A superlinear lower bound on the number of 5-holes

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Abstract

Let P be a finite set of points in the plane in *general position*, that is, no three points of P are on a common line. We say that a set H of five points from P is a 5-*hole in* P if H is the vertex set of a convex 5-gon containing no other points of P. For a positive integer n, let $h_5(n)$ be the minimum number of 5-holes among all sets of n points in the plane in general position.

Despite many efforts in the last 30 years, the best known asymptotic lower and upper bounds for $h_5(n)$ have been of order $\Omega(n)$ and $O(n^2)$, respectively. We show that $h_5(n) = \Omega(n \log^{4/5} n)$, obtaining the first superlinear lower bound on $h_5(n)$.

The following structural result, which might be of independent interest, is a crucial step in the proof of this lower bound. If a finite set P of points in the plane in general position is partitioned by a line ℓ into two subsets, each of size at least 5 and not in convex position, then ℓ intersects the convex hull of some 5-hole in P. The proof of this result is computer-assisted.

1 Introduction

We say that a set of points in the plane is in *general position* if it contains no three points on a common line. A point set is in *convex position* if it is the vertex set of a convex polygon. In 1935, Erdős and Szekeres [16] proved the following theorem, which is a classical result both in combinatorial geometry and Ramsey theory.

Theorem ([16], The Erdős–Szekeres Theorem)**.** *For every integer $k \geq 3$, there is a smallest integer $n = n(k)$ such that every set of at least n points in general position in the plane contains k points in convex position.*

The Erdős–Szekeres Theorem motivated a lot of further research, including numerous modifications and extensions of the theorem. Here we mention only results closely related to the main topic of our paper.

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Let P be a finite set of points in general position in the plane. We say that a set H of k points from P is a *k-hole in P* if H is the vertex set of a convex k-gon containing no other points of P. In the 1970s, Erdős [15] asked whether, for every positive integer k, there is a k-hole in every sufficiently large finite point set in general position in the plane. Harborth [21] proved that there is a 5-hole in every set of 10 points in general position in the plane and gave a construction of 9 points in general position with no 5-hole. After unsuccessful attempts of researchers to answer Erdős' question affirmatively for any fixed integer $k \geq 6$, Horton [22] constructed, for every positive integer n, a set of n points in general position in the plane with no 7-hole. His construction was later generalized to so-called *Horton sets* and *squared Horton sets* [30] and to higher dimensions [31]. The question whether there is a 6-hole in every sufficiently large finite planar point set remained open until 2007 when Gerken [19] and Nicolás [23] independently gave an affirmative answer.

For positive integers n and k, let $h_k(n)$ be the minimum number of k-holes in a set of n points in general position in the plane. Due to Horton's construction, $h_k(n) = 0$ for every n and every $k \geq 7$. Asymptotically tight estimates for the functions $h_3(n)$ and $h_4(n)$ are known. The best known lower bounds are due to Aichholzer et al. [5] who showed that $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$. The best known upper bounds $h_3(n) \leq 1.6196n^2 + o(n^2)$ and $h_4(n) \leq 1.9397n^2 + o(n^2)$ are due to Bárány and Valtr [12].

For $h_5(n)$ and $h_6(n)$, no matching bounds are known. So far, the best known asymptotic upper bounds on $h_5(n)$ and $h_6(n)$ were obtained by Bárány and Valtr [12] and give $h_5(n) \leq 1.0207n^2 + o(n^2)$ and $h_6(n) \leq 0.2006n^2 + o(n^2)$. For the lower bound on $h_6(n)$, Valtr [32] showed $h_6(n) \geq n/229 - 4$.

In this paper we give a new lower bound on $h_5(n)$. It is widely conjectured that $h_5(n)$ grows quadratically in n, but to this date only lower bounds on $h_5(n)$ that are linear in n have been known. As noted by Bárány and Füredi [10], a linear lower bound of $\lfloor n/10 \rfloor$ follows directly from Harborth's result [21]. Bárány and Károlyi [11] improved this bound to $h_5(n) \geq n/6 - O(1)$. In 1987, Dehnhardt [14] showed $h_5(11) = 2$ and $h_5(12) = 3$, obtaining $h_5(n) \geq 3\lfloor n/12 \rfloor$. However, his result remained unknown to the scientific community until recently. García [18] then presented a proof of the lower bound $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$ and a slightly better estimate $h_5(n) \geq \lceil 3/7(n-11) \rceil$ was shown by Aichholzer, Hackl, and Vogtenhuber [6]. Quite recently, Valtr [32] obtained $h_5(n) \geq n/2 - O(1)$. This was strengthened by Aichholzer et al. [5] to $h_5(n) \geq 3n/4 - o(n)$. All improvements on the multiplicative constant were achieved by utilizing the values of $h_5(10)$, $h_5(11)$, and $h_5(12)$. In the bachelor's thesis of Scheucher [27] the exact values $h_5(13) = 3$, $h_5(14) = 6$, and $h_5(15) = 9$ were determined and $h_5(16) \in \{10, 11\}$ was shown. During the preparation of this paper, we further determined the value $h_5(16) = 11$; see the webpage [26]. Table 1 summarizes our knowledge on the values of $h_5(n)$ for $n \leq 20$. The values $h_5(n)$ for $n \leq 16$ can be used to obtain further improvements on the multiplicative constant. By revising the proofs of [5, Lemma 1] and [5, Theorem 3], one can obtain $h_5(n) \geq n - 10$ and $h_5(n) \geq 3n/2 - o(n)$, respectively. We also note that it was shown in [25] that if $h_3(n) \geq (1+\epsilon)n^2 - o(n^2)$, then $h_5(n) = \Omega(n^2)$.

n	9	10	11	12	13	14	15	16	17	18	19	20
$h_5(n)$	0	1	2	3	3	6	9	11	≤ 16	≤ 21	≤ 26	≤ 33

Table 1: The minimum number $h_5(n)$ of 5-holes determined by any set of $n \leq 20$ points.

As our main result, we give the first superlinear lower bound on $h_5(n)$. This solves an open problem, which was explicitely stated, for example, in a book by Brass, Moser, and Pach [13, Chapter 8.4, Problem 5] and in the survey [2].

Theorem 1. *There is an absolute constant $c > 0$ such that for every integer $n \geq 10$ we have $h_5(n) \geq cn \log^{4/5} n$.*

Let P be a finite set of points in the plane in general position and let ℓ be a line that contains no point of P. We say that P is *ℓ-divided* if there is at least one point of P in each of the two halfplanes determined by ℓ. For an ℓ-divided set P, we use $P = A \cup B$ to denote the fact that ℓ partitions P into the subsets A and B. In the rest of the paper, we assume without loss of generality that ℓ is vertical and directed upwards, A is to the left of ℓ, and B is to the right of ℓ.

The following result, which might be of independent interest, is a crucial step in the proof of Theorem 1.

Theorem 2. *Let $P = A \cup B$ be an ℓ-divided set with $|A|, |B| \geq 5$ and with neither A nor B in convex position. Then there is an ℓ-divided 5-hole in P.*

The proof of Theorem 2 is computer-assisted. We reduce the result to several statements about point sets of size at most 11 and then verify each of these statements by an exhaustive computer search. To verify the computer-aided proofs we have implemented two independent programs, which, in addition, are based on different abstractions of point sets; see Subsection 5.2. Some of the tools that we use originate from the bachelor's theses of Scheucher [27, 28].

Using a result of García [18], we adapt the proof of Theorem 1 to provide improved lower bounds on the minimum numbers of 3-holes and 4-holes.

Theorem 3. *The following two bounds are satisfied for every positive integer n:*

(i) $h_3(n) \geq n^2 + \Omega(n \log^{2/3} n)$ *and*

(ii) $h_4(n) \geq \frac{n^2}{2} + \Omega(n \log^{3/4} n)$.

In the rest of the paper, we assume that every point set P is planar, finite, and in general position. We also assume, without loss of generality, that all points in P have distinct x-coordinates. We use $\text{conv}(P)$ to denote the convex hull of P and $\partial \, \text{conv}(P)$ to denote the boundary of the convex hull of P.

A subset Q of P that satisfies $P \cap \text{conv}(Q) = Q$ is called an *island of P*. Note that every k-hole in an island Q of P is also a k-hole in P. For any subset R of the plane, if R contains no point of P, then we say that R is *empty of points of P*.

In Section 2 we derive quite easily Theorem 1 from Theorem 2. Theorem 3 is proved in Section 3. Then, in Section 4, we give some preliminaries for the proof of Theorem 2, which is presented in Section 5. Finally, in Section 6, we give some final remarks. In particular, we show that the assumptions in Theorem 2 are necessary. To provide a better general view, we present a flow summary of the proof of Theorem 1 in Appendix A.

2 Proof of Theorem 1

We now apply Theorem 2 to obtain a superlinear lower bound on the number of 5-holes in a given set of n points. It clearly suffices to prove the statement for the case in which $n = 2^t$ for some integer $t \geq 5^5$.

We prove by induction on $t \geq 5^5$ that the number of 5-holes in an arbitrary set P of $n = 2^t$ points is at least $f(t) := c \cdot 2^t t^{4/5} = c \cdot n \log_2^{4/5} n$ for some absolute constant $c > 0$. For $t = 5^5$, we have $n > 10$ and, by the result of Harborth [21], there is at least one 5-hole in P. If the constant c is sufficiently small, then $f(t) = c \cdot n \log_2^{4/5} n \leq 1$ and we have at least $f(t)$ 5-holes in P, which constitutes our base case.

For the inductive step we assume that $t > 5^5$. We first partition P with a line ℓ into two sets A and B of size $n/2$ each. Then we further partition A and B into smaller sets using the following well-known lemma, which is, for example, implied by a result of Steiger and Zhao [29, Theorem 1].

Lemma 4 ([29])**.** *Let $P' = A' \cup B'$ be an ℓ-divided set and let r be a positive integer such that $r \leq |A'|, |B'|$. Then there is a line that is disjoint from P' and that determines an open halfplane h with $|A' \cap h| = r = |B' \cap h|$.*

We set $r := \lfloor \log_2^{1/5} n \rfloor$, $s := \lfloor n/(2r) \rfloor$, and apply Lemma 4 iteratively in the following way to partition P into islands $P_1, \dots, P_{s+1}$ of P so that for every $i \in \{1, \dots, s\}$, the size of each $P_i \cap A$ and $P_i \cap B$ is exactly r. Let $P'_0 := P$. For every $i = 1, \dots, s$, we consider a line that is disjoint from P'_{i-1} and that determines an open halfplane h with $|P'_{i-1} \cap A \cap h| = r = |P'_{i-1} \cap B \cap h|$. Such a line exists by Lemma 4 applied to the ℓ-divided set P'_{i-1}. We then set $P_i := P'_{i-1} \cap h$, $P'_i := P'_{i-1} \setminus P_i$, and continue with $i+1$. Finally, we set $P_{s+1} := P'_s$.

Let $i \in \{1, \dots, s\}$. If one of the sets $P_i \cap A$ and $P_i \cap B$ is in convex position, then there are at least $\binom{r}{5}$ 5-holes in P_i and, since P_i is an island of P, we have at least $\binom{r}{5}$ 5-holes in P. If this is the case for at least $s/2$ islands P_i, then, given that $s = \lfloor n/(2r) \rfloor$ and thus $s/2 \geq \lfloor n/(4r) \rfloor$, we obtain at least $\lfloor n/(4r) \rfloor \binom{r}{5} \geq c \cdot n \log_2^{4/5} n$ 5-holes in P for a sufficiently small constant $c > 0$.

We thus further assume that for more than $s/2$ islands P_i, neither of the sets $P_i \cap A$ nor $P_i \cap B$ is in convex position. Since $r = \lfloor \log_2^{1/5} n \rfloor \geq 5$, Theorem 2 implies that there is an ℓ-divided 5-hole in each such P_i. Thus there is an ℓ-divided 5-hole in P_i for more than $s/2$ islands P_i. Since each P_i is an island of P and since $s = \lfloor n/(2r) \rfloor$, we have more than $s/2 \geq \lfloor n/(4r) \rfloor$ ℓ-divided 5-holes in P. As $|A| = |B| = n/2 = 2^{t-1}$, there are at least $f(t-1)$ 5-holes in A and at least $f(t-1)$ 5-holes in B by the inductive assumption. Since A and B are separated by the line ℓ, we have at least

$$2f(t-1) + n/(4r) = 2c(n/2) \log_2^{4/5} (n/2) + n/(4r) \geq cn(t-1)^{4/5} + n/(4t^{1/5})$$

5-holes in P. The right side of the above expression is at least $f(t) = cnt^{4/5}$, because the inequality $cn(t-1)^{4/5} + n/(4t^{1/5}) \geq cnt^{4/5}$ is equivalent to the inequality $(t-1)^{4/5}t^{1/5} + 1/(4c) \geq t$, which is true if the constant c is sufficiently small, as $(t-1)^{4/5}t^{1/5} \geq t-1$. This finishes the proof of Theorem 1.

3 Proof of Theorem 3

In this section we improve the lower bounds on the minimum number of 3-holes and 4-holes. To this end we use the notion of generated holes as introduced by García [18].

Given a 5-hole H in a point set P, a 3-hole in P is *generated by H* if it is spanned by the leftmost point p of H and the two vertices of H that are not adjacent to p on the boundary of $\mathrm{conv}(H)$. Similarly, a 4-hole in P is *generated by H* if it is spanned by the vertices of H

with the exception of one of the points adjacent to the leftmost point of H on the boundary of $\text{conv}(H)$. We call a 3-hole or a 4-hole in P *generated* if it is generated by some 5-hole in P. We denote the number of generated 3-holes and generated 4-holes in P by $h_{3|5}(P)$ and $h_{4|5}(P)$, respectively. We also denote by $h_{3|5}(n)$ and $h_{4|5}(n)$ the minimum of $h_{3|5}(P)$ and $h_{4|5}(P)$, respectively, among all sets P of n points.

For an integer $k \geq 3$ and a point set P, let $h_k(P)$ be the number of k-holes in P. We say that a point from P is *extremal* in P if it is a vertex of the polygon $\text{conv}(P)$. A point from P that is not extremal is *inner* in P. García [18] proved the following relationships between $h_3(P)$ and $h_{3|5}(P)$ and between $h_4(P)$ and $h_{4|5}(P)$.

Theorem 5 ([18]). *Let P be a set of n points and let $\gamma(P)$ be the number of extremal points of P. Then the following two equalities are satisfied:*

(i) $h_3(P) = n^2 - 5n + \gamma(P) + 4 + h_{3|5}(P)$ *and*

(ii) $h_4(P) = \frac{n^2}{2} - \frac{7n}{2} + \gamma(P) + 3 + h_{4|5}(P)$.

The proofs of both parts of Theorem 3 are carried out by induction on n similarly to the proof of Theorem 1. The base cases follow from the fact that each set P of $n \geq 10$ points contains at least one 5-hole in P and thus a generated 3-hole in P and a generated 4-hole in P. For the inductive step, let $P = A \cup B$ be an ℓ-divided set of n points with $|A|, |B| \geq \lfloor \frac{n}{2} \rfloor$, where n is a sufficiently large positive integer.

