On the isomorphism of certain primitive Q-polynomial not P-polynomial association schemes

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Abstract

In 2011, Penttila and Williford constructed an infinite new family of primitive Q-polynomial 3-class association schemes, not arising from distance regular graphs, by exploring the geometry of the lines of the unitary polar space $H(3, q^2)$, q even, with respect to a symplectic polar space $W(3, q)$ embedded in it.

In a private communication to Penttila and Williford, H. Tanaka pointed out that these schemes have the same parameters as the 3-class schemes found by Hollmann and Xiang in 2006 by considering the action of $\mathrm{PGL}(2, q^2)$, q even, on a non-degenerate conic of $\mathrm{PG}(2, q^2)$ extended in $\mathrm{PG}(2, q^4)$. Therefore, the question arises whether the above association schemes are isomorphic. In this paper we provide the positive answer. As by product, we get an isomorphism of strongly regular graphs.

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1 Introduction

Let $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ be a (symmetric) association scheme with d classes. For $0 \le i \le d$, let A_i be the adjacency matrix of the relation R_i, and E_i the i-th primitive idempotent of the Bose-Mesner algebra of $\mathfrak{X}$ which projects on the i-th maximal common eigenspace of $A_0, \ldots, A_d$. The matrices P and Q defined by

$$(A_0\ A_1\ \ldots\ A_d) = (E_0\ E_1\ \ldots\ E_d)P$$

and

$$(E_0\ E_1\ \ldots\ E_d) = |X|^{-1}(A_0\ A_1\ \ldots\ A_d)Q$$

are the *first* and the *second eigenmatrix* of $\mathfrak{X}$, respectively.

An association scheme is said to be *P-polynomial*, or *metric*, if, after a reordering of the relations, there are polynomials p_i of degree i such that $A_i = p_i(A_1)$; an association scheme is called *Q-polynomial*, or *cometric*, if, after a reordering of the eigenspaces, there are polynomials q_i of degree i such that $E_i = q_i(E_1)$, where multiplication is done entrywise. The reader is referred to [1, 3] for further information on association schemes.

Two association schemes $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ and $\mathfrak{X}' = (X', \{R'_i\}_{0\le i\le d})$ are *isomorphic* if there exists a bijection φ from X to X' such that for each $i \in \{0, \ldots, d\}$ there exists $j \in \{0, \ldots, d\}$ satisfying $\{(\varphi(x), \varphi(y)) : (x, y) \in R_i\} = R'_j$; the mapping φ is called an *isomorphism* from $\mathfrak{X}$ to $\mathfrak{X}'$.

The idea of P-polynomial and Q-polynomial schemes was introduced by Delsarte in [9], who observed a formal duality between the two notions. Delsarte also noted that $\mathfrak{X}$ is P-polynomial if and only if, after a proper re-ordering of the relations, (X, R_1) is a distance-regular graph [9, Theorems 5.6 and 5.16]. On the other hand, Q-polynomial schemes which are neither P-polynomial nor duals of P-polynomial schemes seem to be quite rare. In [7] van Dam, Martin and Muzychuk constructed an infinite family of such schemes from hemisystems of the unitary polar space $H(3, q^2)$ provided in [16]. In 2011, Penttila and Williford [14] constructed another infinite family of Q-polynomial 3-class association schemes, not P-polynomial nor the dual of a P-polynomial, by considering a relative hemisystem of $H(3, q^2)$, q even, with respect to a symplectic polar space $W(3, q)$ embedded in it. These schemes differ from all those previously known, they being primitive. The known examples of Q-polynomial schemes which are not P-polynomial are listed in [13, 16].

We underline that the Penttila-Williford 3-class schemes are obtained by applying [14, Theorem 2] which provides primitive Q-polynomial subschemes of Q-polynomial Q-bipartite schemes defined on certain generalized quadrangles. This result can be viewed as a reversal of the so-called "extended Q-bipartite double" construction given in [13]. On the other hand, looking at the Krein array of the generic Penttila-Williford scheme, we may note that it comes from a strongly regular graph after splitting one of its relations in two.

In a private communication to the authors of [14], H. Tanaka pointed out that their 3-class schemes have the same parameters as the 3-class schemes provided by Hollmann and Xiang in [11]. The latter, which were previously not noticed to be Q-polynomial, are obtained as fusion of association schemes constructed from the action of the projective group $\mathrm{PGL}(2, q^2)$, q even, on a non-degenerate conic in the Desarguesian projective plane $\mathrm{PG}(2, q^2)$ extended in $\mathrm{PG}(2, q^4)$.

