

NONCROSSING PARTITIONS FOR PERIODIC BRAIDS

EON-KYUNG LEE AND SANG-JIN LEE

ABSTRACT. An element in Artin's braid group B_n is called periodic if it has a power that lies in the center of B_n . The conjugacy problem for periodic braids can be reduced to the following: given a divisor $1 \leq d < n - 1$ of $n - 1$ and an element α in the super summit set of ε^d , find $\gamma \in B_n$ such that $\gamma^{-1}\alpha\gamma = \varepsilon^d$, where $\varepsilon = (\sigma_{n-1} \cdots \sigma_1)\sigma_1$.

In this article we characterize the elements in the super summit set of ε^d in the dual Garside structure by studying the combinatorics of noncrossing partitions arising from periodic braids. Our characterization directly provides a conjugating element γ . And it determines the size of the super summit set of ε^d by using the zeta polynomial of the noncrossing partition lattice.

Keywords: Braid group; dual Garside structure; noncrossing partition; conjugacy problem; periodic braid.

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1. INTRODUCTION

The conjugacy problem in a group has two versions: the conjugacy decision problem (CDP) is to decide whether two given elements are conjugate or not; the conjugacy search problem (CSP) is to find a conjugating element for a given pair of conjugate elements. The conjugacy problem is of great interest for Artin's braid group B_n , which has the well-known presentation [Art25]:

$$(1) \quad B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i - j| \geq 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i - j| = 1. \end{array} \right. \right\rangle.$$

The standard solutions to the conjugacy problem in B_n use Garside structures. The braid group B_n admits two standard Garside structures, the *classical Garside structure* [Gar69, EC+92, EM94] arising from Artin's presentation in (1), and the *dual Garside structure* [BKL98]. Both structures provide efficient solutions to the word problem.

For each element $\alpha \in B_n$, there is a *super summit set* $[\alpha]^S$, which depends on the choice of a particular Garside structure. Every super summit set is finite and nonempty. Two braids are conjugate if and only if their super summit sets are the same. Given a braid, one can compute an element in its super summit set in polynomial time. However, super summit sets are exponentially large at least in the braid index.

Let $\Delta = \sigma_1(\sigma_2\sigma_1) \cdots (\sigma_{n-1} \cdots \sigma_2\sigma_1)$, $\delta = \sigma_{n-1}\sigma_{n-2} \cdots \sigma_1$ and $\varepsilon = \delta\sigma_1$. See Figure 1. The center of B_n is the cyclic group generated by $\Delta^2 = \delta^n = \varepsilon^{n-1}$. An element of B_n is called *periodic* if some power of it lies in the center $\langle \Delta^2 \rangle$. The results of Brouwer [Bro19], Kerékjártó [Ker19] and Eilenberg [Eil34] imply that an n -braid is periodic if and only if it is conjugate to a power of either δ or ε .

The CDP for periodic braids is easy because an n -braid α is periodic if and only if either α^n or α^{n-1} belongs to $\langle \Delta^2 \rangle$ and because two periodic braids are conjugate if and only if

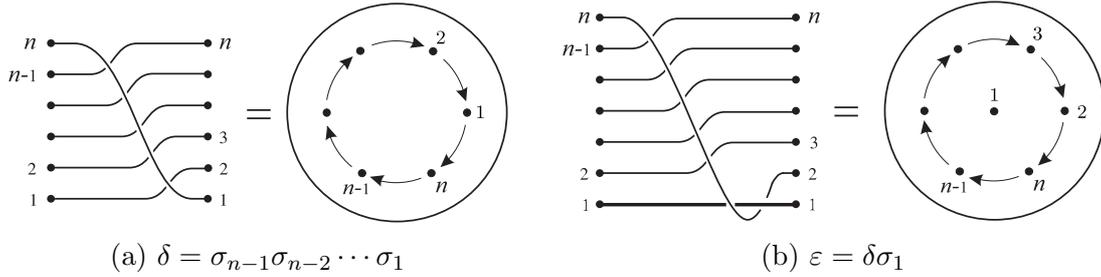


FIGURE 1. The braids δ and ε correspond to rigid rotations of a punctured disk when punctures are located as above.

they have the same exponent sum. The CSP for conjugates of δ^k , $k \in \mathbb{Z}$, is also easy because $[\delta^k]^S$ is the singleton $\{\delta^k\}$ in the dual Garside structure. Therefore the main task for solving the conjugacy problem for periodic braids is to solve the CSP for conjugates of ε^k .

The present work is motivated by the paper [BGG07b] of Birman, Gebhardt and González-Meneses, which provides a polynomial time solution to the conjugacy problems for periodic braids. Let us recall their results.

They first obtain a characterization of the elements in the super summit set $[\varepsilon]^S$ in the *classical* Garside structure. This characterization is fruitful. (i) It shows that the size of $[\varepsilon]^S$ is $(n-2)2^{n-3}$. Because it is exponential in the braid index n , the standard algorithms for the CSP (using the classical Garside structure) applied to periodic n -braids are exponential in n . (ii) It directly provides, given any element α of $[\varepsilon]^S$, an element $\gamma \in B_n$ such that $\gamma^{-1}\alpha\gamma = \varepsilon$. This conjugating element γ has canonical length at most 2. (We remark that the statements in [BGG07b] use ultra summit sets. For periodic braids, the ultra summit set and the super summit set coincide.)

However, it is hard to extend this method to ε^k for $k \geq 2$. They overcame this difficulty by using several known isomorphisms between certain subgroups of the braid groups. By converting the CSP for conjugates of ε^k in B_n to the CSP for conjugates of δ^k in B_{2n-2} , they gave a polynomial time solution to the CSP for conjugates of ε^k . In this case, a conjugating element from an element α of $[\varepsilon^k]^S$ to ε^k can be found only in an algorithmic way, not directly from α as in the case of ε .

As we have seen above, a nice characterization of the elements of $[\varepsilon^k]^S$ would give a better understanding of periodic braids. We ask the following.

Question. Is there a characterization of the elements of $[\varepsilon^k]^S$, which determines the size of $[\varepsilon^k]^S$ and directly provides a conjugating element from any element of $[\varepsilon^k]^S$ to ε^k ?

In this paper, we give an affirmative answer to this question. We give a characterization of the elements of $[\varepsilon^d]^S$ in the *dual* Garside structure, where d is a divisor of $n-1$.

