

BIJECTIVE PROOFS OF SKEW SCHUR POLYNOMIAL FACTORIZATIONS

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ABSTRACT. In a recent paper, Ayer and Behrend present for a wide class of partitions factorizations of Schur polynomials with an even number of variables where half of the variables are the reciprocals of the others into symplectic and/or orthogonal group characters, thereby generalizing results of Ciucu and Krattenthaler for rectangular shapes. Their proofs proceed by manipulations of determinants underlying the characters. The purpose of the current paper is to provide bijective proofs of such factorizations. The quantities involved have known combinatorial interpretations in terms of Gelfand-Tsetlin patterns of various types or half Gelfand-Tsetlin patterns, which can in turn be transformed into perfect matchings of weighted trapezoidal honeycomb graphs. An important ingredient is then Ciucu’s theorem for graphs with reflective symmetry. However, before being able to apply it, we need to employ a certain averaging procedure in order to achieve symmetric edge weights. This procedure is based on a “randomized” bijection, which can however also be turned into a classical bijection. For one type of Schur polynomial factorization, we also need an additional graph operation that almost doubles the underlying graph. Finally, our combinatorial proofs reveal that the factorizations under consideration can in fact also be generalized to skew shapes as discussed at the end of the paper.

1. INTRODUCTION

Schur polynomials $s_\lambda(x_1, \dots, x_n)$ are central objects in algebraic combinatorics with various beautiful properties and numerous applications. In representation theory, they are the irreducible characters of polynomial representations of the *general linear group* $GL_n(\mathbb{C})$. Ayer and Behrend [1, Theorem 1] showed that for two families of partitions, Schur polynomials with $2n$ variables factorize into characters of other classical groups when specializing such that half of the variables are the reciprocals of the others. The two families of partitions are

$$(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1, -\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1) + \lambda_1 \quad (1.1)$$

and

$$(\lambda_1, \dots, \lambda_n, -\lambda_n, \dots, -\lambda_1) + \lambda_1, \quad (1.2)$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition, allowing here and throughout the whole paper zero parts in a partition and “ $+\lambda_1$ ” means that we add λ_1 to each part. An illustration of the families is provided in Figure 1.

In order to state the precise result, we use the following standard notation for characters of classical groups. Let n be a positive integer and $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with at most n non-zero parts, adding trailing zeros if necessary. Recall the following well-known formula for Schur polynomials.

$$s_\lambda(x_1, \dots, x_n) = \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j})}{\det_{1 \leq i, j \leq n} (x_i^{n - j})} \quad (1.3)$$

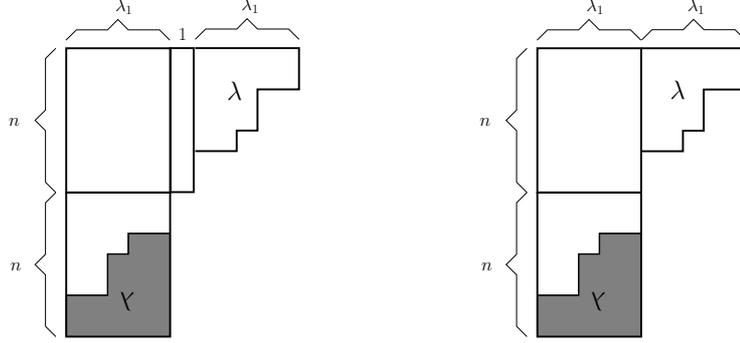


FIGURE 1. The two families of partitions

The other characters appearing in this paper are the following. Throughout the article, we set $\bar{x} = x^{-1}$.

- *Symplectic characters* are the irreducible characters of the *symplectic group* $Sp_{2n}(\mathbb{C})$, and they are given by

$$sp_{\lambda}(x_1, \dots, x_n) = \frac{\det_{1 \leq i, j \leq n} \left(x_i^{\lambda_j + n - j + 1} - \bar{x}_i^{\lambda_j + n - j + 1} \right)}{\det_{1 \leq i, j \leq n} \left(x_i^{n - j + 1} - \bar{x}_i^{n - j + 1} \right)}. \quad (1.4)$$

- *Even orthogonal characters* are the irreducible characters of the *even orthogonal group* $O_{2n}(\mathbb{C})$, and they are given by

$$o_{\lambda}^{\text{even}}(x_1, \dots, x_n) = (1 + [\lambda_n \neq 0]) \frac{\det_{1 \leq i, j \leq n} \left(x_i^{\lambda_j + n - j} + \bar{x}_i^{\lambda_j + n - j} \right)}{\det_{1 \leq i, j \leq n} \left(x_i^{n - j} + \bar{x}_i^{n - j} \right)}, \quad (1.5)$$

where we use the Iverson bracket, i.e., $[S]$ is 1 if S is true and 0 if S is false.

- *Odd orthogonal characters* are the irreducible characters of the *special odd orthogonal group* $SO_{2n+1}(\mathbb{C})$, and they are given by

$$so_{\lambda}^{\text{odd}}(x_1, \dots, x_n) = \frac{\det_{1 \leq i, j \leq n} \left(x_i^{\lambda_j + n - j + 1/2} - \bar{x}_i^{\lambda_j + n - j + 1/2} \right)}{\det_{1 \leq i, j \leq n} \left(x_i^{n - j + 1/2} - \bar{x}_i^{n - j + 1/2} \right)}, \quad (1.6)$$

where δ is the Kronecker delta.

A *half-integer* is an odd integer divided by 2. A *half-integer partition* is a finite weakly decreasing sequence of positive half-integers. In the case of the even orthogonal group, the character formula has a representation theoretic meaning as characters of spin covering groups when λ is a half-integer partition. For more information see, e.g., the book by Fulton and Harris [8, Chap. 24].

The starting point for the research presented in this paper was to provide a combinatorial proof of the following theorem, which appeared in [1, Theorem 1] in a slightly different but equivalent form as also explained there: The case of [1, Theorem 1] in which all parts of λ are half-integers corresponds to the first part of the theorem below (see also [1, Equation (19)]), while the case in which all parts of λ are integers corresponds to the second part (see also [1, Equation (18)]).

For a partition λ and an integer or half-integer ℓ , we denote by $\lambda + \ell$ the tuple obtained by adding ℓ to each part of λ , and we set $\bar{x} = x^{-1}$.

Theorem 1.1. *Let n be a positive integer and $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition.*

(1) *For*

$$\hat{\lambda} = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1, -\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1) + \lambda_1, \quad (1.7)$$

we have

$$s_{\hat{\lambda}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) = sp_{\lambda}(x_1, \dots, x_n) o_{\lambda+1}^{\text{even}}(x_1, \dots, x_n). \quad (1.8)$$

(2) *For*

$$\hat{\lambda} = (\lambda_1, \dots, \lambda_n, -\lambda_n, \dots, -\lambda_1) + \lambda_1, \quad (1.9)$$

we have

$$s_{\hat{\lambda}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) = \prod_{i=1}^n (x_i^{1/2} + \bar{x}_i^{1/2})^{-1} so_{\lambda}^{\text{odd}}(x_1, \dots, x_n) o_{\lambda+\frac{1}{2}}^{\text{even}}(x_1, \dots, x_n). \quad (1.10)$$

Special cases of this theorem were known earlier; those for rectangular shapes in [6] and those for double-staircase shapes were announced in [2, 3]. The known proofs of Theorem 1.1 and its special cases all proceed by manipulations of determinants underlying the characters. In this paper, we interpret the characters as generating functions of Gelfand-Tsetlin patterns, or, equivalently, as matching generating functions of edge-weighted subgraphs of the hexagonal grid, which makes it possible to provide a combinatorial proof. The partitions $\hat{\lambda}$ in Theorem 1.1 are chosen in such a way that the graphs employ a vertical symmetry, which suggests the use of Ciucu's factorization theorem for graphs with reflective symmetry [5]. However, since the edge weights of the graphs are not symmetric, we need to “symmetrize” the weights. This is accomplished by a certain averaging procedure. This procedure is most conveniently explained by what we call a “randomized” bijection (but it can also be turned into a classical bijection). This procedure suffices to fully deal with the identity in Theorem 1.1(1). In the case of Theorem 1.1(2), this procedure results in graphs with symmetric edge weights except for edges incident with the symmetry axis. We resolve this problem by applying a certain graph operation that, in a sense, almost doubles the graph and then Ciucu's factorization theorem is applicable also in this case.

A merit of bijective proofs is often that they reveal more about the relation between two types of objects than “just” the fact that they are counted by the same numbers. In our case, it actually reveals quite naturally the following generalization of Theorem 1.1 to skew shapes. This was not obvious from the previous proofs as they are based on determinantal formulas for the group characters that do not generalize to skew shapes. For the definition of skew symplectic characters and of skew orthogonal characters as well as a discussion of their appearance in the literature as restrictions of straight characters to certain subgroups, we defer to Section 6.

Theorem 1.2. *Let m, n be non-negative integers with $m < n$, and $\mu = (\mu_1, \dots, \mu_m)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ be partitions.*

(1) *For*

$$\hat{\mu} = (\mu_1 + 1, \mu_2 + 1, \dots, \mu_m + 1, -\mu_m, -\mu_{m-1}, \dots, -\mu_1) + \lambda_1, \quad (1.11)$$

$$\hat{\lambda} = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1, -\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1) + \lambda_1, \quad (1.12)$$

we have

$$s_{\widehat{\lambda}/\widehat{\mu}}(x_1, \bar{x}_1, \dots, x_{n-m}, \bar{x}_{n-m}) = sp_{\lambda/\mu}(x_1, \dots, x_{n-m}) o_{(\lambda+1)/(\mu+1)}^{\text{even}}(x_1, \dots, x_{n-m}). \quad (1.13)$$

(2) For

$$\widehat{\mu} = (\mu_1, \mu_2, \dots, \mu_m, -\mu_m, -\mu_{m-1}, \dots, -\mu_1) + \lambda_1, \quad (1.14)$$

$$\widehat{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n, -\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1) + \lambda_1, \quad (1.15)$$

we have

$$s_{\widehat{\lambda}/\widehat{\mu}}(x_1, \bar{x}_1, \dots, x_{n-m}, \bar{x}_{n-m}) = \prod_{i=1}^{n-m} (x_i^{1/2} + \bar{x}_i^{-1/2})^{-1} so_{\lambda/\mu}^{\text{odd}}(x_1, \dots, x_{n-m}) o_{(\lambda+\frac{1}{2})/(\mu+\frac{1}{2})}^{\text{even}}(x_1, \dots, x_{n-m}). \quad (1.16)$$

In both statements, the skew Schur polynomial on the left-hand side has to be interpreted to be zero if the shape $\widehat{\mu}$ is not contained in $\widehat{\lambda}$, and the situation is similar for the characters appearing on the right-hand side.

It would be interesting to find representation-theoretic proofs of Theorems 1.1 and 1.2.

Structure of the paper. The paper is organized as follows. In Section 2, we establish graphical interpretations of the group characters appearing in Theorem 1.1. For Schur polynomials, this is fairly standard. For the other characters, we rely on the work of Proctor [10, 11, 13] where he provides combinatorial interpretations in terms of various types of half (Gelfand-Testlin) patterns. We then use the general idea from the Schur case to obtain equivalent graphical models in terms of honeycomb graphs. However, in these cases there are a few subtleties to take into account. In Section 3, we then perform the above mentioned averaging procedure to achieve symmetric edge weights in the case of Theorem 1.1(1). The proof of this identity is then concluded in Section 4 using Ciucu's factorization result. As for the proof of Theorem 1.1(2), we introduce the above mentioned doubling operation in Section 5 and then complete the proof again by using Ciucu's factorization result. In Section 6, we deal with the case of skew Schur polynomials, where it will be seen that it is straightforward to generalize the proof for straight shapes. Building on previous work by Koike and Terada [12] and others, we briefly discuss the previous appearance of the factors on the right-hand sides of (1.13) and (1.16) in a representation theoretic setting.

Conventions. Throughout the article, we set

$$\bar{x} = x^{-1}. \quad (1.17)$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we always allow zero parts in a partition. Our graphs are edge-weighted in general and if the weight of an edge is not specified, it is 1. Whenever we speak of a matching of a graph, we usually mean a perfect matching unless stated differently.

2. COMBINATORIAL INTERPRETATIONS OF GROUP CHARACTERS AS MATCHING GENERATING FUNCTIONS

The purpose of this section is to provide combinatorial interpretations of the quantities in Theorem 1.1 in terms of matching generating functions.

