gen-Eq and so is in P . When $a \neq 0$, if $c = 0$, it is $\#P$ -hard by Theorem 4.1. In the following we discuss $[1, a, 0, c]$ with $ac \neq 0$.

If $c = \pm 1$, the signature is $[1, a, 0, 1]$ or $[1, a, 0, -1]$. We use G_4 to produce a ternary signature $g = [3a^3 + 1, a^4 + a, a^2, a^3 + 1]$ (both mapped to the same signature, surprisingly). If $a = -1$, it is $[1, 0, -\frac{1}{2}, 0]$ after normalization, which by Theorem 4.1 is $\#P$ -hard and so is the given signature $[1, -1, 0, 1]$. If $a \neq -1$, then g has no zero entry. We then claim that the gadget G_1 works using g . It can be checked that g is non-degenerate since $(a^4 + a)a^2 = (3a^3 + 1)(a^3 + 1)$ has no solution, and that no equation in (2.3) has a solution applied to g . Hence, G_1 works using g and we may apply Theorem 3.3 to g . Using the fact that $a \in \mathbb{Q}$, one can show that g cannot be a Gen-Eq because it has no zero entry, nor can it be affine or degenerate. Thus $[1, a, 0, \pm 1]$ are both $\#P$ -hard.

Now assume $c \neq 0, \pm 1$. We claim that the gadget G_1 works. It can be checked that for the non-degenerate matrix $G_1 = \begin{bmatrix} 1 & 0 \\ a & c \end{bmatrix}$, $\Delta = |1 - c|$, $\lambda/\mu \in \{c, \frac{1}{c}\}$ is not a root of unity. Next we claim that we can obtain $[1, 0]$ on RHS. If $c < 1$ by Lemma 2.9 we can interpolate $[1, x] = [1, 0]$ on RHS with two exceptions to which we already give a dichotomy (see the Remark after Definition 2.10). If $c > 1$, we can interpolate $[y, 1] = [0, 1]$ on LHS and so the gadget in Figure 6 produces $[0, c]$ on RHS, which is not proportional to the row eigenvectors $[1, -y] = [1, 0]$ and $[1, x] = [1, \frac{c-1}{a}]$ of G_1 . By Lemma 2.12, we can interpolate any unary gadget on RHS, including $[1, 0]$. Thus we can always get $[1, 0]$ on RHS. Connect $[1, 0]$ to $[1, a, 0, c]$ and we will get a binary signature $[1, a, 0]$ on LHS, which is $\#P$ -hard by Theorem 2.1. Therefore $[1, a, 0, c]$ is $\#P$ -hard when $c \neq 0, \pm 1$. \square

6 Dichotomy when $abc \neq 0$

We need four lemmas to handle some special cases.

Lemma 6.1. *The problem $[1, -b^2, b, -b^3]$ with $b \in \mathbb{Q}$ is $\#P$ -hard unless $b = 0, \pm 1$, which is in P .*

Proof. If $b = 0, \pm 1$, it is degenerate or affine. Now assume $b \neq 0, \pm 1$. $G_1 = \begin{bmatrix} 1 & b \\ -b^2 & b^3 \end{bmatrix} = \begin{bmatrix} 1 & b \\ -b^2 & b^3 \end{bmatrix} \cdot \begin{bmatrix} 1 & b \\ 1 & b \end{bmatrix}$. Then we can get $[1, -b^2]$ on the LHS similar to the proof in Case 3 of Theorem 4.1. Note that connecting three copies of $[1, b]$ with $[1, -b^2, b, -b^3]$ on LHS produces a global factor $1 - b^6 \neq 0$. Connect $[1, -b^2]$ twice to $[1, 0, 0, 1]$ on RHS, and we get $[1, b^4]$ on RHS. Connect $[1, b^4]$ back to $[1, -b^2, b, -b^3]$ on LHS, and we get a binary signature $g = [1 - b^6, -b^2 + b^5, b - b^7]$, which by Theorem 2.1 is $\#P$ -hard, and so is $[1, -b^2, b, -b^3]$. \square

Lemma 6.2. *The problem $[1, a, -\frac{1}{a}, -1]$ with $a \in \mathbb{Q}$, $a \neq 0$ is #P-hard unless $a = \pm 1$, in which case it is in P.*

Proof. If $a = \pm 1$, $[1, 1, -1, -1]$ is affine and $[1, -1, 1, -1]$ is degenerate, both of which are in P. Now we assume $a \neq \pm 1$ (so the matrix $\begin{bmatrix} a^{-2} & a \\ 1 & 1 \end{bmatrix}$ is invertible). We use the binary straddled gadget G_2 and write down its Jordan Normal Form as

$$G_2 = \begin{pmatrix} 2a^3 + 1 - a^{-3} & -a - a^{-2} \\ a^2 + a^{-1} & a^3 - 1 - 2a^{-3} \end{pmatrix} = \begin{pmatrix} a^{-2} & a \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a^3 - a^{-3} & 0 \\ 0 & 2a^3 - 2a^{-3} \end{pmatrix} \begin{pmatrix} a^{-2} & a \\ 1 & 1 \end{pmatrix}^{-1}$$