To show part (i), it suffices to prove $h_{3|5}(P) \geq \Omega(n \log^{2/3} n)$ as the statement then follows from Theorem 5. We use the recursive approach from the proof of Theorem 1, where we choose $r = \lfloor \log_2^{1/3} n \rfloor$. In each step of the recursion we either obtain $\lfloor \frac{n}{4r} \rfloor$ pairwise disjoint r-holes in P or $\lfloor \frac{n}{4r} \rfloor$ pairwise disjoint ℓ-divided 5-holes in P.

In the first case, each r-hole in P admits $\binom{r}{3}$ 3-holes in P and, by Theorem 5, it contains $\binom{r}{3} - r^2 + 5r - r - 4$ generated 3-holes in P. Thus, in total, we count at least $\frac{n}{4r}\binom{r}{3} - O(nr) \geq \Omega(n \log^{2/3} n)$ generated 3-holes in P.

In the second case, we have at least $\lfloor \frac{n}{4r} \rfloor$ ℓ-divided 5-holes in P. Without loss of generality, we can assume that at least $\frac{1}{2} \lfloor \frac{n}{4r} \rfloor \geq \lfloor \frac{n}{8r} \rfloor$ of those ℓ-divided 5-holes in P contain at least two points to the right of ℓ, as we otherwise continue with the horizontal reflection of P, which has ℓ as the axis of reflection. Therefore we have at least $\lfloor \frac{n}{8r} \rfloor$ ℓ-divided generated 3-holes in P and, analogously as in the proof of Theorem 1, we obtain

$$h_{3|5}(P) \geq 2h_{3|5}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \left\lfloor \frac{n}{4r} \right\rfloor \geq \Omega(n \log^{2/3} n).$$

This finishes the proof of part (i).

The proof of part (ii) is almost identical. We choose $r = \lfloor \log_2^{1/4} n \rfloor$ and use the facts that every r-hole in P contains $\binom{r}{4} - \frac{r^2}{2} + \frac{7r}{2} - r - 3$ generated 4-holes in P and that every ℓ-divided 5-hole in P generates two 4-holes in P, at least one of which is ℓ-divided. This finishes the proof of Theorem 3.

4 Preliminaries for the proof of Theorem 2

Before proceeding with the proof of Theorem 2, we first introduce some notation and definitions, and state some immediate observations.

Let a, b, c be three distinct points in the plane. We denote the line segment spanned by a and b as ab, the ray starting at a and going through b as $\overrightarrow{ab}$, and the line through a and b directed from a to b as $\overline{ab}$. We say c is to the *left* (*right*) of $\overline{ab}$ if the triple (a, b, c) traced in this order is oriented counterclockwise (clockwise). Note that c is to the left of $\overline{ab}$ if and only if c is to the right of $\overline{ba}$, and that the triples (a, b, c), (b, c, a), and (c, a, b) have the same orientation. We say a point set S is to the *left* (*right*) of $\overline{ab}$ if every point of S is to the left (right) of $\overline{ab}$.

Sectors of polygons For an integer $k \geq 3$, let $\mathcal{P}$ be a convex polygon with vertices $p_1, p_2, \ldots, p_k$ traced counterclockwise in this order. We denote by $S(p_1, p_2, \ldots, p_k)$ the open convex region to the left of each of the three lines $\overline{p_1p_2}$, $\overline{p_1p_k}$, and $\overline{p_{k-1}p_k}$. We call the region $S(p_1, p_2, \ldots, p_k)$ a *sector* of $\mathcal{P}$. Note that every convex k-gon defines exactly k sectors. Figure 1(a) gives an illustration.

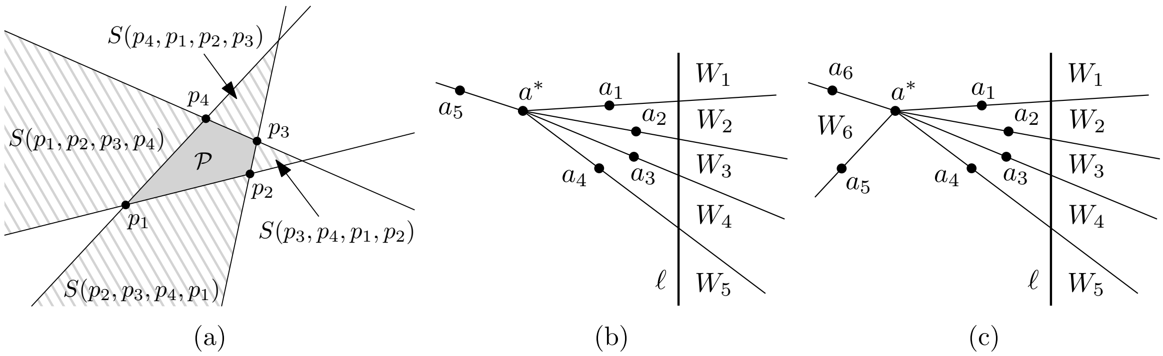

Figure 1: (a) An example of sectors. (b) An example of a^*-wedges with $t = |A| - 1$. (c) An example of a^*-wedges with $t < |A| - 1$.

We use $\triangle(p_1, p_2, p_3)$ to denote the closed triangle with vertices p_1, p_2, p_3. We also use $\square(p_1, p_2, p_3, p_4)$ to denote the closed quadrilateral with vertices p_1, p_2, p_3, p_4 traced in the counterclockwise order along the boundary.

The following simple observation summarizes some properties of sectors of polygons.

Observation 6. *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P. Then the following conditions are satisfied.*

(i) Every sector of an ℓ-divided 4-hole in P is empty of points of P.

(ii) If S is a sector of a 4-hole in A and S is empty of points of A, then S is empty of points of B.

***ℓ*-critical sets and islands** An ℓ-divided set $C = A \cup B$ is called *ℓ-critical* if it fulfills the following two conditions.

(i) Neither A nor B is in convex position.

(ii) For every extremal point x of C, one of the sets $(C \setminus \{x\}) \cap A$ and $(C \setminus \{x\}) \cap B$ is in convex position.

Note that every ℓ-critical set $C = A \cup B$ contains at least four points in each of A and B. Figure 2 shows some examples of ℓ-critical sets. If $P = A \cup B$ is an ℓ-divided set with neither A nor B in convex position, then there exists an ℓ-critical island of P. This can be seen by iteratively removing extremal points so that none of the parts is in convex position after the removal.

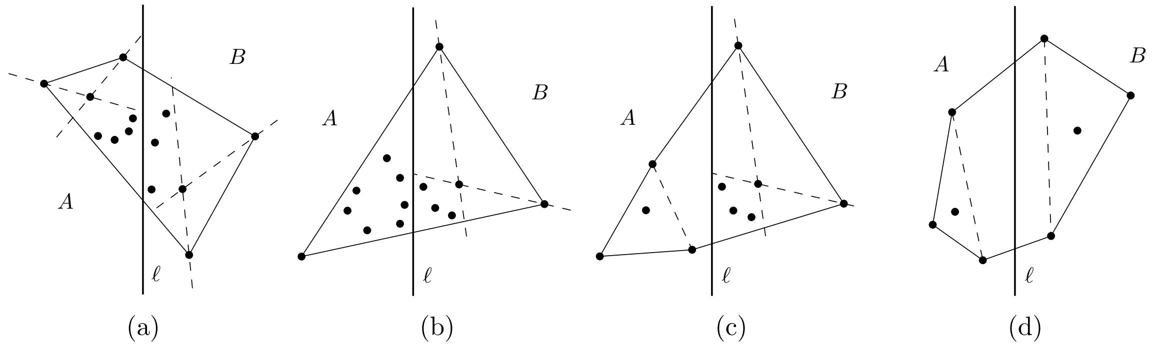

Figure 2: Examples of ℓ-critical sets.

***a*-wedges and *a**-wedges** Let $P = A \cup B$ be an ℓ-divided set. For a point a in A, the rays $\overrightarrow{aa'}$ for all $a' \in A \setminus \{a\}$ partition the plane into $|A| - 1$ regions. We call the closures of those regions *a-wedges* and label them as $W_1^{(a)}, \ldots, W_{|A|-1}^{(a)}$ in the clockwise order around a, where $W_1^{(a)}$ is the topmost a-wedge that intersects ℓ. Let $t^{(a)}$ be the number of a-wedges that intersect ℓ. Note that $W_1^{(a)}, \ldots, W_{t^{(a)}}^{(a)}$ are the a-wedges that intersect ℓ sorted in top-to-bottom order on ℓ. Also note that all a-wedges are convex if a is an inner point of A, and that there exists exactly one non-convex a-wedge otherwise. The indices of the a-wedges are considered modulo $|A| - 1$. In particular, $W_0^{(a)} = W_{|A|-1}^{(a)}$ and $W_{|A|}^{(a)} = W_1^{(a)}$.

If A is not in convex position, we denote the rightmost inner point of A as a^* and write $t := t^{(a^*)}$ and $W_k := W_k^{(a^*)}$ for $k = 1, \ldots, |A| - 1$. Recall that a^* is unique, since all points have distinct x-coordinates. Figures 1(b) and 1(c) give an illustration.

We set $w_k := |B \cap W_k|$ and label the points of A so that W_k is bounded by the rays $\overrightarrow{a^* a_{k-1}}$ and $\overrightarrow{a^* a_k}$ for $k = 1, \ldots, |A| - 1$. Again, the indices are considered modulo $|A| - 1$. In particular, $a_0 = a_{|A|-1}$ and $a_{|A|} = a_1$.

Observation 7. *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position. Then the points $a_1, \ldots, a_{t-1}$ lie to the right of a^* and the points $a_t, \ldots, a_{|A|-1}$ lie to the left of a^*.*

5 Proof of Theorem 2

First, we give a high-level overview of the main ideas of the proof of Theorem 2. We proceed by contradiction and we suppose that there is no ℓ-divided 5-hole in a given ℓ-divided set $P = A \cup B$ with $|A|, |B| \geq 5$ and with neither A nor B in convex position. If $|A|, |B| = 5$, then the statement follows from the result of Harborth [21]. Thus we assume that $|A| \geq 6$ or $|B| \geq 6$. We reduce P to an island Q of P by iteratively removing points from the convex hull until one of the two parts $Q \cap A$ and $Q \cap B$ contains exactly five points or Q is ℓ-critical with $|Q \cap A|, |Q \cap B| \geq 6$. If $|Q \cap A| = 5$ and $|Q \cap B| \geq 6$ or vice versa, then we reduce

Q to an island of Q with eleven points and, using a computer-aided result (Lemma 14), we show that there is an ℓ-divided 5-hole in that island and hence in P. If Q is ℓ-critical with $|Q \cap A|, |Q \cap B| \geq 6$, then we show that $|A \cap \partial \operatorname{conv}(Q)|, |B \cap \partial \operatorname{conv}(Q)| \leq 2$ and that, if $|A \cap \partial \operatorname{conv}(Q)| = 2$, then a^* is the only inner point of $Q \cap A$ and similarly for B (Lemma 19). Without loss of generality, we assume that $|A \cap \partial \operatorname{conv}(Q)| = 2$ and thus a^* is the only inner point of $Q \cap A$. Using this assumption, we prove that $|Q \cap B| < |Q \cap A|$ (Proposition 21). By exchanging the roles of $Q \cap A$ and $Q \cap B$, we obtain $|Q \cap A| \leq |Q \cap B|$ (Proposition 22), which gives a contradiction.

To prove that $|Q \cap B| < |Q \cap A|$, we use three results about the sizes of the parameters $w_1, \ldots, w_t$ for the ℓ-divided set Q, that is, about the numbers of points of $Q \cap B$ in the a^*-wedges $W_1, \ldots, W_t$ of Q. We show that if we have $w_i = 2 = w_j$ for some $1 \leq i < j \leq t$, then $w_k = 0$ for some k with $i < k < j$ (Lemma 12). Further, for any three or four consecutive a^*-wedges whose union is convex and contains at least four points of $Q \cap B$, each of those a^*-wedges contains at most two such points (Lemma 18). Finally, we show that $w_1, \ldots, w_t \leq 3$ (Lemma 20). The proofs of Lemmas 18 and 20 rely on some results about small ℓ-divided sets with computer-aided proofs (Lemmas 15, 16, and 17). Altogether, this is sufficient to show that $|Q \cap B| < |Q \cap A|$.

We now start the proof of Theorem 2 by showing that if there is an ℓ-divided 5-hole in the intersection of P with a union of consecutive a^*-wedges, then there is an ℓ-divided 5-hole in P.

Lemma 8. *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position. For integers i, j with $1 \leq i \leq j \leq t$, let $W := \bigcup_{k=i}^{j} W_k$ and $Q := P \cap W$. If there is an ℓ-divided 5-hole in Q, then there is an ℓ-divided 5-hole in P.*

Proof. If W is convex then Q is an island of P and the statement immediately follows. Hence we assume that W is not convex. The region W is bounded by the rays $\overrightarrow{a^*a_{i-1}}$ and $\overrightarrow{a^*a_j}$ and all points of $P \setminus Q$ lie in the convex region $\mathbb{R}^2 \setminus W$; see Figure 3.

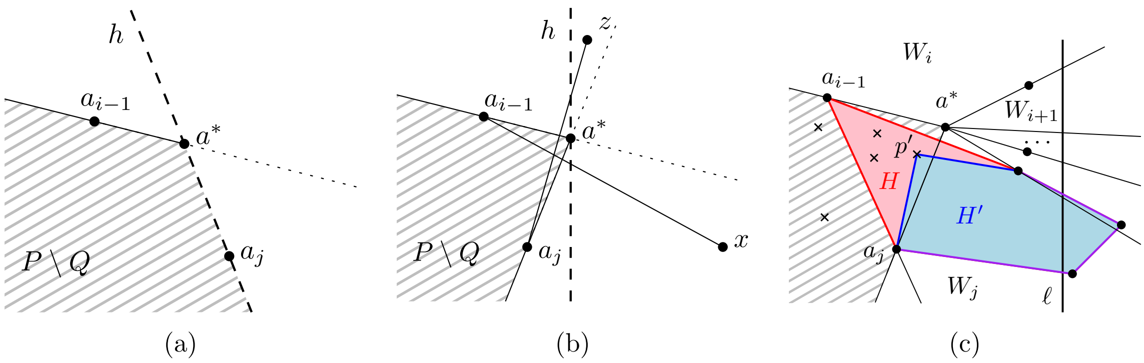

Figure 3: Illustration of the proof of Lemma 8. (a) The point a_j is to the right of a^*. (b) The point a_j is to the left of a^*. (c) The hole H properly intersects the ray $\overrightarrow{a^*a_j}$. The boundary of the convex hull of H is drawn red and the convex hull of H' is drawn blue.

Since W is non-convex and every a^*-wedge contained in W intersects ℓ, at least one of the points a_{i-1} and a_j lies to the left of a^*. Moreover, the points $a_i, \ldots, a_{j-1}$ are to the right of a^* by Observation 7. Without loss of generality, we assume that a_{i-1} is to the left of a^*, as otherwise we consider the vertical reflection of the whole point set P.

If a_j is to the left of a^*, then we let h be the closed halfplane determined by the vertical line through a^* such that a_{i-1} and a_j lie in h. Otherwise, if a_j is to the right of a^*, then we let h be the closed halfplane determined by the line $\overline{a^*a_j}$ such that a_{i-1} lies in h. In either case, $h \cap A \cap Q = \{a^*, a_{i-1}, a_j\}$.