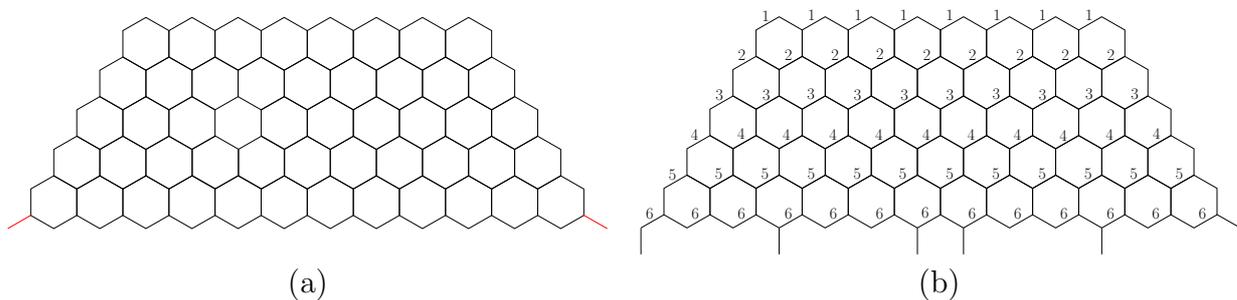


FIGURE 2. The trapezoidal honeycomb graphs (a) $T_{6,8}$ and (b) $T_{6,8}^{1,4,7,8,11,14}(x_1, \dots, x_6)$.

2.1. The general linear group. For a positive integer n and a partition λ , the associated Schur polynomial $s_\lambda(x_1, \dots, x_n)$ is known to be the generating function of *semistandard tableaux of shape λ* ¹ with entries in $\{1, 2, \dots, n\}$ where the weight of a particular semistandard tableau T is

$$x_1^{\# \text{ of 1's in } T} x_2^{\# \text{ of 2's in } T} \dots x_n^{\# \text{ of } n\text{'s in } T}. \quad (2.1)$$

From this interpretation it is obvious that the Schur polynomial $s_\lambda(x_1, \dots, x_n)$ vanishes if λ has more than n non-zero parts. It is fundamental to our combinatorial proof to work with a different interpretation as a generating function which is in terms of the (*perfect*) *matching generating function* of a certain subgraph of the *hexagonal grid* of trapezoidal shape. The graphs relevant for our graphical model are the following.

Definition 2.1 (The graph $T_{n,k}$). *Let n, k be positive integers. The subgraph of the hexagonal grid that consists of $n - 1$ centered rows of consecutive hexagons of lengths $k, k + 1, \dots, k + n - 2$ with two edges added, one incident with the bottom vertex of the leftmost vertical edge and the other incident with the bottom vertex of the rightmost vertical edge, is denoted by $T_{n,k}$. The degenerate case $T_{1,k}$ consists of a zig-zag line with $2k + 1$ vertices.*

See Figure 2(a) for a drawing of $T_{6,8}$, where the two additional edges are marked in red. The graph $T_{n,k}$ is a bipartite graph that has n more vertices in one vertex class than in the other vertex class, thus it has no perfect matching. This can be changed by attaching n vertical edges to a selection of n of the $n + k$ bottommost vertices. In addition, we also introduce edge weights.

Definition 2.2 (The graphs $T_{n,k}^{\mathbf{p}}$ and $T_{n,k}^{\mathbf{p}}(x_1, \dots, x_n)$). *Let n, k be positive integers and $\mathbf{p} = (p_1, \dots, p_n)$ be a sequence of integers with $1 \leq p_1 < p_2 < \dots < p_n \leq n + k$.*

- (1) *The graph $T_{n,k}^{\mathbf{p}}$ is obtained from $T_{n,k}$ by attaching vertical edges to the vertices in positions p_1, p_2, \dots, p_n at the bottom, where the bottommost vertices are numbered from left to right with $1, 2, \dots, n + k$.*
- (2) *The weighted graph $T_{n,k}^{\mathbf{p}}(x_1, \dots, x_n)$ is obtained from $T_{n,k}^{\mathbf{p}}$ as follows: Each edge of type “ \swarrow ” (that is SW-NE edges) in row i of zig-zag lines (counted from the top) carries the weight x_i , while all other edges have weight 1.*

See Figure 2(b) for the graph $T_{6,8}^{1,4,7,8,11,14}$. The weights are also indicated in this figure, where x_i is abbreviated as i , and, by our convention, edges have weight 1 if no weight is indicated.

¹That is fillings of the Young diagram of shape λ which are weakly increasing along rows and strictly increasing along columns.

As usual, the weight of a matching is the product of the weights of all edges that are contained in the matching, and the matching generating function is the sum of all matching weights. In general, the matching generating function is denoted by $M(G)$, where G is an edge-weighted graph, and $\mathcal{M}(G)$ denotes the set of all matchings of G .

We are now in a position to state the different interpretation of $s_\lambda(x_1, \dots, x_n)$ as a generating function. As noted above, the Schur polynomial is zero unless λ has at most n parts. For partitions with less than n parts, it is convenient to fill up λ with zero parts so that it has precisely n parts.

Theorem 2.3. *For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we have*

$$s_\lambda(x_1, \dots, x_n) = M(T_{n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \dots, x_n)). \quad (2.2)$$

This relation between semistandard tableaux and matchings of the trapezoidal honeycomb graph is not new. We illustrate it now with the help of an example where $n = 6$. We consider the following semistandard tableau T of shape $(8, 6, 4, 4, 2, 0)$.

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 5 & 5 & 6 \\ \hline 2 & 2 & 3 & 3 & 5 & 6 & & \\ \hline 3 & 4 & 4 & 4 & & & & \\ \hline 5 & 5 & 6 & 6 & & & & \\ \hline 6 & 6 & & & & & & \\ \hline \end{array} \quad (2.3)$$

By using a standard procedure [14, p. 313ff.], we transform the semistandard tableau into a Gelfand-Tsetlin pattern with n rows as follows: Row i of the Gelfand-Tsetlin pattern is essentially the shape of the entries less than or equal to i in T , written in reverse order and filled up with zeros if necessary so that it has length i . For the semistandard tableau in (2.3), we obtain the following pattern.

$$\begin{array}{cccccccc} & & & & 3 & & & \\ & & & & 2 & 5 & & \\ & & & 1 & 4 & 5 & & \\ & & 0 & 4 & 4 & 5 & & \\ & 0 & 2 & 4 & 5 & 7 & & \\ 0 & 2 & 4 & 4 & 6 & 8 & & \end{array} \quad (2.4)$$

The weight of a Gelfand-Tsetlin pattern with n rows is $\prod_{i=1}^n x_i^{r_i - r_{i-1}}$, where r_i is the sum of the entries in the i -th row and $r_0 = 0$. Now we add i to the i -th \nearrow -diagonal, where we count the diagonals from left to right.

$$\begin{array}{cccccccc} & & & & 4 & & & \\ & & & & 3 & 7 & & \\ & & & 2 & 6 & 8 & & \\ & & 1 & 6 & 7 & 9 & & \\ 1 & 4 & 7 & 9 & 12 & & & \\ 1 & 4 & 7 & 8 & 11 & 14 & & \end{array} \quad (2.5)$$

We translate this pattern into a matching of $T_{6,8}^{1,4,7,8,11,14}$; see also [7, Proposition 2.1]. First note that the positions of the vertical edges added at the bottom of the graph are just $\lambda_6 + 1, \lambda_5 +$

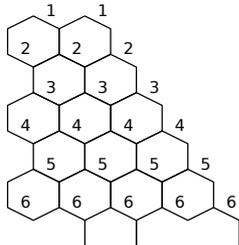


FIGURE 4. The half-trapezoidal honeycomb graph $\text{HT}_{6,2}^{-2,3,5}(x_1, \dots, x_6)$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, denote the set of all $(2n)$ -symplectic patterns with bottom row λ in increasing order as \mathcal{SP}_λ . A combinatorial interpretation of symplectic characters in the form of a generating function is provided next.

Theorem 2.6 ([13, Theorem 4.2]). *Let λ be a partition with n parts. Then*

$$sp_\lambda(x_1, \dots, x_n) = \sum_{P \in \mathcal{SP}_\lambda} w_{\text{sp}}(P). \quad (2.8)$$

Example 2.7. *Let $n = 2$ and $\lambda = (1, 0)$. Then the four 4-symplectic patterns contributing to $sp_{(1,0)}(x_1, x_2)$ and their weights are as follows.*

$$\begin{array}{cccccccc}
 1 & & 0 & & 0 & & 0 & \\
 & 1 & & 1 & & 0 & & 0 \\
 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
 \hline
 & \bar{x}_1 & & x_1 & & \bar{x}_2 & & x_2
 \end{array} \quad (2.9)$$

Analogous to the case of Schur polynomials, we express the symplectic characters as a matching generating function of certain weighted graphs. Let $\text{HT}_{2n,k}^-$ denote the half-trapezoidal honeycomb graph consisting of $2n - 1$ left-justified rows of consecutive hexagons of lengths $k, k, k + 1, k + 1, \dots, k + n - 2, k + n - 2, k + n - 1$, with one edge added incident with the bottom vertex of the rightmost vertical edge; see Figure 4 for an example. Note that the odd and even rows are vertically aligned separately, and we fix the convention that the odd rows are shifted by half a hexagon to the left of the even rows. Just like $T_{n,k}$, the bipartite graph $\text{HT}_{2n,k}^-$ does not have a perfect matching as it has n more vertices in one vertex class than in the other. We attach vertical edges to n of the $n + k$ bottommost vertices at positions $p_1 < \dots < p_n$ (numbered $1, \dots, n + k$ from left to right) to form the graph $\text{HT}_{2n,k}^{-\mathbf{P}}$. We also consider an edge-weighted version: The graph $\text{HT}_{2n,k}^{-\mathbf{P}}(x_1, \dots, x_{2n})$ is formed by weighting each edge of type “ \searrow ” in row i by x_i . All other edges are weighted 1.

Theorem 2.8. *For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we have*

$$sp_\lambda(x_1, \dots, x_n) = M(\text{HT}_{2n, \lambda_1}^{-\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (2.10)$$

In Figure 5, we illustrate Theorem 2.8 with an example where the underlying graph is the one in Figure 4.

Proof. Similar to the case of Schur polynomials, we construct a weight-preserving bijection between the sets corresponding to the two sides of the equation.

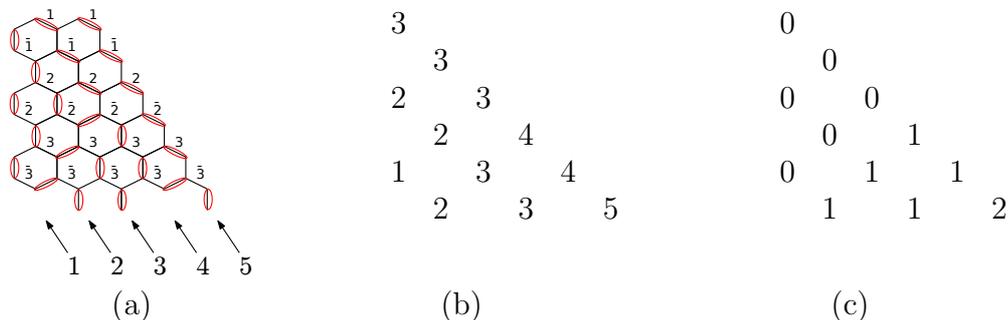


FIGURE 5. (a) A matching of the half-trapezoidal honeycomb graph $\text{HT}_{6,2}^{-2,3,5}(x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ corresponding to the partition $\lambda = (2, 1, 1)$ with weight $x_2 x_3$. The numbering of the vertical edges is indicated below. (b) The positions of the vertical edges in rows 1 through 6 arranged in the shape of a 6-half pattern. (c) A 6-symplectic pattern with bottom row λ having the same weight as the matching on the left.

First of all, note that a matching in $\text{HT}_{2n,k}^{-p_1, \dots, p_n}$ is completely determined by the positions of the vertical edges in each row. Further, the number of vertical edges in the i -th row can be seen to be $\lfloor \frac{i}{2} \rfloor$. Let the positions of the vertical edges in row i be labelled starting from the left with $n + 1 - \lfloor \frac{i}{2} \rfloor$.

Arranging the positions of these vertical edges in the form of a $(2n)$ -half pattern, we see that the \nearrow -diagonals are strictly increasing, and the \searrow -diagonals are weakly increasing. Therefore, decreasing the j -th \searrow -diagonal (counted from the left, starting with 1) by j , we obtain a $(2n)$ -symplectic pattern according to Definition 2.5. This construction is illustrated in Figure 5.