The matrix is non-degenerate and the ratio of its two eigenvalues are $1/2$, so gadget G_2 works. Since here $y = a \neq \pm 1$, we can interpolate $[1, x] = [1, -a^{-2}]$ on RHS. Now connect $[1, x]$ to $[1, a, -\frac{1}{a}, -1]$ on LHS and we can get a binary signature $[1 - \frac{1}{a}, a + \frac{1}{a^3}, -\frac{1}{a} + \frac{1}{a^2}]$, which by Theorem 2.1 is #P-hard, and so is $[1, a, -\frac{1}{a}, -1]$. \square

Before presenting our next lemma, we introduce a ternary gadget G_{aux} in Figure 7. Its signature is $[1 + 2a^3 + b^3, a + 2a^2b + b^2c, a^2 + 2ab^2 + bc^2, a^3 + 2b^3 + c^3]$.

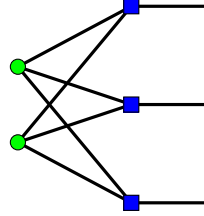


Figure 7: G_{aux}

Lemma 6.3. *The problem $[1, a, b, ab]$ with $a, b \in \mathbb{Q}$ and $a, b \neq 0$ is #P-hard unless it is degenerate or affine, which is in P.*

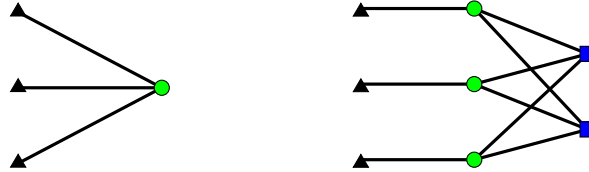


Figure 8: Two gadgets where each triangle represents the unary gadget $[1, a]$

Proof. Using gadget G_1 , we have a degenerate matrix $G_1 = \begin{bmatrix} 1 & b \\ a & ab \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \cdot \begin{bmatrix} 1 & b \end{bmatrix}$. We get $[1, b]$ on RHS if $[1, a]$ can appropriately form some nonzero global factor. Figure 8 indicates two different ways of “absorbing” $[1, a]$ on LHS. The factors they provide are $1 + a^3$ and $1 + 3a^3 + 3a^2b^2 + a^5b^2$ respectively. It is easy to see that at least one of them is nonzero except $a = -1$ and $b = \pm 1$, i.e., $[1, -1, 1, -1]$ which is degenerate or $[1, -1, -1, 1]$ which is affine. Now assume below $[1, a, b, ab]$ is not these two, then we can interpolate $[1, b]$ on RHS. Connect $[1, b]$ back to $[1, a, b, ab]$ on LHS and we get the binary signature $g = [1 + ab, a + b^2, b + bc]$. If $a + b^2 = 0$, since $c = ab$, the given signature is $[1, -b^2, b, -b^3]$ which, according to Lemma 6.1, is #P-hard (as we assumed just now, it is not $[1, -1, \pm 1, \mp 1]$). Now we assume $a + b^2 \neq 0$. Applying Theorem 2.1 to g , it is #P-hard (and so is the given signature $[1, a, b, ab]$) unless

1. $X = 1$. $(1 + ab)(b + bc) = (a + b^2)^2$ implies $(a^2 - b)(b^3 - 1) = 0$. If $a^2 - b = 0$, the given signature is $[1, a, a^2, a^3]$ and is degenerate. If $b^3 - 1 = 0$, since $b \in \mathbb{Q}$, we have $b = 1$ and $a \neq -b^2 = -1$, and the given signature is $[1, a, 1, a]$. We apply G_{aux} in Figure 7 using g to produce a ternary signature $h = [2 + 2a^3, 2a + 2a^2, 2a + 2a^2, 2 + 2a^3]$ on LHS, which has the form $[1, a', a', 1]$ after normalization, as $2 + 2a^3 \neq 0$ for $a \in \mathbb{Q}$ and $a \neq -1$. So, h is #P-hard by Lemma 2.4 unless $a' = 0, \pm 1$, which implies $a = 1$ (as $a \neq 0, -1$) in which case the given signature is $[1, 1, 1, 1]$ and thus in P. Thus, $[1, a, 1, a]$ ($a \neq 0$) is #P-hard unless $a = \pm 1$.
2. $X = Z = 0$. The given signature is $[1, a, -\frac{1}{a}, -1]$ which, by Lemma 6.2, is #P-hard unless $a = \pm 1$.
3. $X = -1, Z = 0$. This turns out to be impossible.
4. $X = -1, Z = -1$. This is also impossible.