Theorem 3.2. *Let B_n be endowed with the dual Garside structure. Let $n = rd + 1$ for integers $r \geq 2$ and $d \geq 1$. Let α be a 1-pure braid in the super summit set of ε^d in B_n . If $d = 1$, then $\alpha = \varepsilon$. If $d \geq 2$, then $\alpha = \delta^d a$ for a simple element a of the form $a = a_{d+1,1} b_0 b_1 \cdots b_{r-1}$, where*

- (1) each b_k is supported on $kd + \{2, 3, \dots, d+1\}$ for $0 \leq k \leq r-1$, and
- (2) $b_0 \tau^{-d}(b_1) \tau^{-2d}(b_2) \cdots \tau^{-(r-1)d}(b_{r-1}) = [d+1, \dots, 3, 2]$.

In this case, $c^{-1}\alpha c = \varepsilon^d$ for a simple element c given by

$$c = \tau^{-d}(b_1 b_2 \cdots b_{r-1}) \tau^{-2d}(b_2 b_3 \cdots b_{r-1}) \cdots \tau^{-(r-1)d}(b_{r-1}).$$

In the above theorem, the decomposition $a = a_{d+1,1} b_0 b_1 \cdots b_{r-1}$ is unique, and is easily obtained from any representation of a in terms of generators of the braid group.

The characterization in the theorem determines the size of $[\varepsilon^d]^S$ as $nZ_d(\frac{n-1}{d}) = n\binom{n-1}{d-1}/d$, where $Z_d(r) = \binom{rd}{d-1}/d$ is the zeta polynomial of the noncrossing partition lattice of $\{1, \dots, d\}$. (See Theorem 3.4.) The characterization directly provides a conjugating element from any element of $[\varepsilon^d]^S$ to ε^d . This conjugating element has canonical length at most 1.

We remark that the condition that d is a divisor of $n - 1$ is a mild restriction. Given any $k \in \mathbb{Z}$, let $d = \gcd(k, n - 1)$. Then ε^k and ε^d generate the same cyclic subgroup in the central quotient $B_n/\langle \Delta^2 \rangle$, hence the CSP for conjugates of ε^k is equivalent to the CSP for conjugates of ε^d . (See Theorem 2.11.)

We hope that the method developed in this paper will be generalized to other groups such as Artin groups of finite type and the braid groups of complex reflection groups.

2. PRELIMINARIES

In this section we recall the dual Garside structure on the braid group B_n and some known results on periodic braids.

2.1. The dual Garside structure. Birman, Ko and Lee [BKL98] introduced the following presentation for B_n :

$$B_n = \left\langle a_{ij}, 1 \leq j < i \leq n \left| \begin{array}{l} a_{kl}a_{ij} = a_{ij}a_{kl} \quad \text{if } (k-i)(k-j)(l-i)(l-j) > 0, \\ a_{ij}a_{jk} = a_{jk}a_{ik} = a_{ik}a_{ij} \quad \text{if } 1 \leq k < j < i \leq n. \end{array} \right. \right\rangle.$$

The generators a_{ij} are called *band generators*. They are related to the classical generators by $a_{ij} = \sigma_{i-1}\sigma_{i-2}\cdots\sigma_{j+1}\sigma_j\sigma_{j+1}^{-1}\cdots\sigma_{i-2}^{-1}\sigma_{i-1}^{-1}$. The *dual Garside structure* on B_n is the triple (B_n, B_n^+, δ) , where B_n^+ is the monoid consisting of the elements represented by positive words in the band generators, and $\delta = a_{n,n-1}a_{n-1,n-2}\cdots a_{32}a_{21}$ is the *Garside element*. (See [DP99, Deh02] for details of Garside groups.)

There is a partial order \preceq on B_n defined as follows: for $\alpha, \beta \in B_n$, $\alpha \preceq \beta$ if and only if $\alpha^{-1}\beta \in B_n^+$. Every pair of elements $\alpha, \beta \in B_n$ admits a unique lcm $\alpha \vee \beta$ and a unique gcd $\alpha \wedge \beta$ with respect to \preceq . The set $[1, \delta] = \{\alpha \in B_n : 1 \preceq \alpha \preceq \delta\}$ generates B_n . Its elements are called *simple elements*. The expression $(1, \delta)$ denotes the set $[1, \delta] \setminus \{1, \delta\}$.

For $\alpha \in B_n$, there are integers $r \leq s$ with $\delta^r \preceq \alpha \preceq \delta^s$. Hence the invariants $\inf(\alpha) = \max\{r \in \mathbb{Z} : \delta^r \preceq \alpha\}$, $\sup(\alpha) = \min\{s \in \mathbb{Z} : \alpha \preceq \delta^s\}$ and $\text{len}(\alpha) = \sup(\alpha) - \inf(\alpha)$ are well-defined. They are called the *infimum*, *supremum* and *canonical length* of α , respectively. There exists a unique expression

$$\alpha = \delta^r a_1 \cdots a_\ell$$

such that $r \in \mathbb{Z}$, $\ell \in \mathbb{Z}_{\geq 0}$, $a_1, \dots, a_\ell \in (1, \delta)$ and $(a_i a_{i+1}) \wedge \delta = a_i$ for $i = 1, \dots, \ell - 1$. It is called the *(left) normal form* of α . In this case, $\inf(\alpha) = r$ and $\sup(\alpha) = r + \ell$, so $\text{len}(\alpha) = \ell$.

Let $\tau : B_n \rightarrow B_n$ be the inner automorphism defined by $\tau(\alpha) = \delta^{-1}\alpha\delta$ for $\alpha \in B_n$. Then $\tau(a_{ij}) = a_{i+1, j+1}$, where indices are taken modulo n and $a_{ji} = a_{ij}$ for $i > j$. Hence

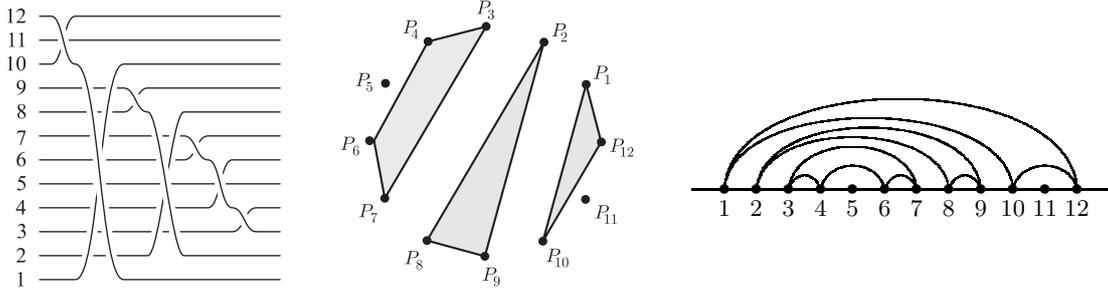


FIGURE 2. The picture on the left is a braid diagram of the simple element $[12, 10, 1][9, 8, 2][7, 6, 4, 3]$ in B_{12} and the other pictures illustrate the corresponding noncrossing partition in two distinct ways.

