FAMILIES OF LATTICE POLYTOPES OF MIXED DEGREE ONE

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Abstract. It has been shown by Soprunov that the normalized mixed volume (minus one) of an n-tuple of n-dimensional lattice polytopes is a lower bound for the number of interior lattice points in the Minkowski sum of the polytopes. He defined n-tuples of mixed degree at most one to be exactly those for which this lower bound is attained with equality, and posed the problem of a classification of such tuples. We give a finiteness result regarding this problem in general dimension $n \geq 4$, showing that all but finitely many n-tuples of mixed degree at most one admit a common lattice projection onto the unimodular simplex Δ_{n-1}. Furthermore, we give a complete solution in dimension $n = 3$. In the course of this we show that our finiteness result does not extend to dimension $n = 3$, as we describe infinite families of triples of mixed degree one not admitting a common lattice projection onto the unimodular triangle Δ_2.

1. Introduction

1.1. **Basic definitions.** A *lattice polytope* $P \subset \mathbb{R}^n$ is a polytope $P \subset \mathbb{R}^n$ whose vertices are elements of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. We call two lattice polytopes $P_1, P_2 \subset \mathbb{R}^n$ *equivalent* if there exists an affine lattice-preserving transformation $U : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $U(P_1) = P_2$. We say that two n-tuples $P_1, \ldots, P_n \subset \mathbb{R}^n$ and $Q_1, \ldots, Q_n \subset \mathbb{R}^n$ are *equivalent* if there is a permutation $\sigma \in S_n$, an affine lattice-preserving transformation $U : \mathbb{R}^n \to \mathbb{R}^n$ and vectors $t_1, \ldots, t_n \in \mathbb{Z}^n$ such that $(P_1, \ldots, P_n) = (U(Q_{\sigma(1)}) + t_1), \ldots, U(Q_{\sigma(n)}) + t_n))$. We denote by $\Delta_n = \mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n) \subset \mathbb{R}^n$ the *standard unimodular simplex* in $\mathbb{R}^n$ and call an n-dimensional simplex *unimodular* if it is equivalent to Δ_n. We write the *Minkowski sum* of two lattice polytopes $P_1, P_2 \subset \mathbb{R}^n$ as $P_1 + P_2 = \{p_1 + p_2 : p_1 \in P_1, p_2 \in P_2\} \subset \mathbb{R}^n$ and denote the *interior* of a lattice polytope $P \subset \mathbb{R}^n$ by P°. If one has $P^\circ \cap \mathbb{Z}^n = \emptyset$, we call the lattice polytope $P \subset \mathbb{R}^n$ *hollow*.

1.2. **Motivation.** In order to give an explicit definition, let us define the (normalized) *mixed volume* of an n-tuple of polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$ via the inclusion-exclusion formula given by $\mathrm{MV}(P_1, \ldots, P_n) := \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{n-|I|} \mathrm{vol}_n(\sum_{i \in I} P_i)$, where vol_n denotes the standard euclidean volume in $\mathbb{R}^n$. Note that there are various equivalent definitions for the mixed volume of an n-tuple of lattice polytopes or, more generally, for an n-tuple of convex bodies in $\mathbb{R}^n$ (see e.g. [Sch14] or [EG15]) and that in our definition the mixed volume is normalized such that $\mathrm{MV}(\Delta_n, \ldots, \Delta_n) = 1$. A central connection of the mixed volume to algebraic geometry is given by the famous Bernstein-Kouchnirenko-Khovanskii theorem ([Ber75]). Combining this theorem with a generalization of the Euler-Jacobi theorem due to Khovanskii ([Kho78]) in the context of sparse polynomial interpolation, Soprunov showed the following lower bound on the number of interior lattice points in the Minkowski sum of an n-tuple of n-dimensional lattice polytopes.

Theorem 1.1 ([Sop07, Nil17])**.** *Let $P_1, \ldots, P_n \subset \mathbb{R}^n$ be n-dimensional lattice polytopes. Then the following inequality holds:*

$$|(P_1 + \cdots + P_n)^\circ \cap \mathbb{Z}^n| \geq \mathrm{MV}(P_1, \ldots, P_n) - 1.$$

Furthermore, equality holds if and only if the Minkowski sum of any choice of $n - 1$ polytopes of the tuple $P_1, \ldots, P_n$ is hollow.

Key words and phrases. mixed degree, lattice polytopes, Minkowski sum, mixed volume.

The n-tuples for which equality holds in the above theorem have been called n-tuples *of mixed degree at most one* by Soprunov in [BNR+08], where a characterization of such tuples has been posed as a problem ([BNR+08, Section 5, Problem 2]). This notion is motivated by a connection to the *degree* of a lattice polytope, which is an intensively studied invariant in Ehrhart Theory (see for example [BN07, DRP09, DRHNP11, BH18]). The degree $\deg(P)$ of an n-dimensional lattice polytope $P \subset \mathbb{R}^n$ is set to equal n if P has at least one interior lattice point, and otherwise is defined as the smallest integer $0 \leq d \leq n-1$ such that the dilated lattice polytope $(n-d)P$ is hollow. Another interpretation is given by the fact that $\deg(P)$ agrees with the degree of the so-called h^*-polynomial of P (see for example [BR15]). Now in the setting $P_1 = \cdots = P_n = P \subset \mathbb{R}^n$, that is for an n-tuple consisting of n copies of the same lattice polytope P, the equality condition from Theorem 1.1 is satisfied if and only if the degree of P is at most one. It is a well-known fact that an n-dimensional lattice polytope has degree 0 if and only if it is equivalent to Δ_n. Also for lattice polytopes of degree one there exists the following complete description by Batyrev-Nill [BN07]. Given an n-dimensional lattice polytope $Q \subset \mathbb{R}^n$, we define the *lattice pyramid* $\mathcal{P}(Q)$ as the $(n+1)$-dimensional polytope

$$\mathcal{P}(Q) := \operatorname{conv}(Q \times \{\mathbf{0}\} \cup \{\mathbf{e}_{n+1}\}) \subset \mathbb{R}^{n+1}.$$

The lattice pyramid construction preserves the degree (it actually preserves the h^*-polynomial). We say that an n-dimensional lattice polytope is an *exceptional simplex* if it is equivalent to the polytope obtained via $n-2$ iterations of the lattice pyramid construction over the polygon $2\Delta_2$. We say that an n-dimensional lattice polytope P is a *Lawrence prism* if P is equivalent to a lattice polytope $\operatorname{conv}(\{\mathbf{0}, a_0\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_1 + a_1\mathbf{e}_n, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_{n-1} + a_{n-1}\mathbf{e}_n\})$ for nonnegative integers $a_0, \ldots, a_{n-1} \in \mathbb{Z}_{\geq 0}$.

Theorem 1.2 ([BN07, Theorem 2.5]). *Let P be a lattice polytope. Then $\deg(P) \leq 1$ (i.e. $(n-1)P$ is hollow) if and only if P is is an exceptional simplex or a Lawrence prism.*

The relation of tuples of mixed degree at most one to the degree of a lattice polytope raises the natural question whether there is a general concept of a mixed degree of an n-tuple of lattice polytopes in $\mathbb{R}^n$ that generalizes both Soprunov's definition of tuples of mixed degree at most one and the degree of a single lattice polytope. A suggestion for such a mixed degree has recently been given by Nill [Nil17]. Let $P_1, \ldots, P_n \subset \mathbb{R}^n$ be an n-tuple of lattice polytopes. The *mixed degree* $\operatorname{md}(P_1, \ldots, P_n)$ is set to equal n if P_i has an interior lattice point for some $1 \leq i \leq n$. Otherwise $\operatorname{md}(P_1, \ldots, P_n)$ is the smallest integer $0 \leq d \leq n-1$ such that the Minkowski sum of any choice of $(n-d)$ polytopes of tuple $P_1, \ldots, P_n$ is hollow. We refer the reader to [Nil17] for additional motivation for this definition.

In this language, Soprunov's problem asks for a characterization of n-tuples of lattice polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$ satisfying $\operatorname{md}(P_1, \ldots, P_n) \leq 1$. The case of $\operatorname{md}(P_1, \ldots, P_n) = 0$ (as this is equivalent to $\operatorname{MV}(P_1, \ldots, P_n) = 1$ by [Nil17, Theorem 2.2]) has already been solved by Cattani et al. in the context of investigating the codimension of so-called mixed discriminants.

Proposition 1.3 ([CCD+13, Proposition 2.7]). *Let $P_1, \ldots, P_n$ be n-dimensional lattice polytopes. Then $\operatorname{md}(P_1, \ldots, P_n) = 0$ if and only if the n-tuple $P_1, \ldots, P_n$ is equivalent to the n-tuple $\Delta_n, \ldots, \Delta_n$.*

We therefore often restrict to tuples with mixed degree equal to one in our approach towards solving Soprunov's problem.