Note that since $a, b \neq 0$, $[1, a, b, ab]$ cannot be Gen-Eq. The lemma is proved. \square

Lemma 6.4. *The problem $[1, a, -a, -1]$ with $a \in \mathbb{Q}$ is #P-hard unless $a = \pm 1, 0$, which is in P.*

Proof. If $a = \pm 1, 0$, it is easy to see the given signature is in P, with $[1, 1, -1, -1]$ being affine. Now we assume $a \neq \pm 1, 0$. We use the gadget G_4 to produce a ternary signature $g = (1 + a)[u, v, v, u]$, where $u = 1 - a + a^2$, and $v = a(1 - a^2)$. Since $u, v \neq 0$ and $u \neq \pm v$, by Lemma 2.4, g is #P-hard and so is the given signature $[1, a, -a, -1]$. \square

Now we prove

Theorem 6.5. *The problem $[1, a, b, c]$ with $a, b, c \in \mathbb{Q}$, $abc \neq 0$, is #P-hard unless it is degenerate, Gen-Eq or affine.*

Proof. By Theorem 3.3 and Lemma 6.3 it suffices to consider the case when the ratio of two eigenvalues in $G_1 = \begin{bmatrix} 1 & b \\ a & c \end{bmatrix}$ is a root of unity and $c \neq ab$. If the ratio of eigenvalues of G_1 is a root of unity, we know at least one condition in (2.3) holds. For convenience, we list the conditions in (2.3) here and label them as R_i where $i = 1, 2, 3, 4, 5$:

$$R = \bigvee_{i=1}^5 R_i, \quad \text{where} \quad \begin{cases} R_1 : c = -1 \\ R_2 : ab + c^2 + c + 1 = 0 \\ R_3 : 2ab + c^2 + 1 = 0 \\ R_4 : 3ab + c^2 - c + 1 = 0 \\ R_5 : 4ab + c^2 - 2c + 1 = 0 \end{cases} \quad (6.7)$$

We apply G_{aux} in Figure 7 on $[1, a, b, c]$, i.e. placing squares to be $[1, a, b, c]$ and circles to be $=_3$, to produce a ternary signature $[w, x, y, z] = [1 + 2a^3 + b^3, a + 2a^2b + b^2c, a^2 + 2ab^2 + bc^2, a^3 + 2b^3 + c^3]$. If $w \neq 0$ and G_1 works on $[w, x, y, z]$, by Theorem 3.3 we have $[w, x, y, z]$ is #P-hard and thus $[1, a, b, c]$ is #P-hard unless at least one condition S_i listed below holds, where $i = 1, 2, 3, 4, 5, 6$:

$$S = \bigvee_{i=1}^6 S_i \quad \text{where} \quad \begin{cases} S_1 : x^2 = wy \wedge y^2 = xz \text{ (degenerate form)} \\ S_2 : x = 0 \wedge y = 0 \text{ (Gen-Eq form)} \\ S_3 : w = y \wedge x = 0 \wedge z = 0 \text{ (affine form } [1, 0, 1, 0]) \\ S_4 : w + y = 0 \wedge x = 0 \wedge z = 0 \text{ (affine form } [1, 0, -1, 0]) \\ S_5 : w = x \wedge w + y = 0 \wedge w + z = 0 \text{ (affine form } [1, 1, -1, -1]) \\ S_6 : w + x = 0 \wedge w + y = 0 \wedge w = z \text{ (affine form } [1, -1, -1, 1]) \end{cases} \quad (6.8)$$

Solve the equation system $R \wedge S$ for variables $a, b, c \in \mathbb{Q}$, we have the following solutions:

- $a = c = -1, b = 1$; the problem $[1, -1, 1, -1]$ is in P since it is degenerate;
- $a = 1, b = c = -1$; the problem $[1, 1, -1, -1]$ is in P since it is affine;
- $a = c = 1, b = -1$; the problem $[1, 1, -1, 1]$ is #P-hard (use the gadget G_4 to produce $[1, 1, -1, 3]$ after flipping 0's and 1's, then use it again to produce $[1, 1, -5, 19]$ which is #P-hard by Theorem 3.3. Note that we need to apply G_4 twice in order that the condition that G_1 works in Theorem 3.3 is satisfied for the newly created ternary signature);
- $a = \frac{1}{2}, b = -\frac{1}{2}, c = -1$; the problem $[1, \frac{1}{2}, -\frac{1}{2}, -1]$ is #P-hard by Lemma 6.4.