τ preserves the monoid B_n^+ and the set $[1, \delta]$ of simple elements. The *cycling* of $\alpha \in B_n$ is defined as $\mathbf{c}(\alpha) = \delta^r a_2 \cdots a_\ell \tau^{-r}(a_1)$.

We denote by $[\alpha]$ the conjugacy class $\{\gamma^{-1}\alpha\gamma : \gamma \in B_n\}$. Define $\inf_s(\alpha) = \max\{\inf(\beta) : \beta \in [\alpha]\}$, $\sup_s(\alpha) = \min\{\sup(\beta) : \beta \in [\alpha]\}$ and $\text{len}_s(\alpha) = \min\{\text{len}(\beta) : \beta \in [\alpha]\}$. The *super summit set* $[\alpha]^S$, the *ultra summit set* $[\alpha]^U$ and the *stable super summit set* $[\alpha]^{St}$ are subsets of the conjugacy class of α defined as follows:

$$\begin{aligned} [\alpha]^S &= \{\beta \in [\alpha] : \text{len}(\beta) = \text{len}_s(\alpha)\}; \\ [\alpha]^U &= \{\beta \in [\alpha]^S : \mathbf{c}^k(\beta) = \beta \text{ for some } k \geq 1\}; \\ [\alpha]^{St} &= \{\beta \in [\alpha]^S : \beta^k \in [\alpha^k]^S \text{ for all } k \in \mathbb{Z}\}. \end{aligned}$$

It is known that $[\alpha]^S$, $[\alpha]^U$ and $[\alpha]^{St}$ are finite and nonempty [EM94, Geb05, BGG07a, LL08]. It is also known that $\beta \in [\alpha]^S$ if and only if $\inf(\beta) = \inf_s(\alpha)$ and $\sup(\beta) = \sup_s(\alpha)$.

2.2. Simple elements. This subsection recalls some properties of simple elements in the dual Garside structure of B_n . See [BKL98, Bes03] for details.

Definition 2.1. For $T = \{i_1, \dots, i_k\}$ with $1 \leq i_1 < \dots < i_k \leq n$, the braid $a_{i_k i_{k-1}} \cdots a_{i_3 i_2} a_{i_2 i_1}$ is denoted by either b_T or $[i_k, \dots, i_1]$. Such a braid b_T is called a *subsimple*. When T is a singleton $\{i_1\}$, the braid b_T is defined to be the identity.

A partition of a set is a collection of pairwise disjoint subsets, called *blocks*, whose union is the entire set. A partition of $\{1, \dots, n\}$ is called a *noncrossing partition* if no two blocks cross each other, that is, if there is no quadruple (i_1, j_1, i_2, j_2) with $1 \leq i_1 < j_1 < i_2 < j_2 \leq n$ such that i_1 and i_2 belong to one block and j_1 and j_2 belong to another. Equivalently, if P_1, \dots, P_n are vertices of a regular n -gon as in the middle picture of Figure 2, then the convex hulls of the respective blocks are pairwise disjoint.

Theorem 2.2 (Theorem 3.4 of [BKL98]). *A braid $a \in B_n$ belongs to $[1, \delta]$ if and only if $a = b_{T_1} \cdots b_{T_k}$ for some noncrossing partition $\{T_1, \dots, T_k\}$ of $\{1, \dots, n\}$.*

In the above theorem, for $a \in [1, \delta]$, the decomposition $a = b_{T_1} \cdots b_{T_k}$ is unique up to reordering the factors because b_{T_i} and b_{T_j} commute for all i, j . Each factor b_{T_i} is called a *subsimple of a* .

Definition 2.3. For a subsimple $b = [i_k, \dots, i_1]$ with $k \geq 2$, the set $\{i_1, \dots, i_k\}$ is called the *support* of b , denoted by $\text{supp}(b)$. We say that a simple element a is *supported on* $S \subseteq \{1, \dots, n\}$ if $\text{supp}(b) \subseteq S$ for each nonidentity subsimple b of a .

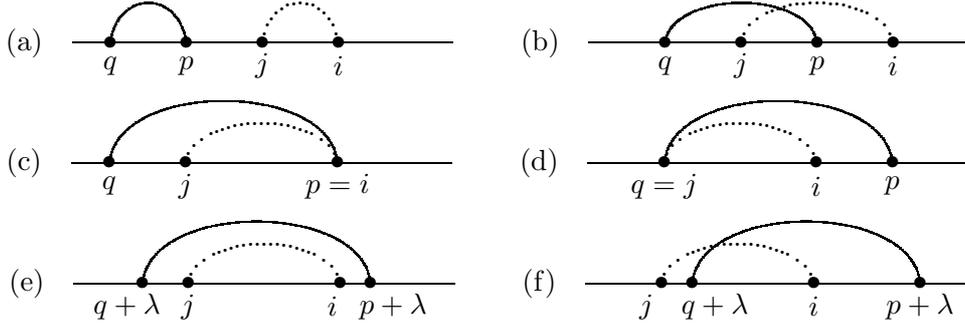


FIGURE 3. The pictures (a) and (c) show the case where $a_{pq}a_{ij}$ is a simple element and the pictures (b) and (d) show the other case. The pictures (e) and (f) show the arcs between $(p + \lambda, 0)$ and $(q + \lambda, 0)$ and between $(i, 0)$ and $(j, 0)$.

Example 2.4. Figure 2 illustrates the noncrossing partition

$$\{ \{12, 10, 1\}, \{9, 8, 2\}, \{7, 6, 4, 3\}, \{11\}, \{5\} \}.$$

The corresponding simple element is $a = [12, 10, 1][9, 8, 2][7, 6, 4, 3]$ in B_{12} . Thus a consists of three subsimples $[12, 10, 1]$, $[9, 8, 2]$ and $[7, 6, 4, 3]$, and it is supported on $\{1, \dots, 12\} \setminus \{5, 11\}$.