Let H be an ℓ-divided 5-hole in Q. We say that H *properly intersects* a ray r if the interior of $\text{conv}(H)$ intersects r. Now we show that if H properly intersects the ray $\overrightarrow{a^*a_j}$, then H contains a_{i-1}. Assume there are points $p, q \in H$ such that the relative interior of pq intersects $r := \overrightarrow{a^*a_j}$. Since r lies in h and neither of p and q lies in r, at least one of the points p and q lies in $h \setminus r$. Without loss of generality, we assume $p \in h \setminus r$. From $h \cap A \cap Q = \{a^*, a_{i-1}, a_j\}$ we have $p = a_{i-1}$. By symmetry, if H properly intersects the ray $\overrightarrow{a^*a_{i-1}}$, then H contains a_j.

Suppose for contradiction that H properly intersects both rays $\overrightarrow{a^*a_{i-1}}$ and $\overrightarrow{a^*a_j}$. Then H contains the points a_{i-1}, a_j, x, y, z for some points $x, y, z \in Q$, where $a_{i-1}x$ intersects $\overrightarrow{a^*a_j}$, and a_jz intersects $\overrightarrow{a^*a_{i-1}}$. Observe that z is to the left of $\overline{a_{i-1}a^*}$ and that x is to the right of $\overline{a_ja^*}$. If a_j lies to the right of a^*, then z is to the left of a^*, and thus z is in A; see Figure 3(a). However, this is impossible as z also lies in h. Hence, a_j lies to the left of a^*; see Figure 3(b). As x and z are both to the right of a^*, the point a^* is inside the convex quadrilateral $\square(a_{i-1}, a_j, x, z)$. This contradicts the assumption that H is a 5-hole in Q.

So assume that H properly intersects exactly one of the rays $\overrightarrow{a^*a_{i-1}}$ and $\overrightarrow{a^*a_j}$, say $\overrightarrow{a^*a_j}$; see Figure 3(c). In this case, H contains a_{i-1}. The interior of the triangle $\triangle(a^*, a_{i-1}, a_j)$ is empty of points of Q, since the triangle is contained in h. Moreover, $\text{conv}(H)$ cannot intersect the line that determines h both strictly above and strictly below a^*. Thus, all remaining points of $H \setminus \{a_{i-1}\}$ lie to the right of $\overline{a_{i-1}a^*}$ and to the right of $\overline{a_ja^*}$. If H is empty of points of $P \setminus Q$, we are done. Otherwise, we let $H' := (H \setminus \{a_{i-1}\}) \cup \{p'\}$ where $p' \in P \setminus Q$ is a point inside $\triangle(a^*, a_{i-1}, a_j)$ closest to $\overline{a_ja^*}$. Note that the point p' might not be unique. By construction, H' is an ℓ-divided 5-hole in P. An analogous argument shows that there is an ℓ-divided 5-hole in P if H properly intersects $\overrightarrow{a^*a_{i-1}}$.

Finally, if H does not properly intersect any of the rays $\overrightarrow{a^*a_{i-1}}$ and $\overrightarrow{a^*a_j}$, then $\text{conv}(H)$ contains no point of $P \setminus Q$ in its interior, and hence H is an ℓ-divided 5-hole in P. □

5.1 Sequences of a^*-wedges with at most two points of B

In this subsection we consider an ℓ-divided set $P = A \cup B$ with A not in convex position. We consider the union W of consecutive a^*-wedges, each containing at most two points of B, and derive an upper bound on the number of points of B that lie in W if there is no ℓ-divided 5-hole in $P \cap W$; see Corollary 13.

Observation 9. *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position. Let W_k be an a^*-wedge with $w_k \geq 1$ and $1 \leq k \leq t$ and let b be the leftmost point in $W_k \cap B$. Then the points a^*, a_{k-1}, b, and a_k form an ℓ-divided 4-hole in P.*

From Observation 6(i) and Observation 9 we obtain the following result.

Observation 10. *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position and with no ℓ-divided 5-hole in P. Let W_k be an a^*-wedge with $w_k \geq 2$ and $1 \leq k \leq t$ and let b be the leftmost point in $W_k \cap B$. For every point b' in $(W_k \cap B) \setminus \{b\}$, the line $\overline{bb'}$ intersects the segment $a_{k-1}a_k$. Consequently, b is inside $\triangle(a_{k-1}, a_k, b')$, to the left of $\overline{a_kb'}$, and to the right of $\overline{a_{k-1}b'}$.*

The following lemma states that there is an ℓ-divided 5-hole in P if two consecutive a^*-wedges both contain exactly two points of B.

Lemma 11. *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position and with $|A|, |B| \geq 5$. Let W_i and W_{i+1} be consecutive a^*-wedges with $w_i = 2 = w_{i+1}$ and $1 \leq i < t$. Then there is an ℓ-divided 5-hole in P.*

Proof. The overall idea of the proof is as follows. We suppose for contradiction that there is no ℓ-divided 5-hole in P. Then we prove a sequence of structural facts on the layout of the points of P forced by this assumption. Eventually we show that the structure of the point set P resembles the point set from Figure 6(a). In particular, we arrive at the conclusion that $|B| = 4$, which contradicts our assumption $|B| \geq 5$.

Suppose for contradiction that there is no ℓ-divided 5-hole in P. Let $W := W_i \cup W_{i+1}$ and let $Q := P \cap W$. By Lemma 8, there is also no ℓ-divided 5-hole in Q. We label the points in $B \cap W_i$ as b_{i-1} and b_i so that b_{i-1} is to the right of b_i. Similarly, we label the points in $B \cap W_{i+1}$ as b_{i+1} and b_{i+2} so that b_{i+2} is to the right of b_{i+1}. By Observation 10, the point a_i is to the right of $\overline{b_i b_{i-1}}$ and to the left of $\overline{b_{i+1} b_{i+2}}$. If the points $b_{i-1}, b_i, b_{i+1}, b_{i+2}$ are in convex position, then $a_i, b_{i+1}, b_{i+2}, b_{i-1}, b_i$ form an ℓ-divided 5-hole in P; see Figure 4(a). Thus, we assume the points $b_{i-1}, b_i, b_{i+1}, b_{i+2}$ are not in convex position. Without loss of generality, we assume that the line $\overline{b_i b_{i-1}}$ intersects the segment $b_{i+1} b_{i+2}$, as otherwise we consider the vertical reflection of the whole point set P.

Claim 11.1. *The segments $a_i b_{i-1}$ and $b_i b_{i+1}$ intersect.*

As $\overline{b_i b_{i-1}}$ intersects $a_i a_{i-1}$ and $b_{i+1} b_{i+2}$, the point b_{i-1} lies in the triangle $\triangle(b_i, b_{i+1}, b_{i+2})$. Moreover, b_{i-1} is to the right of $\overline{b_{i+1} b_i}$, a_i is to the left of $\overline{b_{i+1} b_i}$, b_i is to the left of $\overline{a_i b_{i-1}}$, and b_{i+1} is to the right of $\overline{a_i b_{i-1}}$. Consequently, the points $a_i, b_{i+1}, b_{i-1}, b_i$ form an ℓ-divided 4-hole in P, and, in particular, the segments $a_i b_{i-1}$ and $b_i b_{i+1}$ intersect; see Figure 4(b). This finishes the proof of Claim 11.1.

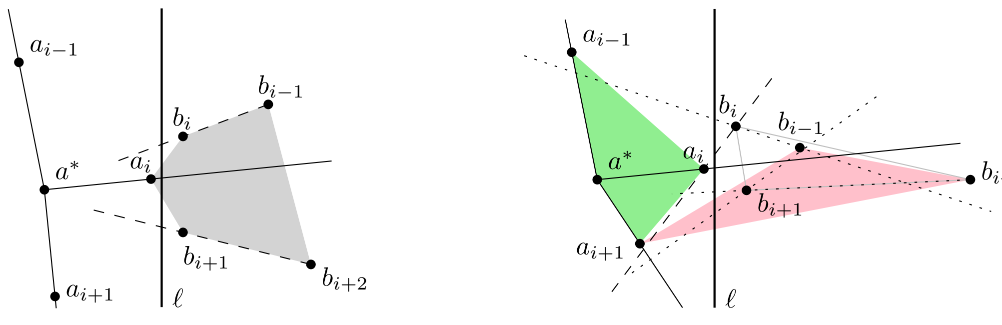

Figure 4: (a) If $b_{i-1}, b_i, b_{i+1}, b_{i+2}$ are in convex position, then there is an ℓ-divided 5-hole in P. (b) The points $a^*, a_{i+1}, a_i, a_{i-1}$ form a 4-hole in P.

The points $a_{i-1}, b_i, b_{i-1}, b_{i+2}$ are in convex position because a_{i-1} is the leftmost and b_{i+2} is the rightmost of those four points and because both a_{i-1} and b_{i+2} lie to the left of $\overline{b_i b_{i-1}}$. Moreover, the points $a_{i-1}, b_i, b_{i-1}, b_{i+2}$ form an ℓ-divided 4-hole in P as $\square(a_{i-1}, b_i, b_{i-1}, b_{i+2})$ lies in W and $w_i = w_{i+1} = 2$.

Claim 11.2. *Among the four points $b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1}$, the clockwise order around b_{i+2} is $a_{i+1}, b_{i+1}, b_{i-1}$.*

The point b_{i+2} is the rightmost of those four points. By Observation 10, b_{i+1} lies to the right of $\overline{a_i b_{i+2}}$ and a_{i+1} lies to the right of $\overline{b_{i+1} b_{i+2}}$. Since $b_{i-1} \in W_i$ and $b_{i+2} \in W_{i+1}$, the point b_{i-1} lies to the left of $\overline{a_i b_{i+2}}$. This finishes the proof of Claim 11.2.

Claim 11.3. *The points $b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1}$ are not in convex position.*

Suppose for contradiction that the points $b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1}$ form a convex quadrilateral. Due to the clockwise order around b_{i+2}, the convex quadrilateral is $\square(b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1})$. The only points of P that can lie in the interior of this quadrilateral are a^*, a_{i-1}, a_i, and b_i. Since the triangle $\triangle(b_{i+2}, b_{i+1}, a_{i+1})$ is contained in W_{i+1}, it contains neither of the points a^*, a_{i-1}, a_i, and b_i. Since the triangle $\triangle(b_{i+2}, b_{i-1}, b_{i+1})$ is contained in the convex hull of B, it does not contain a^*, a_{i-1}, nor a_i. Moreover, as b_{i-1} lies in the triangle $\triangle(b_i, b_{i+1}, b_{i+2})$, the triangle $\triangle(b_{i+2}, b_{i-1}, b_{i+1})$ also does not contain b_i. Thus the quadrilateral $\square(b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1})$ is empty of points of P. By Observation 6(i), the two sectors $S(a_{i-1}, b_i, b_{i-1}, b_{i+2})$ and $S(b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1})$ contain no point of P. Since every point of $B \setminus \{b_{i-1}, b_i, b_{i+1}, b_{i+2}\}$ is either in $S(a_{i-1}, b_i, b_{i-1}, b_{i+2})$ or in $S(b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1})$, we have $B = \{b_{i-1}, b_i, b_{i+1}, b_{i+2}\}$. This contradicts the assumption that $|B| \geq 5$ and finishes the proof of Claim 11.3.

In particular, the point b_{i+1} lies in the triangle $\triangle(b_{i-1}, a_{i+1}, b_{i+2})$, since a_{i+1} is the leftmost and b_{i+2} is the rightmost of the points $b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1}$ and since b_{i-1} lies in W_i. The red area in Figure 4(b) gives an illustration.

Consequently, the point a_{i+1} lies to the left of $\overline{b_{i+1} b_{i-1}}$. By Observation 6(i), the point a_{i+1} is not in the sector $S(b_{i+1}, b_{i-1}, b_i, a_i)$, as otherwise the points $b_{i+1}, b_{i-1}, b_i, a_i, a_{i+1}$ form an ℓ-divided 5-hole in P. Thus the point a_{i+1} lies to the left of $\overline{a_i b_i}$; see Figure 4(b).

Claim 11.4. *The points $a^*, a_{i+1}, a_i, a_{i-1}$ are not in convex position.*

The points $a^*, a_{i+1}, a_i, a_{i-1}$ do not form a 4-hole in P because otherwise b_i lies in the sector $S(a_{i-1}, a^*, a_{i+1}, a_i)$ and forms a 5-hole together with $a_{i-1}, a^*, a_{i+1}, a_i$, which is impossible by Observation 6(ii). This finishes the proof of Claim 11.4.

Claim 11.5. *The point a^* is inside the triangle $\triangle(a_{i-1}, a_{i+1}, a_i)$.*

The point a_i is not inside $\triangle(a_{i-1}, a_{i+1}, a^*)$, since, by Observation 7, a_i is to the right of a^* and since a^* is the rightmost inner point of A. Since a_{i-1} is to the left of $\overline{a^* a_i}$ and a_{i+1} is to the right of $\overline{a^* a_i}$, a^* is the inner point of $a^*, a_{i+1}, a_i, a_{i-1}$. Figure 5 gives an illustration. This finishes the proof of Claim 11.5.

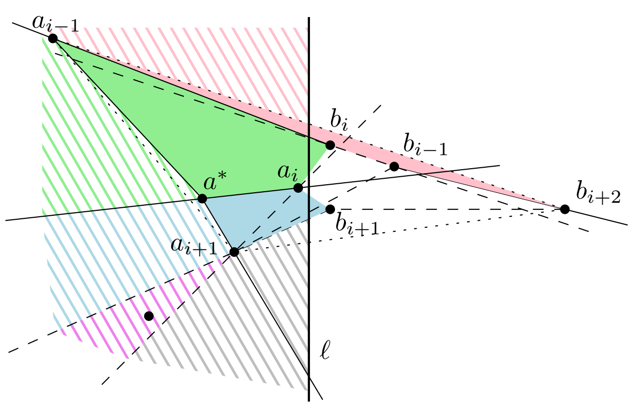

Figure 5: Location of the points of $A \setminus Q$.

Claim 11.6. *All points of $B \setminus Q$ lie in a^*-wedges below W_{i+1}.*

Since $|B| \geq 5$, there is another a^*-wedge besides W_i and W_{i+1} that intersects ℓ. Now we show that all points of $B \setminus Q$ lie in a^*-wedges below W_{i+1}. The rays $\overrightarrow{b_i a_{i-1}}$ and $\overrightarrow{b_{i-1} b_{i+2}}$ both start in W_i and then leave W_i. Moreover, the segment $b_i a_{i-1}$ intersects ℓ and $b_{i-1} b_{i+2}$ intersects $\overrightarrow{a^* a_i}$. As both b_i and b_{i-1} lie to the right of $\overline{a_{i-1} b_{i+2}}$, all points of $B \setminus Q$ that lie in an a^*-wedge above W_i also lie in the sector $S(a_{i-1}, b_i, b_{i-1}, b_{i+2})$. We recall that, by Observation 6(i), the sector $S(a_{i-1}, b_i, b_{i-1}, b_{i+2})$ is empty of points of P. This finishes the proof of Claim 11.6.

Claim 11.7. *We have $i = 1$. That is, W_i is the topmost a^*-wedge that intersects ℓ.*

By Observation 7, a_{i+1} lies to the right of a^*. Since a_i and a_{i+1} are both to the right of a^* and since a^* is inside the triangle $\triangle(a_{i-1}, a_{i+1}, a_i)$, the point a_{i-1} is to the left of a^*. By Observation 7, we have $i = 1$. This proves Claim 11.7.