1.3. **Results.** The contribution of this paper is to partially solve Soprunov's problem by presenting a finiteness result for dimension $n \geq 4$ and to give a complete characterization of triples of 3-dimensional lattice polytopes of mixed degree one.

In order to describe a trivial class of n-tuples of mixed degree (at most) one, let us introduce the concept of lattice projections. By *lattice projection*, we denote a surjective affine-linear map

$\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ satisfying $\varphi(\mathbb{Z}^n) = \mathbb{Z}^m$. The kernel of such a projection is affinely generated by lattice points of $\mathbb{Z}^n$ and we consider two projections to be equal if and only if they have the same kernel up to lattice translations.

The trivial class of n-tuples of mixed degree (at most) one is now given by the following example.

Example 1.4. Let $P_1, \dots, P_n \subset \mathbb{R}^n$ be n-dimensional lattice polytopes and $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ a lattice projection satisfying $\varphi(P_i) = \Delta_{n-1}$ for all $1 \le i \le n$. Then $P_1, \dots, P_n$ has mixed degree at most one, as any Minkowski sum of $n-1$ polytopes of the tuple $P_1, \dots, P_n$ is projected onto the hollow polytope $(n-1)\Delta_{n-1} \subset \mathbb{R}^{n-1}$ by φ. An example of such a trivial tuple in dimension $n = 3$ is shown in Figure 1.

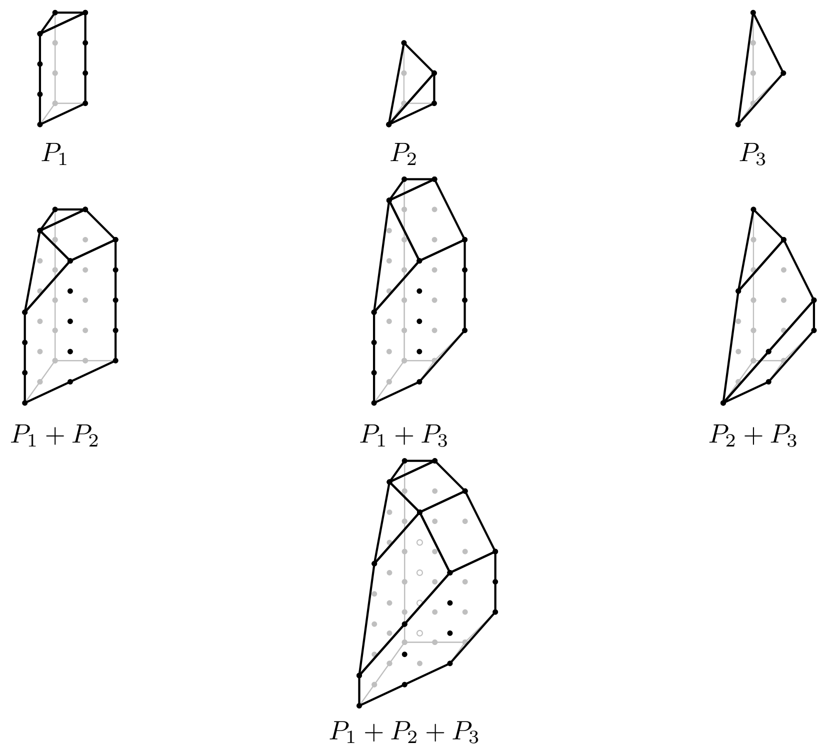

FIGURE 1. A triple $P_1, P_2, P_3 \subset \mathbb{R}^3$ having mixed degree one, where P_1, P_2, P_3 all project onto Δ_2 under the projection along the vertical axis.

One can view n-tuples from Example 1.4 as consisting of n lattice polytopes that all are Lawrence prisms and additionally satisfy that they extend into the same height-direction over the same unimodular $(n-1)$-dimensional simplex. Clearly we cannot expect this to be the only class of n-tuples of mixed degree one, as already the unmixed setting of Theorem 1.2 additionally yields n-tuples of copies of the same exceptional simplex as having mixed degree one. Unlike in the unmixed case there actually exist many more such non-trivial examples (see our classification result for $n = 3$ in Theorem 1.6, one example is shown in Figure 2).

This raises the question whether there is any chance to make reasonable statements about n-tuples of mixed degree one at all. Our main result is to provide a positive answer to this question by showing that, for any dimension n at least 4, all but finitely many exceptions of n-tuples of mixed degree one are actually of the trivial type described in Example 1.4.

Theorem 1.5. *Fix $n \ge 4$ and let $P_1, \dots, P_n \subset \mathbb{R}^n$ be n-dimensional lattice polytopes with $\operatorname{md}(P_1, \dots, P_n) = 1$. Then, up to equivalence, the n-tuple $P_1, \dots, P_n$ either belongs to a finite list of exceptions or there is a lattice projection $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $\varphi(P_i) = \Delta_{n-1}$ for all $1 \le i \le n$.*

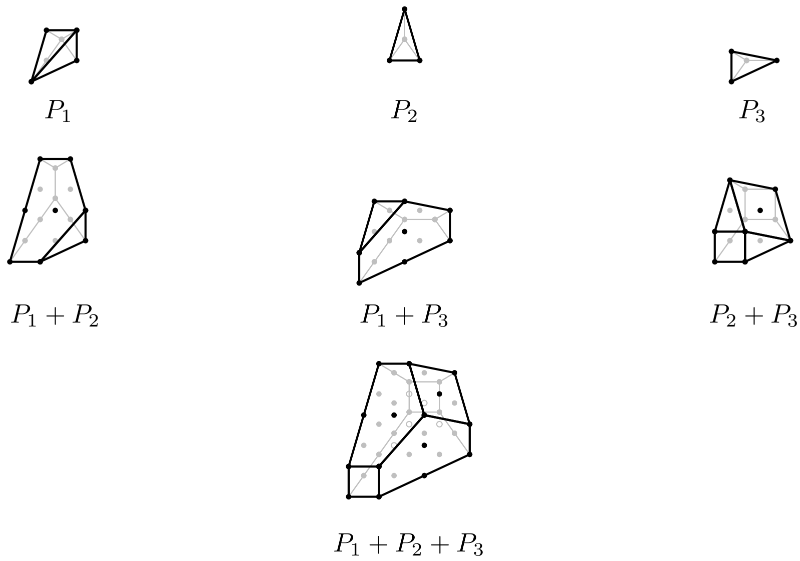

FIGURE 2. A triple $P_1, P_2, P_3 \subset \mathbb{R}^3$ having mixed degree one for which no lattice projection exists commonly mapping P_1, P_2, P_3 onto translates of Δ_2 (see (d) of Corollary 4.8).

We refer the reader to Section 3 for the proof of Theorem 1.5.

Theorem 1.5 is not true for dimension $n \in \{2, 3\}$. This fact is straightforward to see for $n = 2$, as pairs of lattice polygons $P_1, P_2 \subset \mathbb{R}^2$ are of mixed degree (at most) one if and only if both P_1 and P_2 are hollow. Fixing P_1 to be any hollow polygon and letting P_2 range through all polygons that are equivalent to a fixed hollow polygon will clearly yield infinitely many non-equivalent pairs of mixed degree one without there being a projection commonly mapping both polytopes onto the segment Δ_1.

For $n = 3$, however, we find that only a very specific class of triples of mixed degree one contains an infinite number of exceptions and we can explicitly describe a finite number of 1-parameter families covering this class. This is part of the following classification result, which essentially gives a complete answer to Soprunov's problem for dimension $n = 3$. Note that we say that a k-tuple of n-dimensional lattice polytopes $P_1, \dots, P_k$ *admits a common lattice projection* onto translates of an $(n-1)$-dimensional lattice polytope Q if there exists a lattice projection $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ satisfying $\varphi(P_i) = Q + t_i$ for all $1 \le i \le k$ and some $t_i \in \mathbb{Z}^{n-1}$.

Theorem 1.6. *Let $P_1, P_2, P_3 \subset \mathbb{R}^3$ be a triple of 3-dimensional lattice polytopes that satisfies $\operatorname{md}(P_1, P_2, P_3) = 1$. Then either there is a lattice projection $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $\varphi(P_i) = \Delta_2$ for all $1 \le i \le 3$, or one of the following holds.*

- *(i) There is no pair in P_1, P_2, P_3 admitting a common lattice projection onto translates of Δ_2 and P_1, P_2, P_3 is equivalent to one out of 29 possible triples.*
- *(ii) There is exactly one pair in P_1, P_2, P_3 admitting a common lattice projection onto translates of Δ_2 and P_1, P_2, P_3 is equivalent to one out of 141 possible triples.*
- *(iii) There are exactly two pairs in P_1, P_2, P_3 admitting a common lattice projection onto translates of Δ_2 and P_1, P_2, P_3 is equivalent to one out of 82 possible triples.*
- *(iv) All pairs in P_1, P_2, P_3 admit a common lattice projection onto translates of Δ_2 and*
  - *(a) the kernels of the projections cannot be shifted into a common hyperplane and the triple P_1, P_2, P_3 is equivalent to one out of 27 possible triples.*
  - *(b) the kernels of the projections can be shifted into a common hyperplane and P_1, P_2, P_3 belongs, up to equivalence, to one out of finitely many infinite 1-parameter families of triples.*

We refer the reader to Section 4 for a proof of Theorem 1.6.