For a noncrossing partition $\mathcal{T} = \{T_1, \dots, T_k\}$ of $\{1, \dots, n\}$, we denote by $b_{\mathcal{T}}$ the braid $b_{T_1} \cdots b_{T_k}$. Because two braids b_{T_i} and b_{T_j} commute for $i \neq j$, the product $b_{T_1} \cdots b_{T_k}$ is well-defined.

By Theorem 2.2, the set of simple elements in the dual Garside structure of B_n is in one-to-one correspondence with the set of noncrossing partitions of $\{1, \dots, n\}$, where the nonidentity subsimples correspond to the blocks containing at least two numbers. For partitions \mathcal{T} and \mathcal{T}' of $\{1, \dots, n\}$, it is known that $b_{\mathcal{T}} \preceq b_{\mathcal{T}'}$ if and only if \mathcal{T} is a refinement of \mathcal{T}' , i.e. each block of \mathcal{T} is a subset of some block of \mathcal{T}' .

Lemma 2.5. *Let b be a subsimple.*

- (1) $a_{ij} \preceq b$ if and only if $\{i, j\} \subseteq \text{supp}(b)$.
- (2) For a simple element a , $a \preceq b$ if and only if a is supported on $\text{supp}(b)$.

The following lemmas provide easy ways to check that a given positive braid is a simple element.

Lemma 2.6 (Lemma 3.3 and Corollary 3.6 in [BKL98]). *$a_{pq}a_{ij}$ is a simple element if and only if, for $0 < \lambda < 1$, we can connect $(p + \lambda, 0)$ to $(q + \lambda, 0)$ and $(i, 0)$ to $(j, 0)$ by nonintersecting arcs in the upper half plane $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. (See Figure 3.)*

Lemma 2.7 (Lemma 3.3 and Corollary 3.6 in [BKL98]). *Let $a = x_1 \cdots x_k \in B_n^+$ for some band generators x_i . Then a is a simple element if and only if $x_i x_j$ is a simple element for all $1 \leq i < j \leq k$.*

An immediate consequence of the above lemma is the following.

Lemma 2.8. *If $a_1 b_1 a_2 b_2 \cdots a_k b_k$ is a simple element with $a_i, b_i \in B_n^+$ for $1 \leq i \leq k$, then $a_1 a_2 \cdots a_k$ is a simple element.*

2.3. Periodic braids. This subsection recalls some results on periodic braids in the dual Garside structure of B_n from [LL11].

Lemma 2.9 (Lemma 3.6 in [LL11]). *If $\alpha \in B_n$ is periodic, then $\text{len}_s(\alpha) \in \{0, 1\}$.*

By the above lemma, if α is a periodic braid then $[\alpha]^S = [\alpha]^U$.

The notion of cycling can be generalized as follows.

Definition 2.10. Let $\alpha = \delta^u a_1 a_2 \cdots a_\ell \in B_n$ be in normal form. Let a'_1 and a''_1 be simple elements such that $a_1 = a'_1 a''_1$. The conjugation

$$\tau^{-u}(a'_1)^{-1} \alpha \tau^{-u}(a'_1) = \delta^u a''_1 a_2 \cdots a_\ell \tau^{-u}(a'_1)$$

is called a *partial cycling* of α by a'_1 .

The following theorem shows that the conjugacy problem for conjugates of ε^k is equivalent to that for conjugates of ε^d , where $d = \gcd(k, n-1)$, and then it lists some properties of $[\varepsilon^d]^S$. It comes from Lemmas 3.18, 3.21 and Theorem 3.24 in [LL11].

Theorem 2.11. *Let B_n be endowed with the dual Garside structure.*

- (1) *Let $k \neq 0$. Suppose d, p and q are integers such that $d = \gcd(k, n-1)$ and $(n-1)p + kq = d$. Let $\alpha \in B_n$ and $\alpha' = \delta^{np} \alpha^q$. Then α is conjugate to ε^k if and only if α' is conjugate to ε^d . Moreover, for $\gamma \in B_n$, $\gamma^{-1} \alpha \gamma = \varepsilon^k$ if and only if $\gamma^{-1} \alpha' \gamma = \varepsilon^d$.*
- (2) *Let $1 \leq d < n-1$ be a divisor of $n-1$, hence $n = rd + 1$ for some $r \geq 2$.*
 - (a) $[\varepsilon^d]^S = [\varepsilon^d]^U = [\varepsilon^d]^{St}$;
 - (b) $[\varepsilon^d]^S$ is closed under partial cyclings;
 - (c) every element of $[\varepsilon^d]^S$ is of the form $\delta^d a$ for some $a \in (1, \delta)$ satisfying $a \tau^{-d}(a) \tau^{-2d}(a) \cdots \tau^{-(r-1)d}(a) = \delta$.

3. THE SUPER SUMMIT SET OF ε^d

Throughout this section, we assume that the braid group B_n is endowed with the dual Garside structure and that $1 \leq d < n-1$ is a divisor of $n-1$, hence $n = dr + 1$ for some $r \geq 2$.

Definition 3.1. For a braid $\alpha \in B_n$, let π_α denote the induced permutation of α . For $1 \leq i \leq n$, a braid $\alpha \in B_n$ is called *i -pure* if $\pi_\alpha(i) = i$.

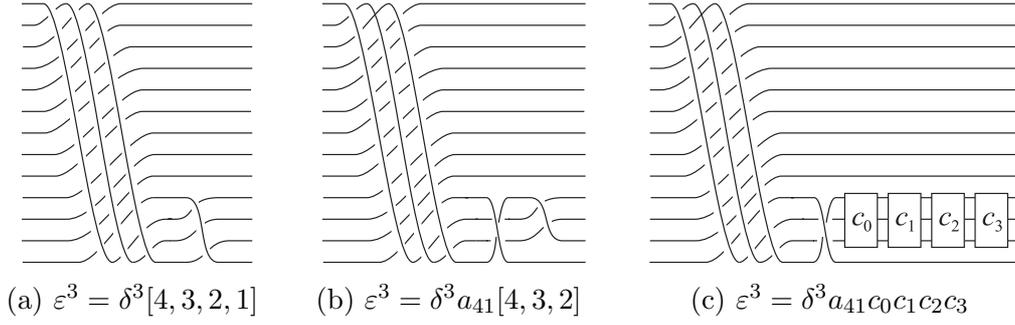
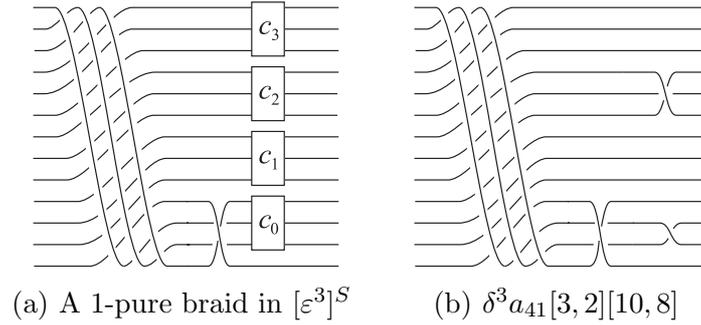
For example, $\pi_\delta(i) = i-1 \pmod{n}$ for $1 \leq i \leq n$. A noncentral ε^k is i -pure if and only if $i = 1$.