Continuing the discussion for the ternary signature $[w, x, y, z]$, it remains to consider the case when $w = 0$ or G_1 does not work on $[w, x, y, z]$. For $w \neq 0$ we normalize $[w, x, y, z]$ to be $[1, \frac{x}{w}, \frac{y}{w}, \frac{z}{w}]$ and substituting $\frac{x}{w}, \frac{y}{w}, \frac{z}{w}$ into a, b, c respectively in (2.3), we get at least one condition T_i listed below, where $i = 1, 2, 3, 4, 5, 6$:

$$T = \bigvee_{i=1}^6 T_i, \text{ where } \begin{cases} T_1 : zw + w^2 = 0 \\ T_2 : xy + z^2 + zw + w^2 = 0 \\ T_3 : 2xy + z^2 + w^2 = 0 \\ T_4 : 3xy + z^2 - zw + w^2 = 0 \\ T_5 : 4xy + z^2 - 2zw + w^2 = 0 \\ T_6 : xy = wz \end{cases} \quad (6.9)$$

Note that T_1 incorporates the case when $w = 0$. So we have the condition $R \wedge T$. We now apply G_{aux} once again using $[w, x, y, z]$ to produce another new ternary signature $[w_2, x_2, y_2, z_2]$ where $w_2 = w^3 + 2x^3 + y^3$, $x_2 = w^2x + 2x^2y + y^2z$, $y_2 = wx^2 + 2xy^2 + yz^2$, $z_2 = x^3 + 2y^3 + z^3$. Similarly as the previous argument, if $w_2 \neq 0$ and G_1 works on $[w_2, x_2, y_2, z_2]$, we know $[w_2, x_2, y_2, z_2]$ is #P-hard and thus $[1, a, b, c]$ is #P-hard unless at least one condition U_i listed below holds, where $i = 1, 2, 3, 4, 5, 6$:

$$U = \bigvee_{i=1}^6 U_i, \text{ where } \begin{cases} U_1 : x_2^2 = w_2y_2 \wedge y_2^2 = x_2z_2 \text{ (degenerate form)} \\ U_2 : x_2 = 0 \wedge y_2 = 0 \text{ (Gen-Eq form)} \\ U_3 : w_2 = y_2 \wedge x_2 = 0 \wedge z_2 = 0 \text{ (affine form } [1, 0, 1, 0]) \\ U_4 : w_2 + y_2 = 0 \wedge x_2 = 0 \wedge z_2 = 0 \text{ (affine form } [1, 0, -1, 0]) \\ U_5 : w_2 = x_2 \wedge w_2 + y_2 = 0 \wedge w_2 + z_2 = 0 \text{ (affine form } [1, 1, -1, -1]) \\ U_6 : w_2 + x_2 = 0 \wedge w_2 + y_2 = 0 \wedge w_2 = z_2 \text{ (affine form } [1, -1, -1, 1]) \end{cases} \quad (6.10)$$

Solve the equation system $R \wedge T \wedge U$ for rational-valued variables a, b, c , we have the following solutions:

- $a = c = -1, b = 1$; the problem $[1, -1, 1, -1]$ is in P since it is degenerate;
- $a = 1, b = c = -1$; the problem $[1, 1, -1, -1]$ is in P since it is affine;
- $a = -1, b = c = 1$; the problem $[1, -1, 1, 1]$ is #P-hard (use the gadget G_4 to produce $[1, 1, -1, 3]$, use it again to produce $[1, 1, -5, 19]$ which is #P-hard by Theorem 3.3);
- $a = c = 1, b = -1$; the problem $[1, 1, -1, 1]$ is #P-hard (this is the reversal of $[1, -1, 1, 1]$);
- $a = \frac{1}{2}, b = -\frac{1}{2}, c = -1$; the problem $[1, \frac{1}{2}, -\frac{1}{2}, -1]$ is #P-hard by Lemma 6.4.

Otherwise, we know $w_2 = 0$ or G_1 does not work on $[w_2, x_2, y_2, z_2]$. Similarly, we know at least

one condition V_i listed below holds, where $i = 1, 2, 3, 4, 5, 6$:

$$V = \bigvee_{i=1}^6 V_i, \text{ where } \begin{cases} V_1 : z_2 w_2 + w_2^2 = 0 \\ V_2 : x_2 y_2 + z_2^2 + z_2 w_2 + w_2^2 = 0 \\ V_3 : 2x_2 y_2 + z_2^2 + w_2^2 = 0 \\ V_4 : 3x_2 y_2 + z_2^2 - z_2 w_2 + w_2^2 = 0 \\ V_5 : 4x_2 y_2 + z_2^2 - 2z_2 w_2 + w_2^2 = 0 \\ V_6 : x_2 y_2 = w_2 z_2 \end{cases} \quad (6.11)$$

Finally, solve the equation system $R \wedge T \wedge V$ for variables $a, b, c \in \mathbb{Q}$, we have the following solutions:

- $a = -1, b = c = 1$; the problem $[1, -1, 1, 1]$ is #P-hard (see the case above for $R \wedge T \wedge U$);
- $a = c = 1, b = -1$; the problem $[1, 1, -1, 1]$ is #P-hard (this is the reversal of $[1, -1, 1, 1]$);
- $a = -b, c = -1$; the problem $[1, a, -a, -1]$ is #P-hard unless $a = \pm 1, 0$ by Lemma 6.4.

The proof is now complete. □

7 Dichotomy for $[0, a, b, 0]$

We quickly finish the discussion for $[0, a, b, 0]$ with the help of previous theorems on $[1, a, b, c]$.