In the following example we present one of the 1-parameter families from Theorem 1.6 (iv). Let us denote $\square_2 := \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2) \subset \mathbb{R}^2$.

Example 1.7. Let $P_1^k, P_2^k, P_3 \subset \mathbb{R}^3$ be the triple given by

$$P_1^k := \operatorname{conv}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_2, k\mathbf{e}_2 + \mathbf{e}_3),$$
$$P_2^k := \operatorname{conv}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_1, k\mathbf{e}_1 + \mathbf{e}_3),$$
$$P_3 := \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3),$$

for some $k \in \mathbb{Z}_{\geq 0}$. Then $\operatorname{md}(P_1^k, P_2^k, P_3) = 1$ and, while all pairs in P_1^k, P_2^k, P_3 admit a common lattice projection onto translates of Δ_2, there is no lattice projection commonly mapping the whole triple P_1^k, P_2^k, P_3 onto translates of Δ_2. Note that P_1^k, P_2^k and P_3 as single lattice polytopes are all equivalent to $\mathcal{P}(\square_2)$ for all $k \in \mathbb{Z}_{\geq 0}$ (see Figure 3).

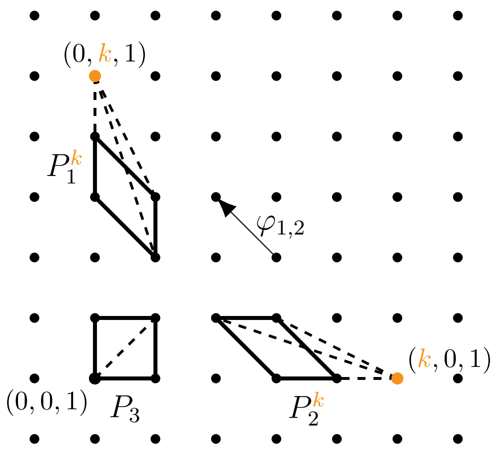

FIGURE 3. Top view of the infinite family from Example 1.7. The arrow labeled $\varphi_{1,2}$ shows the direction of the common projection of P_1^k and P_2^k onto translates of Δ_2. The common projections of P_3 and P_1^k as well as P_3 and P_2^k are given by the projection along the second and the first coordinate respectively.

All computations have been carried out using Magma [BCP97] and the code can be found at `https://github.com/gabrieleballetti/mixed_degree_one`.

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2. Cayley sums and projections

While there is a well-known way of determining whether two lattice polytopes are equivalent by comparing their normal forms (see e.g. [KS98]), the task of checking two tuples for equivalence seems a priori more complicated. However, we will show that in our setting the construction of the

Cayley sum of a tuple allows us to reduce the problem of checking equivalence of tuples to checking equivalence of two higher-dimensional lattice polytopes. The Cayley sum (or Cayley polytope) construction occurs in various contexts in the literature, for example in the construction of mixed subdivisions of Minkowski sums ([DLRS10]), in the study of A-discriminants ([FI16, GKZ94]) and in structural results on lattice polytopes of high dimension and small degree ([DRHNP11, DRP09]).

Let us recall the definition and some basic properties of the Cayley sum of a tuple of lattice polytopes.

Definition 2.1. Let $P_1, \dots, P_k \subset \mathbb{R}^n$ be lattice polytopes. We define the *Cayley sum* $P_1 * \cdots * P_k$ as

$$P_1 * \cdots * P_k := \operatorname{conv}((P_1 \times \{\mathbf{0}\}) \cup (P_2 \times \{\mathbf{e}_1\}) \cup \cdots \cup (P_k \times \{\mathbf{e}_{k-1}\})) \subset \mathbb{R}^{n+k-1}.$$

We call a Cayley sum $P_1 * \cdots * P_k$ *proper* if $P_i \neq \emptyset$ for all $1 \leq i \leq k$. In this case one has $\dim(P_1 * \cdots * P_k) = \dim(P_1 + \cdots + P_k) + k - 1$.

Proposition 2.2 ([BN08, Proposition 2.3])**.** *Let $P \subset \mathbb{R}^n$ be a lattice polytope. Then the following are equivalent.*

(i) There exists a lattice projection $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{k-1}$ with $\varphi(P) = \Delta_{k-1}$,
(ii) there exist lattice polytopes $P_1, \dots, P_k \subset \mathbb{R}^{n-k+1}$ such that $P \cong P_1 * \cdots * P_k$.

In particular if $k = n$, P is the Cayley sum of n segments.

Remark 2.3. Let $P = P_1 * \cdots * P_k$. Then all faces of P are of the form $F = F_1 * \cdots * F_k$ for (possibly empty) faces $F_i \subseteq P_i$. One has $\dim(F) = \dim(F_1 + \cdots + F_k) + l - 1$ where l is the number of F_i that satisfy $F_i \neq \emptyset$. In particular, each of the P_i corresponds to a face of P which we will denote by $\hat{P}_i$. Furthermore $\hat{P}_i^c$, by which we denote the convex hull of all vertices of P that are not contained in $\hat{P}_i$, is always a proper face of P.

Let us now formulate the lemma which allows us to determine equivalence of certain n-dimensional tuples by comparing the normal forms of their Cayley sums.

Lemma 2.4. *Let $P_1, \dots, P_k$ and $Q_1, \dots, Q_k$ be k-tuples of n-dimensional lattice polytopes in $\mathbb{R}^n$. Assume that there is no lattice projection $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{k-1}$ mapping $P_1, \dots, P_k$ onto translates of Δ_{k-1}. Then the k-tuples $P_1, \dots, P_k$ and $Q_1, \dots, Q_k$ are equivalent if and only if their Cayley sums $P_1 * \cdots * P_k$ and $Q_1 * \cdots * Q_k$ are equivalent.*

Proof. The fact that Cayley sums of equivalent k-tuples of lattice polytopes are equivalent is a straightforward consequence of the definition. Suppose now that the k-tuple $P_1, \dots, P_k$ does not admit a common projection onto translates of Δ_{k-1} and suppose that $P_1 * \cdots * P_k$ and $Q_1 * \cdots * Q_k$ are equivalent. Let $P_1', \dots, P_k'$ be the images of $\hat{P}_1, \dots, \hat{P}_k$ under the lattice-preserving transformation yielding the equivalence, respectively. We will first show that, up to renumbering, $P_i' = \hat{Q}_i$ for all $1 \leq i \leq k$. Suppose without loss of generality $P_1' \neq \hat{Q}_i$ for all $1 \leq i \leq k$. Note that P_1' cannot properly contain any $\hat{Q}_i$ (by Remark 2.3 any face of $Q_1 * \cdots * Q_k$ that properly contains a $\hat{Q}_i$ is of dimension greater or equal to $n+1$). This also implies that P_1' cannot be disjoint to some $\hat{Q}_i$ due to the following. If P_1' was disjoint to, say, $\hat{Q}_1$, its complement $(P_1')^c$ would contain $\hat{Q}_1$. As however P_1' does not contain any $\hat{Q}_i$, the complement $(P_1')^c$ contains at least one point of $\hat{Q}_i$ for each $1 \leq i \leq k$. Therefore, if we pick points $p_2, \dots, p_k$ in $\hat{Q}_2 \setminus P_1', \dots, \hat{Q}_k \setminus P_1'$ respectively, we have $\dim(\hat{P}_1^c) = \dim((P_1')^c) \geq \dim(\hat{Q}_1 * \{p_2\} * \cdots * \{p_k\}) = n + k - 1$, which is a contradiction to $\hat{P}_1^c$ being a proper face of the Cayley sum $P_1 * \cdots * P_k$ (see Remark 2.3).

As P_1' was chosen arbitrarily this argumentation implies that all P_i' have non-empty intersection with all $\hat{Q}_j$. Therefore the natural projection of $Q_1 * \cdots * Q_k$ onto Δ_{k-1} remains surjective when restricted to the affine hull of P_i' for each $1 \leq i \leq k$. As the affine hulls of $P_1', \dots, P_k'$ are by construction pairwise parallel n-dimensional affine subspaces of $\mathbb{R}^{n+k-1}$ the natural projection yields a lattice projection $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{k-1}$ mapping P_i onto a translate of Δ_{k-1} for each $1 \leq i \leq k$. This contradicts our assumption.