3.1. Typical super summit elements of ε^d . Let us consider typical elements of $[\varepsilon^d]^S$. Recall that $n = rd + 1$ with $r \geq 2$ and $d \geq 1$. Suppose $d \neq 1$.

- (1) Notice that $\varepsilon^d = \delta^d[d+1, \dots, 2, 1] = \delta^d a_{d+1,1}[d+1, \dots, 2]$. Let c_0, c_1, \dots, c_{r-1} be simple elements such that

$$[d+1, \dots, 2] = c_0 c_1 \cdots c_{r-1}.$$

See Figure 4, where $n = 13$, $d = 3$ and $r = 4$. In this case, the simple elements c_i are 13-braids supported on the set $\{2, 3, 4\}$. Each $\boxed{c_i}$ in Figure 4 (and also in Figures 5 and 7 later) means the 3-braid obtained from c_i by deleting all the j -th strands for $j \notin \{2, 3, 4\}$.


 FIGURE 4. Braid diagrams of ε^3 in B_{13}

 FIGURE 5. Elements of $[\varepsilon^3]^S$ in B_{13} .

(2) $\varepsilon^d = \delta^d a_{d+1,1} c_0 c_1 \cdots c_{r-1}$ is conjugate to

$$\alpha = \delta^d a_{d+1,1} b_0 b_1 \cdots b_{r-1},$$

where $b_k = \tau^{kd}(c_k)$ for $0 \leq k \leq r-1$. See Figure 5(a). If we let

$$x = \tau^{-d}(b_1 b_2 \cdots b_{r-1}) \tau^{-2d}(b_2 b_3 \cdots b_{r-1}) \cdots \tau^{-(r-1)d}(b_{r-1}),$$

then $x^{-1} \alpha x = \varepsilon^d$. Note that each b_k is supported on $\{kd+2, \dots, kd+(d+1)\}$. For an example, see Figure 5(b), where $c_0 = [3, 2]$, $c_1 = c_3 = 1$, $c_2 = [4, 2]$, hence $b_0 = [3, 2]$, $b_1 = b_3 = 1$, $b_2 = [10, 8]$.

(3) Notice that $\alpha \in [\varepsilon^d]^S$ because $\text{len}(\alpha) = 1 = \text{len}_s(\varepsilon^d)$.

3.2. Super summit elements of ε^d . In this subsection, we give a characterization of the super summit elements of ε^d . For $0 \leq k \leq r-1$, define

$$S_k = kd + \{2, 3, \dots, d+1\} = \{kd+2, kd+3, \dots, kd+(d+1)\}.$$

Then $\{\{1\}, S_0, S_1, \dots, S_{r-1}\}$ is a partition of $\{1, \dots, n\}$. The braid α in §3.1 is 1-pure such that each b_k is supported on S_k .

The following is the main theorem of this paper, which shows that every 1-pure super summit element of ε^d can be obtained using the construction in §3.1.

Theorem 3.2. *Let B_n be endowed with the dual Garside structure. Let $n = rd + 1$ for integers $r \geq 2$ and $d \geq 1$. Let α be a 1-pure braid in the super summit set of ε^d in B_n . If $d = 1$, then $\alpha = \varepsilon$. If $d \geq 2$, then $\alpha = \delta^d a$ for a simple element a of the form $a = a_{d+1,1} b_0 b_1 \cdots b_{r-1}$, where*

- (1) each b_k is supported on $kd + \{2, 3, \dots, d+1\}$ for $0 \leq k \leq r-1$, and
- (2) $b_0 \tau^{-d}(b_1) \tau^{-2d}(b_2) \cdots \tau^{-(r-1)d}(b_{r-1}) = [d+1, \dots, 3, 2]$.

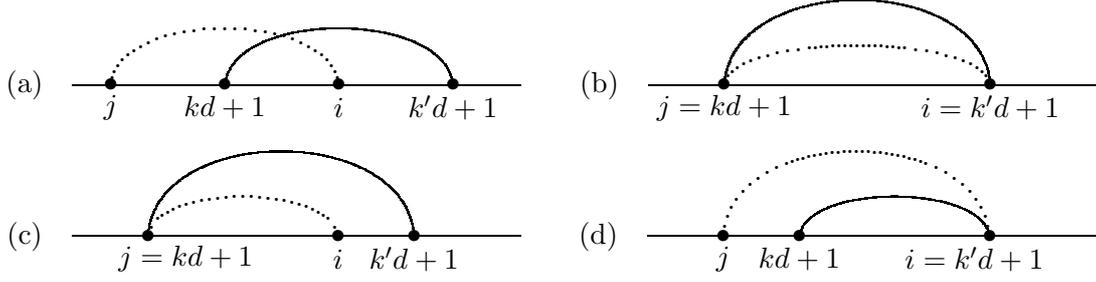


FIGURE 6. $a_{k'd+1, kd+1} a_{ij}$ is not a simple element in these four cases.

In this case, $c^{-1}\alpha c = \varepsilon^d$ for a simple element c given by

$$c = \tau^{-d}(b_1 b_2 \cdots b_{r-1}) \tau^{-2d}(b_2 b_3 \cdots b_{r-1}) \cdots \tau^{-(r-1)d}(b_{r-1}).$$

Proof. By Theorem 2.11, $\alpha = \delta^d a$ for some $a \in (1, \delta)$.