Theorem 7.1. *The problem $[0, a, b, 0]$ with $a, b \in \mathbb{Q}$ is #P-hard unless $a = b = 0$, in which case the Holant value is 0.*

Proof. We apply the gadget G_4 on $[0, a, b, 0]$ to produce the ternary signature $g = [3a^2b^2, a(a^3 + 2b^3), b(2a^3 + b^3), 3a^2b^2]$.

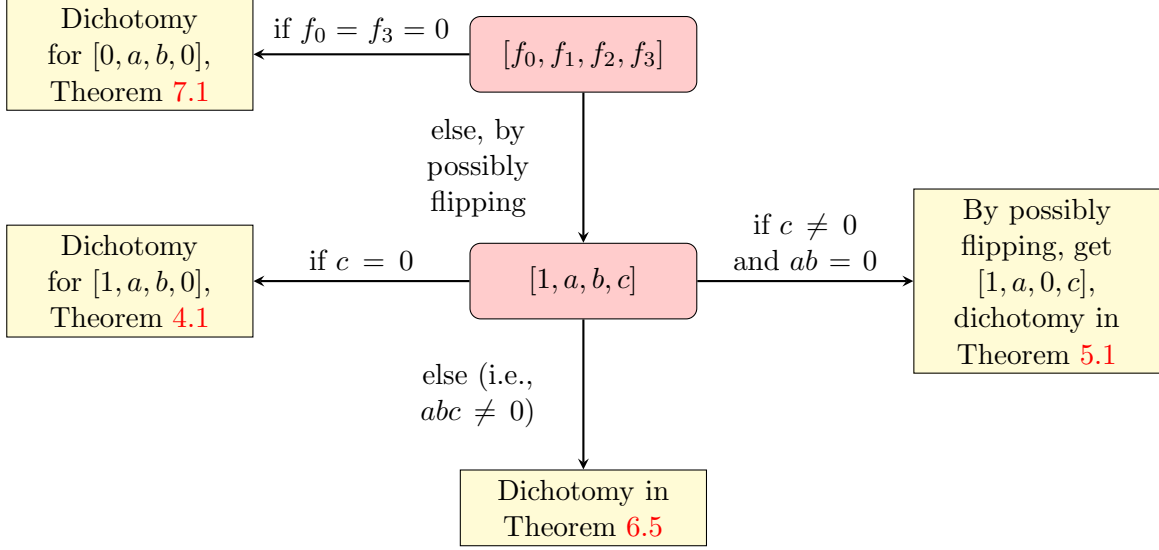
If $ab \neq 0$, we can normalize g to be the form $[1, a', b', c']$. Since $a, b, c \in \mathbb{Q}$, we have $a'b'c' \neq 0$. By Theorem 6.5, we know $[1, a', b', c']$ is #P-hard (and so is $[0, a, b, 0]$) unless it is degenerate, Gen-Eq or affine. However, that $[1, a', b', c']$ in P implies $b = a$, i.e. the given signature is $[0, a, a, 0]$. It suffices to consider $[0, 1, 1, 0]$. We apply the gadget G_{aux} on $[0, 1, 1, 0]$ and get the ternary signature $[3, 2, 2, 3]$ which is #P-hard by Lemma 2.4, so $[0, 1, 1, 0]$ is #P-hard.

Now if exactly one of a and b is 0, it suffices to consider the problem $[0, 1, 0, 0]$. This problem is to count the number of exact set covers in a 3-regular 3-uniform set system. This problem is #P-hard by Lemma 6.1 in [15]. □

8 Main Theorem

We are now ready to prove our main theorem. The following is a flowchart of the logical structure for the proof of Theorem 8.1.

Flowchart of proof structure:



Theorem 8.1. *The problem $\text{Holant}\{[f_0, f_1, f_2, f_3] \mid (=3)\}$ with $f_i \in \mathbb{Q}$ ($i = 0, 1, 2, 3$) is $\#P$ -hard unless the signature $[f_0, f_1, f_2, f_3]$ is degenerate, Gen-Eq or belongs to the affine class.*

Proof. First, if $f_0 = f_3 = 0$, by Theorem 7.1, we know that it is $\#P$ -hard unless it is $[0, 0, 0, 0]$ which is degenerate. Note that in all other cases, $[0, f_1, f_2, 0]$ is not Gen-Eq, degenerate or affine.

Assume now at least one of f_0 and f_3 is not 0. By flipping the role of 0 and 1, we can assume $f_0 \neq 0$, then the signature becomes $[1, a, b, c]$ after normalization.

If $c = 0$, the dichotomy for $[1, a, b, 0]$ is proved in Theorem 4.1.

If in $[1, a, b, c]$, $c \neq 0$, then a and b are symmetric by flipping. Now if $ab = 0$, we can assume $b = 0$ by the afore-mentioned symmetry, i.e., the signature becomes $[1, a, 0, c]$. By Theorem 5.1, it is $\#P$ -hard unless $a = 0$, in which case it is Gen-Eq. In all other cases, it is not Gen-Eq or degenerate or affine.