This shows that, if $P_1, \dots, P_k$ do not have a common projection onto translates of Δ_{k-1}, any affine lattice-preserving transformation $U\colon \mathbb{R}^{n+k-1} \to \mathbb{R}^{n+k-1}$ mapping $P_1 * \cdots * P_k$ to $Q_1 * \cdots * Q_k$ yields (up to renumbering) a bijection mapping the face $\hat{P}_i$ to the face $\hat{Q}_i$. We may without loss of generality assume that U preserves the origin. Restricting U to a map from the affine hull of P_1 to the affine hull of Q_1 we obtain a linear lattice-preserving transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$. It is straightforward to verify that restricting U to the affine hull of any other P_i results in a map $x \mapsto L(x) + t_i$ from $\mathbb{R}^n \to \mathbb{R}^n$ for some $t_i \in \mathbb{Z}^n$. □

3. Proof of the main theorem

From Proposition 2.2 we can easily deduce the following lemma. We call a facet F of a lattice polytope $P \subset \mathbb{R}^n$ *unimodular* if F is a unimodular simplex inside the affine lattice defined as the intersection of the affine hull of F with $\mathbb{Z}^n$.

Lemma 3.1. *Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope and $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ a lattice projection that projects P onto Δ_{n-1}. Then $\ker\varphi = \mathbb{R}e$ where e is a vector parallel to an edge between vertices v_1, v_2, where $v_1 \in F_1$ and $v_2 \in F_2$ for two different unimodular facets $F_1 \neq F_2$ of P.*

We now study n-dimensional polytopes projecting onto Δ_{n-1} along multiple directions. Recall that we denote by $\mathcal{P}^{n-2}(\square_2)$ the $(n-2)$-fold lattice pyramid formed over the square $\square_2 = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2) \subset \mathbb{R}^2$.

Lemma 3.2. *Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope such that there are different lattice projections $\varphi_1, \varphi_2\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ that map P onto Δ_{n-1}. Then P is equivalent either to the unimodular simplex Δ_n or to $\mathcal{P}^{n-2}(\square_2)$. If there exists another projection $\varphi_3\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ mapping P onto Δ_{n-1}, then P is necessarily equivalent to Δ_n.*

Proof. As P has one projection onto Δ_{n-1}, by Proposition 2.2 we may assume that P is of the form $P = I_1 * \cdots * I_n$ for n segments $I_i = [0, a_i]$ with $a_i \in \mathbb{Z}_{\geq 0}$. Two facets of P are given by $\Delta_{n-1} \times \{0\}$ and $\{a_1\} * \cdots * \{a_n\}$. All other facets of P are of a form we denote by F_k for $1 \leq k \leq n$, which is the Cayley sum of all I_i excluding I_k. As there exists another lattice projection φ_2 mapping P onto Δ_{n-1}, by Lemma 3.1 the facet F_k has to be unimodular for some $1 \leq k \leq n$. Assume without loss of generality that F_1 is unimodular and $a_2 = 1$ and $a_3 = \cdots = a_n = 0$. Furthermore, a_1 cannot be greater than one as otherwise P would have an edge of lattice length at least 2. Therefore any projection which is not along this edge direction could not be projecting P onto Δ_{n-1}. If $a_1 = 0$, then P is equivalent to Δ_n, otherwise $a_1 = 1$ and P is equivalent to $\mathcal{P}^{n-2}(\square_2)$. One easily verifies that $\mathcal{P}^{n-2}(\square_2)$ does not have more than two different projections onto Δ_{n-1}. □

Lemma 3.3. *Let $S_1, S_2 \subset \mathbb{R}^n$ be two unimodular n-dimensional simplices, $u_1, u_2 \in \mathbb{Z}^n$ be linearly independent edge directions for S_1 and S_2 respectively, and $C_1, C_2 \subset \mathbb{R}^n$ be the infinite prisms $S_1 + \mathbb{R}u_1$ and $S_2 + \mathbb{R}u_2$ respectively. Given $z \in \mathbb{Z}^n$, denote by P_z the intersection* $\operatorname{conv}(C_1 \cap (C_2 + z) \cap \mathbb{Z}^n)$*. Let $v, w \in \mathbb{Z}^n$ such that P_v and P_w are both n-dimensional. Then P_v and P_w are the same lattice polytope up to translation.*

Proof. By Lemma 3.2 we know that, if there exists $v \in \mathbb{Z}^n$ such that P_v is n-dimensional, then P_v is equivalent either to Δ_n or to $\mathcal{P}^{n-2}(\square_2)$ having two edges parallel to the directions u_1 and u_2. In either of the two cases, we can assume P_v to be exactly Δ_n or $\mathcal{P}^{n-2}(\square_2)$.

If $P_v = \mathcal{P}^{n-2}(\square_2)$ then, up to reordering and changes of signs, $u_1 = \mathbf{e}_1$ and $u_2 = \mathbf{e}_2$. In particular, $C_1 = \operatorname{conv}(\mathbf{0}, \mathbf{e}_2, \dots, \mathbf{e}_n) + \mathbb{R}\mathbf{e}_1$ and $C_2 + v = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, \dots, \mathbf{e}_n) + \mathbb{R}\mathbf{e}_2$. One easily verifies that $C_1 \cap (C_2 + w)$ is full-dimensional if and only if $w - v \in \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2$. In all these cases $C_1 \cap (C_2 + w)$ is a translation of P_v.

On the other hand, if $P_v = \Delta_n$, then there is another case distinction. If u_1 and u_2 are parallel to adjacent edges of D_n, then we can assume $u_1 = \mathbf{e}_1$ and $u_2 = \mathbf{e}_2$. But in this case C_1 and $C_2 + v$

must intersect in $\mathcal{P}^{n-2}(\square_2)$ instead of in Δ_n, hence we have a contradiction. Therefore u_1 and u_2 are parallel to non-adjacent edges of Δ_n and we can assume $u_1 = \mathbf{e}_1$ and $u_2 = \mathbf{e}_2 - \mathbf{e}_3$. In particular, $C_1 = \operatorname{conv}(\mathbf{0}, \mathbf{e}_2, \dots, \mathbf{e}_n) + \mathbb{R}\mathbf{e}_1$ and $C_2 + v = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, \dots, \mathbf{e}_n) + \mathbb{R}(\mathbf{e}_2 - \mathbf{e}_3)$. Again, one easily verifies that $C_1 \cap (C_2 + w)$ is full-dimensional if and only if $w - v \in \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}(\mathbf{e}_2 - \mathbf{e}_3)$. In all these cases $C_1 \cap (C_2 + w)$ is a translation of Δ_n. ☐

Lemma 3.4. *Let $P \subset \mathbb{R}^n$ be a unimodular n-simplex and $\varphi_1, \varphi_2 \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be two different lattice projections such that, for each $1 \le i \le 2$, the images $\varphi_i(P)$ and $\varphi_i(\Delta_n)$ are translates of Δ_{n-1}. Then, up to translation and coordinate permutation, P is contained in $\mathcal{P}^{n-2}(\square_2)$. If there exists another projection $\varphi_3 \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ mapping P and Δ_n onto translates of Δ_{n-1}, then P is necessarily a translate of Δ_n.*

Proof. By Lemma 3.1, φ_1 and φ_2 are projections along the directions u_1 and u_2 of two edges of Δ_n. If u_1 and u_2 are the directions of two adjacent edges of Δ_n, we can suppose that $u_1 = \mathbf{e}_1$ and $u_2 = \mathbf{e}_2$. Then P is contained in the intersection $(C_1 + z_1) \cap (C_2 + z_2)$ where $C_1 := \Delta_n + \mathbb{R}\mathbf{e}_1$ and $C_2 := \Delta_n + \mathbb{R}\mathbf{e}_2$, for some $z_1, z_2 \in \mathbb{Z}^n$. By Lemma 3.3, P is, up to translation, contained in $C_1 \cap C_2 = \mathcal{P}^{n-2}(\square_2)$. If u_1 and u_2 are the directions of two non-adjacent edges of Δ_n then we can suppose that $u_1 = \mathbf{e}_1$ and $u_2 = \mathbf{e}_2 - \mathbf{e}_3$. Then P is contained in the intersection $(C_1 + z_1) \cap (C_2 + z_2)$ where $C_1 := \Delta_n + \mathbb{R}\mathbf{e}_1$ and $C_2 := \Delta_n + \mathbb{R}(\mathbf{e}_2 - \mathbf{e}_3)$, for some $z_1, z_2 \in \mathbb{Z}^n$. By Lemma 3.3, P is, up to translation, contained in $C_1 \cap C_2 = \Delta_n$, therefore P is a translate of $\Delta_n \subset \mathcal{P}^{n-2}(\square_2)$. This proves the first part of the statement.