The proof that this map is weight-preserving is very similar to that for the general linear group. (We give more details on this in a similar situation in the proof of Lemma 2.12.) \square

2.3. The even orthogonal group. We now define Gelfand-Tsetlin patterns for the even orthogonal groups. For an n -half pattern $P = (P_{i,j})$ as described in Definition 2.4, the *absolute row sum* of the i -th row is given by $r_i^+ = \sum_j |P_{i,j}|$. Let $\text{sgn}(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0. \end{cases}$

Definition 2.9 (Orthogonal patterns). *Let n be a positive integer. A $(2n - 1)$ -orthogonal (Gelfand-Tsetlin) pattern P is a $(2n - 1)$ -half pattern whose entries are either all integers or all half-integers, and which satisfy the following conditions:*

- All entries except the odd starters are non-negative.
- The odd starters satisfy $|P_{2i-1,1}| \leq \min(P_{2i-2,1}, P_{2i,1})$ (with $P_{0,1}, P_{2n,1} = +\infty$).

The weight $w_{\circ}^{\text{even}}(P)$ of a $(2n - 1)$ -orthogonal pattern P is given by

$$w_{\circ}^{\text{even}}(P) = \prod_{i=1}^n x_i^{\text{sgn}(P_{2i-1,1}) \text{sgn}(P_{2i-3,1}) (r_{2i-1}^+ - 2r_{2i-2}^+ + r_{2i-3}^+)}, \quad (2.11)$$

where we set $r_0^+ = r_{-1}^+ = 0$ and $P_{-1,1} = 0$.

For an integer partition or a half-integer partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we set $\lambda^- = (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$. Denote the set of all $(2n-1)$ -orthogonal patterns with bottom row λ or λ^- in increasing order as \mathcal{OP}_λ or \mathcal{OP}_{λ^-} , respectively.

Theorem 2.10 ([13, First part of Theorem 7.3]). *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be either an integer partition or a half-integer partition. Then*

$$o_\lambda^{\text{even}}(x_1, \dots, x_n) = \sum_{P \in \mathcal{OP}_\lambda \cup \mathcal{OP}_{\lambda^-}} w_o^{\text{even}}(P). \quad (2.12)$$

Example 2.11. *Let $n = 2$ and $\lambda = (1, 1)$. Then the six 3-orthogonal patterns contributing to $o_{(1,1)}^{\text{even}}(x_1, x_2)$ and their weights are as follows.*

$$\begin{array}{cccccc} -1 & & 0 & & 1 & & -1 & & 0 & & 1 \\ & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\ 1 & & 1 & & 1 & & 1 & & -1 & & 1 & & -1 & & 1 \\ \hline & \bar{x}_1 \bar{x}_2 & & 1 & & x_1 x_2 & & \bar{x}_1 x_2 & & 1 & & x_1 \bar{x}_2 \end{array} \quad (2.13)$$

Again we aim to relate the even orthogonal characters to the matching generating functions of certain honeycomb graphs. This relation is not as straightforward as in the case of symplectic characters in Theorem 2.8. Let $\text{HT}_{+2n,k}$ be a half-trapezoidal honeycomb graph consisting of $2n-1$ right-justified rows of consecutive hexagons of length $k, k+1, k+1, \dots, k+n-1, k+n-1$ with three extra edges (see Figure 6 for an example): one at the end of the top zig-zag row, and two at the beginning and end of the bottom zig-zag row. Again, the odd and even rows are aligned separately, and we fix the even rows to be aligned half a hexagon to the right of the odd rows.

We attach n vertical edges to vertices in the bottom row at positions $p_1 < \dots < p_n$ numbered $0, 1, \dots, n+k$ from *right to left* to form the graph $\text{HT}_{+2n,k}^{\mathbf{P}}$. In the edge-weighted variant $\text{HT}_{+2n,k}^{\mathbf{P}}(x_1, \dots, x_n)$, each edge of type “ \nearrow ” in row i is weighted x_i and rightmost vertical edges are weighted by $\frac{1}{2}$. The version where the rightmost vertical edge are weighted by 1 is denoted by $\widehat{\text{HT}}_{+2n,k}^{\mathbf{P}}(x_1, \dots, x_n)$.²

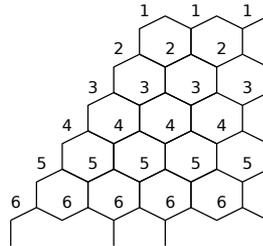


FIGURE 6. The half-trapezoidal honeycomb graph $\text{HT}_{+6,2}^{2,3,5}(x_1, \dots, x_6)$. The blue edges on the right are weighted $\frac{1}{2}$.

To prove the main result of this section, we need a preliminary identity involving a subset of orthogonal patterns. We say that an orthogonal pattern is *non-negative* if all its entries are non-negative.

²Note that combining $\text{HT}_{+2n,k}^{p_1, \dots, p_n}$ with $\text{HT}_{-2n,k}^{q_1, \dots, q_n}$ by adding one edge to each of the $2n$ leftmost vertices of the latter graph, we obtain the trapezoidal graph $\text{T}_{2n, 2k+1}^{n+k+1-p_n, n+k+1-p_{n-1}, \dots, n+k+1-p_1, n+k+1+q_1, \dots, n+k+1+q_n}$.

Lemma 2.12. *For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we have*

$$\sum_{\substack{P \in \mathcal{OP}_\lambda \\ P \text{ non-negative}}} w_o^{\text{even}}(P) = M(\widehat{\text{HT}}_{2n, \lambda_1 - 1}^{\lambda_n, \lambda_{n-1} + 1, \dots, \lambda_1 + n - 1})(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n). \quad (2.14)$$

We illustrate Lemma 2.12 with an example in Figure 7 where the underlying graph is the one given in Figure 6.

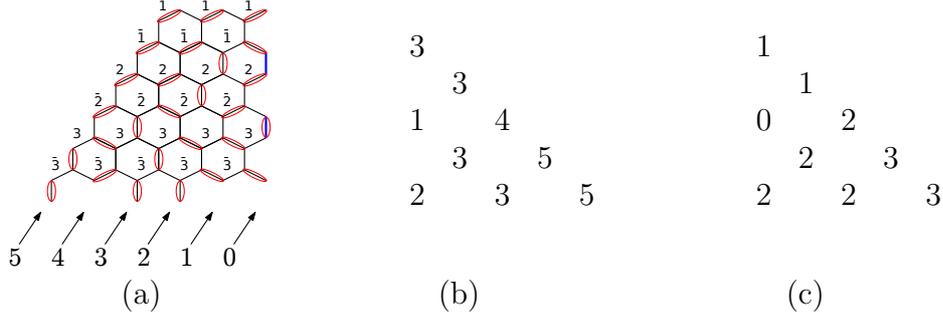


FIGURE 7. (a) A matching of the half-trapezoidal honeycomb graph $\text{HT}+_{6,2}^{2,3,5}(x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3)$ corresponding to the partition $\lambda = (3, 2, 2)$ with weight $(x_1 x_2 \bar{x}_3)/2$. The positions of the vertical edges are marked below. (b) The positions of the vertical edges in rows 2 through 6 arranged in the shape of a 5-half pattern. (c) The corresponding 5-orthogonal pattern with bottom row λ is displayed which is obtained by subtracting i from the i -th \searrow -diagonal, where diagonals are counted from the left starting with 0.

Proof. We construct a weight-preserving bijection between $(2n - 1)$ -orthogonal patterns with all odd starters being non-negative and $\mathcal{M}(\widehat{\text{HT}}_{2n, \lambda_1 - 1}^{\lambda_n, \lambda_{n-1} + 1, \dots, \lambda_1 + n - 1})(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)$. This proceeds similar to the proof of Theorem 2.8.

As usual, each matching is uniquely determined by the positions of the vertical edges in each row, with there being $\lfloor \frac{i}{2} \rfloor$ matching edges in the i -th row. (Note that the edges in the top zig-zag row are forced because of the jutting edge “/” on the right, and, therefore, there are no vertical matching edges in the top row.) Label the positions of the vertical edges starting from the left in each row with $n + k$ and decreasing to the right, and construct an array that has the form of a $(2n - 1)$ -half pattern by listing the positions of the vertical edges in each row in increasing order.

We then see that the \nearrow -diagonals are strictly increasing, while the \searrow -diagonals are weakly increasing. Subtract i from the i -th \searrow -diagonal, counting from the left starting with 0. The odd starters are non-negative by construction.

We now show that this bijection is weight-preserving. The formula for the weight of the pattern in (2.11) simplifies since $r_i^+ = r_i$ for each i and the signs are all 1. In the graphical model, let the positions of the vertical matching edges in the i -th row in increasing order (i.e. right to left) be $T_{i,1}, \dots, T_{i, \lfloor i/2 \rfloor}$ for $i = 1, 2, \dots, 2n$ (see the middle column of Figure 7). The corresponding row of the orthogonal pattern P is given by $P_{i-1,1}, \dots, P_{i-1, \lfloor i/2 \rfloor}$, where

$$P_{i-1,j} = T_{i,j} - (n - \lfloor i/2 \rfloor + j - 1). \quad (2.15)$$

To compare the exponents of x_i , we need to consider the matching edges of type “ \nearrow ” in rows $2i - 1$ and $2i$ of zig-zags. To this end, we need consider the vertically matching edges in rows $2i - 2$, $2i - 1$ and $2i$. These are given by the tuples $(T_{2i-2,1}, \dots, T_{2i-2,i-1})$, $(T_{2i-1,1}, \dots, T_{2i-1,i-1})$ and $(T_{2i,1}, \dots, T_{2i,i})$. The number of matching edges of type “ \nearrow ” in row $2i - 1$ is

$$\begin{aligned} & (T_{2i-2,1} - (n - i + 1)) + (T_{2i-2,2} - T_{2i-1,1} - 1) + \dots + (T_{2i-2,i-1} - T_{2i-1,i-2} - 1) + (n + \lambda_1 - 1 - T_{2i-1,i-1}) \\ & = \sum_{j=1}^{i-1} (T_{2i-2,j} - T_{2i-1,j}) + \lambda_1. \end{aligned} \quad (2.16)$$

Similarly, the number of matching edges of type “ $/$ ” in row $2i$ is

$$\begin{aligned} & (T_{2i-1,1} - T_{2i,1} - 1) + \dots + (T_{2i-1,i-1} - T_{2i,i-1} - 1) + (n + \lambda_1 - 1 - T_{2i,i}) \\ & = \sum_{j=1}^{i-1} T_{2i-1,j} - \sum_{j=1}^i T_{2i,j} - i + n + \lambda_1. \end{aligned} \quad (2.17)$$

Therefore, the total exponent of x_i is the difference of these expressions, which is

$$\sum_{j=1}^{i-1} T_{2i-2,j} - 2 \sum_{j=1}^{i-1} T_{2i-1,j} + \sum_{j=1}^i T_{2i,j} - n + i. \quad (2.18)$$

Using (2.15), this can be shown to be equal to $r_{2i-1} - 2r_{2i-2} + r_{2i-3}$, as desired. \square

We need to consider the following operation on orthogonal patterns. This is a variation of the Bender-Knuth involution [4].

Proposition 2.13. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition. Define the map J_i on orthogonal patterns \mathcal{OP}_λ for $2 \leq i \leq n$ as follows. For $P \in \mathcal{OP}_\lambda$, $J_i(P)$ leaves all the rows of P , except for row $2i - 2$ unchanged, and*

$$(J_i(P))_{2i-2,j} = \max(|P_{2i-3,j}|, |P_{2i-1,j}|) + \min(P_{2i-3,j+1}, P_{2i-1,j+1}) - P_{2i-2,j}, \quad 1 \leq j \leq i - 1, \quad (2.19)$$

where we fix $P_{2i-3,i} = +\infty$. Then the following properties hold for J_i .

- $J_i(P) \in \mathcal{OP}_\lambda$.
- J_i is an involution.
- If either of the odd starters in rows $2i - 3$ or $2i - 1$ is 0, then the exponent of x_i in $w_{\circ}^{\text{even}}(J_i(P))$ is the negative of the exponent of x_i in $w_{\circ}^{\text{even}}(P)$.

We will only apply the operation J_i to orthogonal patterns such that either of the odd starters in rows $2i - 3$ or $2i - 1$ is 0. The proof of Proposition 2.13 is a routine calculation and left to the interested reader.

We denote by \mathcal{I}_n the set of permutations of $\{x_1, \bar{x}_1, x_2, \bar{x}_2, \dots, x_n, \bar{x}_n\}$ that are generated by the transpositions $(x_1, \bar{x}_1), (x_2, \bar{x}_2), \dots, (x_n, \bar{x}_n)$. This group is isomorphic to \mathbb{Z}_2^n . For $\sigma \in \mathcal{I}_n$, $\sigma \text{ HT} +_{2n,k}^{p_1, \dots, p_n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)$ denotes the edge weighted graph $\text{HT} +_{2n,k}^{p_1, \dots, p_n}$ whose row-weights are permuted according to σ .