Claim 11.8. *All points of $A \setminus Q$ lie to the left of $\overline{a_{i+1} a_i}$, to the right of $\overline{a_{i+1} b_{i+1}}$, and to the right of $\overline{a^* a_{i+1}}$.*

The violet area in Figure 5 gives an illustration where the remaining points of $A \setminus Q$ lie. We recall that the sector $S(a_{i-1}, b_i, b_{i-1}, b_{i+2})$ (red shaded area in Figure 5) is empty of points of P. By Observation 9, both sets $\{a^*, a_i, b_i, a_{i-1}\}$ and $\{a^*, a_{i+1}, b_{i+1}, a_i\}$ form ℓ-divided 4-holes in P. By Observation 6(i), the two sectors $S(a^*, a_i, b_i, a_{i-1})$ (green shaded area in Figure 5) and $S(a^*, a_{i+1}, b_{i+1}, a_i)$ (blue shaded area in Figure 5) are thus empty of points of P. Therefore, no point of $A \setminus Q$ lies to the left of $\overline{a_{i+1} b_{i+1}}$. Since W is non-convex, every point of P that is to the left of $\overline{a^* a_{i+1}}$ lies in Q. Thus every point of $A \setminus Q$ lies to the right of $\overline{a^* a_{i+1}}$. Moreover, no point a of $A \setminus Q$ lies to the right of $\overline{a_{i+1} a_i}$ (gray area in Figure 5) because otherwise, a_{i+1} is an inner point of $\triangle(a_i, a^*, a)$, which is impossible since a^* is the rightmost inner point of A and a_{i+1} is to the right of a^*. This finishes the proof of Claim 11.8.

Now we have restricted where the points of $A \setminus Q$ lie. In the rest of the proof we show the following claim. We will then use the sectors $S(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2})$ and $S(a_{i-1}, b_i, b_{i-1}, b_{i+2})$ to argue that $|B| = |B \cap Q| = 4$, which then contradicts the assumption $|B| \geq 5$.

Claim 11.9. *The points $b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2}$ form an ℓ-divided 4-hole in P.*

We consider a_{i+2} and show that the points $a_{i+1}, a^*, a_{i-1}, a_{i+2}$ are in convex position. It suffices to show that a_{i+2} does not lie in the triangle $\triangle(a^*, a_{i-1}, a_{i+1})$ because of the cyclic order of $A \setminus \{a^*\}$ around a^*. Recall that a^* lies inside the triangle $\triangle(a_{i-1}, a_{i+1}, a_i)$, that b_{i+1} lies inside the triangle $\triangle(a_i, a_{i+1}, b_{i+2})$, and that b_{i-1} lies inside the triangle $\triangle(a_{i-1}, a_i, b_{i+2})$. Since the triangles $\triangle(a_{i-1}, a_{i+1}, a_i)$, $\triangle(a_i, a_{i+1}, b_{i+2})$, and $\triangle(a_{i-1}, a_i, b_{i+2})$ are oriented counterclockwise along the boundary, the point a_i lies inside $\triangle(a_{i-1}, a_{i+1}, b_{i+2})$. Thus also the points a^*, b_i, b_{i+1} lie in the triangle $\triangle(a_{i-1}, a_{i+1}, b_{i+2})$. Consequently, the triangle $\triangle(a^*, a_{i-1}, a_{i+1})$ is contained in the union of the sectors $S(a_{i+1}, b_{i+1}, a_i, a^*)$ (blue shaded area in Figure 5) and $S(a^*, a_i, b_i, a_{i-1})$ (green shaded area in Figure 5). Thus a_{i+2} does not lie in the triangle $\triangle(a^*, a_{i-1}, a_{i+1})$ and the points $a_{i+1}, a^*, a_{i-1}, a_{i+2}$ are in convex position.

We now show that the sector $S(a_{i+1}, a^*, a_{i-1}, a_{i+2})$ is empty of points of P. If the quadrilateral $\square(a_{i+1}, a^*, a_{i-1}, a_{i+2})$ is not empty of points of P, then there is a point a'_{i-1} of A in $\triangle(a^*, a_{i-1}, a_{i+2})$. This is because $\triangle(a^*, a_{i+2}, a_{i+1})$ is empty of points of A due to the cyclic order of $A \setminus \{a^*\}$ around a^*. We can choose a'_{i-1} to be a point that is closest to the line $\overline{a^* a_{i+2}}$

among the points of A inside $\triangle(a^*, a_{i+2}, a_{i+1})$. If the quadrilateral $\square(a_{i+1}, a^*, a_{i-1}, a_{i+2})$ is empty of points of P, then we set $a'_{i-1} := a_{i-1}$.

By the choice of a'_{i-1}, the quadrilateral $\square(a_{i+1}, a^*, a'_{i-1}, a_{i+2})$ is empty of points of P. Since a_{i+1} and a_{i+2} are consecutive in the order around a^*, no point of A lies in the sector $S(a_{i+1}, a^*, a'_{i-1}, a_{i+2})$. By Observation 6(ii), the sector $S(a_{i+1}, a^*, a'_{i-1}, a_{i+2})$ (gray shaded area in Figure 6(a)) is empty of points of P. Since the sector $S(a_{i+1}, a^*, a_{i-1}, a_{i+2})$ is a subset of $S(a_{i+1}, a^*, a'_{i-1}, a_{i+2})$, the sector $S(a_{i+1}, a^*, a_{i-1}, a_{i+2})$ is empty of points of P.

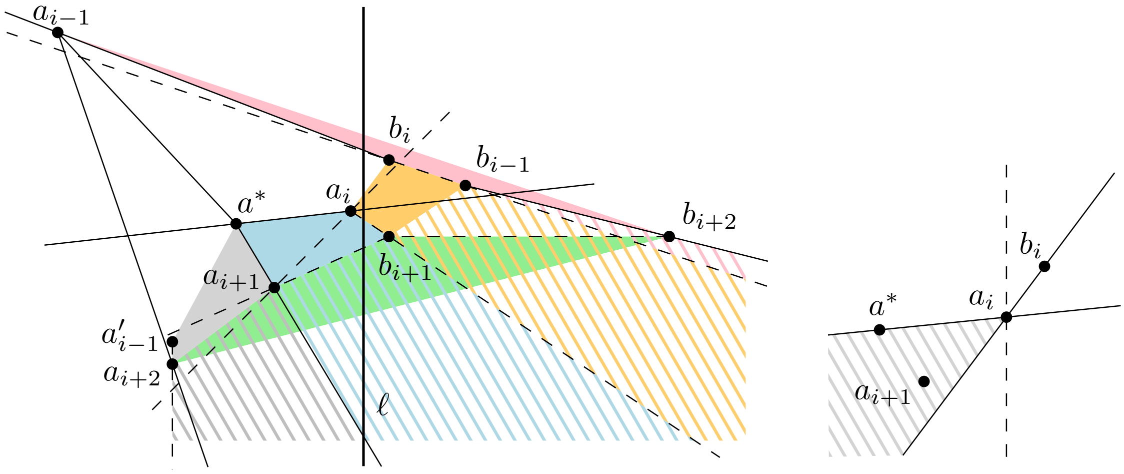

Figure 6: (a) Location of the points of $B \setminus Q$. (b) The point a_{i+1} lies to the left of a_i.

We show that a_{i+1} is to the left of a_i and to the right of a_{i+2}. Recall that a_i lies to the right of a^* and to the left of b_i. The point b_i lies to the left of $\overline{a^*a_i}$ and the point a_{i+1} lies to the right of this line; see Figure 6(b). The point a_{i+1} then lies to the left of a_i, since we know already that a_{i+1} lies to the left of $\overline{a_ib_i}$. Recall that a_{i+1} is to the right of a^*. Consequently, the point a_{i+2} lies to the left of a_{i+1}, as a_{i+2} lies to the right of $\overline{a^*a_{i+1}}$ and to the left of $\overline{a_{i+1}a_i}$ by Claim 11.8.

Now we are ready to prove that the points $b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2}$ form an ℓ-divided 4-hole in P (green area in Figure 6(a)). Recall that b_{i+2} and a_{i+2} both lie to the right of $\overline{a_{i+1}b_{i+1}}$, and that a_{i+2} is the leftmost and b_{i+2} is the rightmost of those four points. Altogether, we see that the points $b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2}$ are in convex position. The four sectors $S(b_{i+2}, a_{i-1}, b_i, b_{i-1})$ (red shaded area in Figure 6(a)), $S(b_{i-1}, b_i, a_i, b_{i+1})$ (orange shaded area in Figure 6(a)), $S(b_{i+1}, a_i, a^*, a_{i+1})$ (blue shaded area in Figure 6(a)), and $S(a_{i+1}, a^*, a'_{i-1}, a_{i+2})$ (gray shaded area in Figure 6(a)) contain the quadrilateral $\square(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2})$ (green area in Figure 6(a)). The sectors are empty of points of P by Observation 6(i). Consequently, the convex quadrilateral $\square(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2})$ is an ℓ-divided 4-hole in P. This concludes the proof of Claim 11.9.

To finish the proof of Lemma 11, recall that all points of $B \setminus Q$ lie in a^*-wedges below W_{i+1} by Claim 11.6. Since a_{i+2} is to the left of a_{i+1}, the line $\overline{a_{i+2}a_{i+1}}$ intersects ℓ above $\ell \cap W_{i+2}$. The line $\overline{a_{i+1}b_{i+1}}$ also intersects ℓ above $\ell \cap W_{i+2}$, since a_{i+1} and b_{i+1} both lie in W_{i+1}. From $i = 1$, every point of $B \setminus Q$ is to the right of $\overline{a_{i+2}a_{i+1}}$ and to the right of $\overline{a_{i+1}b_{i+1}}$. Since the points $b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2}$ form an ℓ-divided 4-hole in P by Claim 11.9, Observation 6(i) implies that the sector $S(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2})$ is empty of points of P. Thus every point of

$B \setminus Q$ lies to the left of $\overline{b_{i+1}b_{i+2}}$. Since $\overline{b_{i+1}b_{i+2}}$ intersects $\ell \cap W_{i+1}$ above $\ell \cap a_{i+1}b_{i+1}$ and since b_{i-1} lies to the left of b_{i+2} and to the left of $\overline{b_{i+1}b_{i+2}}$, every point of $B \setminus Q$ lies to the left of $\overline{b_{i-1}b_{i+2}}$ and to the right of b_{i+2}, and thus in the sector $S(a_{i-1}, b_i, b_{i-1}, b_{i+2})$. However, by Observation 6(i), this sector is empty of points of P. Thus we obtain $B = \{b_{i-1}, b_i, b_{i+1}, b_{i+2}\}$, which contradicts the assumption $|B| \geq 5$. This concludes the proof of Lemma 11. $\square$

Next we show that if there is a sequence of consecutive a^*-wedges where the first and the last a^*-wedge both contain two points of B and every a^*-wedge in between them contains exactly one point of B, then there is an ℓ-divided 5-hole in P.

Lemma 12. *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position and with $|A| \geq 5$ and $|B| \geq 6$. Let $W_i, \ldots, W_j$ be consecutive a^*-wedges with $1 \leq i < j \leq t$, $w_i = 2 = w_j$, and $w_k = 1$ for every k with $i < k < j$. Then there is an ℓ-divided 5-hole in P.*

Proof. For $i = j - 1$, the statement follows by Lemma 11. Thus we assume $j \geq i + 2$. That is, we have at least three consecutive a^*-wedges. Suppose for contradiction that there is no ℓ-divided 5-hole in P. Let $W := \bigcup_{k=i}^{j} W_k$ and $Q := P \cap W$. By Lemma 8, there is also no ℓ-divided 5-hole in Q. Note that $|Q \cap B| = j - i + 3$. Also observe that $|Q \cap A| = j - i + 2$ if $a_{i-1} = a_j = a_t$ and $|Q \cap A| = j - i + 3$ otherwise. We label the points in $B \cap W_i$ as b_{i-1} and b_i so that b_{i-1} is to the right of b_i. Further, we label the unique point in $B \cap W_k$ as b_k for each $i < k < j$, and the two points in $B \cap W_j$ as b_j and b_{j+1} so that b_{j+1} is to the right of b_j; see Figure 7.

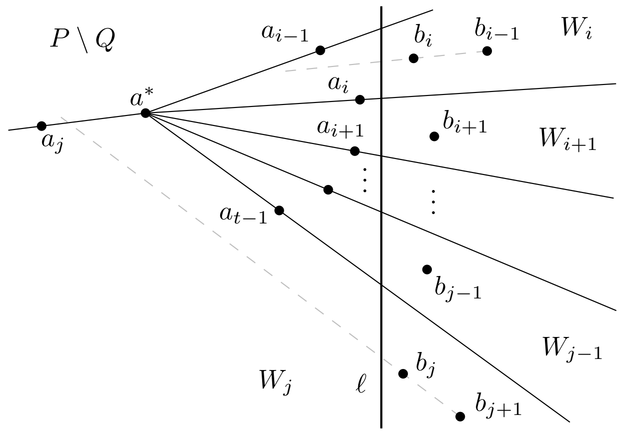

Figure 7: An illustration of a^*-wedges $W_i, \ldots, W_j$ in the proof of Lemma 12.

Claim 12.1. *All points of $B \cap (W_{k-1} \cup W_k \cup W_{k+1})$ are to the right of $\overline{a_k a_{k-1}}$ for every k with $i < k < j$.*

The claim clearly holds for points from $B \cap W_k$. Thus it suffices to prove the claim only for points from $B \cup W_{k-1}$, as for points from $B \cup W_{k+1}$ it follows by symmetry. Since $i < k < j$, Observation 7 implies that the points a_{k-1} and a_k are both to the right of a^*.

We now distinguish the following two cases.

1. The point a_{k-2} is to the left of $\overline{a^* a_k}$; see Figure 8(a). Since a^* is the rightmost inner point of A, a_{k-1} does not lie inside the triangle $\triangle(a^*, a_k, a_{k-2})$ and thus $\square(a_{k-2}, a^*, a_k, a_{k-1})$ is a 4-hole in P. All points of $B \cap W_{k-1}$ lie to the right of $\overline{a^* a_{k-2}}$ and to the left of $\overline{a_{k-2}a_{k-1}}$.

By Observation 6(ii), no point of $B \cap W_{k-1}$ lies in the sector $S(a_{k-2}, a^*, a_k, a_{k-1})$ (red shaded area in Figure 8(a)) and thus all points of $B \cap W_{k-1}$ are to the right of $\overline{a_k a_{k-1}}$.

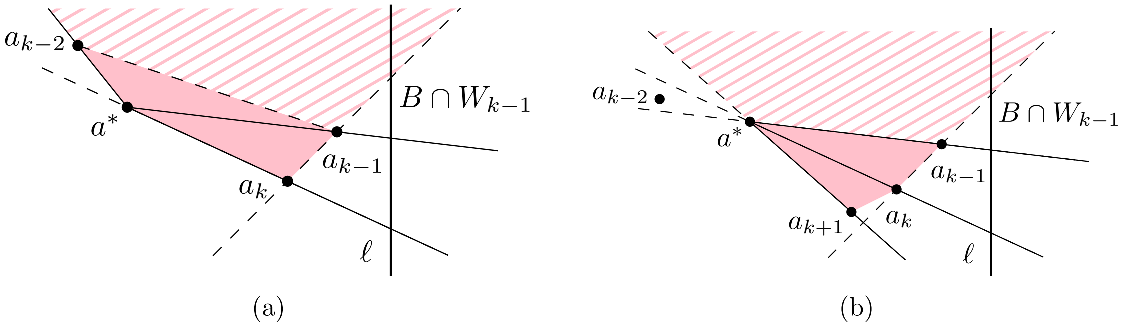

Figure 8: An illustration of the proof of Claim 12.1.