For the second part of the statement we note that φ_3 must also be a projection along the direction u_3 of an edge of Δ_n. The only case we need to check is when the edges parallel to u_1, u_2 and u_3 form a triangle in Δ_n. Indeed, if this is not the case either two of these edges are non-adjacent and P must be a translate of Δ_n as above, or u_1,u_2 and u_3 share a vertex. In the latter case we may assume $u_i = \mathbf{e}_i$ for $1 \le i \le 3$. As deduced above from Lemma 3.3, this in particular yields that P is contained in the intersection of a translation of the square pyramid $\operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_n)$ with the flipped square pyramid $\operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_4, \dots, \mathbf{e}_n)$. This implies that P is a translate of Δ_n. Let us therefore assume that $u_1 = \mathbf{e}_1$, $u_2 = \mathbf{e}_2$ and $u_3 = \mathbf{e}_1 - \mathbf{e}_2$. In this case P is a translate of one of the four n-dimensional subpolytopes of $\mathcal{P}^{n-2}(\square_2)$. It is easy to verify that Δ_n is the only one of them that is projected by φ_3 onto a translate of $\varphi_3(\Delta_n)$. ☐

Definition 3.5. Let $P_1, \dots, P_{n-1} \subset \mathbb{R}^n$ be n-dimensional polytopes with the Minkowski sum $P_1 + \cdots + P_{n-1}$ being hollow. We call the $(n-1)$-tuple $P_1, \dots, P_{n-1}$ *exceptional*, if there exists no projection $\varphi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $\varphi(P_1 + \cdots + P_{n-1}) \subset \mathbb{R}^{n-1}$ is a hollow polytope.

Remark 3.6. By [NZ11, Theorem 1.2] there exist only finitely many n-dimensional lattice polytopes not admitting a lattice projection onto a hollow $(n-1)$-dimensional lattice polytope, up to equivalence. So in particular, up to equivalence, there exist only finitely many exceptional $(n-1)$-tuples of n-dimensional lattice polytopes.

Furthermore, by Proposition 1.3, for any non-exceptional $(n-1)$-tuple $P_1, \dots, P_{n-1}$ of n-dimensional lattice polytopes there exists a lattice projection $\varphi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ mapping all P_i onto translates of Δ_{n-1}.

Proof of Theorem 1.5. Let $n \ge 4$. Given $1 \le k \le n$, denote by I_k the set $\{1, ..., n\} \setminus \{k\}$, and by $[P]_k$ the $(n-1)$-tuple given by all P_i for $i \in I_k$. Denote furthermore by P_{I_k} the Minkowski sum $\sum_{i \in I_k} P_i$ of the polytopes in $[P]_k$. Since $\operatorname{md}(P_1, \dots, P_n) = 1$, the Minkowski sum P_{I_k} is hollow for any $1 \le k \le n$. Recall that, if $[P]_k$ is not exceptional, then by Remark 3.6 there exists a projection $\varphi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ mapping all polytopes in $[P]_k$ onto translates of Δ_{n-1}. We treat cases separately, depending on the number of exceptional $(n-1)$-subtuples of the tuple $P_1, \dots, P_n$.

(0) If $P_1, \dots, P_n$ has no exceptional $(n-1)$-subtuples then either there exists a projection $\varphi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ mapping $P_1, \dots, P_n$ onto translates of Δ_{n-1} (and in this case there is nothing to prove), or each of the P_i admits $n-1$ pairwise different projections onto Δ_{n-1}.

Indeed if two of these projections were the same, then we would be in the previous case. Suppose there exist $n-1$ pairwise different projections. As $n \geq 4$, Lemma 3.2 yields that each of the P_i is a unimodular n-dimensional simplex. Without loss of generality we assume $P_1 = \Delta_n$. Given $2 \leq i \leq n$, there exist $n-2$ pairwise different projections mapping P_1 and P_i onto translates of Δ_{n-1}. If $n \geq 5$, by Lemma 3.4, we can immediately deduce that, up to translations, $P_1 = P_2 = \ldots = P_n = \Delta_n$. If $n = 4$, Lemma 3.4 only ensures that $P_2, \ldots, P_n$ are, up to translation and coordinate permutation, contained in $\mathcal{P}^{n-2}(\square_2)$. This yields finitely many cases and checking them computationally we find among them only 4-tuples admitting a common projection onto Δ_3.

(1) $P_1, \ldots, P_n$ has exactly one exceptional $(n-1)$-subtuple, which we can assume to be $[P]_n$. As $[P]_n$ is an exceptional $(n-1)$-tuple, the Minkowski sum P_{I_n} belongs to a finite list of hollow n-dimensional polytopes. This means that there are, up to equivalence, finitely many exceptional tuples to choose $[P]_n$ from. We now show, that given $[P]_n$ there are finitely many possible choices for P_n that lead to the n-tuple $P_1, \ldots, P_n$ having exactly $[P]_n$ as an exceptional $(n-1)$-subtuple, which shows the finiteness of this case.

Let therefore $\varphi_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a lattice projection mapping the lattice polytopes in $[P]_2$ to translates of Δ_{n-1}. Similarly, let $\varphi_2 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a lattice projection mapping the lattice polytopes in $[P]_1$ to translates of Δ_{n-1}. The existence of such projections follows from the fact that $[P]_2$ and $[P]_1$ are non-exceptional. We remark that there exist finitely many such projections. Let C_i be the infinite prism $P_i + \ker \varphi_i$, for $1 \leq i \leq 2$. Then we know that any possible choice of P_n is contained in $(C_1 + z_1) \cap (C_2 + z_2)$ for some $z_1, z_2 \in \mathbb{Z}^n$. By Lemma 3.3, for any choices of lattice points $z_1, z_2, z_1', z_2' \in \mathbb{Z}^n$ such that $\dim((C_1 + z_1) \cap (C_2 + z_2) \cap \mathbb{Z}^n) = \dim((C_1 + z_1') \cap (C_2 + z_2') \cap \mathbb{Z}^n) = n$ we find that $(C_1 + z_1') \cap (C_2 + z_2')$ is a translate of $(C_1 + z_1) \cap (C_2 + z_2)$. Therefore, up to translations, all possible choices for P_n are contained in $(C_1 + z_1) \cap (C_2 + z_2)$ for fixed $z_1, z_2 \in \mathbb{Z}^n$. Note that the intersection $(C_1 + z_1) \cap (C_2 + z_2)$ is either equivalent to Δ_n or $\mathcal{P}^{n-2}(\square_2)$ by Lemma 3.2, where the choice of the equivalence class depends entirely on $[P]_n$. This implies that P_n must be one element of a finite list of lattice polytopes fully determined by $[P]_n$.

(2+) If $P_1, \ldots, P_n$ has two or more exceptional $(n-1)$-subtuples, then we can suppose that $[P]_n$ and $[P]_{n-1}$ are exceptional. In particular, there exists an upper bound depending only on n for the volume of the Minkowski sums P_{I_n} and $P_{I_{n-1}}$ and therefore (since $n > 2$) for the volume of $P_1 + P_i$ for any $2 \leq i \leq n$. Recall that, by [LZ91, Theorem 2], there are, up to equivalence, only finitely many lattice polytopes of any fixed volume $K \in \mathbb{Z}_{\geq 0}$. Therefore, as in particular the volume of P_1 is bounded, there exist only finitely many choices for P_1 up to equivalence. Furthermore, fixing P_1 determines, up to translation, finitely many possibilities for each P_i with $2 \leq i \leq n$ due to the volume bound on $P_1 + P_i$. This yields that there are only finitely many n-tuples $P_1, \ldots, P_n$ in this case, up to equivalence.

□

Note, that the assumption $n > 3$ is only used in case (0) of the previous proof.

The unmixed result of Theorem 1.2 also gives an explicit description of lattice polytopes of degree one that are not Lawrence prisms, in fact, up to equivalence and the lattice pyramid construction, there exists only one such exception over all dimensions. Such an explicit description of the list of exceptions from the statement of Theorem 1.5 is not known in dimension $n \geq 4$.

Question 3.7. *For dimension $n \geq 4$, what are the n-tuples of n-dimensional lattice polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$ with $\operatorname{md}(P_1, \ldots, P_n) = 1$ that are not of the trivial type described in Example 1.4? Is there a finite description over all dimensions as there is in the unmixed case?*

In [Nil17] the mixed degree is actually treated in a more general way, also being defined for m-tuples of n-dimensional lattice polytopes, with $m \neq n$. In particular, an m-tuple of lattice polytopes

$P_1, \ldots, P_m \subset \mathbb{R}^n$ satisfies $\operatorname{md}(P_1, \ldots, P_m) \leq 1$ if and only if $m \geq n-1$ and the Minkowski sum of each $(n-1)$-subtuple is hollow. For $m = n-1$ we obtain an analogous result to Theorem 1.5 (even for $n \in \{2, 3\}$) immediately from [NZ11, Theorem 1.2]. We remark that Theorem 1.5 also inductively extends to the case of $m > n$ as follows.