Claim 1. $a_{d+1,1} \preccurlyeq a$

Proof of Claim 1. Since $\alpha = \delta^d a$ is 1-pure and $1 = \pi_\alpha(1) = (\pi_{\delta^d} \circ \pi_a)(1) = \pi_a(1) - d$, we have $\pi_a(1) = d+1$. Hence there exists a subsimple b of a of the form $b = [d+1, i_k, \dots, i_1, 1]$. Since $[d+1, i_k, \dots, i_1, 1] = a_{d+1,1}[d+1, i_k, \dots, i_1]$, we have $a_{d+1,1} \preccurlyeq b \preccurlyeq a$. \square

By the above claim, $a = a_{d+1,1} a'$ for a simple element a' .

If $d = 1$, then $\alpha = \varepsilon$ because $\varepsilon = \delta a_{21}$ and $\alpha = \delta a_{21} a'$ have the same exponent sum.

Now we assume $d \geq 2$.

Claim 2. $[rd+1, (r-1)d+1, \dots, d+1, 1] a'$ is a simple element.

Proof of Claim 2. By Theorem 2.11, $a \tau^{-d}(a) \cdots \tau^{-d(r-1)}(a) = \delta$. Applying $\tau^{(r-1)d}$ to both sides, we have

$$\tau^{(r-1)d}(a) \cdots \tau^d(a) a = \delta.$$

Because $\tau^k(a) = \tau^k(a_{d+1,1}) \tau^k(a')$ for each k , the braid $\tau^{(r-1)d}(a_{d+1,1}) \cdots \tau^d(a_{d+1,1}) a_{d+1,1} a'$ is a simple element by Lemma 2.8. Since

$$\begin{aligned} & \tau^{(r-1)d}(a_{d+1,1}) \cdots \tau^d(a_{d+1,1}) a_{d+1,1} \\ &= a_{rd+1, (r-1)d+1} \cdots a_{2d+1, d+1} a_{d+1,1} \\ &= [rd+1, (r-1)d+1, \dots, d+1, 1], \end{aligned}$$

we are done. \square

Claim 3. Each subsimple of a' is supported on S_k for some $0 \leq k \leq r-1$.

Proof of Claim 3. Assume not. By Lemma 2.5, there is a generator $a_{ij} \preccurlyeq a'$ such that $\{i, j\}$ is not contained in any S_k . Then there exist $0 \leq k < k' \leq r$ such that

$$j \leq kd+1 < i \leq k'd+1.$$

There are four cases: (a) $j < kd+1 < i < k'd+1$; (b) $j = kd+1 < i = k'd+1$; (c) $j = kd+1 < i < k'd+1$; (d) $j < kd+1 < i = k'd+1$. In any case, $a_{k'd+1, kd+1} a_{ij}$ is not a simple element by Lemma 2.6. See Figure 6.

On the other hand, $a_{k'd+1, kd+1} \preccurlyeq [rd+1, (r-1)d+1, \dots, d+1, 1]$ by Lemma 2.5. Hence $a_{k'd+1, kd+1} a_{ij}$ is a simple element by Claim 2 and Lemma 2.8, which is a contradiction. \square

For $0 \leq k \leq r-1$, let b_k be the product of subsimples of a' which are supported on S_k . (Hence each b_k is a simple element supported on S_k .) By the above claim, we have

$$a = a_{d+1,1}b_0b_1 \cdots b_{r-1}.$$

Claim 4. $c = \tau^{-d}(b_1b_2 \cdots b_{r-1})\tau^{-2d}(b_2b_3 \cdots b_{r-1}) \cdots \tau^{-(r-1)d}(b_{r-1})$ is a simple element.

Proof of Claim 4. In the proof of Claim 2, we have seen $a\tau^{-d}(a) \cdots \tau^{-(r-1)d}(a) = \delta$. Then c is a simple element by Lemma 2.8 because $b_k b_{k+1} \cdots b_{r-1} \preceq a$ for $1 \leq k \leq r-1$. \square

We remark that $a_{d+1,1}b_0\tau^{-d}(b_1) \cdots \tau^{-kd}(b_k)$ is a simple element for $0 \leq k \leq r-1$ by the same reason as above.

Claim 5. $a_{d+1,1}b_0\tau^{-d}(b_1)\tau^{-2d}(b_2) \cdots \tau^{-(r-1)d}(b_{r-1}) = [d+1, \dots, 2, 1]$ and $c^{-1}\alpha c = \varepsilon^d$.

Proof of Claim 5. Let $x_k = \tau^{-kd}(b_k b_{k+1} \cdots b_{r-1})$ and $\alpha_k = (x_1 \cdots x_k)^{-1}\alpha(x_1 \cdots x_k)$ for $1 \leq k \leq r-1$. Then $c = x_1 x_2 \cdots x_{r-1}$.

Since each b_k is supported on S_k for $1 \leq k \leq r-1$, it commutes with $a_{d+1,1}b_0$. Since

$$\begin{aligned} \alpha &= \delta^d(a_{d+1,1}b_0) \cdot (b_1 \cdots b_{r-1}) = \delta^d(b_1 \cdots b_{r-1}) \cdot (a_{d+1,1}b_0) \\ &= \tau^{-d}(b_1 \cdots b_{r-1}) \cdot \delta^d a_{d+1,1}b_0 = x_1 \cdot \delta^d a_{d+1,1}b_0, \end{aligned}$$

we have

$$\begin{aligned} \alpha_1 &= x_1^{-1}\alpha x_1 = \delta^d a_{d+1,1}b_0 x_1 = \delta^d a_{d+1,1}b_0 \tau^{-d}(b_1 \cdots b_{r-1}) \\ &= \delta^d a_{d+1,1}b_0 \tau^{-d}(b_1) \cdot \tau^{-d}(b_2 \cdots b_{r-1}). \end{aligned}$$

Notice that $a_{d+1,1}b_0\tau^{-d}(b_1)$ is supported on $\{1, 2, \dots, d+1\}$ and $\tau^{-d}(b_2 \cdots b_{r-1})$ is supported on $S_1 \cup \cdots \cup S_{r-2}$. Therefore they commute. By a similar computation as above,

$$\alpha_2 = x_2^{-1}\alpha_1 x_2 = \delta^d a_{d+1,1}b_0 \tau^{-d}(b_1)\tau^{-2d}(b_2) \cdot \tau^{-2d}(b_3 \cdots b_{r-1}).$$