2. The point a_{k-2} is to the right of $\overline{a^* a_k}$; see Figure 8(b). Since a_{k-1} and a_k are to the right of a^* and since a_{k-2} is to the left of $\overline{a^* a_{k-1}}$ and to the right of $\overline{a^* a_k}$, the point a_{k-2} is to the left of a^*. By Observation 7, we have $k = 2$. That is, W_{k-1} is the topmost a^*-wedge that intersects ℓ.

   There is another a^*-wedge below W_{k+1}, since otherwise $|B| = |B \cap (W_{k-1} \cup W_k \cup W_{k+1})| \leq 2 + 1 + 2 = 5$, which is impossible according to the assumption $|B| \geq 6$. By Observation 7, the point a_{k+1} is to the right of a^*. Moreover, since a^* is the rightmost inner point of A, the point a_k does not lie inside the triangle $\triangle(a^*, a_{k+1}, a_{k-1})$. The points $a^*, a_{k+1}, a_k, a_{k-1}$ then form a 4-hole in P, which has a^* as the leftmost point.

   By definition, all points of $B \cap W_{k-1}$ lie to the left of $\overline{a^* a_{k-1}}$. As the ray $\overrightarrow{a^* a_{k+1}}$ intersects ℓ, all points of $B \cap W_{k-1}$ lie also to the left of $\overline{a^* a_{k+1}}$. By Observation 6(ii), no point of $B \cap W_{k-1}$ lies in the sector $S(a^*, a_{k+1}, a_k, a_{k-1})$. Thus all points of $B \cap W_{k-1}$ lie to the right of $\overline{a_k a_{k-1}}$.

This finishes the proof of Claim 12.1.

We say that points p_1, p_2, p_3, p_4 form a *counterclockwise-oriented convex quadrilateral* if every triple (p_x, p_y, p_z) with $1 \leq x < y < z \leq 4$ is oriented counterclockwise.

Claim 12.2. *The points $b_{i-1}, b_i, a_i, a_{i+1}$ form a counterclockwise-oriented convex quadrilateral.*

Due to Claim 12.1, the points b_{i-1} and b_i are both to the right of $\overline{a_{i+1} a_i}$. Thus the points a_i and a_{i+1} are both extremal points of those four points. Also the point b_{i-1} is extremal, since it is the rightmost of those four points. The point b_i does not lie inside the triangle $\triangle(a_{i+1}, a_i, b_{i-1})$, since, by Observation 10, b_i lies to the left of $\overline{a_i b_{i-1}}$. To finish the proof of Claim 12.2, it suffices to observe that the triples (b_{i-1}, b_i, a_i), (b_{i-1}, b_i, a_{i+1}), (b_{i-1}, a_i, a_{i+1}), and (b_i, a_i, a_{i+1}) are all oriented counterclockwise.

Claim 12.3. *The point b_{i+1} lies to the right of $\overline{b_i b_{i-1}}$.*

Suppose for contradiction that b_{i+1} lies to the left of $\overline{b_i b_{i-1}}$. We consider the five points $a_{i-1}, a_i, b_{i-1}, b_i, b_{i+1}$; see Figure 9. By Claim 12.1, the points b_{i-1}, b_i, and b_{i+1} lie to the right of $\overline{a_i a_{i-1}}$. Moreover, since b_{i-1} and b_i lie in W_i and since b_{i+1} lies in W_{i+1}, the points b_{i-1} and b_i both lie to the left of $\overline{a_i b_{i+1}}$. By Observation 10, the point a_{i-1} lies to the left of

$\overline{b_i b_{i-1}}$ and b_{i+1} is to the right of b_{i-1}. Consequently, the points b_{i-1} and b_i lie in the triangle $\triangle(a_{i-1}, a_i, b_{i+1})$. Altogether, the points a_{i-1}, b_i, b_{i-1}, and b_{i+1} are in convex position.

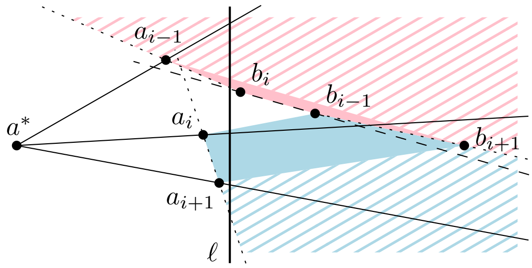

Figure 9: An illustration of the proof of Claim 12.3.

By Claim 12.1, the points b_{i-1} and b_{i+1} lie to the right of $\overline{a_{i+1} a_i}$. Moreover, since b_{i-1} is to the left of b_{i+1} and to the left of $\overline{a_i b_{i+1}}$, the points b_{i+1}, b_{i-1}, a_i, and a_{i+1} are in convex position. Since there are no further points in W_i and W_{i+1}, the sets $\{a_{i-1}, b_i, b_{i-1}, b_{i+1}\}$ and $\{b_{i+1}, b_{i-1}, a_i, a_{i+1}\}$ are ℓ-divided 4-holes in P. By Observation 6(i), the point b_{i+2} lies neither in $S(a_{i-1}, b_i, b_{i-1}, b_{i+1})$ nor in $S(b_{i+1}, b_{i-1}, a_i, a_{i+1})$. Recall that the ray $\overrightarrow{b_{i-1} b_{i+1}}$ intersects $\overrightarrow{a^* a_i}$ and the ray $\overrightarrow{b_i a_{i-1}}$ does not intersect $\overrightarrow{a^* a_i}$. Therefore b_{i+2} is to the right of $\overline{a_i a_{i+1}}$. This contradicts Claim 12.1 and finishes the proof of Claim 12.3.

Claim 12.4. *For each k with $i < k < j$, the point b_k lies to the left of $\overline{a_k b_{i-1}}$ and to the left of b_{i-1}.*

Recall the labeling of the points in $B \cap W$; see Figure 7. We show by induction on k that

(i) the points b_{i-1}, b_{k-1}, a_{k-1}, and a_k form a counterclockwise-oriented convex quadrilateral, which has b_{i-1} as the rightmost point, and

(ii) the point b_k lies inside this convex quadrilateral and, in particular, to the left of $\overline{a_k b_{i-1}}$.

Claim 12.4 then clearly follows.

For the base case, we consider $k = i + 1$. By Claim 12.2, the points b_{i-1}, b_i, a_i, and a_{i+1} form a counterclockwise-oriented convex quadrilateral. By definition, b_{i-1} is the rightmost of those four points. Figure 10(a) gives an illustration. The point b_{i+1} lies to the right of $\overline{a_{i+1} a_i}$ and, by Claim 12.3, to the right of $\overline{b_i b_{i-1}}$. Moreover, since b_{i+1} lies in W_{i+1}, it lies to the right of $\overline{a_i b_i}$. By Observation 6(i), b_{i+1} does not lie in the sector $S(b_{i-1}, b_i, a_i, a_{i+1})$. Consequently, b_{i+1} lies inside the quadrilateral $\square(b_{i-1}, b_i, a_i, a_{i+1})$.

For the inductive step, let $i + 1 < k < j$. By the inductive assumption, the point b_{k-1} lies to the left of $\overline{a_{k-1} b_{i-1}}$ and to the left of b_{i-1}. By Claim 12.1, b_{k-1} lies to the right of $\overline{a_k a_{k-1}}$. Hence, the points a_k and b_{i-1} both lie to the right of $\overline{a_{k-1} b_{k-1}}$. Recall that the points $b_{i-1}, b_{k-1}, a_{k-1}, a_k$ lie to the right of a^*. Since b_{i-1} is the first and a_k is the last in the clockwise order around a^*, the points $b_{i-1}, b_{k-1}, a_{k-1}, a_k$ form a counterclockwise-oriented convex quadrilateral,

Recall that the points b_{k-1} and b_k both lie to the right of $\overline{a_k a_{k-1}}$ and that b_{k-1} is to the left of $\overline{a_{k-1} b_{i-1}}$. Since $b_k \in W_k$, the point b_k lies to the right of $\overline{a_{k-1} b_{i-1}}$. Therefore the clockwise order of $\{b_{k-1}, b_{i-1}, b_k\}$ around a_{k-1} is b_{k-1}, b_{i-1}, b_k. Since b_{i-1} is not contained in $W_{k-1} \cup W_k$, the point b_{i-1} is not contained in the triangle $\triangle(a_{k-1}, b_k, b_{k-1})$. Consequently, the points $a_{k-1}, b_k, b_{i-1}, b_{k-1}$ form a convex quadrilateral and, in particular, b_k lies to the right of

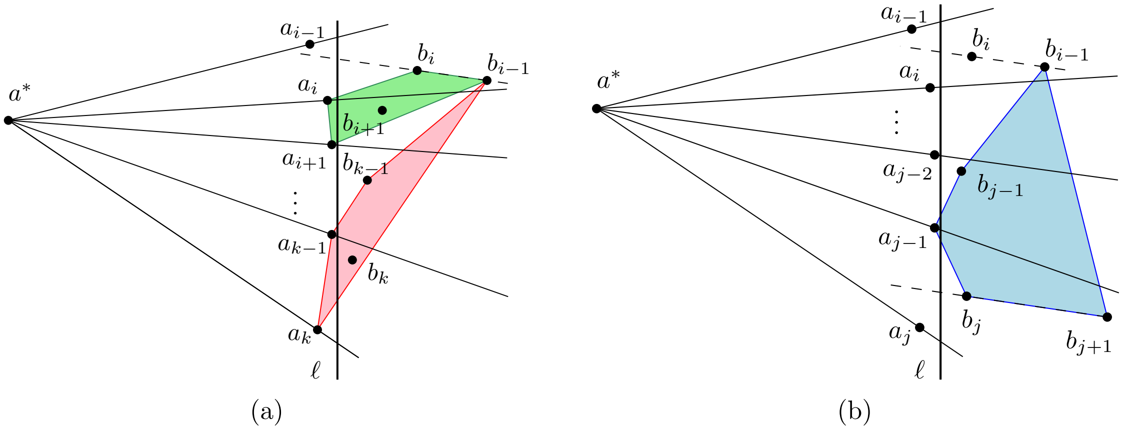

Figure 10: (a) An illustration of the proof of Claim 12.4. (b) An illustration of the proof of Lemma 12.

$\overline{b_{k-1}b_{i-1}}$. Figure 10(a) gives an illustration. Since b_k lies in W_k, it lies to the right of $\overline{a_{k-1}b_{k-1}}$. By Observation 6(i), the point b_k does not lie in the sector $S(b_{i-1}, b_{k-1}, a_{k-1}, a_k)$. Thus b_k lies inside the quadrilateral $\square(b_{i-1}, b_{k-1}, a_{k-1}, a_k)$. This finishes the proof of Claim 12.4.

Using Claim 12.4, we now finish the proof of Lemma 12, by finding an ℓ-divided 5-hole in the island Q and thus obtaining a contradiction with the assumption that there is no ℓ-divided 5-hole in P. In the following, we assume, without loss of generality, that b_{j+1} is to the right of b_{i-1}. Otherwise we can consider a vertical reflection of P.

We consider the polygon $\mathcal{P}$ through the points $b_{i-1}, b_{j-1}, a_{j-1}, b_j, b_{j+1}$ and we show that $\mathcal{P}$ is convex and empty of points of Q. See Figure 10(b) for an illustration. This will give us an ℓ-divided 5-hole in Q.

We show that $\mathcal{P}$ is convex by proving that every point of $\{b_{i-1}, b_{j-1}, a_{j-1}, b_j, b_{j+1}\}$ is a convex vertex of $\mathcal{P}$. The point a_{j-1} is a convex vertex of $\mathcal{P}$ because it is the leftmost point in $\mathcal{P}$. The point b_{i-1} is a convex vertex of $\mathcal{P}$ because all points of $\mathcal{P}$ lie to the right of a^* and b_{i-1} is the topmost point in the clockwise order around a^*. The point b_{j+1} is a convex vertex of $\mathcal{P}$ because b_{j+1} is the rightmost point in $\mathcal{P}$ by Claim 12.4 and by the assumption that b_{j+1} is to the right of b_{i-1}. The point b_{j-1} is a convex vertex of $\mathcal{P}$ because b_{j-1} lies to the left of $\overline{a_{j-1}b_{i-1}}$ by Claim 12.4 while b_j and b_{j+1} both lie to the right of this line. The point b_j is a convex vertex of $\mathcal{P}$ because, by Observation 10, b_j lies to the right of $\overline{a_{j-1}b_{j+1}}$ while b_{j-1} and b_{i-1} both lie to the right of this line. Consequently, $\mathcal{P}$ is a convex pentagon with vertices from both A and B. Moreover, by Claim 12.4, all points b_k with $i < k < j$ lie to the left of $\overline{a_k b_{i-1}}$. Since b_i is to the left of $\overline{b_{j-1}b_{i-1}}$, $\mathcal{P}$ is thus empty of points of Q, which gives us a contradiction with the assumption that there is no ℓ-divided 5-hole in P. $\square$

We now use Lemma 12 to show the following upper bound on the total number of points of B in a sequence $W_i, \dots, W_j$ of consecutive a^*-wedges with $w_i, \dots, w_j \le 2$.

Corollary 13. *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole, with A not in convex position, and with $|A| \ge 5$ and $|B| \ge 6$. For $1 \le i \le j \le t$, let $W_i, \dots, W_j$ be consecutive a^*-wedges with $w_k \le 2$ for every k with $i \le k \le j$. Then $\sum_{k=i}^{j} w_k \le j - i + 2$.*

Proof. Let n_0, n_1, and n_2 be the number of a^*-wedges from $W_i, \dots, W_j$ with 0, 1, and 2 points of B, respectively. Due to Lemma 12, we can assume that between any two a^*-wedges from $W_i, \dots, W_j$ with two points of B each, there is an a^*-wedge with no point of B. Thus

$n_2 \leq n_0 + 1$. Since $n_0 + n_1 + n_2 = j - i + 1$, we have $\sum_{k=i}^{j} w_k = 0n_0 + 1n_1 + 2n_2 = (j - i + 1) + (n_2 - n_0) \leq j - i + 2$. □

5.2 Computer-assisted results

We now provide lemmas that are key ingredients in the proof of Theorem 2. All these lemmas have computer-aided proofs. Each result was verified by two independent implementations, which are also based on different abstractions of point sets; see below for details.