Remark 3.8. Fix $n \geq 4$ and let $P_1, \ldots, P_{n+k} \subset \mathbb{R}^n$ be n-dimensional lattice polytopes with $\operatorname{md}(P_1, \ldots, P_{n+k}) = 1$. Then, up to equivalence, the $(n+k)$-tuple $P_1, \ldots, P_{n+k}$ either belongs to a finite list of exceptions or there is a lattice projection $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $\varphi(P_i) = \Delta_{n-1}$ for all $1 \leq i \leq n+k$.

One can see this with an induction argument on k, where the base case is given by Theorem 1.5. Indeed, let $P_1, \ldots, P_{n+k+1} \subset \mathbb{R}^n$ be an $(n+k+1)$-tuple of n-dimensional lattice polytopes with $\operatorname{md}(P_1, \ldots, P_{n+k+1}) = 1$. One easily verifies that this implies that any $(n+k)$-subtuple of $P_1, \ldots, P_{n+k+1}$ has mixed degree at most 1. Analogously to the proof of Theorem 1.5 one can distinguish three cases depending on how many $(n+k)$-subtuples of $P_1, \ldots, P_{n+k+1}$ do not admit a common lattice projection onto translates of Δ_{n-1}, and use the induction hypothesis.

4. The 3-dimensional case

This section is devoted to the proof of Theorem 1.6, giving a classification of triples of n-dimensional lattice polytopes $P_1, P_2, P_3 \subset \mathbb{R}^3$ of mixed degree one. Note that from the proof of Theorem 1.5 it follows that the number of such triples is finite, if we assume at least one of the subpairs of P_1, P_2, P_3 to be exceptional. Here we first classify, up to equivalence, these finitely many triples. In Proposition 4.7 we show that there are non-trivial infinite 1-parameter families of triples.

As an intermediate step towards the classification of triples of lattice polytopes of mixed degree one with at least one exceptional subpair we calculate all (equivalence classes of) exceptional pairs of 3-dimensional lattice polytopes. In order to do that we consider the list of maximal hollow 3-dimensional lattice polytopes classified by Averkov–Wagner–Weismantel [AWW11] (see also [AKW17]), and compute all subpolytopes of the maximal hollow lattice polytopes that have lattice width greater than one.

Proposition 4.1 ([AWW11, Corollary 1])**.** *Let $P \subset \mathbb{R}^3$ be a hollow 3-dimensional lattice polytope of width at least two. Then, up to equivalence, P is contained either in the unbounded polyhedron $2\Delta_2 \times \mathbb{R}$ or in one of 12 maximal hollow lattice polytopes.*

As we are interested in obtaining a list of exceptional pairs $P, Q \subset \mathbb{R}^3$ we use an implementation in Magma in order to compute the decompositions of all subpolytopes of the 12 maximal hollow polytopes into Minkowski sums of two 3-dimensional lattice polytopes. Afterwards we determine those pairs that actually do not admit a common projection onto translates of Δ_2 and then determine equivalent pairs using Lemma 2.4.

Corollary 4.2. *There are, up to equivalence, 32 pairs of 3-dimensional lattice polytopes whose Minkowski sum is hollow and that do not admit a common projection onto translates of Δ_2.*

We use this classification in order to compute all triples of lattice polytopes $P_1, P_2, P_3 \subset \mathbb{R}^3$ of mixed degree one with at least two exceptional subpairs as follows.

Assume that P_1, P_2 and P_1, P_3 are exceptional pairs. Then there exist two pairs A, B and C, D out of the 32 of Corollary 4.2 such that A, B is equivalent to P_1, P_2 and C, D is equivalent to P_1, P_3. We can suppose that P_1 is equal to A and equivalent to C. Thus there exists an affine lattice-preserving transformation φ mapping C to $A = P_1$ such that the triple P_1, P_2, P_3 is equivalent to the triple $A, B, \varphi(D)$.

This justifies the following algorithm to construct all the triples P_1, P_2, P_3 containing at least two exceptional subpairs: we iterate over all the pairs of ordered pairs A, B and C, D of Corollary 4.2, and, whenever there exists an affine lattice-preserving transformation φ mapping C to A, check if the triple $A, B, \varphi(\psi(D))$ has mixed degree one, where ψ ranges among all the possible affine automorphisms of C (and therefore $\varphi \circ \psi$ ranges among all affine lattice-preserving transformations sending C to A). Equivalent triples can be removed using the criterion following from Lemma 2.4. An implementation in Magma yields the following result proving parts (i)-(ii) of Theorem 1.6.

Proposition 4.3. *There are, up to equivalence, 170 triples of 3-dimensional lattice polytopes of mixed degree one having two or three exceptional subpairs. In the first case there are 29 triples, in the latter there are 141.*

We now discuss the case of triples of lattice polytopes of mixed degree one having exactly one exceptional subpair. Specifically, $P_1, P_2, P_3 \subset \mathbb{R}^3$ is a triple of 3-dimensional lattice polytopes with $\operatorname{md}(P_1, P_2, P_3) = 1$ and (without loss of generality) there are two different lattice projections $\varphi_2 \colon \mathbb{R}^3 \to \mathbb{R}^2$ and $\varphi_3 \colon \mathbb{R}^3 \to \mathbb{R}^2$ where φ_k maps P_i and P_j to translates of Δ_2 whenever $i, j \neq k$. In particular P_1 is a lattice polytope with two different lattice projections onto Δ_2 (and therefore by Lemma 3.2 is equivalent either to Δ_3 or to $\mathcal{P}(\square_2)$) and P_2, P_3 is an exceptional pair. Note that P_1 must be contained in both the infinite prisms $C_2 := P_3 + \ker \varphi_2 + u$ and $C_3 := P_2 + \ker \varphi_3 + v$, for some translation vectors $u, v \in \mathbb{Z}^3$.

In order to classify all such triples we use the fact that we may choose P_2, P_3 from the list of 32 exceptional pairs of Corollary 4.2. Given an exceptional pair P_2, P_3, we iterate over all the possible pairs of lattice projections φ_3, φ_2, such that $\varphi_3(P_2)$ and $\varphi_2(P_3)$ are unimodular triangles. Each such choice determines two infinite prisms $C_3 := P_2 + \ker \varphi_3$ and $C_2 := P_3 + \ker \varphi_2$. We know that any lattice polytope $P_1 \subset \mathbb{R}^3$, such that $\varphi_3(P_1)$ and $\varphi_2(P_1)$ are translates of $\varphi_3(P_2)$ and $\varphi_2(P_3)$ respectively, is contained in both the infinite prisms $C_2 := P_3 + \ker \varphi_2 + u$ and $C_3 := P_2 + \ker \varphi_3 + v$, for some translation vectors $u, v \in \mathbb{Z}^3$. Up to translation of P_1 we may assume $u = \mathbf{0}$. By Lemma 3.3 it suffices to find one choice of $v \in \mathbb{Z}^3$ such that C_2 and C_3 intersect in a full-dimensional lattice polytope, in order to determine the inclusion-maximal choice for P_1 up to translation. Furthermore, there are only finitely many choices for $v \in \mathbb{Z}^3$ to check for the existence of a full-dimensional intersection of C_2 and C_3 as we may suppose P_2 and P_3 to have a common vertex. This is due to the fact that, if C_2 and C_3 intersect in a full-dimensional lattice polytope, then one may translate P_2 along $\ker \varphi_3$ and P_3 along $\ker \varphi_2$ without changing the infinite prisms. It therefore suffices to restrict to translation vectors v that map a vertex of P_2 to a vertex of P_3. Thus we can determine, up to equivalence, all inclusion-maximal P_1 as above, form triples for all subpolytopes of P_1 and remove equivalent triples using Lemma 2.4. An implementation in Magma yields the following result proving part (iii) of Theorem 1.6.

Proposition 4.4. *There are, up to equivalence, 82 triples of 3-dimensional lattice polytopes of mixed degree one having exactly one exceptional subpair.*

In the remaining part of this section we are going to deal with non-trivial triples not having any exceptional subpair in order to prove part (iv) of Theorem 1.6.