Theorem 2.14. *For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we have*

$$o_{\lambda}^{\text{even}}(x_1, \dots, x_n) = \sum_{\sigma \in \mathcal{I}_n} M(\sigma \text{ HT} +_{2n, \lambda_1 - 1}^{\lambda_n, \lambda_{n-1} + 1, \dots, \lambda_1 + n - 1}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (2.20)$$

Proof. We use Theorem 2.10 to interpret the left-hand side. To be more precise, we need to refine this interpretation in the following sense: each orthogonal pattern P appearing on the left-hand side that has precisely k odd starters equal to 0 is replaced by 2^k copies of P , the weight of each copy being $\frac{1}{2^k} w_o^{\text{even}}(P)$, and we accompany each copy with a (different) $\{0, 1\}$ -sequence of length k to remedy this modification.

Fix a pair (m, σ) , where m is a matching of $\text{HT}^+_{2n, \lambda_1-1}^{\lambda_n, \lambda_{n-1}+1, \dots, \lambda_1+n-1}$ and $\sigma \in \mathcal{I}_n$. We give a weight-preserving bijection between such pairs (where the weight is the weight of the matching in $\sigma \text{HT}^+_{2n, \lambda_1-1}^{\lambda_n, \lambda_{n-1}+1, \dots, \lambda_1+n-1}$) and objects from the left-hand side as described in the previous paragraph.

First, let P be the orthogonal non-negative pattern corresponding to m by Lemma 2.12. Now we change inductively, for each $i \in \{1, 2, \dots, n\}$, the odd starter P_{2i-1} (by changing the sign) or row $2i-2$ of P (by applying J_i) according to σ such that the map is weight-preserving. Here the crucial observation is that the weight of a matching in

$$(x_i, \bar{x}_i) \sigma \text{HT}^+_{2n, \lambda_1-1}^{\lambda_n, \lambda_{n-1}+1, \dots, \lambda_1+n-1}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) \quad (2.21)$$

is obtained from its weight in

$$\sigma \text{HT}^+_{2n, \lambda_1-1}^{\lambda_n, \lambda_{n-1}+1, \dots, \lambda_1+n-1}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) \quad (2.22)$$

by replacing x_i with \bar{x}_i .

Let $\sigma = (x_1, \bar{x}_1)^{\epsilon_1} \dots (x_n, \bar{x}_n)^{\epsilon_n}$ for appropriately chosen $\epsilon_i \in \{0, 1\}$. For $i = 1$, we do the following: If $P_{1,1} > 0$, we change the sign of $P_{1,1}$ iff $\epsilon_1 = 1$; if $P_{1,1} = 0$, we leave P unchanged and the corresponding position in the accompanying $\{0, 1\}$ -sequence is ϵ_1 .

Now suppose we have reached $i > 1$. First assume $P_{2i-1,1} > 0$. If $\epsilon_i = 0$, we give $P_{2i-1,1}$ the appropriate sign so that it has the same sign as $P_{2i-3,1}$, otherwise so that it has the opposite sign. If, however, $P_{2i-1,1} = 0$ we apply J_i to P if and only if either $\epsilon_i = 1$ and $P_{2i-3,1} \geq 0$, or $\epsilon_i = 0$ and $P_{2i-3,1} < 0$. If we have applied J_i , we record 1 in the appropriate position of the accompanying $\{0, 1\}$ -sequence and 0 otherwise.

To give an example, we consider the matching in Figure 7 and $\sigma = (x_1, \bar{x}_1)(x_3, \bar{x}_3)$ to it. The weight is then $(\bar{x}_1 x_2 x_3)/2$. Applying the procedure just described, we obtain the following 5-orthogonal pattern with the same weight

$$\begin{array}{ccccc} & & -1 & & \\ & & & 2 & \\ & & 0 & & 2 \\ & & & 2 & & 2 \\ & & 2 & & 2 & & 3 \end{array} \quad (2.23)$$

and the accompanying $\{0, 1\}$ -sequence of length 1 is (1). \square

We now consider the case when the bottom row of the $(2n-1)$ -orthogonal pattern consists of half-integers. For this purpose, the following variation of Lemma 2.12 is helpful.

Lemma 2.15. *For a half-integer partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we have*

$$\sum_{\substack{P \in \mathcal{OP}_\lambda \\ P \text{ non-negative}}} w_o^{\text{even}}(P) = \prod_{i=1}^n x_i^{1/2} M(\widehat{\text{HT}}^+_{2n, (\lambda_1 - \frac{1}{2})-1}^{(\lambda_n - \frac{1}{2}), (\lambda_{n-1} - \frac{1}{2})+1, \dots, (\lambda_1 - \frac{1}{2})+n-1}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (2.24)$$

Let $\widehat{\text{HT}}_{-2n,k}^{p_1, \dots, p_n}(x_1, \dots, x_{2n})$ denote the weighted graph obtained from $\text{HT}_{-2n,k}^{p_1, \dots, p_n}(x_1, \dots, x_{2n})$ by adding 1 to the weight of the first edge in the even rows of zigzags (which is always an edge of type “ \searrow ”), so that this edge in row $2i$ then has weight $x_{2i} + 1$. Similarly to Theorem 2.8, one can show the following.

Theorem 2.20. *For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we have*

$$so_{\lambda}^{\text{odd}}(x_1, \dots, x_n) = M(\widehat{\text{HT}}_{-2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (2.30)$$

Proof. For a $(2n)$ -split orthogonal pattern, we add $\frac{1}{2}$ to the odd starters that are half-integers to obtain a $(2n)$ -symplectic pattern. Using Theorem 2.8, we then obtain the corresponding matching of

$$\text{HT}_{-2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n). \quad (2.31)$$

This is clearly a 2^r -to-1 mapping, where r is the number of non-zero odd starters. The weight $\bar{x}_i + 1$ of the first edge of type “ \searrow ” in row $2i$ of

$$\widehat{\text{HT}}_{-2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) \quad (2.32)$$

remedies this for the following reason. This edge is in the matching if and only if the starter in row $2i - 1$ is non-zero. The weight of a $(2n)$ -split orthogonal pattern where the odd starter in row $2i - 1$ is a half integer is obtained from the weight of the pattern with this odd starter rounded up by multiplying with x_i . \square

3. SYMMETRIZING THE WEIGHTS

Now that we have combinatorial realizations of all the quantities in Theorem 1.1, we can proceed with our combinatorial proof. The proof consists of two major steps. The purpose of this section is to provide necessary ingredients for the first step. The second step is then the application of Ciucu’s *factorization theorem for graphs with reflective symmetry* [5]. The unweighted graphs $\text{T}_{n,k}^{\text{P}}$ that are relevant for Theorem 1.1 are indeed already endowed with such a reflective symmetry. However, the edge weights of $\text{T}_{n,k}^{\text{P}}(x_1, \dots, x_n)$ are not symmetric and Ciucu’s theorem is only applicable if they are. In this section, we show how, in the context of the first part of Theorem 1.1, it is possible to reduce the problem to graphs where the edge weights are also symmetric. (Later we will see that the underlying procedure is also useful for proving the second part of Theorem 1.1.) For this purpose, it is necessary to introduce the following different edge weights of $\text{T}_{n,k}^{\text{P}}$.

Definition 3.1 ($\text{ST}_{n,k,j}^{\text{P}}(x_1, \dots, x_n)$). *Let ℓ_j denote the vertical line that contains the j -th vertex in the top row, counted from the left, of $\text{T}_{n,k}^{\text{P}, \dots, p_n}$. To the left of ℓ_j , the weights of $\text{ST}_{n,k,j}^{\text{P}}(x_1, \dots, x_n)$ coincide with the weights of $\text{T}_{n,k,j}^{\text{P}}(x_1, \dots, x_n)$, and to the right of ℓ_j the edges of type “ \searrow ” are assigned the weight x_i in row i , while all other edges are assigned the weight 1.*

See Figure 8 for $\text{ST}_{6,8,4}^{1,4,7,8,11,14}(x_1, \dots, x_6)$.

In the following, it is convenient to use a concept which we call *randomized bijection*: Suppose A and B are two finite sets, then a randomized map from A to B is a randomized algorithm that assigns to each element $a \in A$ an element $b \in B$ with some probability $p_{a,b}$ such that

$$\sum_{b \in B} p_{a,b} = 1 \quad (3.1)$$

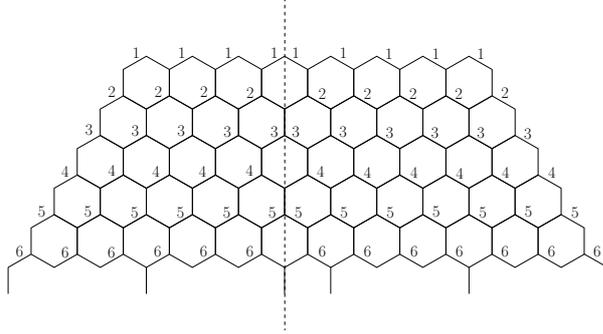


FIGURE 8. The graph $\text{ST}_{6,8,4}^{1,4,7,8,11,14}(x_1 \dots, x_6)$.

for all $a \in A$. We say that a randomized map is a randomized bijection, if there exists a randomized map from B to A such that the corresponding randomized algorithm sends $b \in B$ to $a \in A$ with probability $p_{a,b}$ (which implies then also $\sum_{a \in A} p_{a,b} = 1$ for all $b \in B$). A randomized bijection can only exist if A and B have the same cardinality, as

$$|A| = \sum_{a \in A} 1 = \sum_{a \in A} \sum_{b \in B} p_{a,b} = \sum_{b \in B} \sum_{a \in A} p_{a,b} = \sum_{b \in B} 1 = |B|. \quad (3.2)$$

This concept can also be used to show that the generating functions of two sets A and B are the same: if a randomized bijection from A and B is weight-preserving (that is $p_{a,b} = 0$ unless $w(a) = w(b)$), then it is a randomized bijection between the subset of elements of A that have a prescribed weight w and the subset of elements of B that have the same weight w , for every possible weight w .

Lemma 3.2. *Let $1 \leq j \leq k$. Then the matching generating function*

$$\text{M}(\text{TP}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) \quad (3.3)$$

is equal to

$$\frac{1}{2^n} \sum_{\sigma \in \mathcal{I}_n} \text{M}(\sigma \text{ST}_{2n,k,j}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (3.4)$$

Proof. By Theorem 2.3 and the symmetry of the Schur polynomial,

$$\text{M}(\sigma \text{TP}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) = \text{M}(\text{TP}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) \quad (3.5)$$

for any $\sigma \in \mathcal{I}_n$. It follows that

$$\text{M}(\text{TP}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) = \frac{1}{2^n} \sum_{\sigma \in \mathcal{I}_n} \text{M}(\sigma \text{TP}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) \quad (3.6)$$

and therefore it suffices to show

$$\sum_{\sigma \in \mathcal{I}_n} \text{M}(\sigma \text{TP}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) = \sum_{\sigma \in \mathcal{I}_n} \text{M}(\sigma \text{ST}_{2n,k,j}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (3.7)$$

We construct a weight-preserving randomized bijection from

$$\bigcup_{\sigma \in \mathcal{I}_n} \mathcal{M}(\sigma \text{TP}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) \quad (3.8)$$

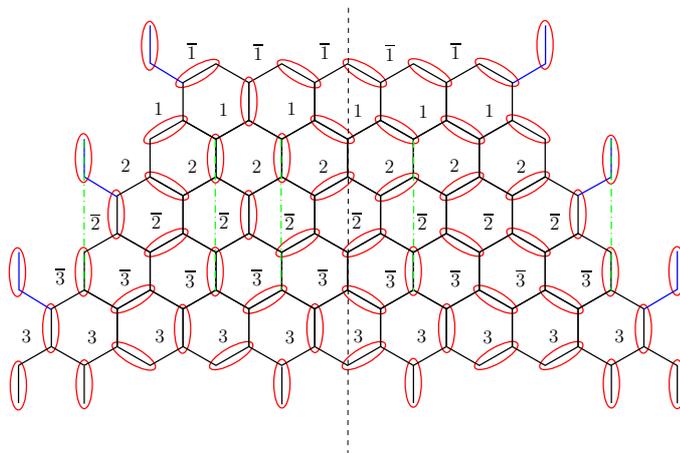


FIGURE 9. The running example in the proof of Lemma 3.2: a matching of $T_{6,5}^{1,2,5,7,10,11}(\bar{x}_1, x_1, x_2, \bar{x}_2, \bar{x}_3, x_3)$

to

$$\bigcup_{\sigma \in \mathcal{I}_n} \mathcal{M}(\sigma \text{ STP}_{2n,k,j}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (3.9)$$

Note that each element in the two unions consists essentially of a pair of a matching of $T_{2n,k}^{\mathbf{P}}$ and a certain weighting of the edges of the graph which is prescribed by σ .