Finally, for the problem $[1, a, b, c]$ where $abc \neq 0$, Theorem 6.5 proves the dichotomy that it is $\#P$ -hard unless the signature is degenerate or Gen-Eq or affine. \square

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Appendix

In this paper we use Mathematica™ to carry out symbolic computations. In particular, the function `CylindricalDecomposition` is used heavily. It is an implementation of Tarski's theorem on the decidability of the theory of real-closed fields, and can prove the non-existence solutions of polynomial systems.

- In the proof of Lemma 2.3, we use `CylindricalDecomposition` to show that the intersection of $f_1 = f_2 = f_3 = f_4 = 0$ is empty for $a, b, c \in \mathbb{Q}$ and $x \in \mathbb{C}$, where $f_1 = cx^3 + 3bx^2 + 3ax + 1$, $f_2 = (ab + c)x^3 + (3bc + 2a^2 + b)x^2 + (2b^2 + ac + 3a)x + ab + 1$, $f_3 = (a^3 + b^3 + c^3)x^3 + 3(a^2 + 2ab^2 + bc^2)x^2 + 3(a + 2a^2b + b^2c)x + 1 + 2a^3 + b^3$ and $f_4 = (ab + 2abc + c^3)x^3 + (2a^2 + b + 2a^2c + 3ab^2 + bc + 3b^2c)x^2 + (3a + 3a^2b + ac + 2b^2 + 2b^2c + ac^2)x + 1 + 2ab + abc$. We write x as $u + vi$ with $u, v \in \mathbb{R}$, and set both the real and imaginary parts of each f_i to 0.

```

1 Clear["Global`*"];
2 (* below x = u + vi is complex,
3 r3 is real part of x^3, i3 is imaginary part of x^3 ,
4 r2 is real part of x^2, i2 is imaginary part of x^2,
5 r1 is real part of x, i1 is imaginary part of x *)
6
7 r3 = u^3 - 3 u v v;
8 i3 = 3 u u v - v^3;
9 r2 = u u - v v;
10 i2 = 2 u v;
11 r1 = u;
12 i1 = v;
13 (* below f_k1 is real part of f_k, f_k2 is imaginary part of f_k,
14 where k = 1,2,3,4 *)
15 f11 = c r3 + 3 b r2 + 3 a r1 + 1;
16 f12 = c i3 + 3 b i2 + 3 a i1;
17 f21 = (a b + c) r3 + (3 b c + 2 a a + b) r2 + (2 b b + a c + 3 a) r1 +
18     a b + 1;
19 f22 = (a b + c) i3 + (3 b c + 2 a a + b) i2 + (2 b b + a c + 3 a) i1;
20 f31 = (a^3 + b^3 + c^3) r3 + 3 (a a + 2 a b b + b c c) r2 +
21     3 (a + 2 a a b + b b c) r1 + 2 a^3 + b^3 + 1;
22 f32 = (a^3 + b^3 + c^3) i3 + 3 (a a + 2 a b b + b c c) i2 +
23     3 (a + 2 a a b + b b c) i1;
24 f41 = (a b + 2 a b c + c^3) r3 + (2 a a + b + 3 a b b + 2 a a c +
25     b c + 3 b c c) r2 + (3 a + 3 a a b + 2 b b + a c + 2 b b c +
26     a c c) r1 + 1 + 2 a b + a b c;
27 f42 = (a b + 2 a b c + c^3) i3 + (2 a a + b + 3 a b b + 2 a a c +
28     b c + 3 b c c) i2 + (3 a + 3 a a b + 2 b b + a c + 2 b b c +
29     a c c) i1;
30
31 CylindricalDecomposition[
32     f11 == 0 && f12 == 0 && f21 == 0 && f22 == 0 && f31 == 0 &&
33     f32 == 0 && f41 == 0 && f42 == 0, {a, b, c, u, v}]