Lemma 4.5. *Let $P_1, P_2, P_3 \subset \mathbb{R}^n$ be lattice polytopes, and $\varphi_1, \varphi_2, \varphi_3 \colon \mathbb{R}^3 \to \mathbb{R}^2$ be lattice projections such that, for all $1 \leq i, j, k \leq 3$, the images $\varphi_k(P_i)$ and $\varphi_k(P_j)$ are translates of Δ_2 if and only if $i, j \neq k$. Let $v_i \in \mathbb{Z}^3$ be the projection direction of φ_i for $1 \leq i \leq 3$. Then v_i and v_j are part of a lattice basis of $\mathbb{Z}^3$, for all $1 \leq i, j \leq 3$. Moreover, if v_1, v_2, v_3 linearly span $\mathbb{R}^3$, then they form a lattice basis of $\mathbb{Z}^3$.*

Proof. For $1 \leq k \leq 3$ let C_k be the infinite prism $\varphi_k(P_i) + \mathbb{R}v_k$, for some $i \neq k$. Note that, up to translation, this does not depend on the choice of i as both P_i and P_j are contained in different translates of C_k, whenever $i, j \neq k$. We now fix any of the infinite prisms, say C_1. For simplicity we suppose $C_1 = (\{0\} \times \Delta_2) + \mathbb{R}\mathbf{e}_1$ and $P_2, P_3 \subset C_1$. In this way we avoid dealing with translations. Note that v_2 is parallel to an edge of P_3, and v_3 is parallel to an edge of P_2. Since both edges are contained in C_1, they project along $\mathbf{e}_1$ either to the same side of the triangle $\varphi_1(P_2) = \varphi_1(P_3) = \Delta_2$, or to two adjacent sides. In the second case $\mathbf{e}_1$, v_2 and v_3 linearly span $\mathbb{R}^3$

and it is easy to verify that they form a lattice basis of $\mathbb{Z}^3$. In the first case $\mathbf{e}_1$, v_2 and v_3 span a plane, and from Lemma 3.2 it follows that any two of them are part of a lattice basis of $\mathbb{Z}^3$. $\square$

Proposition 4.6. *There are, up to equivalence, 27 triples of 3-dimensional lattice polytopes $P_1, P_2, P_3 \subset \mathbb{R}^3$ satisfying the hypotheses of Lemma 4.5 for projection directions $v_1, v_2, v_3 \in \mathbb{Z}^3$ that linearly span $\mathbb{R}^3$. All of them are, up to equivalence, contained in one of the following three inclusion-maximal triples of mixed degree one:*

- *the maximal triple given by the following three reflections of $\mathcal{P}(\square_2)$*
  $\mathrm{conv}(\mathbf{0}, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1)$,
  $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2)$,
  $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3) = \mathcal{P}(\square_2)$,
- *the maximal triple*
  $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2)$,
  $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3)$,
  $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3) = \mathcal{P}(\square_2)$.
- *and the maximal triple*
  $\mathrm{conv}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3)$,
  $\mathrm{conv}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3)$,
  $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3) = \mathcal{P}(\square_2)$.

Proof. By Lemma 4.5 we may assume v_1, v_2, v_3 to be $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively, and that two primitive segments parallel to the directions $\mathbf{e}_2$ and $\mathbf{e}_3$ are contained in C_1. This restricts C_1 to be, up to translation, one of the four infinite prisms of the form $\mathrm{conv}(\mathbf{0}, \pm\mathbf{e}_2, \pm\mathbf{e}_3) + \mathbb{R}\mathbf{e}_1$. In particular, up to translation, C_1 is contained in the infinite prism $\mathrm{conv}(\mathbf{0}, \mathbf{e}_2) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_3) + \mathbb{R}\mathbf{e}_1$. Similarly, $C_2 \subset \mathrm{conv}(\mathbf{0}, \mathbf{e}_1) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_3) + \mathbb{R}\mathbf{e}_2$ and $C_3 \subset \mathrm{conv}(\mathbf{0}, \mathbf{e}_1) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_2) + \mathbb{R}\mathbf{e}_3$. In particular all the P_i are, up to translations, subpolytopes of the unit cube $\square_3 = \mathrm{conv}(\mathbf{0}, \mathbf{e}_1) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_2) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_3)$, which leaves finitely many cases that we check computationally. $\square$

From the proof of Proposition 4.6 it is clear that all the maximal triples from Proposition 4.6 are actually contained inside the triple consisting of three copies of the unit cube $\square_3$. Note however that one has $\mathrm{md}(\square_3, \square_3, \square_3) > 1$, as the Minkowski sum $\square_3 + \square_3$ has an interior lattice point.

Proposition 4.7. *There are, up to equivalence, infinitely many triples of 3-dimensional lattice polytopes $P_1, P_2, P_3 \subset \mathbb{R}^3$ satisfying the hypothesis of Lemma 4.5 for projection directions $v_1, v_2, v_3 \in \mathbb{Z}^3$ that linearly span $\mathbb{R}^2 \times \{0\}$. All of them, up to equivalence, are contained in one of the following triples of mixed degree two given by the parallelepipeds $Q_k, R_k, \square_3$ for some $k \in \mathbb{Z}_{\geq 0}$, where*

$$\begin{aligned} Q_k &:= \mathrm{conv}(\mathbf{0}, \mathbf{e}_1 - \mathbf{e}_2) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_2) + \mathrm{conv}(\mathbf{0}, k\mathbf{e}_2 + \mathbf{e}_3), \\ R_k &:= \mathrm{conv}(\mathbf{0}, \mathbf{e}_2 - \mathbf{e}_1) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_1) + \mathrm{conv}(\mathbf{0}, k\mathbf{e}_1 + \mathbf{e}_3), \\ \square_3 &:= \mathrm{conv}(\mathbf{0}, \mathbf{e}_1) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_2) + \mathrm{conv}(\mathbf{0}, \mathbf{e}_3). \end{aligned}$$

They can be covered by a finite number of 1-parameter families. In particular, if we denote by ψ_1^k the shearing $(x, y, z) \mapsto (x, y + kz, z)$ and by ψ_2^k the shearing $(x, y, z) \mapsto (x + kz, y, z)$, one may choose 1-parameter families of the form $\left\{\psi_1^k(P_1^0), \psi_2^k(P_2^0), P_3\right\}_{k \in \mathbb{Z}_{\geq 0}}$ for all subpolytopes $P_1^0 \subset Q_0, P_2^0 \subset R_0$ and $P_3 \subset \square_3$ satisfying $\mathrm{md}(P_1^0, P_2^0, P_3) = 1$.

Proof. By Lemma 4.5 we may assume v_1, v_2, v_3 to be $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2$. Here, the assumption $v_3 = \mathbf{e}_1 - \mathbf{e}_2$ follows from the fact that both the pairs $\mathbf{e}_1, v_3$ and $\mathbf{e}_2, v_3$ need to be part of a lattice basis of $\mathbb{Z}^3$, and the projection directions v_i may be chosen with arbitrary sign. By Lemma 3.2 the polytope P_3, which projects onto Δ_2 along the directions $\mathbf{e}_1$ and $\mathbf{e}_2$, can be fixed to be in the unit cube $\square_3$. Consequently we can assume C_1, C_2 to be in the infinite prisms $\square_3 + \mathbb{R}\mathbf{e}_1$ and $\square_3 + \mathbb{R}\mathbf{e}_2$, respectively. Finally, we assume P_1 and P_2 to be in the infinite prisms C_2 and C_1, respectively. Now consider the linear functional f defined by $(x, y, z) \mapsto x + y$. Consider a lattice point $v_0 \in P_1 \cap (\mathbb{R}^2 \times \{0\})$ minimizing f. Since P_1 projects onto Δ_2 along the direction $\mathbf{e}_1 - \mathbf{e}_2$, one verifies that for any other point $u_0 \in P_1 \cap (\mathbb{R}^2 \times \{0\})$ one has $f(v_0) \le f(u_0) \le f(v_0) + 1$. Analogously, if v_1 is a lattice point in $P_1 \cap (\mathbb{R}^2 \times \{1\})$ minimizing f, then $f(v_1) \le f(u_1) \le f(v_1) + 1$ for all $u_1 \in P_1 \cap (\mathbb{R}^2 \times \{1\})$. Since we are free to translate P_1 along $\mathbf{e}_2$, we can suppose $f(v_0) = 0$ and we denote $k = f(v_1)$. As a consequence, $P_1 \cap (\mathbb{R}^2 \times \{0\})$ is contained in the parallelogram $q_0 := \operatorname{conv}(\mathbf{0}, \mathbf{e}_1 - \mathbf{e}_2) + \operatorname{conv}(\mathbf{0}, \mathbf{e}_2)$. Analogously $P_1 \cap (\mathbb{R}^2 \times \{1\})$ is contained in the parallelogram $q_1 := q_0 + k\mathbf{e}_2 + \mathbf{e}_3$. In particular P_1 is contained in the parallelepiped $\operatorname{conv}(q_0 \cup q_1) = Q_k$. Therefore C_3 is contained in the infinite prism $Q_k + \mathbb{R}(\mathbf{e}_1 - \mathbf{e}_2)$. This completely determines the parallelepiped $R_k = C_1 \cap C_3$, satisfying $R_k \supset P_2$. It is easy to verify that the triple $Q_k, R_k, \square_3$ is equivalent to the triple $Q_{-k}, R_{-k}, \square_3$, so one can always assume $k \in \mathbb{Z}_{\ge 0}$.