Lemma 14. *Let $P = A \cup B$ be an ℓ-divided set with $|A| = 5$, $|B| = 6$, and with A not in convex position. Then there is an ℓ-divided 5-hole in P.*

Lemma 15. *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P, $|A| = 5$, $4 \leq |B| \leq 6$, and with A in convex position. Then for every point a of A, every convex a-wedge contains at most two points of B.*

Lemma 16. *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P, $|A| = 6$, and $|B| = 5$. Then for each point a of A, every convex a-wedge contains at most two points of B.*

Lemma 17. *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P, $5 \leq |A| \leq 6$, $|B| = 4$, and with A in convex position. Then for every point a of A, if the non-convex a-wedge is empty of points of B, every a-wedge contains at most two points of B.*

To prove these lemmas, we employ an exhaustive computer search through all combinatorially different sets of $|P| \leq 11$ points in the plane. Since none of these statements depends on the actual coordinates of the points but only on the relative positions of the points, we distinguish point sets only by orientations of triples of points as proposed by Goodman and Pollack [20]. That is, we check all possible equivalence classes of point sets in the plane with respect to their triple-orientations, which are known as *order types*.

We wrote two independent programs to verify Lemmas 14 to 17. Both programs are available online [8, 26].

The first implementation is based on programs from the two bachelor's theses of Scheucher [27, 28]. For our verification purposes we reduced the framework from there to a very compact implementation [26]. The program uses the order type database [3, 7], which stores all order types realizable as point sets of size up to 11. The order types realizable as sets of ten points are available online [1] and the ones realizable as sets of eleven points need about 96 GB and are available upon request from Aichholzer. The running time of each of the programs in this implementation does not exceed two hours on a standard computer.

The second implementation [8] neither uses the order type database nor the program used to generate the database. Instead it relies on the description of point sets by so-called *signature functions* [9, 17]. In this description, points are sorted according to their x-coordinates and every unordered triple of points is represented by a sign from $\{-, +\}$, where the sign is $-$ if the triple traced in the order by increasing x-coordinates is oriented clockwise and the sign is $+$ otherwise. Every 4-tuple of points is then represented by four signs of its triples, which are ordered lexicographically. There are only eight 4-tuples of signs that we can obtain (out of 16 possible ones); see [9, Theorem 3.2] or [17, Theorem 7] for details. In our algorithm, we generate all possible signature functions using a simple depth-first search algorithm and verify the conditions from our lemmas for every signature. The running time of each of the programs in this implementation takes up to a few hundreds of hours.

5.3 Applications of the computer-assisted results

Here we present some applications of the computer-assisted results from Section 5.2.

Lemma 18. *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P, with $|A| \geq 6$, and with A not in convex position. Then the following two conditions are satisfied.*

(i) Let W_i, W_{i+1}, W_{i+2} be three consecutive a^*-wedges whose union is convex and contains at least four points of B. Then $w_i, w_{i+1}, w_{i+2} \leq 2$.

(ii) Let $W_i, W_{i+1}, W_{i+2}, W_{i+3}$ be four consecutive a^*-wedges whose union is convex and contains at least four points of B. Then $w_i, w_{i+1}, w_{i+2}, w_{i+3} \leq 2$.

Proof. To show part (i), let $W := W_i \cup W_{i+1} \cup W_{i+2}$, $A' := A \cap W$, $B' := B \cap W$, and $P' := A' \cup B'$. Since W is convex, P' is an island of P and thus there is no ℓ-divided 5-hole in P'. Note that $|A'| = 5$ and A' is in convex position. If $|B'| \leq 5$, then every convex a^*-wedge in P' contains at most two points of B' by Lemma 15 applied to P'. So assume that $|B'| \geq 6$. If necessary, we remove points from P' from the right to obtain $P'' = A' \cup B''$, where B'' contains exactly six points of B'. Note that there is no ℓ-divided 5-hole in P'', since P'' is an island of P'. By Lemma 15, each a^*-wedge in P'' contains exactly two points of B''. Let $\tilde{B}$ be the set of points of B that are to the left of the rightmost point of B'', including this point, and let $\tilde{P} := A \cup \tilde{B}$. Note that $B'' \subseteq \tilde{B}$. Since $|B''| = 6$ and since $W \cap \tilde{B} = B''$, each of the a^*-wedges W_i, W_{i+1}, W_{i+2} contains exactly two points of $\tilde{B}$. The a^*-wedges W_i, W_{i+1}, and W_{i+2} are also a^*-wedges in $\tilde{P}$. Thus, Lemma 11 applied to $\tilde{P}$ and W_i, W_{i+1} then gives us an ℓ-divided 5-hole in $\tilde{P}$. From the choice of $\tilde{P}$, we then have an ℓ-divided 5-hole in P, a contradiction.

To show part (ii), let $W := W_i \cup W_{i+1} \cup W_{i+2} \cup W_{i+3}$, $A' := A \cap W$, $B' := B \cap W$, and $P' := A' \cup B'$. Since W is convex, P' is an island of P and thus there is no ℓ-divided 5-hole in P'. Note that $|A'| = 6$ and A' is in convex position. If $|B'| = 4$, then the statement follows from Lemma 17 applied to P' since a^* is an extremal point of P'. If $|B'| = 5$, then the statement follows from Lemma 16 applied to P' and thus we can assume $|B'| \geq 6$. Suppose for contradiction that $w_j \geq 3$ for some $i \leq j \leq i+3$. If necessary, we remove points from P from the right to obtain P'' so that $B'' := P'' \cap B$ contains exactly six points of $W \cap B$. By applying part (i) for P'' and $W_i \cup W_{i+1} \cup W_{i+2}$ and $W_{i+1} \cup W_{i+2} \cup W_{i+3}$, we obtain that $|B'' \cap W_i|, |B'' \cap W_{i+3}| = 3$ and $|B'' \cap W_{i+1}|, |B'' \cap W_{i+2}| = 0$. Let b be the rightmost point from $P'' \cap W$. By Lemma 16 applied to $W \cap (P'' \setminus \{b\})$, there are at most two points of $B'' \setminus \{b\}$ in every a^*-wedge in $W \cap (P'' \setminus \{b\})$. This contradicts the fact that either $|(B'' \cap W_i) \setminus \{b\}| = 3$ or $|(B'' \cap W_{i+3}) \setminus \{b\}| = 3$. $\square$

5.4 Extremal points of ℓ-critical sets

Recall the definition of ℓ-critical sets: An ℓ-divided point set $C = A \cup B$ is called ℓ-critical if neither $C \cap A$ nor $C \cap B$ is in convex position and if for every extremal point x of C, one of the sets $(C \setminus \{x\}) \cap A$ and $(C \setminus \{x\}) \cap B$ is in convex position.

In this section, we consider an ℓ-critical set $C = A \cup B$ with $|A|, |B| \geq 5$. We first show that C has at most two extremal points in A and at most two extremal points in B. Later, under the assumption that there is no ℓ-divided 5-hole in C, we show that $|B| \leq |A| - 1$ if A contains two extremal points of C (Section 5.4.1) and that $|B| \leq |A|$ if B contains two extremal points of C (Section 5.4.2).

Lemma 19. *Let $C = A \cup B$ be an ℓ-critical set. Then the following statements are true.*

(i) If $|A| \geq 5$, then $|A \cap \partial\,\mathrm{conv}(C)| \leq 2$.

(ii) If $A \cap \partial\,\mathrm{conv}(C) = \{a, a'\}$, then a^* is the only inner point in A and every point of $A \setminus \{a, a'\}$ lies in the convex region spanned by the lines $\overline{a^*a}$ and $\overline{a^*a'}$ that does not have any of a and a' on its boundary.

(iii) If $A \cap \partial\,\mathrm{conv}(C) = \{a, a'\}$, then the a^*-wedge that contains a and a' contains no point of B.

By symmetry, analogous statements hold for B.

Proof. To show statement (i), suppose for contradiction that $|A \cap \partial\,\mathrm{conv}(C)| \geq 3$. Let a, a', and a'' be three points from $A \cap \partial\,\mathrm{conv}(C)$ that are consecutive vertices of the convex hull $\mathrm{conv}(C)$. If there is no point of A in the triangle $\triangle(a, a', a'')$ spanned by the points a, a', and a'', then $A \setminus \{a'\}$ is not in convex position. This is impossible, since C is an ℓ-critical set. If there is at least one point $a^{(1)}$ in $\triangle(a, a', a'')$, then we consider an arbitrary point $a^{(2)}$ from $A \setminus \{a, a', a'', a^{(1)}\}$. Such a point $a^{(2)}$ exists, since $|A| \geq 5$. The point $a^{(1)}$ lies inside one of the triangles $\triangle(a, a', a^{(2)})$, $\triangle(a, a'', a^{(2)})$, or in $\triangle(a', a'', a^{(2)})$ and thus one of the sets $A \setminus \{a''\}$, $A \setminus \{a'\}$, or $A \setminus \{a\}$ is not in convex position, which is again impossible. In any case, C cannot be ℓ-critical and we obtain a contradiction.

To show statement (ii), assume that $A \cap \partial\,\mathrm{conv}(C) = \{a, a'\}$. Every triangle in A with a point of A in its interior has a and a' as vertices, as otherwise $A \setminus \{a\}$ or $A \setminus \{a'\}$ is not in convex position, which is impossible. Consider points $a^{(1)}$ and $a^{(2)}$ from A such that $\triangle(a, a', a^{(1)})$ contains $a^{(2)}$. Denote by R the region bounded by $\overline{aa^{(2)}}$ and $\overline{a'a^{(2)}}$ that contains $a^{(1)}$. If there is a point $a^{(3)}$ in $A \setminus (R \cup \{a, a'\})$ then $a^{(2)}$ lies in one of $\triangle(a, a^{(1)}, a^{(3)})$ and $\triangle(a', a^{(1)}, a^{(3)})$, implying that $A \setminus \{a\}$ or $A \setminus \{a'\}$ is not in convex position. Hence all points of $A \setminus \{a, a', a^{(2)}\}$ lie in R. Moreover, any further inner point $a^{(4)}$ from $A \cap R$ lies in some triangle $\triangle(a, a', a^{(5)})$ for some $a^{(5)} \in A \cap R$. Thus, $a^{(4)}$ also lies in one of the triangles $\triangle(a, a^{(2)}, a^{(5)})$ or $\triangle(a', a^{(2)}, a^{(5)})$. This implies that $A \setminus \{a\}$ or $A \setminus \{a'\}$ is not in convex position. Hence $a^{(2)}$ is the only inner point of A.

To show statement (iii), assume that $A \cap \partial\,\mathrm{conv}(C) = \{a, a'\}$. Let W_i be the wedge that contains a and a'. Since a and a' are the only extremal points of C contained in A, the segment aa' is an edge of $\mathrm{conv}(C)$. The points a, a', and a^* all lie in A and thus the triangle $\triangle(a, a', a^*)$ contains no points of B. Since all points of C lie in the closed halfplane that is determined by the line $\overline{aa'}$ and that contains a^*, the wedge W_i contains no points of B. □

We remark that the assumption $|A| \geq 5$ in part (i) of Lemma 19 is necessary. In fact, arbitrarily large ℓ-critical sets with only four points in A and with three points of A on $\partial\,\mathrm{conv}(C)$ exist, and analogously for B. Figure 2(c) gives an illustration.

Lemma 20. *Let $C = A \cup B$ be an ℓ-critical set with no ℓ-divided 5-hole in C and with $|A| \geq 6$. Then $w_i \leq 3$ for every $1 < i < t$. Moreover, if $|A \cap \partial\,\mathrm{conv}(C)| = 2$, then $w_1, w_t \leq 3$.*

Proof. Recall that, since C is ℓ-critical, we have $|B| \geq 4$. Let i be an integer with $1 \leq i \leq t$. First, assume that $A \cap \partial\,\mathrm{conv}(C) \subset W_i$. By Lemma 19(i), we have $|A \cap \partial\,\mathrm{conv}(C)| \in \{1, 2\}$. If $|A \cap \partial\,\mathrm{conv}(C)| = 1$, then $i \in \{1, t\}$, so there is nothing to prove for w_i. In the remaining case $|A \cap \partial\,\mathrm{conv}(C)| = 2$, by Lemma 19(iii) we have $W_i \cap B = \emptyset$, and thus $w_i = 0$.

In the remaining case there is a point $a \in A \cap \partial\,\mathrm{conv}(C) \setminus W_i$. We consider $C' := C \setminus \{a\}$. Since C is an ℓ-critical set, $A' := C' \cap A$ is in convex position. Thus, there is a non-convex a^*-wedge W' of C'. Since W' is non-convex, all other a^*-wedges of C' are convex. Moreover, since W' is the union of the two a^*-wedges of C that contain a, all other a^*-wedges of C' are also a^*-wedges of C. Let W be the union of all a^*-wedges of C that are not contained in W'. Note that W is convex and contains at least $|A| - 3 \geq 3$ a^*-wedges of C. Since $|A| \geq 6$, the statement follows from Lemma 18(i). □

5.4.1 Two extremal points of C in A

Proposition 21. *Let $C = A \cup B$ be an ℓ-critical set with no ℓ-divided 5-hole in C, with $|A|, |B| \geq 6$, and with $|A \cap \partial\,\mathrm{conv}(C)| = 2$. Then $|B| \leq |A| - 1$.*

Proof. Since $|A \cap \partial\,\mathrm{conv}(C)| = 2$, Lemma 20 implies that $w_i \leq 3$ for every $1 \leq i \leq t$. Let a and a' be the two points in $A \cap \partial\,\mathrm{conv}(C)$. By Lemma 19(ii), all points of $A \setminus \{a, a'\}$ lie in the convex region R that is bounded by the lines $\overline{a^*a}$, $\overline{a^*a'}$, and ℓ, and does not have any of a and a' on its boundary. That is, without loss of generality, $a = a_{h-1}$ and $a' = a_h$ for some $1 \leq h \leq |A| - 1$ and, by Lemma 19(iii), we have $w_h = 0$. Since all points of $A \setminus \{a, a'\}$ lie in the convex region R, the regions $W := \mathrm{cl}(\mathbb{R}^2 \setminus (W_{h-1} \cup W_h))$ and $W' := \mathrm{cl}(\mathbb{R}^2 \setminus (W_h \cup W_{h+1}))$ are convex; see Figure 11. Here $\mathrm{cl}(X)$ denotes the closure of a set $X \subseteq \mathbb{R}^2$. Recall that the indices of the a^*-wedges are considered modulo $|A| - 1$ and that $\mathbb{R}^2$ is the union of all a^*-wedges.

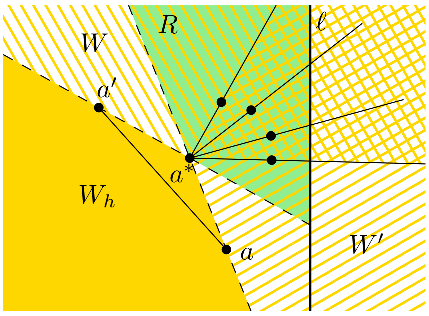

Figure 11: An illustration of the proof of Proposition 21.

First, suppose for contradiction that $|A| = 6$. There are exactly five a^*-wedges $W_1, \ldots, W_5$, and only four of them can contain points of B, since $w_h = 0$. We apply Lemma 18(i) to W and to W'. An easy case analysis shows that either $w_i \leq 2$ for every $1 \leq i \leq t$ or $w_{h-1}, w_{h+1} = 3$ and $w_i = 0$ for every $i \notin \{h-1, h+1\}$. In the first case, Corollary 13 implies that $|B| \leq 5$ and in the latter case Lemma 16 applied to $P \setminus \{b\}$, where b is the rightmost point of B, gives $|B| \leq 5$, a contradiction to $|B| \geq 6$. Hence, we assume $|A| \geq 7$.