We explain the weight-preserving randomized bijection also with the help of the example given in Figure 9. The first modification is already indicated in blue there: with the exception of the top row and the bottom row, each row of hexagons has *precisely* two vertices of degree 2 (top and bottom row have more vertices of degree two); corresponding vertices of degree 2 also exist in the top and bottom row. We add paths of length 2 at these vertices in rows 1, 3, \dots , $2n - 1$ and extend the matching accordingly.

We consider the subgraph R_{2i-1} that consists of a fixed odd row $2i - 1$ of hexagons together with the edges just added so that R_{2i-1} has a half hexagon on the left side and a half hexagon on the right side. Further we consider the following subdivision of R_{2i-1} into sections that start and end with half hexagons: The dividing lines are the vertical lines that go through two vertices of R_{2i-1} , where at least one of them is matched to a vertex not contained in R_{2i-1} . In Figure 9, these dividing lines are indicated in green for row 3. In principle, there are three types of sections as indicated in Figure 10.

We now perform certain operations to the rows 1, 3, \dots , $2n - 1$. More specifically, we choose for each of these rows either the part left of the line ℓ_j or the part right of the line ℓ_j to which we apply the operations then. The side is fixed if the two vertices of the row that lie on the line ℓ_j are matched to vertices that lie in the row and on the same side of ℓ_j ; in that case we choose the other side. Otherwise the two sides are chosen with equal probability $\frac{1}{2}$. The line ℓ_j serves then also as a dividing line of the sections of R_{2i-1} (possibly in addition to the lines that were already identified).

For each odd row $2i - 1$ and each section of type 1 that is situated on the chosen side of that particular row, we now perform the following operation; see also Figure 11. If the section contains t vertical edges and the unique vertical matching edge is in position s , counted from the left, then we transform this section into another section of type 1, where the unique vertical

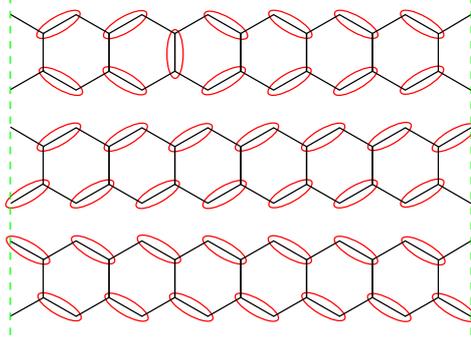


FIGURE 10. The three types of sections of rows: the first type has precisely one vertical matching edge, in the second type the matching edges are precisely those of type “ / ”, while in the third type the matching edges are precisely those of type “ \ ”.

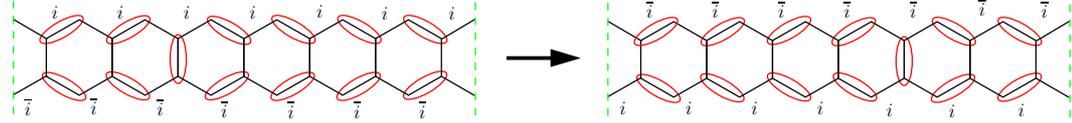


FIGURE 11. Reflecting the matching edges vertically and the edge weights horizontally leaves the total weight of \bar{x}_i^2 unchanged.

matching edge is in position $t + 1 - s$. This surely changes the weight, however, this can be compensated by changing the weights in this section, more specifically we simply interchange weights x_i with \bar{x}_i . We can change the weights accordingly in sections of type 2 and 3 on the chosen side, as the weight is 1 for both choices in these sections.

Finally, we modify the weights on the right side of the line ℓ_j in such a way that the matching generating function is unchanged; see Figure 12 for an illustration. The modification is based on the following simple observation: Suppose G is an edge-weighted graph and v is a vertex of V . Let G' be the edge-weighted graph obtained from G by multiplying the weights of all edges incident with v with w . Then we have the following relation between the two matching generating functions:

$$M(G') = w M(G). \quad (3.10)$$

Now consider all vertical edges e that are (fully) contained in row R_{2i-1} and located right of the line ℓ_j . Let a and b be the two endpoints of e . Both vertices have precisely two incident edges with weight 1 (one of them is e), while the third edge has weight x_i for one vertex and weight \bar{x}_i for the other vertex (these edges are both of type “ / ”). We assume without loss of generality that there is an edge incident with a that has weight x_i . Now we multiply the weights of each edge incident with a by \bar{x}_i and the weights of each edge incident with b by x_i . Now a has precisely one vertex incident with it that has weight \bar{x}_i , while b has precisely one vertex incident with it that has weight x_i . These edges are both of type “ \ ”, and we have reached an element of

$$\bigcup_{\sigma \in \mathcal{I}_n} \mathcal{M}(\sigma \text{ST}_{2n,k,j}^{\text{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (3.11)$$

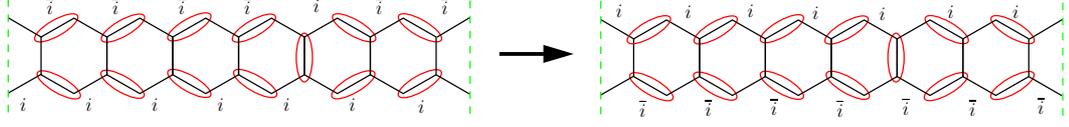


FIGURE 12. Modification of edge weights on the right leaves the weights unchanged.

Each step of the procedure is obviously invertible and thus this establishes a randomized bijection. The probabilities for transforming one element of the domain of the bijection into an element of the codomain is $\frac{1}{2^m}$ or 0, where m is the number of odd rows such that one of its vertices on the line ℓ_j is matched to the left while the other is matched to the right. It is crucial that the procedure of the matching does not change this number m . \square

Remark 3.3. *The randomized bijection can be transformed into a classical bijection from*

$$\{0, 1\}^n \times \bigcup_{\sigma \in \mathcal{I}_n} \mathcal{M}(\sigma \text{TP}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) \quad (3.12)$$

to

$$\{0, 1\}^n \times \bigcup_{\sigma \in \mathcal{I}_n} \mathcal{M}(\sigma \text{STP}_{2n,k,j}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)), \quad (3.13)$$

where the i -th letter of the $\{0, 1\}$ -sequence in $\{0, 1\}^n$ encodes which side of ℓ_j we choose in row $2i - 1$, $1 \leq i \leq n$, whenever there is actually a choice. When there is no choice in row $2i - 1$, then the i -th letter of the word has no effect. The $\{0, 1\}$ -sequence itself is unchanged in the procedure.

4. APPLICATION OF CIUCU'S FACTORIZATION THEOREM AND A COMBINATORIAL PROOF OF THEOREM 1.1(1)

4.1. Ciucu's theorem. For convenience, we recall Ciucu's factorization theorem for graphs with reflective symmetry [5]. We assume that the edge-weighted graph G has the following properties:

- It is planar, bipartite and connected.
- It exhibits symmetry with respect to a vertical symmetry axis ℓ (including that the edge-weights are symmetric).
- Removing the vertices on ℓ disconnects the graph.

Without loss of generality, we assume that there are an even number of vertices on the symmetry axis. If not, G has an odd number of vertices by symmetry and thus no perfect matching. We denote by $n(G)$ half of the number of vertices on the symmetry axis and by $a_1, b_1, \dots, a_{n(G)}, b_{n(G)}$ the vertices on ℓ as they appear from top to bottom. We refer to the vertices in one vertex class of G as the *positive vertices*, while we refer to the vertices in the other vertex class as the *negative vertices*.

For a vertex v on ℓ , we define two cutting operations: "Cutting left of v " means that we delete all incident edges left of ℓ , while "cutting right of v " means that we delete all incident edges right of ℓ . Now we define two subgraphs of G as follows: We perform the cutting operation right of all positive a_i 's and negative b_i 's, and left of all negative a_i 's and positive b_i 's. Reduce the weights of the edges on ℓ by half and leave all other weights unchanged. We obtain two

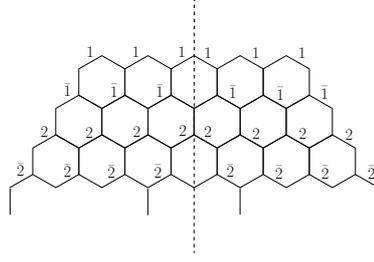


FIGURE 13. The graph $\text{ST}_{4,5,3}^{1,4,6,9}(x_1, \bar{x}_1, x_2, \bar{x}_2)$.

disconnected graphs, and denote by G^+ the left graph and by G^- the right graph. We are now able to state Ciucu's factorization theorem.

Theorem 4.1. [5, Theorem 1.2] *With the notations introduced above, we have*

$$M(G) = 2^{n(G)} M(G^+) M(G^-). \quad (4.1)$$

4.2. Proof of Theorem 1.1(1). We now apply the gathered information to our problem. Recall that we consider the following specialization of the Schur function $s_{\hat{\lambda}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)$ for

$$\hat{\lambda} = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1, -\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1) + \lambda_1. \quad (4.2)$$

Using Theorem 2.3, this is equal to the following matching generating function.

$$M(\text{T}_{2n, 2\lambda_1+1}^{\mathbf{p}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) \quad (4.3)$$

with

$$\mathbf{p} = (1 - \lambda_1, 2 - \lambda_2, \dots, n - \lambda_n, n + 2 + \lambda_n, n + 3 + \lambda_{n-1}, \dots, 2n + 1 + \lambda_1) + \lambda_1. \quad (4.4)$$

Now we observe that with this particular choice of parameters, $\text{T}_{2n, 2\lambda_1+1}^{\mathbf{p}}$ is symmetric with respect to the axis l_{λ_1+1} . However, the weights are not symmetric. To remedy this issue, we apply Lemma 3.2 to see that (4.3) is equal to

$$\frac{1}{2^n} \sum_{\sigma \in \mathcal{I}_n} M(\sigma \text{ST}_{2n, 2\lambda_1+1, \lambda_1+1}^{\mathbf{p}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (4.5)$$

Now the weighted graphs $\sigma \text{ST}_{2n, 2\lambda_1+1, \lambda_1+1}^{\mathbf{p}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)$ have symmetric edge weights and we may apply Ciucu's factorization theorem. For $n = 2$ and $\lambda = (2, 0)$, we have $\mathbf{p} = (1, 4, 6, 9)$ and the graph $\text{ST}_{4,5,3}^{1,4,6,9}(x_1, \bar{x}_1, x_2, \bar{x}_2)$ is displayed in Figure 13.

We fix the top vertex on the symmetry axis to be a positive vertex (which is a_1 in the setting of the factorization theorem). The positive vertices alternate with the negative vertices on the symmetry axis. By Ciucu's construction, we cut right of all the vertices on the symmetry axis. It follows that G^+ is $\text{HT}_{2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}$, while G^- is $\text{HT}_{2n, \lambda_1}^{-\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}$. Thus we obtain

$$2^n M(\sigma \text{HT}_{2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) M(\sigma \text{HT}_{2n, \lambda_1}^{-\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) \quad (4.6)$$

for the summand of $\sigma \in \mathcal{I}_n$ in (4.5).

As for $M(\text{HT}_{2n, \lambda_1}^{-\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n))$, we know from Theorem 2.8 that it is equal to $sp_{\lambda}(x_1, \dots, x_n)$. Now it is crucial that $sp_{\lambda}(x_1, \dots, x_n)$ is invariant under replacing x_i by \bar{x}_i for any i . This follows easily, for instance, from the determinantal expression of $sp_{\lambda}(x_1, \dots, x_n)$,

but can also be seen from the combinatorial interpretations provided in this paper: Using Theorem 2.6, one has to employ an operation analogous to J_i as defined in Proposition 2.13 for odd rows, or, alternatively, relying on Theorem 2.8, one has to employ an operation analogous to one from Lemma 3.2. Now, since $M(\sigma \text{HT}_{-2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n))$ is obtained from $M(\text{HT}_{-2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n))$ by replacing x_i by \bar{x}_i for all i such that (x_i, \bar{x}_i) “appears” in σ , it follows that

$$sp_\lambda(x_1, \dots, x_n) = M(\sigma \text{HT}_{-2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) \quad (4.7)$$

for all σ . We can conclude that (4.5) is equal to

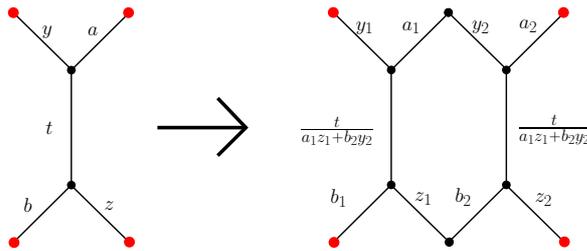
$$sp_\lambda(x_1, \dots, x_n) \sum_{\sigma \in \mathcal{I}_n} M(\sigma \text{HT}_{+2n, \lambda_1}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (4.8)$$

The first part of Theorem 1.1 now follows from Theorem 2.14.