In order to see that the set of triples that are subtriples of $Q_k, R_k, \square_3$ for some $k \in \mathbb{Z}_{\ge 0}$ can be covered by 1-parameter families as claimed it suffices to notice that any subtriple of $Q_k, R_k, \square_3$ can be written as $\psi_1^k(P_1^0), \psi_2^k(P_2^0), \square_3$ for subpolytopes $P_1^0 \subset Q_0, P_2^0 \subset R_0$ and $P_3 \subset \square_3$. The fact that any family $\left\{\psi_1^k(P_1^0), \psi_2^k(P_2^0), P_3\right\}_{k \in \mathbb{Z}_{\ge 0}}$ for subpolytopes $P_1^0 \subset Q_0, P_2^0 \subset R_0$ and $P_3 \subset \square_3$ actually contains infinitely many non-equivalent triples can be verified by picking edges $E_i \subset P_i$ for $1 \le i \le 3$ between vertices on height 0 and 1, and noticing that the volume of the parallelepiped $E_1 + E_2 + E_3$ grows quadratically in k. An example of one of these inifinite 1-parameter families is given in Example 1.7. $\square$

A computer assisted search for mixed degree one triples in $Q_k, R_k, \square_3$ for small values of k shows that there are 51 non-equivalent triples when $k = 0$, and 36 for larger values of k, where, for each k, the overlaps that occur for preceding values of k are excluded.

Let us finally mention that the 252 triples classified in Theorem 1.6 *(i) –(iii)* can all be found as subtriples of six special triples which are maximal with respect to inclusion. We have verified this computationally by enumerating subtriples of the six special ones.

Corollary 4.8. *All triples $P_1, P_2, P_3 \subset \mathbb{R}^3$ of 3-dimensional lattice polytopes of mixed degree one of types (i) –(iii) from Theorem 1.6 are, up to equivalence, contained in one of the following 6 maximal triples:*

(a) the maximal triple $\mathcal{P}(2\Delta_2), \mathcal{P}(2\Delta_2), \mathcal{P}(2\Delta_2)$,

(b) the maximal triple $2\Delta_3, \Delta_3, \Delta_3$,

(c) the maximal triple $\{\operatorname{conv}(\mathbf{0}, 2\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) : i, j, k \in [3]$ *pairwise different*$\}$,

(d) the maximal triple

$$\begin{aligned}&\operatorname{conv}(\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2) * \operatorname{conv}(\mathbf{0}, \mathbf{e}_1),\\&\operatorname{conv}(\mathbf{0}, \mathbf{e}_1, -\mathbf{e}_2) * \{-\mathbf{e}_2\},\\&\operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2) * \{\mathbf{e}_2\},\end{aligned}$$

(e) the maximal triple

$$\begin{aligned}&\operatorname{conv}(\mathbf{0}, 2\mathbf{e}_2) * \operatorname{conv}(\mathbf{0}, \mathbf{e}_1),\\&\operatorname{conv}(\mathbf{0}, -\mathbf{e}_1, -\mathbf{e}_1 - \mathbf{e}_2) * \{-\mathbf{e}_1 - 2\mathbf{e}_2\},\\&\operatorname{conv}(\mathbf{0}, \mathbf{e}_2, -\mathbf{e}_1) * \{\mathbf{e}_1\},\end{aligned}$$

(f) the maximal triple

$$\begin{aligned}&\operatorname{conv}(\mathbf{0}, 2\mathbf{e}_2) * \operatorname{conv}(\mathbf{0}, \mathbf{e}_1),\\&\operatorname{conv}(\mathbf{0}, -\mathbf{e}_1, -\mathbf{e}_1 - \mathbf{e}_2) * \{-\mathbf{e}_1 - 2\mathbf{e}_2\},\\&\operatorname{conv}(\mathbf{0}, -\mathbf{e}_2, -\mathbf{e}_1) * \{\mathbf{e}_1 - 2\mathbf{e}_2\}.\end{aligned}$$

Note that the maximal triples (a) and (b) of Corollary 4.8 admit direct generalizations to an arbitrary dimension n that are of mixed degree one. Furthermore it follows from the proof of

Theorem 1.5 that there are no n-tuples of a type analogous to type (iv) of Theorem 1.6 for $n \geq 4$. A bold guess for an answer to Question 3.7 would be that for arbitrary $n \geq 4$ (or n large enough) all exceptions of n-tuples of mixed degree one are contained in one of these generalizations, that is either in $\mathcal{P}^{n-2}(2\Delta_2), \ldots, \mathcal{P}^{n-2}(2\Delta_2)$ or $2\Delta_n, \Delta_n, \ldots, \Delta_n$ (as it can be easily verified that the straightforward generalization of the maximal family (c) to dimension $n \geq 4$ does not yield n-tuples of mixed degree one).

References

[AKW17] Gennadiy Averkov, Jan Krümpelmann, and Stefan Weltge. Notions of maximality for integral lattice-free polyhedra: the case of dimension three. *Math. Oper. Res.*, 42(4):1035–1062, 2017.

[AWW11] Gennadiy Averkov, Christian Wagner, and Robert Weismantel. Maximal lattice-free polyhedra: finiteness and an explicit description in dimension three. *Math. Oper. Res.*, 36(4):721–742, 2011.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[Ber75] D. N. Bernstein. The number of roots of a system of equations. *Funkcional. Anal. i Priložen.*, 9(3):1–4, 1975.

[BH18] Gabriele Balletti and Akihiro Higashitani. Universal inequalities in ehrhart theory. *Israel Journal of Mathematics*, 227(2):843–859, Aug 2018.

[BN08] Victor Batyrev and Benjamin Nill. Combinatorial aspects of mirror symmetry. In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 35–66. Amer. Math. Soc., Providence, RI, 2008.

[BN07] Victor Batyrev and Benjamin Nill. Multiples of lattice polytopes without interior lattice points. *Mosc. Math. J.*, 7(2):195–207, 349, 2007.

[BNR+08] Matthias Beck, Benjamin Nill, Bruce Reznick, Carla Savage, Ivan Soprunov, and Zhiqiang Xu. Let me tell you my favorite lattice-point problem .... In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 179–187. Amer. Math. Soc., Providence, RI, 2008.

[BR15] Matthias Beck and Sinai Robins. *Computing the continuous discretely*. Undergraduate Texts in Mathematics. Springer, New York, second edition, 2015. Integer-point enumeration in polyhedra, With illustrations by David Austin.

[CCD+13] Eduardo Cattani, María Angélica Cueto, Alicia Dickenstein, Sandra Di Rocco, and Bernd Sturmfels. Mixed discriminants. *Math. Z.*, 274(3-4):761–778, 2013.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations*, volume 25 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2010. Structures for algorithms and applications.

[DRHNP11] Sandra Di Rocco, Christian Haase, Benjamin Nill, and Andreas Paffenholz. Polyhedral adjunction theory. *Algebra and Number Theory*, 7, 05 2011.

[DRP09] Alicia Dickenstein, Sandra Di Rocco, and Ragni Piene. Classifying smooth lattice polytopes via toric fibrations. *Advances in Mathematics*, 222(1):240 – 254, 2009.

[EG15] Alexander Esterov and Gleb Gusev. Systems of equations with a single solution. *J. Symbolic Comput.*, 68(part 2):116–130, 2015.

[FI16] K. Furukawa and A. Ito. A combinatorial description of dual defects of toric varieties. *ArXiv e-prints*, May 2016.

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[HT19] Takayuki Hibi and Akiyoshi Tsuchiya. *Algebraic and Geometric Combinatorics on Lattice Polytopes*. WORLD SCIENTIFIC, 2019.

[Kho78] A. G. Khovanskiĭ. Newton polyhedra and the Euler-Jacobi formula. *Uspekhi Mat. Nauk*, 33(6(204)):237–238, 1978.

[KS98] Maximilian Kreuzer and Harald Skarke. Classification of reflexive polyhedra in three dimensions. *Adv. Theor. Math. Phys.*, 2(4):853–871, 1998.

[LZ91] Jeffrey C. Lagarias and Günter M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canad. J. Math.*, 43(5):1022–1035, 1991.

[Nil17] Benjamin Nill. The mixed degree of families of lattice polytopes. http://arxiv.org/abs/1708.03250, 2017, to appear in Annals of Combinatorics.

[NZ11] Benjamin Nill and Günter M. Ziegler. Projecting lattice polytopes without interior lattice points. *Math. Oper. Res.*, 36(3):462–467, 2011.

[Sch14] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[Sop07] Ivan Soprunov. Global residues for sparse polynomial systems. *J. Pure Appl. Algebra*, 209(2):383–392, 2007.

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