```

- In the proof of Lemma 3.1, we use `CylindricalDecomposition` several times to solve a polynomial system in a, b, c . There, $(con1)$ is $a^3 - b^3 - ab(1 - c) = 0$ and $(con2)$ is $a^3 + ab + 2b^3 = 0$. Together with a third condition, we solve for $a, b, c \in \mathbb{Q}$. The third equation is among one in (3.6) (note that we use the function `Factor` here to get an irreducible polynomial f_1 over \mathbb{Q}), or one in (2.5), or $(a + b^2)(a^2 + bc) = (1 + ab)(ab + c^2)$.

```

1 Clear["Global`*"];
2 (* dsq means \Delta^2 *)
3
4 dsq = 1 + 4 a^3 + 4 a^2 b^2 + 4 a b c + 4 b^3 c - 2 c^2 + c^4;
5 d = 1 - c c;
6 e = 2 (a + b b);
7
8 (*y^3 + y^2 b + y a + c\[Equal]0 is transformed to LHS = RHS below,
9 by eliminating the square root part
10 i.e., we give a function in real domain *)
11
12 LHS = dsq*(dsq + 3 d d + 2 d e b + e e a)^2;
13 RHS = (dsq*(3 d + e b) + d d d + d d e b + e e a d + c e e e)^2;
14 Factor[LHS - RHS]
15 (* the factor result is f1 * (a + b b) ^3 , where f1 is below *)
16
17 f1 = a^3 + 4 a^6 + 3 a^5 b^2 + a^3 b^3 - c - 4 a^3 c + 6 a^4 b c -
18     6 a^2 b^2 c - b^3 c - 3 a^2 b^5 c - 3 a^3 c^2 - 3 a b c^2 -
19     4 b^3 c^2 - a^3 b^3 c^2 - 6 a b^4 c^2 - 4 b^6 c^2 + 3 c^3 +
20     4 a^3 c^3 + 6 a^2 b^2 c^3 + 3 b^3 c^3 + a^3 c^4 + 3 a b c^4 +
21     4 b^3 c^4 - 3 c^5 - b^3 c^5 + c^7;
22 f2 = a + b b;
23 f3 = c - a b;
24 f4 = a^3 - b^3 c + a b (c c - 1);
25 con1 = a a a - b b b - a b (1 - c) == 0;
26 con2 = a a a + a b + 2 b b b == 0;
27 (* the following four commands correspond to (3.7) in paper,
28 the exceptional cases of that G3 works but may not interpolate any \
29 unary,
30 *)
31 CylindricalDecomposition[con1 && con2 && f1 == 0, {a, b, c}]
32 CylindricalDecomposition[con1 && con2 && f2 == 0, {a, b, c}]
33 CylindricalDecomposition[con1 && con2 && f3 == 0, {a, b, c}]
34 CylindricalDecomposition[con1 && con2 && f4 == 0, {a, b, c}]
35
36 A = 1 + 2 a b + c^2;
37 B = 4 a^3 + 4 a^2 b^2 + 4 a b c + 4 b^3 c + (-1 + c^2)^2;
38 h1 = A == 0;
39 h2 = B == 0;
40 h3 = A A + B == 0;
41 h4 = A A + 3 B == 0;
42 h5 = 3 A A + B == 0;
43 (* G3 doesn't work *)
44 CylindricalDecomposition[con1 && con2 && h1, {a, b, c}]
45 CylindricalDecomposition[con1 && con2 && h2, {a, b, c}]
46 CylindricalDecomposition[con1 && con2 && h3, {a, b, c}]
47 CylindricalDecomposition[con1 && con2 && h4, {a, b, c}]
48 CylindricalDecomposition[con1 && con2 && h5, {a, b, c}]
49 CylindricalDecomposition[
50   con1 && con2 && a^3 - a b + a b c + b^3 c - c^2 - a b c^2 == 0, {a,
51   b, c}]

```

- In the proof of Lemma 3.2, similarly, we have the code below.

```

1 Clear["Global`*"];
2 (* note that the following con2 is not the same one as in the lemma,

```

```

3 however, according to the proof of the lemma,
4 it suffices to consider this new con2 with the original con1 *)
5
6 con1 = a a a - b b b - a b (1 - c) == 0;
7 con2 = a^9 + a^4 b^4 + a^3 b^6 + b^9 == 0;
8
9 f1 = a^3 + 4 a^6 + 3 a^5 b^2 + a^3 b^3 - c - 4 a^3 c + 6 a^4 b c -
10 6 a^2 b^2 c - b^3 c - 3 a^2 b^5 c - 3 a^3 c^2 - 3 a b c^2 -
11 4 b^3 c^2 - a^3 b^3 c^2 - 6 a b^4 c^2 - 4 b^6 c^2 + 3 c^3 +
12 4 a^3 c^3 + 6 a^2 b^2 c^3 + 3 b^3 c^3 + a^3 c^4 + 3 a b c^4 +
13 4 b^3 c^4 - 3 c^5 - b^3 c^5 + c^7;
14 f2 = a + b b;
15 f3 = c - a b;
16 f4 = a^3 - b^3 c + a b (c c - 1);
17
18 CylindricalDecomposition[con1 && con2 && f1 == 0, {a, b, c}]
19 CylindricalDecomposition[con1 && con2 && f2 == 0, {a, b, c}]
20 CylindricalDecomposition[con1 && con2 && f3 == 0, {a, b, c}]
21 CylindricalDecomposition[con1 && con2 && f4 == 0, {a, b, c}]
22
23 A = 1 + 2 a b + c^2;
24 B = 4 a^3 + 4 a^2 b^2 + 4 a b c + 4 b^3 c + (-1 + c^2)^2;
25
26 h1 = A == 0;
27 h2 = B == 0;
28 h3 = A A + B == 0;
29 h4 = A A + 3 B == 0;
30 h5 = 3 A A + B == 0;
31
32 CylindricalDecomposition[con1 && con2 && h1, {a, b, c}]
33 CylindricalDecomposition[con1 && con2 && h2, {a, b, c}]
34 CylindricalDecomposition[con1 && con2 && h3, {a, b, c}]
35 CylindricalDecomposition[con1 && con2 && h4, {a, b, c}]
36 CylindricalDecomposition[con1 && con2 && h5, {a, b, c}]
37 CylindricalDecomposition[
38 con1 && con2 && a^3 - a b + a b c + b^3 c - c^2 - a b c^2 == 0, {a,
39 b, c}]