Therefore the question arises whether there exists an isomorphism that takes the Pentilla-Williford association schemes to the Hollmann-Xiang fusion schemes. In this paper, we provide the answer by proving the following result:

Main Theorem. *The Penttila-Williford 3-class association schemes and the Hollmann-Xiang fusion association schemes are isomorphic.*

The proof essentially uses geometric arguments. We start off with an explicit description of the Penttila-Williford relative hemisystems in terms of coordinates in the projective space $\mathrm{PG}(3, q^2)$. Via the Klein correspondence from the lines of $\mathrm{PG}(3, q^2)$ to the points of the Klein quadric of $\mathrm{PG}(5, q^2)$, we obtain a geometric representation of the Penttila-Williford association schemes in the orthogonal polar space $Q^-(5, q)$ whose points are the image of the lines in $H(3, q^2)$ [10]. Thanks to this representation we are able to find a desired isomorphism.

In [11] it was pointed out that a further fusion scheme of the 3-class Hollmann-Xiang scheme produces a strongly regular graph with parameters $v = q^2(q^2 - 1)/2$, $k = (q^2 + 1)(q - 1)$, $\lambda = q^2 + q - 2$, $\nu = 2(q^2 - q)$. These graphs have the same parameters of the ones found by R. Metz [8], which can be also constructed from a fusion of the Penttila-Williford schemes; see also [4, p.189]. These graphs are denoted by $NO^-(5, q)$ in Brouwer's table of strongly regular graphs [2].

The paper [11] announced an alleged isomorphism between the above graphs in a forthcoming paper. To the best of our knowledge, such a paper appears to have never been published. Anyway, the Main Theorem confirms the conjectured isomorphism.

The paper is structured as follows: in Section 2 we recall the construction of the Hollmann-Xiang and Penttila-Williford association schemes. In Sections 3 we give a coordinatization of the relative hemisystems of Penttila and Williford together with their representation in $Q^-(5, q)$. Finally, Section 4 contains the proof of the Main Theorem.

2 Preliminaries

For any given n-dimensional vector space $V = V(n, F)$ over a field F, the *projective geometry defined by* V is the partially ordered set of all subspaces of V, and it will be denoted by $\mathrm{PG}(V)$. If F is the finite field $\mathbb{F}_q$ with q elements, then we may write $V = V(n, q)$ and $\mathrm{PG}(n-1, q)$ instead of $\mathrm{PG}(V)$. The 1-dimensional subspaces are called *points*, the 2-dimensional subspaces are called *lines*, and the $(n-1)$-dimensional subspaces are called *hyperplanes* of $\mathrm{PG}(V)$. For a nonzero $v \in V$, $\langle v \rangle$ will denote the point of $\mathrm{PG}(V)$ spanned by v. In order to simplify notation, for each subspace U of V, that is an element of $\mathrm{PG}(V)$, we will use the same letter for the projective geometry defined by U. If V is endowed with a non-degenerate alternating, quadratic or hermitian form of Witt index m, the set $\mathcal{P}$ of totally isotropic (or singular, in the case of quadratic form) subspaces of V is a *polar space of rank* m of $\mathrm{PG}(V)$, which is called *symplectic, orthogonal* or *unitary*, respectively. Our principal reference on projective geometries and polar spaces is [15].

2.1 The Hollmann-Xiang association schemes

A *non-degenerate conic* $\mathcal{C}$ of $\mathrm{PG}(2, q^2)$ is an orthogonal polar space (of rank 1) arising from a non-degenerate quadratic form Q on $V(3, q^2)$. A line ℓ of $\mathrm{PG}(2, q^2)$ is called a *passant*, *tangent* or *secant* of $\mathcal{C}$ according as $|\ell \cap \mathcal{C}| = 0$, 1 or 2.

Embed $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(2, q^4)$. Concretely this can be done by extending the scalars in $V(3, q^2)$. It follows that each point of $\mathrm{PG}(2, q^2)$ extends to a point of $\mathrm{PG}(2, q^4)$. Similarly, each line ℓ of $\mathrm{PG}(2, q^2)$ extends to a line $\bar{\ell}$ of $\mathrm{PG}(2, q^4)$. The extension $\overline{Q}$ of Q in $V(3, q^4)$ is a non-degenerate quadratic form, and it defines a (non-degenerate) conic $\overline{\mathcal{C}}$ in $\mathrm{PG}(2, q^4)$. While the extension $\bar{\ell}$ of a tangent (or secant) line ℓ of $\mathcal{C}$ is a tangent (or secant) of $\overline{\mathcal{C}}$, the extension of a passant line of $\mathcal{C}$ is a secant of $\overline{\mathcal{C}}$. Such a line is called an *elliptic* line of $\overline{\mathcal{C}}$, and we will denote by $\mathcal{E}$ the set of these lines. Note that $\mathcal{E}$ has size $(q^4 - q^2)/2$.