Continuing this computation, we obtain

$$\alpha_{r-1} = x_{r-1}^{-1}\alpha_{r-2}x_{r-1} = \delta^d a_{d+1,1}b_0 \tau^{-d}(b_1)\tau^{-2d}(b_2) \cdots \tau^{-(r-1)d}(b_{r-1}).$$

Write $b = a_{d+1,1}b_0\tau^{-d}(b_1)\tau^{-2d}(b_2) \cdots \tau^{-(r-1)d}(b_{r-1})$. Notice that b is a simple element supported on $\{1, 2, \dots, d+1\}$, hence $b \preceq [d+1, \dots, 2, 1]$ by Lemma 2.5. Because both elements have the same exponent sum, they must be equal, i.e. $b = [d+1, \dots, 2, 1]$. Hence $\alpha_{r-1} = \delta^d[d+1, \dots, 2, 1] = \varepsilon^d$. Since $\alpha_{r-1} = (x_{r-1}^{-1} \cdots x_1^{-1})\alpha(x_1 \cdots x_{r-1}) = c^{-1}\alpha c$, we are done. \square

Claims 1–5 complete the proof of Theorem 3.2. \square

Every element of $[\varepsilon^d]^S$ can be conjugated to a 1-pure braid in $[\varepsilon^d]^S$ by δ^u for some $0 \leq u \leq n-1$. By Theorem 3.2 any 1-pure braid of $[\varepsilon^d]^S$ can be conjugated to ε^d by a simple element c . Therefore every element of $[\varepsilon^d]^S$ can be conjugated to ε^d by $\delta^u c$ for some $0 \leq u \leq n-1$ and a simple element c .

3.3. The size of the super summit set of ε^d . Using the characterization of the elements of $[\varepsilon^d]^S$ in Theorem 3.2, we can figure out the size of $[\varepsilon^d]^S$.

Let $Z_d(r)$ be the number of multi-chains $1 \preceq a_1 \preceq a_2 \preceq \cdots \preceq a_{r-1} \preceq [d, d-1, \dots, 1]$. $Z_d(r)$ is called the *zeta polynomial of the noncrossing partition lattice* of $\{1, 2, \dots, d\}$ [Ede80].

Theorem 3.3 (Edelman [Ede80]). $Z_d(r) = \binom{dr}{d-1}/d$ for $d, r \geq 1$.

An r -tuple (c_0, \dots, c_{r-1}) of simple elements in B_n is called an r -composition of a simple element a if $a = c_0 \cdots c_{r-1}$.

There is a one-to-one correspondence between the set of r -compositions of $[d, d-1, \dots, 1]$ and the set of multi-chains $1 \preceq a_1 \preceq a_2 \preceq \cdots \preceq a_{r-1} \preceq [d, d-1, \dots, 1]$. For an r -composition (c_0, \dots, c_{r-1}) , let $a_k = c_0 c_1 \cdots c_{k-1}$ for $1 \leq k \leq r-1$. Then (a_1, \dots, a_{r-1}) is a multi-chain. Conversely, for a multi-chain (a_1, \dots, a_{r-1}) , let $c_k = a_k^{-1} a_{k+1}$ for $0 \leq k \leq r-1$ where $a_0 = 1$ and $a_r = [d, d-1, \dots, 1]$. Then $(c_0, c_1, \dots, c_{r-1})$ is an r -composition.

Theorem 3.4. *Let $n = rd + 1$ for $r \geq 2$ and $d \geq 1$. The cardinality of $[\varepsilon^d]^S$ is $nZ_d(r) = n \binom{dr}{d-1} / d = n \binom{n-1}{d-1} / d$.*

Proof. It suffices to show that $Z_d(r)$ is equal to the number of 1-pure braids in $[\varepsilon^d]^S$ because any element of $[\varepsilon^d]^S$ is uniquely expressed as $\tau^u(\alpha)$ for $0 \leq u \leq n-1$ and a 1-pure braid α in $[\varepsilon^d]^S$.

If $d = 1$, then $Z_d(r) = 1$ and ε is the only 1-pure braid in $[\varepsilon]^S$ by Theorem 3.2. Now we assume $d \geq 2$. Notice that there is a one-to-one correspondence between r -compositions of $[d, d-1, \dots, 1]$ and those of $[d+1, d, \dots, 2]$.

Given an r -composition $(c_0, c_1, \dots, c_{r-1})$ of $[d+1, d, \dots, 2]$, let

$$\alpha = \delta^d a_{d+1,1} b_0 b_1 \cdots b_{r-1},$$

where $b_k = \tau^{kd}(c_k)$ for $0 \leq k \leq r-1$. Then α is a 1-pure braid in $[\varepsilon^d]^S$ as we have seen in §3.1. Combining this observation with Theorem 3.2, we can see that there is a one-to-one correspondence between the set of 1-pure braids in $[\varepsilon^d]^S$ and the set of r -compositions of $[d, d-1, \dots, 1]$. Hence $Z_d(r)$ is equal to the number of 1-pure braids in $[\varepsilon^d]^S$. \square

3.4. The stable super summit sets of ε^d and ε^k . The CSP for conjugates of ε^k is equivalent to the CSP for conjugates of ε^d for $d = \gcd(k, n-1)$. (See Theorem 2.11.) However, their super summit sets look different. For example, ε^k does not have such a nice characterization of the super summit elements as ε^d .

For the study of conjugacy classes of periodic braids, the stable super summit set looks more natural than the super summit set. In Proposition 3.5 we will see that the stable super summit set of ε^k is in one-to-one correspondence with that of ε^d . Moreover, the correspondence is given by taking powers and by multiplying by central elements. In Corollary 3.7 we will see that all the 1-pure braids in $[\varepsilon^k]^{St}$ have a common reduction system when $d \geq 2$.

Proposition 3.5. *For an integer k , let $d = \gcd(k, n-1)$ and $(n-1)p + kq = d$. Let $f : [\varepsilon^d] \rightarrow [\varepsilon^k]$ and $g : [\varepsilon^k] \rightarrow [\varepsilon^d]$ be the maps defined by*

$$f(\alpha) = \alpha^{k/d} \quad \text{and} \quad g(\beta) = \delta^{np} \beta^q,$$

where $[\varepsilon^d]$ and $[\varepsilon^k]$ denote the conjugacy classes of ε^d and ε^k , respectively.