```

• In the proof of Theorem 6.5, again we use `CylindricalDecomposition` to solve equation systems $R \wedge S$, $R \wedge T \wedge U$ and $R \wedge T \wedge V$. In solving the system $R \wedge T \wedge V$ which requires a significant amount of computation, we apply `CylindricalDecomposition` to each sub-system $R_i \wedge T_j \wedge V_k$ where $i \in \{1, 2, 3, 4, 5\}$, $j, k \in \{1, 2, 3, 4, 5, 6\}$ separately by using the function `Manipulate` and combine their solutions. There are a total of $5 \times 6 \times 6 = 180$ sub-systems in $R \wedge T \wedge V$.

```

1 Clear["Global`*"];
2 w = 1 + 2 a a a + b b b;
3 x = a + 2 a a b + b b c;
4 y = a a + 2 a b b + b c c;
5 z = a a a + 2 b b b + c c c;
6
7 w2 = w^3 + 2 x^3 + y^3;
8 x2 = w w x + 2 x x y + y y z;
9 y2 = w x x + 2 x y y + y z z;
10 z2 = x x x + 2 y y y + z z z;
11

```

```

12 tricon = a b c != 0;
13 (* R: G1 doesn't work on [1,a,b,c] *)
14 R1 = c == -1;
15 R2 = a b + c c + c + 1 == 0;
16 R3 = 2 a b + c c + 1 == 0;
17 R4 = 3 a b + c c - c + 1 == 0;
18 R5 = 4 a b + c c - 2 c + 1 == 0;
19 R = R1 || R2 || R3 || R4 || R5;
20
21 (* T: G1 doesn't work on [w,x,y,z] *)
22 T1 = z w + w w == 0;
23 T2 = x y + z z + z w + w w == 0;
24 T3 = 2 x y + z z + w w == 0;
25 T4 = 3 x y + z z - z w + w w == 0;
26 T5 = 4 x y + z z - 2 z w + w w == 0;
27 T6 = x y == w z;
28 T = T1 || T2 || T3 || T4 || T5 || T6;
29
30
31 (* S:[w,x,y,z] in P *)
32 S1 = x x == w y && y y == x z;
33 S2 = x == 0 && y == 0;
34 S3 = w == y && x == 0 && z == 0;
35 S4 = w + y == 0 && x == 0 && z == 0;
36 S5 = w == x && w + y == 0 && w + z == 0;
37 S6 = w + x == 0 && w + y == 0 && w == z;
38 S = S1 || S2 || S3 || S4 || S5 || S6;
39
40 (* U:[w2,x2,y2,z2] in P *)
41 U1 = x2 x2 == w2 y2 && y2 y2 == x2 z2;
42 U2 = x2 == 0 && y2 == 0;
43 U3 = w2 == y2 && x2 == 0 && z2 == 0;
44 U4 = w2 + y2 == 0 && x2 == 0 && z2 == 0;
45 U5 = w2 == x2 && w2 + y2 == 0 && w2 + z2 == 0;
46 U6 = w2 + x2 == 0 && w2 + y2 == 0 && w2 == z2;
47 U = U1 || U2 || U3 || U4 || U5 || U6;
48
49 CylindricalDecomposition[tricon && R && S, {a, b, c}]
50
51 CylindricalDecomposition[tricon && R && T && U, {a, b, c}]
52
53
54 (* Below, con1, con2, con3 corresponds to R, T, V in paper, \
55 respectively *)
56 Manipulate[
57   CylindricalDecomposition[
58     con1 && con2 && con3 && (a b c != 0), {a, b, c}], {con1, {c == -1,
59     a b + c c + c + 1 == 0, 2 a b + c c + 1 == 0,
60     3 a b + c c - c + 1 == 0,
61     4 a b + c c - 2 c + 1 == 0}}, {con2, {z w + w w == 0,
62     x y + z z + z w + w w == 0, 2 x y + z z + w w == 0,
63     3 x y + z z - z w + w w == 0, 4 x y + z z - 2 z w + w w == 0,
64     x y == w z}}, {con3, {z2 w2 + w2 w2 == 0,
65     x2 y2 + z2 z2 + z2 w2 + w2 w2 == 0, 2 x2 y2 + z2 z2 + w2 w2 == 0,
66     3 x2 y2 + z2 z2 - z2 w2 + w2 w2 == 0,
67     4 x2 y2 + z2 z2 - 2 z2 w2 + w2 w2 == 0, x2 y2 == w2 z2}}]

```