Claim 21.1. *For $1 \leq k \leq t - 3$, if one of the four consecutive a^*-wedges W_k, W_{k+1}, W_{k+2}, or W_{k+3} contains 3 points of B, then $w_k + w_{k+1} + w_{k+2} + w_{k+3} = 3$.*

There are $|A| - 1 \geq 6$ a^*-wedges and, in particular, W and W' are both unions of at least four a^*-wedges. For every W_i with $w_i = 3$ and $1 \leq i \leq t$, the a^*-wedge W_i is either contained in W or in W'. Thus we can find four consecutive a^*-wedges $W_k, W_{k+1}, W_{k+2}, W_{k+3}$ whose union is convex and contains W_i. Lemma 18(ii) implies that each of $W_k, W_{k+1}, W_{k+2}, W_{k+3}$ except of W_i is empty of points of B. This finishes the proof of Claim 21.1.

Claim 21.2. *For all integers i and j with $1 \le i < j \le t$, we have $\sum_{k=i}^{j} w_k \le j - i + 2$.*

Let $S := (w_i, \dots, w_j)$ and let S' be the subsequence of S obtained by removing every 1-entry from S. If S contains only 1-entries, the statement clearly follows. Thus we can assume that S' is non-empty. Recall that, by Lemma 20, S' contains only 0-, 2-, and 3-entries, since $w_i \le 3$ for all $1 \le i \le t$. Due to Claim 21.1, there are at least three consecutive 0-entries between every pair of nonzero entries of S' that contains a 3-entry. Together with Lemma 12, this implies that there is at least one 0-entry between every pair of 2-entries in S'.

By applying the following iterative procedure, we show that $\sum_{s \in S'} s \le |S'| + 1$. While there are at least two nonzero entries in S', we remove the first nonzero entry s from S'. If $s = 2$, then we also remove the 0-entry from S' that succeeds s in S. If $s = 3$, then we also remove the two consecutive 0-entries from S' that succeed s in S'. The procedure stops when there is at most one nonzero element s' in the remaining subsequence S'' of S'. If $s' = 3$, then S'' contains at least one 0-entry and thus S'' contains at least $s' - 1$ elements. Since the number of removed elements equals the sum of the removed elements in every step of the procedure, we have $\sum_{s \in S'} s \le |S'| + 1$. This implies

$$\sum_{k=i}^{j} w_k = \sum_{s \in S} s = |S| - |S'| + \sum_{s \in S'} s \le |S| - |S'| + |S'| + 1 = j - i + 2$$

and finishes the proof of Claim 21.2.

If W_h does not intersect ℓ, that is, $t < h \le |A| - 1$, then the statement follows from Claim 21.2 applied with $i = 1$ and $j = t$. Otherwise, we have $h = 1$ or $h = t$ and we apply Claim 21.2 with $(i, j) = (2, t)$ or $(i, j) = (1, t-1)$, respectively. Since $t \le |A| - 1$ and $w_h = 0$, this gives us $|B| \le |A| - 1$. □

5.4.2 Two extremal points of C in B

Proposition 22. *Let $C = A \cup B$ be an ℓ-critical set with no ℓ-divided 5-hole in C, with $|A|, |B| \ge 6$, and with $|B \cap \partial \operatorname{conv}(C)| = 2$. Then $|B| \le |A|$.*

Proof. If $w_k \le 2$ for all $1 \le k \le t$, then the statement follows from Corollary 13, since $|B| = \sum_{k=1}^{t} w_k \le t + 1 \le |A|$. Therefore we assume that there is an a^*-wedge W_i that contains at least three points of B. Let b_1, b_2, and b_3 be the three leftmost points in $W_i \cap B$ from left to right. Without loss of generality, we assume that b_3 is to the left of $\overline{b_1 b_2}$. Otherwise we can consider a vertical reflection of P. Figure 12 gives an illustration.

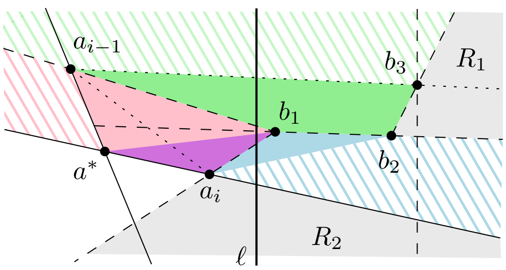

Figure 12: An illustration of the proof of Proposition 22.

Let R_1 be the region that lies to the left of $\overline{b_1 b_2}$ and to the right of $\overline{b_2 b_3}$ and let R_2 be the region that lies to the right of $\overline{a_i b_1}$ and to the right of $\overline{a^* a_i}$. Let $B' := B \setminus \{b_1, b_2, b_3\}$.

Claim 22.1. *Every point of B' lies in $R_1 \cup R_2$.*

We first show that every point of B' that lies to the left of $\overline{b_1b_2}$ lies in R_1. Then we show that every point of B' that lies to the right of $\overline{b_1b_2}$ lies in R_2.

By Observation 10, both lines $\overline{b_1b_2}$ and $\overline{b_1b_3}$ intersect the segment $a_{i-1}a_i$. Since the segment $a_{i-1}b_1$ intersects ℓ and since b_1 is the leftmost point of $W_i \cap B$, all points of B' that lie to the left of $\overline{b_1b_2}$ lie to the left of $\overline{a_{i-1}b_1}$. The four points a_{i-1}, b_1, b_2, b_3 form an ℓ-divided 4-hole in P, since a_{i-1} is the leftmost and b_3 is the rightmost point of a_{i-1}, b_1, b_2, b_3 and both a_{i-1} and b_3 lie to the left of $\overline{b_1b_2}$. By Observation 6(i), the sector $S(a_{i-1}, b_1, b_2, b_3)$ is empty of points of P (green shaded area in Figure 12). Altogether, all points of B' that lie to the left of $\overline{b_1b_2}$ are to the right of $\overline{b_2b_3}$ and thus lie in R_1.

Since the segment a_ib_1 intersects ℓ and since b_1 is the leftmost point of $W_i \cap B$, all points of B' that lie to the right of $\overline{b_1b_2}$ lie to the right of $\overline{a_ib_1}$. By Observation 6(i), the sector $S(b_1, b_2, b_3, a_{i-1})$ is empty of points of P. Combining this with the fact that a^* is to the right of $\overline{a_{i-1}b_3}$, we see that a^* lies to the right of $\overline{b_1b_2}$. Since b_1 and b_2 both lie to the left of $\overline{a^*a_i}$ and since a^* and a_i both lie to the right of $\overline{b_1b_2}$, the points b_2, b_1, a^*, a_i form an ℓ-divided 4-hole in P. By Observation 6(i), the sector $S(b_2, b_1, a^*, a_i)$ (blue shaded area in Figure 12) is empty of points of P. Altogether, all points of B' that lie to the right of $\overline{b_1b_2}$ are to the right of $\overline{a^*a_i}$ and to the right of $\overline{a_ib_1}$ and thus lie in R_2. This finishes the proof of Claim 22.1.

Claim 22.2. *If b_4 is a point from $B' \setminus R_1$, then b_2 lies inside the triangle $\triangle(b_3, b_1, b_4)$.*

By Claim 22.1, b_4 lies in R_2 and thus to the right of $\overline{a_ib_1}$ and to the right of $\overline{a^*a_i}$. We recall that b_4 lies to the right of $\overline{b_1b_2}$.

We distinguish two cases. First, we assume that the points b_2, b_3, b_1, a_i are in convex position. Then b_2, b_3, b_1, a_i form an ℓ-divided 4-hole in P and, by Observation 6(i), the sector $S(b_2, b_3, b_1, a_i)$ is empty of points from P. Thus b_4 lies to the right of $\overline{b_2b_3}$ and the statement follows.

Second, we assume that the points b_2, b_3, b_1, a_i are not in convex position. Due to Observation 10, b_2 and b_3 both lie to the right of $\overline{a_ib_1}$. Moreover, since b_3 is the rightmost of those four points, b_2 lies inside the triangle $\triangle(b_3, b_1, a_i)$. In particular, a_i lies to the right of $\overline{b_2b_3}$. Therefore, since b_2 and b_3 are to the left of $\overline{a^*a_i}$, the line $\overline{b_2b_3}$ intersects ℓ in a point p above $\ell \cap \overline{a^*a_i}$. Let q be the point $\ell \cap \overline{b_1b_2}$. Note that q is to the left of $\overline{a^*a_i}$. The point b_4 is to the right of $\overline{b_2b_3}$, as otherwise b_4 lies in $\triangle(p, q, b_2)$, which is impossible because the points p, q, b_2 are in W_i while b_4 is not. Altogether, b_2 is inside $\triangle(b_3, b_1, b_4)$ and this finishes the proof of Claim 22.2.

Claim 22.3. *Either every point of B' is to the right of b_3 or b_3 is the rightmost point of B.*

By Observation 6(i), the sector $S(b_3, a_{i-1}, b_1, b_2)$ is empty of points of P and thus all points of $B' \cap R_1$ lie to the left of $\overline{a_{i-1}b_3}$ and, in particular, to the right of b_3.

Suppose for contradiction that the claim is not true. That is, there is a point $b_4 \in B'$ that is the rightmost point in B and there is a point $b_5 \in B'$ that is to the left of b_3. Note that b_4 is an extremal point of C. By Claim 22.1 and by the fact that all points of $B' \cap R_1$ lie to the right of b_3, b_5 lies in $R_2 \setminus R_1$. By Claim 22.2, b_2 lies in the triangle $\triangle(b_1, b_5, b_3)$, and thus $B \setminus \{b_4\}$ is not in convex position. This contradicts the assumption that C is an ℓ-critical set. This finishes the proof of Claim 22.3.

Claim 22.4. *The point b_3 is the third leftmost point of B. In particular, W_i is the only a^*-wedge with at least three points of B.*

Suppose for contradiction that b_3 is not the third leftmost point of B. Then by Claim 22.3, b_3 is the rightmost point of B and therefore an extremal point of B. This implies that $B' \subseteq R_2 \setminus R_1$, since all points of $B' \cap R_1$ lie to the right of b_3. By Claim 22.2, each point of B' then forms a non-convex quadrilateral together with b_1, b_2, and b_3. Since neither b_1 nor b_2 are extremal points of C and since $|B \cap \partial \operatorname{conv}(C)| = 2$, there is a point $b_4 \in B$ that is an extremal point of C. Since $|B| \geq 5$, the set $C \setminus \{b_4\}$ has none of its parts separated by ℓ in convex position, which contradicts the assumption that C is an ℓ-critical set. Since W_i is an arbitrary a^*-wedge with $w_i \geq 3$, Claim 22.4 follows.

Claim 22.5. *Let W be a union of four consecutive a^*-wedges that contains W_i. Then $|W \cap B| \leq 4$.*

Suppose for contradiction that $|W \cap B| \geq 5$. Let $C' := C \cap W$. Note that $|C' \cap A| = 6$ and that a^*, a_{i-1}, a_i lie in C'. By Lemma 8, there is no ℓ-divided 5-hole in C'. We obtain C'' by removing points from C' from the right, if necessary, until $|C'' \cap B| = 5$. Since C'' is an island of C', there is no ℓ-divided 5-hole in C''. From Claim 22.4 we know that b_1, b_2, b_3 are the three leftmost points in C and thus lie in C''. We apply Lemma 16 to C'' and, since b_1, b_2, b_3 lie in a convex a^*-wedge of C'', we obtain a contradiction. This finishes the proof of Claim 22.5.

We now complete the proof of Proposition 22. First, we assume that $1 \leq i \leq 4$. Let $W := W_1 \cup W_2 \cup W_3 \cup W_4$. By Claim 22.5, $|W \cap B| \leq 4$. Claim 22.4 implies that $w_k \leq 2$ for every k with $5 \leq k \leq t$. By Corollary 13, we have

$$|B| = \sum_{k=1}^{4} w_k + \sum_{k=5}^{t} w_k \leq 4 + (t-3) = t + 1 \leq |A|.$$

The case $t - 3 \leq i \leq t$ follows by symmetry.

Finally, we assume that $5 \leq i \leq t-4$. Let $W := W_{i-3} \cup W_{i-2} \cup W_{i-1} \cup W_i$. Note that W is convex, since $2 \leq i-3$ and $i < t$. By Lemma 18(ii), we have $w_{i-3} + w_{i-2} + w_{i-1} + w_i \leq 3$ and $w_i + w_{i+1} + w_{i+2} + w_{i+3} \leq 3$. By Claim 22.4, $w_k \leq 2$ for all k with $1 \leq k \leq i-4$. Thus, by Corollary 13, $\sum_{k=1}^{i-4} w_k \leq i-3$. Similarly, we have $\sum_{k=i+4}^{t} w_k \leq t-i-2$. Altogether, we obtain that

$$|B| = \sum_{k=1}^{i-4} w_k + \sum_{k=i-3}^{i-1} w_k + w_i + \sum_{k=i+1}^{i+3} w_k + \sum_{k=i+4}^{t} w_k \leq (i-3) + 3 + (t-i-2) = t-2 \leq |A| - 3.$$

□

5.5 Finalizing the proof of Theorem 2

We are now ready to prove Theorem 2. Namely, we show that for every ℓ-divided set $P = A \cup B$ with $|A|, |B| \geq 5$ and with neither A nor B in convex position there is an ℓ-divided 5-hole in P.

Suppose for the sake of contradiction that there is no ℓ-divided 5-hole in P. By the result of Harborth [21], every set P of ten points contains a 5-hole in P. In the case $|A|, |B| = 5$, the statement then follows from the assumption that neither of A and B is in convex position.

So assume that at least one of the sets A and B has at least six points. We obtain an island Q of P by iteratively removing extremal points so that neither part is in convex position after the removal and until one of the following conditions holds.

(i) One of the parts $Q \cap A$ and $Q \cap B$ has only five points.

(ii) Q is an ℓ-critical island of P with $|Q \cap A|, |Q \cap B| \geq 6$.

In case (i), we have $|Q \cap A| = 5$ or $|Q \cap B| = 5$. We can assume by symmetry that $|Q \cap A| = 5$ and $|Q \cap B| \geq 6$. We let Q' be the union of $Q \cap A$ with the six leftmost points of $Q \cap B$. Since $Q \cap A$ is not in convex position, Lemma 14 implies that there is an ℓ-divided 5-hole in Q', which is also an ℓ-divided 5-hole in Q, since Q' is an island of Q. However, this is impossible as then there is an ℓ-divided 5-hole in P because Q is an island of P.

In case (ii), we have $|Q \cap A|, |Q \cap B| \geq 6$. There is no ℓ-divided 5-hole in Q, since Q is an island of P. By Lemma 19(i), we can assume without loss of generality that $|A \cap \partial \operatorname{conv}(Q)| = 2$, as $|A \cap \partial \operatorname{conv}(Q)| + |B \cap \partial \operatorname{conv}(Q)| \geq 3$ and thus $|A \cap \partial \operatorname{conv}(Q)|$ and $|B \cap \partial \operatorname{conv}(Q)|$ cannot be both smaller than 2. Then it follows from Proposition 21 that $|Q \cap B| < |Q \cap A|$. By exchanging the roles of $Q \cap A$ and $Q \cap B$ and by applying Proposition 22, we obtain that $|Q \cap A| \leq |Q \cap B|$, a contradiction. This finishes the proof of Theorem 2.