5. “DOUBLING” THE GRAPH AND A COMBINATORIAL PROOF OF THEOREM 1.1(2)

As for the second part of Theorem 1.1, the largest part of the partition $\hat{\lambda}$ is even, and therefore we need to work with a graph $T_{n,k}^{\mathbf{P}}$ where there are an even number k of hexagons in the top row (since k is just the largest part of $\hat{\lambda}$ by Theorem 2.3). Therefore Lemma 3.2 cannot be applied directly to achieve symmetric edge weights, since the symmetry axis of the unweighted graph is not a line ℓ_j as described in the statement of Lemma 3.2. Instead it will turn out to be useful to apply the operations provided in the following lemma, simultaneously at various places, with the effect that the graph is (almost) doubled.

Lemma 5.1. *The following replacement rule in a weighted graph leaves the matching generating function invariant, where in the replacement the degree of the black vertices does not change and the red vertices are the connecting points. In the drawings, the label next to the edge indicates the weight and we assume $a_1 \cdot a_2 = a$, $b_1 \cdot b_2 = b$, $y_1 \cdot y_2 = y$ and $z_1 \cdot z_2 = z$.*



Proof. The proof proceeds by considering the various (not necessarily perfect) matchings of the subgraphs that cover all black vertices. \square

We will apply the rules in Lemma 5.1 to the graph $T_{n,k}^{\mathbf{P}}(x_1, \dots, x_n)$ at various places. If we say that we apply the rule to a particular vertical edge then we mean that the vertical edge on the left side of the rule corresponds to this chosen vertical edge. By $\text{DT}_{n,k}^{\mathbf{P}}(x_1, \dots, x_n)$ we denote the graph that is obtained from $T_{n,k}^{\mathbf{P}}(x_1, \dots, x_n)$ by applying the rules from Lemma 5.1 to the vertical edges in all odd rows $1, 3, 5, \dots$. In row $2i-1$, we specify the weights as follows: $a_1 = x_{2i-1}^{1/2}$, $b_1 = b_2 = x_{2i}^{1/2}$, $y_2 = 1$, $z_1 = z_2 = 1$, $t = 1$, and $a_2 \in \{0, x_{2i-1}^{1/2}\}$ and $y_1 \in \{0, 1\}$, where we choose $y_1 = 0$ if we are on the left boundary and $a_2 = 0$ if we are on the right boundary.

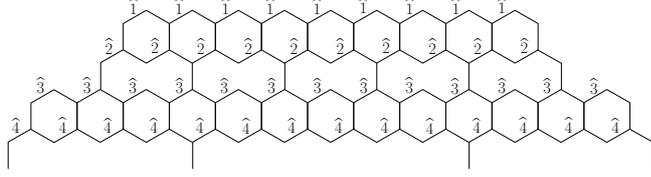


FIGURE 14. The graph $\text{DT}_{4,4}^{1,3,6,8}(x_1, x_2, x_3, x_4)$. The vertical edges in row 1 carry the weight $(x_1^{1/2} + x_2^{1/2})^{-1}$, while the vertical edges in row 3 carry the weight $(x_3^{1/2} + x_4^{1/2})^{-1}$, and \hat{i} stands for $x_i^{1/2}$.

The graph $\text{DT}_{4,4}^{1,3,6,8}$ is displayed in Figure 14. Lemma 5.1 implies

$$\text{M}(\text{T}_{n,k}^{\mathbf{P}}(x_1, \dots, x_n)) = \text{M}(\text{DT}_{n,k}^{\mathbf{P}}(x_1, \dots, x_n)). \quad (5.1)$$

Again we consider also different edge weights of $\text{DT}_{n,k}^{\mathbf{P}}(x_1, \dots, x_n)$: Let ℓ_j denote the vertical line that contains the j -th vertex in the top row (counted from the left). Left of ℓ_j , the weights are unchanged, but right of ℓ_j the edges of type “ \searrow ” are assigned the weight $x_i^{1/2}$ in row i , while edges of type “ \swarrow ” are assigned the weight 1; the weights of the vertical edges do not change. This edge-weighted graph is denoted by $\text{SDT}_{n,k,j}^{\mathbf{P}}(x_1, \dots, x_n)$. Now, a slight modification³ and of Lemma 3.2 shows that

$$\text{M}(\text{DT}_{2n,k}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)) = \frac{1}{2^n} \sum_{\sigma \in \mathcal{I}_n} \text{M}(\sigma \text{SDT}_{2n,k,j}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (5.2)$$

In the second part of Theorem 1.1, we consider the specialization of the Schur polynomial

$$s_{\hat{\lambda}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n), \quad (5.3)$$

where

$$\hat{\lambda} = (\lambda_1, \dots, \lambda_n, -\lambda_n, \dots, -\lambda_1) + \lambda_1. \quad (5.4)$$

By Theorem 2.3, this is equal to the matching generating function

$$\text{M}(\text{T}_{2n,2\lambda_1}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)), \quad (5.5)$$

with

$$\mathbf{p} = (1 - \lambda_1, 2 - \lambda_2, \dots, n - \lambda_n, n + 1 + \lambda_n, \dots, 2n + \lambda_1) + \lambda_1. \quad (5.6)$$

Using (5.1) and (5.2), this is equal to

$$\frac{1}{2^n} \sum_{\sigma \in \mathcal{I}_n} \text{M}(\sigma \text{SDT}_{2n,2\lambda_1,2\lambda_1+1}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)). \quad (5.7)$$

Note that $\text{SDT}_{2n,2\lambda_1,2\lambda_1+1}^{\mathbf{P}}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)$ has $4\lambda_1 + 1$ hexagons in the top row and thus $\ell_{2\lambda_1+1}$ is its symmetry axis.

Now we apply Ciucu’s factorization theorem to each summand in (5.7). We declare the top vertex a_1 on the symmetry axis to be a positive vertex (in the context of Ciucu’s theorem).

³Only the graphs differ insofar as some edges that are irrelevant for the procedure have been deleted, since they only appear in even rows. Also the edge weights of the vertical edges do not cause any difficulties as they are constant on rows.

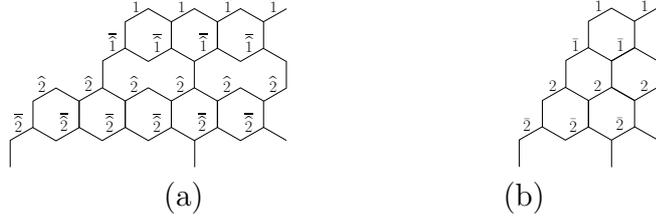


FIGURE 15. (a) The graph G^+ of $\text{SDT}_{4,4,5}^{1,3,6,8}(x_1, \bar{x}_1, x_2, \bar{x}_2)$. The vertical edge-weights in odd rows are as in Figure 14. (b) The graph G^+ after application of Lemma 5.1 in reverse direction.

It follows that all vertices a_i on the symmetry axis are positive, while all vertices b_i on the symmetry axis are negative. Thus we need to cut right of all vertices on the symmetry axis.

We first consider the left graphs G^+ . The vertices on the former symmetry axis are all of degree 1. We delete the incident edges as well as all edges adjacent to these edges to obtain a new graph whose matching generating function differs from the original by a multiplicative factor of

$$\sigma \prod_{i=1}^n x_i^{1/2}, \quad (5.8)$$

which needs to be multiplied to the generating function of the reduced graph to obtain the generating function of the former graph. For our example, the graph is displayed in Figure 15(a) for $\sigma = \text{Id}$. Next we apply Lemma 5.1 in the reverse direction, i.e., we shrink hexagons to vertical edges. To be more precise, we shrink the hexagons in the odd rows that are in odd positions, if counted from the left. We obviously obtain the graph

$$\sigma \widehat{\text{HT}}_{2n, \lambda_1 - 1}^{\lambda_n, \lambda_{n-1} + 1, \dots, \lambda_1 + n - 1}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) \quad (5.9)$$

in Figure 15(b) and so the matching generating function of G^+ is

$$\left[\sigma \prod_{i=1}^n x_i^{1/2} \right] M \left(\sigma \widehat{\text{HT}}_{2n, \lambda_1 - 1}^{\lambda_n, \lambda_{n-1} + 1, \dots, \lambda_1 + n - 1}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) \right). \quad (5.10)$$

We now consider the right graphs G^- . For our example, the graph is displayed in Figure 16 when $\sigma = \text{Id}$. Again we apply Lemma 5.1 in reverse direction to every other hexagon in odd rows, more precisely to the hexagons in odd positions if counted from the *right*. The weights of the vertical edges are then 1 again, except for the leftmost vertical edges of the odd rows which still carry the weight $x_i^{1/2} + x_i^{-1/2}$ in row $2i - 1$. We multiply the weights of the edges incident with the vertices along the left horizontal line in positions $2i$, $1 \leq i \leq n$, counted from the top with $x_i^{1/2} + x_i^{-1/2}$. We obtain the graph

$$\sigma \widehat{\text{HT}}_{-2n, \lambda_1}^{\lambda_n + 1, \lambda_{n-1} + 2, \dots, \lambda_1 + n}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n). \quad (5.11)$$

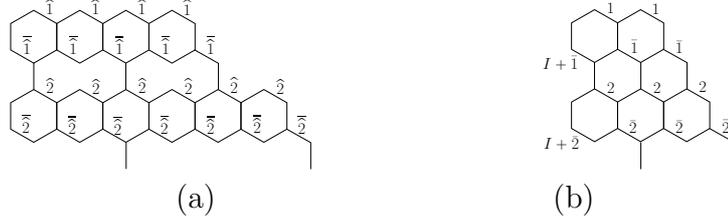


FIGURE 16. (a) The graph G^- of $\text{SDT}_{4,4,5}^{1,3,6,8}(x_1, \bar{x}_1, x_2, \bar{x}_2)$. The vertical edge-weights in odd rows are as in Figure 14. (b) The graph G^- after application of Lemma 5.1 in reverse direction. Here we denote the number 1 by I .

Using Theorem 2.20, the matching generating function is equal to $\sigma so_\lambda^{\text{odd}}(x_1, \dots, x_n)$ and thus independent of σ as $so_\lambda^{\text{odd}}(x_1, \dots, x_n)$ is invariant under replacing x_i with \bar{x}_i . In total, we obtain

$$\prod_{i=1}^n \left(x_i^{1/2} + x_i^{-1/2} \right)^{-1} so_\lambda^{\text{odd}}(x_1, \dots, x_n) \times \sum_{\sigma \in \mathcal{I}_n} \left[\sigma \prod_{i=1}^n x_i^{1/2} \right] M \left(\sigma \widehat{\text{HT}}_{2n, \lambda_1 - 1}^{\lambda_n, \lambda_{n-1} + 1, \dots, \lambda_1 + n - 1} (x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) \right). \quad (5.12)$$

Theorem 1.1(2) now follows from Theorem 2.16.

6. THE SKEW CASE: PROOF OF THEOREM 1.2

Our bijective proof of Theorem 1.1 actually reveals that the theorem can be generalized to skew Schur polynomials, provided that also the inner shape of the skew shape satisfies a certain symmetry property as laid down in Theorem 1.2. This follows merely from the observation that edges can also stick out at the top of the trapezoidal graph $T_{n,k}$, just as they stick out at the bottom according to the outer shape. These top edges encode the inner shape of the skew shape. The symmetry property of the inner shape has to guarantee that also the distribution of the top edges is symmetric with respect to the vertical symmetry axis of the graph so that Ciucu's factorization theorem can be applied. Phrased differently, this can also be seen as a refinement of Theorem 1.1, where we fix in a certain fixed row, say, m of the graph $T_{n,k}$ the vertical matching edges. (In the associated semistandard tableaux this corresponds to fixing the shape of the entries that are less than or equal to m .) From this point of view, the skew case concerns the subgraph of $T_{n,k}$ consisting of what is below this fixed row and adding edges sticking out at the (new) top according to the fixed matching in (the old) row m . Since our "procedures" (in a sense) do not mix between different rows of $T_{n,k}$ (especially those used in the proof of Lemma 3.2), the proofs of the straight cases generalize easily to the skew cases. The only additional effort is in finding the correct skew generalizations of the objects such as patterns and honeycomb graphs, which is also straightforward as we only need to chop off the appropriate number of top rows. This considerably increases the notational complexity because we also need to involve the positions of the extra edges sticking out at the top of the honeycomb graphs. A detailed proof of Theorem 1.2 will only obscure the main ideas. We therefore chose to give a detailed proof in the straight case and highlight a few major steps of the skew case in this section.