Then f and g are inverses to each other such that $f([\varepsilon^d]^{St}) = [\varepsilon^k]^{St}$ and $g([\varepsilon^k]^{St}) = [\varepsilon^d]^{St}$. In particular,

$$\#[\varepsilon^d]^S = \#[\varepsilon^d]^{St} = \#[\varepsilon^k]^{St} \leq \#[\varepsilon^k]^S.$$

Proof. First, observe that f and g are inverses to each other. Let α be conjugate to ε^d . Then $\alpha^{(n-1)/d} = \delta^n$ because $\alpha^{(n-1)/d}$ is conjugate to $(\varepsilon^d)^{(n-1)/d} = \varepsilon^{n-1} = \delta^n$ and δ^n is central. Hence

$$g(f(\alpha)) = \delta^{np} (\alpha^{k/d})^q = \alpha^{\frac{p(n-1)}{d}} \alpha^{\frac{kq}{d}} = \alpha^{\frac{(n-1)p+kq}{d}} = \alpha.$$

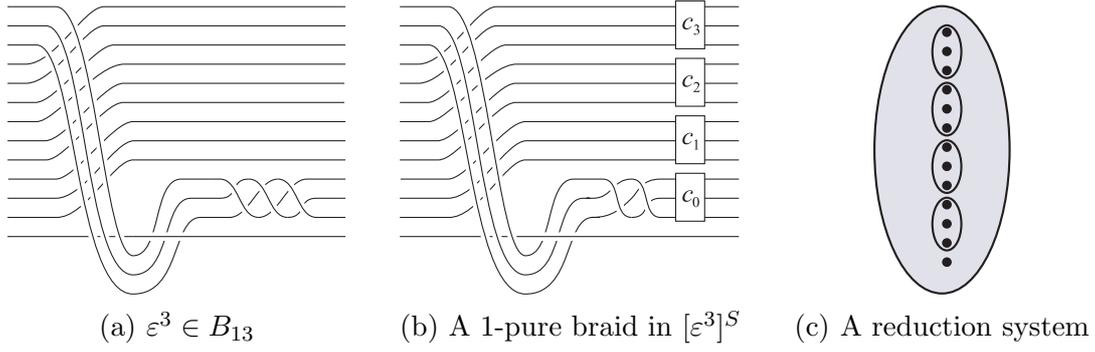


FIGURE 7. The braid diagrams (a) and (b) show that 1-pure braids in $[\varepsilon^3]^S = [\varepsilon^3]^{St}$ are reducible with a *round* reduction system shown in (c).

Let β be conjugate to ε^k . Then $\beta^{(n-1)/d} = \delta^{kn/d}$ because $\beta^{(n-1)/d}$ is conjugate to $\varepsilon^{k(n-1)/d} = (\varepsilon^{n-1})^{k/d} = (\delta^n)^{k/d}$ and δ^n is central. Hence

$$f(g(\beta)) = (\delta^{np} \beta^q)^{\frac{k}{d}} = \delta^{\frac{kn p}{d}} \beta^{\frac{k q}{d}} = \beta^{\frac{(n-1)p}{d}} \beta^{\frac{k q}{d}} = \beta^{\frac{(n-1)p + k q}{d}} = \beta.$$

Notice that, for $\alpha \in B_n$, if $\beta \in [\alpha]^{St}$, then $\beta^m \in [\alpha^m]^{St}$ and $\delta^{mn} \beta \in [\delta^{mn} \alpha]^{St}$ for every $m \in \mathbb{Z}$. Thus $f([\varepsilon^d]^{St}) \subseteq [\varepsilon^k]^{St}$ and $g([\varepsilon^k]^{St}) \subseteq [\varepsilon^d]^{St}$. Then it follows that $f([\varepsilon^d]^{St}) = [\varepsilon^k]^{St}$ and $g([\varepsilon^k]^{St}) = [\varepsilon^d]^{St}$ because $f \circ g$ and $g \circ f$ are the identity functions on the conjugacy classes $[\varepsilon^k]$ and $[\varepsilon^d]$, respectively. In particular, $\#[\varepsilon^d]^{St} = \#[\varepsilon^k]^{St}$. Because $[\varepsilon^d]^S = [\varepsilon^d]^{St}$ by Theorem 2.11 and $[\varepsilon^k]^{St} \subseteq [\varepsilon^k]^S$ by definition, we are done. \square

Example 3.6. We give an example showing that $[\varepsilon^k]^S$ can be strictly larger than $[\varepsilon^k]^{St}$. Consider $\varepsilon = \delta[2, 1]$ and $\varepsilon^2 = \delta^2[3, 2, 1] = \delta^2[2, 1][3, 1]$ in B_6 . Let

$$\alpha = [3, 1]\varepsilon^2[3, 1]^{-1} = \delta^2[5, 3][2, 1].$$

Then $\alpha \in [\varepsilon^2]^S$ as $\text{len}(\alpha) = 1 = \text{len}_s(\varepsilon^2)$. But $\alpha^2 = \delta^4[5, 4, 3, 1] \cdot [2, 1] \notin [\varepsilon^4]^{St}$ as $\text{len}(\alpha^2) = 2 \neq 1 = \text{len}_s(\varepsilon^4)$. Therefore $\alpha \in [\varepsilon^2]^S \setminus [\varepsilon^2]^{St}$.

Let $D_n \subseteq \mathbb{C}$ be the closed disk of radius $n + 1$, centered at the origin, with punctures at $\{1, 2, \dots, n\}$. The n -braids are identified with the mapping classes of D_n . A braid α is called *reducible* if it preserves setwise a family of nondegenerate simple closed curves, called a *reduction system* for α . A reduction system is called *round* if each component is homotopic to a geometric circle in D_n .

If $d = \gcd(k, n - 1) \geq 2$, then the periodic braids ε^d and ε^k are reducible with a round reduction system. See Figure 7(a) for a braid diagram of $\varepsilon^3 \in B_{13}$. Figure 7(c) illustrates a round reduction system for ε^3 . Figure 7(b) is another braid diagram for the 1-pure braid of $[\varepsilon^3]^S$ in Figure 5(a), from which we can see that Figure 7(c) is also a reduction system for this braid. This happens for any 1-pure braid of $[\varepsilon^k]^{St}$ by Theorem 3.2 and Proposition 3.5.

Corollary 3.7. *If $d = \gcd(k, n - 1) \geq 2$, then every 1-pure braid in $[\varepsilon^k]^{St}$ has a round reduction system, consisting of $(n - 1)/d$ circles each of which encloses d punctures, as in Figure 7(c).*

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