6 Final Remarks

At a first glance, it might seem that a similar approach could be used to derive stronger lower bounds also on the minimum number of 6-holes $h_6(n)$. However, since there are point sets of 29 points with no 6-hole [24], one would need to investigate point sets of size at least 30 in order to find an ℓ-divided 6-hole. This task is too demanding for our implementations, since the number of combinatorially different point sets grows too rapidly. Moreover, the case analysis in several steps of our proof would become much more complicated.

6.1 Necessity of the assumptions in Theorem 2

In the statement of Theorem 2 we require that the ℓ-divided set $P = A \cup B$ satisfies $|A|, |B| \geq 5$. We now show that those requirements are necessary in order to guarantee an ℓ-divided 5-hole in P by constructing an arbitrarily large ℓ-critical set $C = A \cup B$ with $|A| = 4$ and with no ℓ-divided 5-hole in C.

Proposition 23. *For every integer $n \geq 5$, there exists an ℓ-critical set $C = A \cup B$ with $|A| = 4$, $|B| = n$, and with no ℓ-divided 5-hole in C.*

Proof. First, we consider the case where n is odd. Let $p^+ = (0, 1)$ and $p^- = (0, -1)$ be two auxiliary points and let $\ell^+ = \{(x, y) \in \mathbb{R}^2 : y = x/4\}$ and $\ell^- = \{(x, y) \in \mathbb{R}^2 : y = -x/4\}$ be two auxiliary lines. We place the point $b'_1 = (2, -1/2)$ on the line ℓ^- and the auxiliary point $q = (2, 1/2)$ on the line ℓ^+. For $i = 2, \ldots, n$, we iteratively let b'_i be the intersection of the line ℓ^+ with the segment $p^+b'_{i-1}$ if i is even and the intersection of ℓ^- with $p^-b'_{i-1}$ if i is odd. We place two points a_1 and a_2 sufficiently close to p^+ so that a_1 is above a_2, the segment a_1a_2 is vertical with the midpoint p^+, and all non-collinear triples (b'_i, b'_j, p^+) have the same orientation as (b'_i, b'_j, a_1) and (b'_i, b'_j, a_2). Similarly, we place two points a_3 and a_4 sufficiently close to p^- so that a_3 is to the left of a_4, the segment a_3a_4 lies on the line $\overline{p^-q}$ and has p^- as its midpoint, the point a_4 is to the left of b'_n, and all non-collinear triples (b'_i, b'_j, p^-) have the same orientation as (b'_i, b'_j, a_3) and (b'_i, b'_j, a_4). Figure 13 gives an illustration.

We let A, B', and B'_3 be the sets $\{a_1, a_2, a_3, a_4\}$, $\{b'_1, \ldots, b'_n\}$, and $B' \setminus \{b'_3\}$, respectively. Note that the line $\overline{a_3a_4}$ intersects the segment $b'_1b'_3$. Since $\max_{a \in A} x(a) < \min_{b' \in B'} x(b')$, the sets A and B' are separated by a vertical line ℓ.

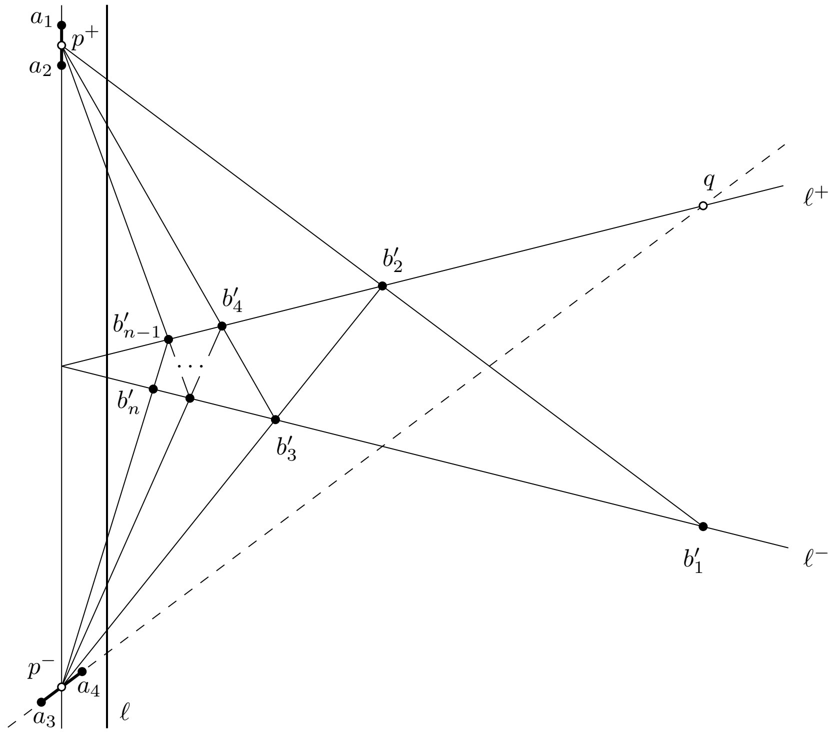

Figure 13: The set C constructed in the proof of Proposition 23 for n odd.

Next we slightly perturb b'_3 to obtain a point b_3 such that b_3 lies above ℓ^- and all non-collinear triples (b_3, c, d) with $c, d \in A \cup B'_3$ have the same orientation as (b'_3, c, d). Note that the point b_3 lies in the interior of $\text{conv}(B'_3)$, since $n \geq 5$.

To ensure general position, we transform every point $b'_i = (x, y) \in B'_3 \cap \ell^+$ to $b_i = (x, y - \varepsilon x^2)$ and every point $b'_i = (x, y) \in B'_3 \cap \ell^-$ to $b_i = (x, y + \varepsilon x^2)$ for some $\varepsilon > 0$. The remaining points in $A \cup \{b_3\}$ remain unchanged. We choose ε sufficiently small so that all non-collinear triples of points from $A \cup B'_3 \cup \{b_3\}$ have the same orientations as their images after the perturbation. Finally, let B be the set $\{b_1, \dots, b_n\}$ and set $B_3 := B \setminus \{b_3\}$.

Since the points from B_3 lie on two parabolas, the set B is in general position. In particular, points from B_3 are in convex position and the point b_3 lies inside $\text{conv}(B_3)$. Also observe that the line ℓ separates A and B and that a_1, a_3, and b_1 are the extremal points of $C := A \cup B$. Since neither of the sets A and B is in convex position, and removal of any of the extremal points a_1, a_3, b_1 leaves either A or B in convex position, the set $C = A \cup B$ is ℓ-critical.

We now show that C contains no ℓ-divided 5-hole. Suppose for contradiction that there is an ℓ-divided 5-hole H in C. We set $A^+ := \{a_1, a_2\}$, $A^- := \{a_3, a_4\}$, $B^+ := \{b_2, b_4, \dots, b_{n-1}\}$, and $B^- := \{b_1, b_3, \dots, b_n\}$. First we assume that H contains points from both A^+ and A^-. Then $H \cap B \subseteq \{b_{n-1}, b_n\}$, since if there is a point b_i in H with $i < n - 1$, then b_n lies in the interior of $\text{conv}(H)$. Note that if $H \cap B = \{b_{n-1}, b_n\}$, then neither a_4 nor a_1 lies in H and thus $|H| < 5$. Hence $|H \cap B| = 1$, which is again impossible, as H cannot contain all points from A. Therefore we either have $H \cap A \subseteq A^+$ or $H \cap A \subseteq A^-$ and, in particular, $1 \leq |H \cap A| \leq 2$.

We now distinguish the following two cases.

1. $|H \cap A| = 2$. If $H \cap A = A^+$, then the hole H can contain only the point b_n from B^-. This is because if there is a point b_i in $H \cap B^-$ with $i < n$, then the point b_{i+1} lies in the interior of conv(H). Additionally, H contains at most two points from B^+, since otherwise H is not in convex position. Consequently, b_n lies in H and $|H \cap B^+| = 2$, which is impossible, as H would not be in convex position.

   If $H \cap A = A^-$, then the hole H contains no point from B^+. This is because if there is a point b_i in $H \cap B^+$, then the point b_{i+1} lies in the interior of conv(H). The point b_1 cannot lie in H because otherwise H is not in convex position as the line $\overline{a_3a_4}$ separates b_1 from $B \setminus \{b_1\}$. Additionally, H contains at most two points from B^-, since otherwise H is not in convex position. Thus H contains at most four points of C, which is impossible.

2. $|H \cap A| = 1$. Assume first that $H \cap A \subseteq A^+$. Note that for $b_i, b_j \in B^-$ with $i < j \le n$, the point b_{i+1} lies inside the triangle $\triangle(a_1, b_i, b_j)$ and, if $j < n$, the point b_{j+1} lies inside $\triangle(a_2, b_i, b_j)$. Thus H contains at most one point from B^- or we have $H \cap B^- = \{b_{n-2}, b_n\}$ and $H \cap A = \{a_2\}$. The latter case does not occur, since for every $b_i \in B^+$ with $i < n-1$ the point b_{n-1} lies in the interior of conv$(\{a_2, b_i, b_{n-2}, b_n\})$. Therefore we consider the case $|H \cap B^-| \le 1$. However, $|H \cap B^+| \ge 3$ is impossible since H would not be in convex position. Altogether, we obtain $|H| < 5$, which is impossible.

   Now we assume that $H \cap A \subseteq A^-$. Note that for $b_i, b_j \in B^+$ with $i < j < n$, the point b_{i+1} lies inside the triangle $\triangle(a_4, b_i, b_j)$ and the point b_{j+1} lies inside $\triangle(a_3, b_i, b_j)$. Thus H contains at most one point from B^+. Consequently, H contains at least three points from B^-, which is possible only if $H \cap B^- = \{b_1, b_3, b_5\}$. However, then H contains a point b_i from B^+ and b_3 lies in the interior of conv(H).

Thus, in any case, H is not an ℓ-divided 5-hole in C, a contradiction.

To finish the proof, we consider the case where n is even. Let $\tilde{C} = A \cup \tilde{B}$ be the set constructed above with $|A| = 4$ and $|\tilde{B}| = n+1$. We set $B := \tilde{B} \setminus \{b_2\}$ and $C := A \cup B$. Note that C is ℓ-critical.

It remains to show that C contains no ℓ-divided 5-hole. Suppose for contradiction that there is an ℓ-divided 5-hole H in C. There is no ℓ-divided 5-hole in $\tilde{C}$ and thus b_2 lies in the interior of conv(H). Since b_1 is the only point from C to the right of b_2, the point b_1 lies in H. Since a_1 is the only point of C to the left of $\overline{b_2b_1}$, all other points of H lie to the right of $\overline{b_2b_1}$. Then, however, the set $(H \setminus \{a_1\}) \cup \{b_2\}$ is a 5-hole in $\tilde{C}$, which gives a contradiction. □

6.2 Necessity of the assumptions in Lemmas 14 to 17

We remark that all the assumptions in the statements of Lemmas 14 to 17 are necessary; Figure 14(a) shows that the conditions $|B| = 5$ in Lemma 16 and the convexity of A in Lemma 17 are both necessary. The horizontal reflection of Figure 14(a) also shows the necessity of the assumption $|A| = 5$ in Lemma 14. It follows from the example in Figure 14(b) that the condition $|B| = 4$ cannot be omitted in Lemma 17, since there is an a-wedge with three points of B. The same point set without the point a' shows that the assumption $|B| \ge 4$ in Lemma 15 is necessary. The example from Figure 14(c) shows that the conditions $|B| = 6$ in Lemma 14, the convex position of A in Lemma 15, and $|A| = 6$ in Lemma 16 are all necessary. The same set without the point a shows that $|A| = 5$ in Lemma 15 is also needed and, if we remove the points a and a', then the resulting point set shows that we need $5 \le |A|$

in Lemma 17. We can make statements only about convex a-wedges in Lemmas 15 and 16, as there are counterexamples for the corresponding statements without the convexity condition. It suffices to consider so-called *double-chains*, which are point sets obtained by placing n points on each of the two branches of a hyperbola. Double-chains also show, that A cannot be in convex position in Lemma 14, and, that the non-convex a-wedge must be empty of points in B in Lemma 17.

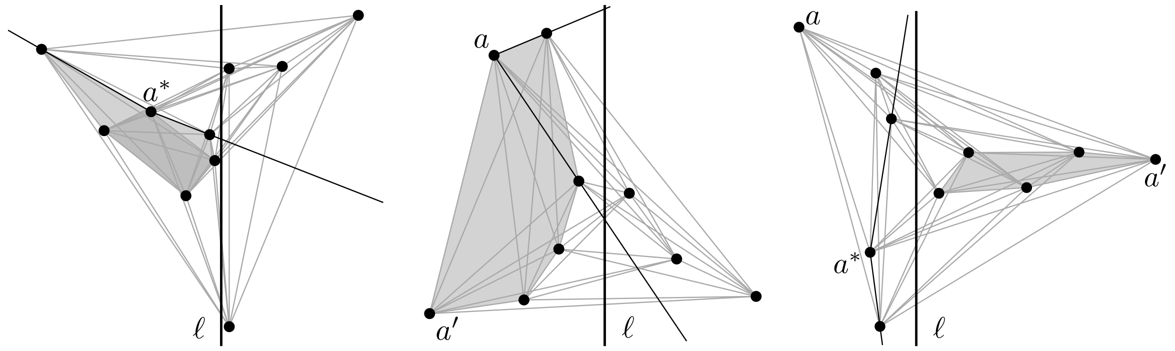

Figure 14: Examples of points sets that witness tightness of Lemmas 14 to 17. All k-holes in these sets with $k \geq 5$ are highlighted in gray.

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A Flow summary

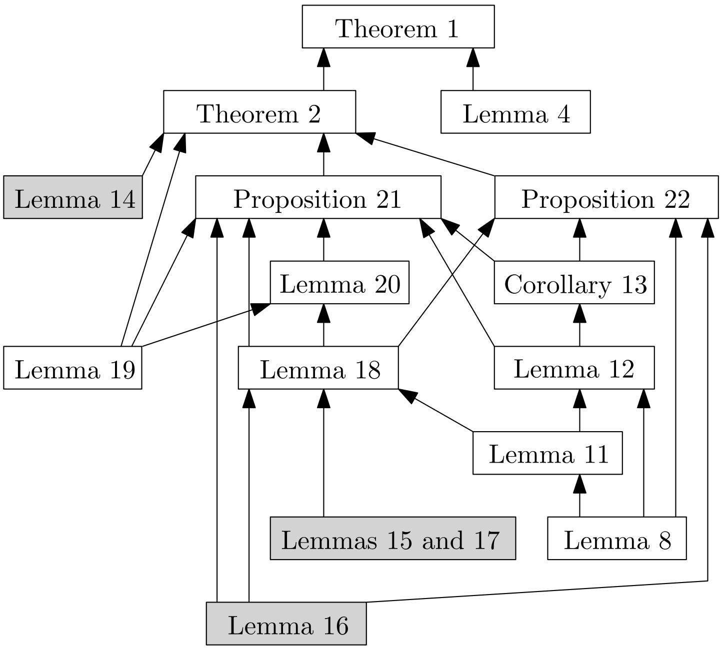

Figure 15: Flow summary. The shaded boxes correspond to computer-assisted results.