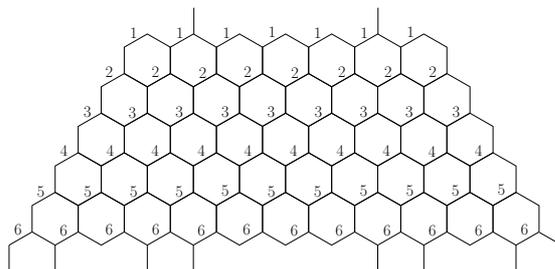


FIGURE 17. The graph ${}^{2,6}T_{6,7}^{1,2,4,5,9,10,12,13}(x_1, x_2, x_3, x_4, x_5, x_6)$

6.1. Graphical model of skew Schur polynomials. Theorem 2.3 can be extended to *skew Schur polynomials*. For this purpose, we only need to generalize $T_{n,k}^{\mathbf{p}}(x_1, \dots, x_n)$ so that we allow also additional vertical edges attached to the vertices in the topmost row. Let n, k be a positive integer and m be a non-negative integers with $m < n$ and $m \leq k$, and consider $T_{n-m,k}$. Adding n vertical edges to a selection of the $n - m + k$ vertices in the bottommost row, while adding m vertical edges to a selection of the k vertices in the topmost row results in a bipartite graph that has the same number of vertices in each vertex class, and may as such possess a perfect matching. Let $\mathbf{p} = (p_1, \dots, p_n)$, $1 \leq p_1 < p_2 < \dots < p_n \leq n - m + k$, be these positions at the bottom, and, $\mathbf{q} = (q_1, \dots, q_m)$, $1 \leq q_1 < q_2 < \dots < q_m \leq k$, be these positions at the top, then ${}^{\mathbf{q}}T_{n-m,k}^{\mathbf{p}}$ denotes the corresponding (unweighted) graph, while ${}^{\mathbf{q}}T_{n-m,k}^{\mathbf{p}}(x_1, \dots, x_{n-m})$ denotes the corresponding weighted graph. An example is given in Figure 17. Recall that for two partitions μ, λ such that the Young diagram of μ is contained in the Young diagram of λ , the skew Schur polynomial $s_{\lambda/\mu}(x_1, \dots, x_{n-m})$ is the generating function of semistandard fillings of the Young diagram of shape λ/μ with respect to the weight in (2.1) (replacing n by $n - m$ there). The generalization of Theorem 2.3 is as follows.

Theorem 6.1. *Let m, n be non-negative integers with $m < n$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_m)$ be partitions such that μ is contained in λ . Then*

$$s_{\lambda/\mu}(x_1, \dots, x_{n-m}) = M({}^{\mu_m+1, \mu_{m-1}+2, \dots, \mu_1+m}T_{n-m, \lambda_1+m}^{\lambda_n+1, \lambda_{n-1}+2, \dots, \lambda_1+n}(x_1, \dots, x_{n-m})). \quad (6.1)$$

Note that there is the following ambiguity: Since we can add any number of zeros to μ and λ , there is of course an infinite family of graphs that can be used for a particular skew Schur polynomial. However, adding a zero to both μ and λ means that we add a \nearrow -diagonal of hexagons left of the graph and we have an edge sticking out at the first position of the top row as well as at the first position of the bottom row. By forcing, the leftmost vertical edge of each row is a matching edge, thus the leftmost \nearrow -diagonal can be deleted again without changing the matching generating function.

6.2. Combinatorial definitions of skew characters of other classical groups. Here we first define skew versions of the characters of symplectic groups and of the orthogonal groups as they are suggested by our combinatorial proof, thereby clarifying the right-hand sides of the identities in Theorem 1.2. More specifically, we need to generalize the patterns provided in Definitions 2.5, 2.9, 2.17 and their weights. The skew characters are then the generating functions of these patterns with respect to the weights. The general principle is very simple as laid down in the introductory paragraph of this section: In all cases, a pattern associated with the parameters m, n (m, n are as usual the number of parts of μ, λ , respectively) is obtained by

deleting at the top of a pattern of “order” n a pattern of order m , except for the bottom row of the pattern of order m which remains and corresponds to the inner shape. The generalized patterns are then obviously of trapezoidal shape.

We then also give references to appearances of these skew characters in representation theory, where they show up when restricting the straight characters to a certain subgroup and combinatorial interpretations are given in terms of various skew tableaux by Koike and Terada [12]. Using standard arguments, these tableaux representations can be transformed into pattern representations, thereby establishing the connection to the patterns appearing in our proof of Theorem 1.2. To unify the relationship between our graphical models, patterns and tableaux, we will use notations that will differ from theirs and we will point to the difference in each case separately. It is worth noting that Hamel [9] has given determinantal formulas for skew symplectic and skew odd orthogonal characters.

6.2.1. *Skew symplectic characters.* In this section, we assume $m < n$ to be non-negative integers, and μ, λ to be partitions with m and n parts, respectively, where we allow (as usual) also zero parts. The proof of Theorem 1.2 suggests the following definition, which is a generalization of Definition 2.5.

Definition 6.2 (Trapezoidal symplectic patterns). *Let m, n, μ, λ be as above.*

- (1) *A $(2n)/(2m)$ -symplectic pattern has the shape of a $(2n)$ -symplectic pattern with a $(2m)$ -symplectic pattern deleted from the top, except for the bottom row of the $(2m)$ -symplectic pattern that remains, such that the entries are non-negative integers and weakly increase along \nearrow -diagonals and \searrow -diagonals.*
- (2) *The rows are indexed from 0 to $2n - 2m$, starting with the top row, and r_i is the sum of entries in row i . The weight of the pattern P is then*

$$w_{\text{sp}}(P) = \prod_{i=1}^{n-m} x_i^{r_{2i} - 2r_{2i-1} + r_{2i-2}}. \quad (6.2)$$

- (3) *Denote the set of all $(2n)/(2m)$ -symplectic patterns with top row μ and bottom row λ , both written in increasing order, as $\mathcal{SP}_{\lambda/\mu}$.*

Note that our definitions are consistent with the straight case, which is obtained by setting $m = 0$. In this case, the 0-indexed row is empty, which is consistent with setting $r_0 = 0$ in Definition 2.5. The skew symplectic character $sp_{\lambda/\mu}(x_1, \dots, x_{n-m})$ appearing in Theorem 1.2 is the generating function

$$\sum_{P \in \mathcal{SP}_{\lambda/\mu}} w_{\text{sp}}(P), \quad (6.3)$$

as can be seen when generalizing our combinatorial proof of Theorem 1.1 to skew shapes. By Proposition 6.6, this combinatorially motivated definition coincides with the representation theoretic definition.

Remark 6.3. *Since only now we have clarified the right-hand side of (1.13), let us point out the following subtlety: letting $m = 2, n = 3, \mu = (1, 1), \lambda = (3, 2, 2)$, the shape $\hat{\mu}$ is not contained in $\hat{\lambda}$, so the left-hand side of (1.13) is zero. However, this is also true for the right-hand side, because $\mathcal{SP}_{\lambda/\mu}$ is empty in this case.*

In representation theory, the skew symplectic character $sp_{\lambda/\mu}(x_1, \dots, x_{n-m})$ appears when restricting the (straight) symplectic character to a certain subgroup. In [12, Proposition 4.1], Koike and Terada provide a combinatorial interpretation of this skew symplectic character in terms of the generating function of the following skew tableaux. The barred and unbarred symbols are interchanged in our convention.

Definition 6.4. Let m, n, μ, λ be as above and assume $\mu \subseteq \lambda$. A skew symplectic semistandard tableau T of λ/μ with entries

$$\bar{1} < 1 < \bar{2} < 2 < \dots < \overline{n-m} < n-m \quad (6.4)$$

is a filling of the shape with these entries satisfying the following conditions:

- the entries increase weakly along rows,
- the entries increase strictly along columns,
- the entries in the $(i+m)$ -th row must be greater than or equal to \bar{i} .

For α being any of the entries above, let $n_\alpha(T)$ be the number of occurrences of α in T . Then the weight of such a tableau is given by

$$w_{\text{sp}}(T) = \prod_{i=1}^{n-m} x_i^{n_i(T) - n_{\bar{i}}(T)}. \quad (6.5)$$

Let $\mathcal{SPT}_{\lambda/\mu}$ be the set of skew symplectic semistandard tableau of shape λ/μ filled with entries $\bar{1}, 1, \dots, \overline{n-m}, n-m$. For example, choosing $n = 5, m = 3, \mu = (3, 2, 1)$ and $\lambda = (5, 4, 4, 2, 2)$, then

$$\begin{array}{cccc} & & & \bar{1} & 1 \\ & & & 1 & 1 \\ & & 1 & 2 & 2 \\ \bar{1} & \bar{2} & & & \\ \bar{2} & 2 & & & \end{array} \quad (6.6)$$

is a skew symplectic tableau of shape λ/μ with weight $x_1^2 x_2$.

Theorem 6.5 ([12, Proposition 4.1]). Let m, n, μ, λ be as above and assume $\mu \subseteq \lambda$. The skew symplectic character of shape λ/μ is given by

$$sp_{\lambda/\mu}(x_1, \dots, x_{n-m}) = \sum_{T \in \mathcal{SPT}_{\lambda/\mu}} w_{\text{sp}}(T). \quad (6.7)$$

The following result generalizes Theorem 2.6 and should be well-known although we could not find an explicit reference.

Proposition 6.6. Let m, n, μ, λ be as above.

$$sp_{\lambda/\mu}(x_1, \dots, x_{n-m}) = \sum_{P \in \mathcal{SP}_{\lambda/\mu}} w_{\text{sp}}(P). \quad (6.8)$$

Proof. Using Theorem 6.5, it suffices to find a weight preserving bijection between skew symplectic tableaux and trapezoidal symplectic patterns. The strategy of proof is standard, using the general principle used to transform a semistandard tableau into a Gelfand-Tsetlin pattern as explained after Proposition 2.3 with appropriate modifications as follows. Recall

Remark 6.8. *Continuing the theme of Remark 6.3, observe that for $m = 2, n = 3, \mu = (1, 1), \lambda = (3, 2, 2)$, the right-hand side of (1.13) is also zero because $\mathcal{OP}_{(\lambda+1)/(\mu+1)}$ is empty in this case.*

Assuming μ, λ to be integer partitions, Koike and Terada [12, Proposition 4.3] show that the representation theoretical skew even orthogonal characters $o_{\lambda/\mu}^{\text{even}}(x_1, \dots, x_{n-m})$, which appear when restricting the (straight) even orthogonal characters to a certain subgroup, have a combinatorial interpretation in terms of the generating function of the following skew tableaux. Their symbols $\#_i, \flat_i, \hat{i}, \check{i}$ correspond to our symbols $\hat{i}, \check{i}, \bar{i}, i$ respectively.

Definition 6.9. *Let m, n, λ, μ be as above such that μ, λ are integer partitions and $\mu \subseteq \lambda$. A skew even orthogonal semistandard tableau T of shape λ/μ with entries*

$$\hat{1} < \check{1} < \bar{1} < 1 < \hat{2} < \check{2} < \bar{2} < 2 < \dots < \widehat{n-m} < \widetilde{n-m} < \overline{n-m} < n-m, \quad (6.13)$$

is a filling of the shape with these entries satisfying the following conditions:

- *the entries increase weakly along rows,*
- *the entries increase strictly along columns,*
- *the entries in the $(i+m)$ -th row must be greater than or equal to \check{i} ,*
- *the entry \hat{i} can only appear in the first column of row $i+m-1$, and \check{i} can only appear in the first column of row $i+m$, and either of them only appears if the other does,*
- *if \bar{i} appears in the first column of row $i+m$ and also i appear in that row, then there is an \bar{i} immediately above this i .*

The weight of such a tableau is given by

$$w_o^{\text{even}}(T) = \prod_{i=1}^{n-m} x_i^{n_i(T) - n_{\bar{i}}(T)}. \quad (6.14)$$

The set of skew even orthogonal semistandard tableau of shape λ/μ filled with entries $\hat{1}, \dots, n-m$ is denoted by $\mathcal{EOT}_{\lambda/\mu}$.

Example 6.10. *Let $n = 3$ and $m = 1$ and consider the skew shape $(1, 1, 1)/(1)$. Then we have the following tableaux in $\mathcal{EOT}_{(1,1,1)/(1)}$ with entries in $\{\hat{1}, \check{1}, \bar{1}, 1, \hat{2}, \check{2}, \bar{2}, 2\}$:*

$$\begin{array}{|c|} \hline \emptyset \\ \hline \bar{1} \\ \hline \bar{2} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \emptyset \\ \hline \bar{1} \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline \emptyset \\ \hline 1 \\ \hline \bar{2} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \emptyset \\ \hline 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline \emptyset \\ \hline \bar{2} \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline \emptyset \\ \hline \hat{2} \\ \hline \check{2} \\ \hline \end{array} \quad (6.15)$$

Here \emptyset means that the cell is unoccupied. It follows that $o_{(1,1,1)/(1)}^{\text{even}}(x_1, x_2) = 2 + x_1x_2 + x_1\bar{x}_2 + \bar{x}_1x_2 + \bar{x}_1\bar{x}_2$.

Theorem 6.11 ([12, Proposition 4.3]). *Let m, n, λ, μ be as above such that μ and λ are integer partitions and $\mu \subseteq \lambda$. The skew even orthogonal character of the shape λ/μ is given by*

$$o_{\lambda/\mu}^{\text{even}}(x_1, \dots, x_{n-m}) = \sum_{T \in \mathcal{EOT}_{\lambda/\mu}} w_o^{\text{even}}(T). \quad (6.16)$$

This theorem implies the following. Although this should be well-known as well, we again sketch the proof because we were unable to find an explicit reference.

Proposition 6.12. *Let m, n, λ, μ be as above and $\mu \subseteq \lambda$. Then*

$$o_{\lambda/\mu}^{\text{even}}(x_1, \dots, x_{n-m}) = \sum_{P \in \mathcal{OP}_{\lambda/\mu}} w_o^{\text{even}}(P). \quad (6.17)$$

Proof. As in the proof of Proposition 6.6, we prove this by constructing a weight-preserving bijection $\phi : \mathcal{OP}_{\lambda/\mu} \rightarrow \mathcal{EOT}_{\lambda/\mu}$. Let $P \in \mathcal{OP}_{\lambda/\mu}$ and let λ^i be the i -th row of P , written in decreasing order. For all $1 \leq i \leq n - m$:

- If $\text{sgn}(P_{2i-3,1}) \text{sgn}(P_{2i-1,1}) > 0$, fill the cells of $\lambda^{2i-2}/\lambda^{2i-3}$ by \bar{i} and those of $\lambda^{2i-1}/\lambda^{2i-2}$ by i .
- If $\text{sgn}(P_{2i-3,1}) \text{sgn}(P_{2i-1,1}) < 0$, fill the cells of $\lambda^{2i-2}/\lambda^{2i-3}$ by i and those of $\lambda^{2i-1}/\lambda^{2i-2}$ by \bar{i} . Suppose, at this stage, \bar{i} occurs below i , in column c . Then i should be replaced by \hat{i} and \bar{i} , by \check{i} , if $c = 1$, and these two entries should be interchanged if $c > 1$.

We now claim that $\phi(P)$ satisfies all the conditions in Definition 6.9. The first, second and fourth conditions are not difficult to prove. The entries i and \bar{i} are filled in $\phi(P)$ only while parsing rows $2i-2$ and $2i-1$ of P . The lengths of these rows are $m+i-1$ and $m+i$ respectively. Therefore, these entries cannot occur after row $m+i$. This proves the third condition. Now, suppose $\phi(P)_{i+m,1} = \bar{i}$, then $\text{sgn}(P_{2i-3,1}) \text{sgn}(P_{2i-1,1}) < 0$ and cell $(i+m, 1)$ is contained in $\lambda^{2i-1}/\lambda^{2i-2}$. If, moreover, $\phi(P)_{i+m,j} = i$ for some $j > 1$, and $\phi(P)_{i+m-1,j} \neq \bar{i}$, that means cell $(i+m, j)$ is contained in $\lambda^{2i-2}/\lambda^{2i-3}$, which is clearly impossible. Hence, the fifth condition holds. It is not difficult to construct the inverse map. \square

We illustrate Proposition 6.12 with the following example.

Example 6.13. *The bijection ϕ maps the 9/3-orthogonal pattern on the left onto the skew even orthogonal tableau on the right:*

$$\begin{array}{cccccc}
 1 & 2 & & & & \\
 & 1 & 3 & & & \\
 -1 & 2 & 3 & & & \\
 & 1 & 2 & 4 & & \\
 0 & 1 & 3 & 4 & & \\
 & 1 & 2 & 4 & 4 & \\
 -1 & 1 & 2 & 4 & 4 &
 \end{array}
 \xrightarrow{\phi}
 \begin{array}{ccc}
 & & \boxed{1} \ \boxed{2} \\
 & & \boxed{\bar{1}} \ \boxed{\bar{2}} \ \boxed{3} \\
 \boxed{\bar{1}} & \boxed{3} & \\
 \boxed{\hat{3}} & & \\
 \boxed{\check{3}} & &
 \end{array}
 \quad (6.18)$$

6.2.3. Skew odd orthogonal characters. In this section, we assume $m < n$ to be non-negative integers, and μ, λ to be integer partitions or half-integer partitions with m and n parts, respectively. The proof of Theorem 1.2 suggests the following definition, which is a generalization of Definition 2.17.

Definition 6.14 (Trapezoidal split orthogonal patterns). *Let m, n, μ, λ be as above.*

- (1) *A $(2n)/(2m)$ -split orthogonal pattern has the shape of a $(2n)$ -split orthogonal pattern with a $2m$ -split orthogonal pattern deleted from the top, except for the bottom row of the $2m$ -split orthogonal pattern that remains, such that the following conditions are satisfied:*
 - *the entries except for the odd starters are either all integers or all half-integers,*
 - *the entries are non-negative,*
 - *the entries weakly increase along \nearrow -diagonals and \searrow -diagonals.*

- (2) The rows are indexed from 0 to $2n - 2m$, starting with the top, and r_i is the sum of entries in row i . The weight of a pattern is then

$$w_{\text{so}}(P) = \prod_{i=1}^{n-m} x_i^{r_{2i} - 2r_{2i-1} + r_{2i-2}}. \quad (6.19)$$

- (3) Denote the set of $(2n)/(2m)$ -split orthogonal patterns with top row μ and bottom row λ , both written in increasing order, as $\mathcal{SOP}_{\lambda/\mu}$.

For integer partitions λ, μ , the odd orthogonal character $so_{\lambda/\mu}^{\text{odd}}(x_1, \dots, x_{n-m})$ appearing in Theorem 1.2 is the generating function

$$\sum_{P \in \mathcal{SOP}_{\lambda/\mu}} w_{\text{so}}(P). \quad (6.20)$$

The symbols b_i, i, \bar{i} in [12] correspond to our symbols \hat{i}, \bar{i}, i respectively.

Definition 6.15. Let m, n, λ, μ be as above and $\mu \subseteq \lambda$. A skew odd orthogonal semistandard tableau T of shape λ/μ with entries

$$\hat{1} < \bar{1} < 1 < \hat{2} < \bar{2} < 2 < \dots < \widehat{n-m} < \overline{n-m} < n-m, \quad (6.21)$$

is a filling of λ/μ with these entries satisfying the following conditions:

- the entries increase weakly along rows,
- the entries increase strictly along columns,
- the entries in the $(i+m)$ -th row must be greater than or equal to \hat{i} ,
- \hat{i} can only appear in the first column of row $i+m$.

The weight of such a tableau is given by

$$w_{\text{so}}(T) = \prod_{i=1}^{n-m} x_i^{n_i(T) - n_{\bar{i}}(T)}. \quad (6.22)$$

Let $\mathcal{OOT}_{\lambda/\mu}$ be the set of skew odd orthogonal semistandard tableau of shape λ/μ filled with entries $\hat{1}, \dots, n-m$.

Example 6.16. Let $n = 3$ and $m = 1$ and consider the skew shape $(1, 1, 1)/(1)$. Then we have the following tableaux in $\mathcal{OOT}_{(1,1,1)/(1)}$ with entries in $\{\hat{1}, \bar{1}, 1, \hat{2}, \bar{2}, 2\}$:

$$\begin{array}{|c|} \hline \emptyset \\ \hline \hat{1} \\ \hline \hat{2} \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline \hat{1} \\ \hline \bar{2} \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline \hat{1} \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline \bar{1} \\ \hline \hat{2} \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline \bar{1} \\ \hline \bar{2} \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline \bar{1} \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline 1 \\ \hline \hat{2} \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline 1 \\ \hline \bar{2} \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline 1 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline \emptyset \\ \hline \bar{2} \\ \hline 2 \\ \hline \end{array} \quad (6.23)$$

Here, \emptyset means that the cell is unoccupied. It follows that

$$so_{(1,1,1)/(1)}^{\text{odd}}(x_1, x_2) = (1 + \bar{x}_1 + x_1)(1 + \bar{x}_2 + x_2) + 1. \quad (6.24)$$

Theorem 6.17 ([12, Proposition 4.2]). Let m, n, λ, μ be as above such μ, λ are integer partitions and $\mu \subseteq \lambda$. The skew odd orthogonal character of the shape λ/μ is given by

$$so_{\lambda/\mu}^{\text{odd}}(x_1, \dots, x_{n-m}) = \sum_{T \in \mathcal{OOT}_{\lambda/\mu}} w_{\text{so}}(T). \quad (6.25)$$

Proposition 6.18. *Let m, n, λ, μ be as above such μ, λ are integer partitions and $\mu \subseteq \lambda$. Then*

$$so_{\lambda/\mu}^{\text{odd}}(x_1, \dots, x_{n-m}) = \sum_{P \in \mathcal{SOP}_{\lambda/\mu}} w_{\text{so}}(P). \quad (6.26)$$

Proof. We construct a weight-preserving bijection $\phi : \mathcal{SOP}_{\lambda/\mu} \rightarrow \mathcal{OOT}_{\lambda/\mu}$. The strategy is very similar to the proof of Proposition 6.6.

Let $P \in \mathcal{SOP}_{\lambda/\mu}$. The i -th row of P , when sorted in weakly decreasing order is a partition λ_i , where odd starters are rounded up. The interlacing conditions ensure that $\lambda_{i-1} \subseteq \lambda_i$, with $\lambda_0 = \mu$ and $\lambda_{2n-2m} = \lambda$. We now construct a tableau T of shape λ/μ as follows:

- Fill the entries of $\lambda_{2i-1}/\lambda_{2i-2}$ by \bar{i} for $1 \leq i \leq n-m$. If the starter in row $2i-1$ is a half-integer, replace the first entry in row $m+i$ by \hat{i} .
- Fill the entries of $\lambda_{2i}/\lambda_{2i-1}$ by i for $1 \leq i \leq n-m$.

See Example 6.19 for an example of this map. It is easily seen that the map is weight-preserving and the inverse map is easily constructed as well, establishing the result. \square

Example 6.19. *The bijection ϕ of Proposition 6.18 maps the 10/4-split orthogonal pattern on the left onto the skew odd orthogonal tableau on the right:*

$$\begin{array}{cccccc}
 & 1 & 2 & & & \\
 0 & & 2 & 2 & & \\
 & 2 & 2 & 3 & & \\
 \frac{3}{2} & & 2 & 3 & 3 & \\
 & 2 & 2 & 3 & 4 & \\
 1 & & 2 & 2 & 3 & 4 \\
 & 2 & 2 & 2 & 3 & 4
 \end{array}
 \xrightarrow{\phi}
 \begin{array}{|c|c|}
 \hline
 & 1 & 2 \\
 \hline
 & \bar{1} & \bar{2} \\
 \hline
 1 & 1 \\
 \hline
 \hat{2} & \bar{2} \\
 \hline
 \bar{3} & 3 \\
 \hline
 \end{array} \quad (6.27)$$

with weight $x_1^2 \bar{x}_2^1 x_3^0$.

ACKNOWLEDGEMENTS

We thank R. Behrend for useful discussions. We also thank the anonymous referees for very useful comments and references. We acknowledge the hospitality of the Institut Mittag Leffler where part of this was done. Arvind Ayyer was partially supported by UGC centre for Advanced Study grant and by Department of Science and Technology grant EMR/2016/006624. Ilse Fischer acknowledges support from the Austrian Science Foundation FWF, START grant Y463 and SFB grant F50.

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