Since all non-degenerate quadratic forms on $V(3, q^2)$ are equivalent, we may assume

$$\begin{array}{rccc} \overline{Q}: & V(3, q^4) & \to & \mathbb{F}_{q^4} \\ & (x, y, z) & \mapsto & y^2 - xz. \end{array}$$

Therefore,

$$\overline{\mathcal{C}} = \{\langle (1, t, t^2) \rangle : t \in \mathbb{F}_{q^4}\} \cup \{\langle (0, 0, 1) \rangle\}$$

and

$$\mathcal{C} = \{\langle (1, t, t^2) \rangle : t \in \mathbb{F}_{q^2}\} \cup \{\langle (0, 0, 1) \rangle\}.$$

Therefore, for every elliptic line $\bar{\ell}$ of $\overline{\mathcal{C}}$ we have $\bar{\ell} \cap \overline{\mathcal{C}} = \{\langle(1, t, t^2)\rangle, \langle(1, t^{q^2}, t^{2q^2})\rangle\}$, for some $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. The reader is referred to [11] for more details.

Under the identification of $\mathbb{F}_{q^4} \cup \{\infty\}$ with $\overline{\mathcal{C}}$ given by

$$\xi : \quad t \leftrightarrow \langle(1, t, t^2)\rangle, \qquad \infty \leftrightarrow \langle(0, 0, 1)\rangle, \tag{1}$$

the pair $\mathbf{t} = \{t, t^{q^2}\}$, with $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, may be associated with the elliptic line intersecting $\overline{\mathcal{C}}$ at $\{\langle(1, t, t^2)\rangle, \langle(1, t^{q^2}, t^{2q^2})\rangle\}$. We will use $\bar{\ell}_{\mathbf{t}}$ to denote this line.

We assume q is even. For any given pair of distinct elliptic lines $\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}$, let

$$\widehat{\rho}(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) = \widehat{\rho}(\mathbf{s}, \mathbf{t}) = \frac{1}{\rho(s,t) + \rho(s,t)^{-1}}, \tag{2}$$

where

$$\rho(s,t) = \frac{(s+t)(s^{q^2} + t^{q^2})}{(s + t^{q^2})(s^{q^2} + t)}. \tag{3}$$

It is evident that $\operatorname{Im} \widehat{\rho}$ is a subset of $\mathbb{F}_{q^2}$. The following result is straigtforward.

Lemma 2.1. [11, Lemma 5.1]

$$\widehat{\rho}(\mathbf{s}, \mathbf{t}) = \frac{(s+t)(s^{q^2} + t^{q^2})(s + t^{q^2})(s^{q^2} + t)}{(s + s^{q^2})^2 (t + t^{q^2})^2} = \left(\frac{1}{\rho(s,t) + 1}\right)^2 + \left(\frac{1}{\rho(s,t) + 1}\right).$$

Set $q = 2^h$. For $r \in \{1, 2\}$, let $\mathbf{T}_0(q^r)$ be the set of elements of $\mathbb{F}_{q^r}$ with absolute trace zero:

$$\mathbf{T}_0(q^r) = \left\{ x \in \mathbb{F}_{q^r} \, : \, \sum_{i=0}^{rh-1} x^{2^i} = 0 \right\}.$$

In [11] Hollmann and Xiang consider the following sets to construct a 3-class association scheme:

$$\mathbf{T}_0 = \mathbf{T}_0(q^2), \qquad \mathbf{S}_0^* = \mathbf{T}_0(q) \setminus \{0\}, \qquad \mathbf{S}_1 = \mathbb{F}_q \setminus \mathbf{S}_0.$$

Note that $\mathbf{T}_0 = \{\alpha + \alpha^2 : \alpha \in \mathbb{F}_{q^2}\}$ as q is even. By Lemma 2.1, $\operatorname{Im} \widehat{\rho}$ is contained in $\mathbf{T}_0$.

Theorem 2.2. [11] *On the set of the elliptic lines $\mathcal{E}$ define the following relations:*

R_1*: $(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in R_1$ if and only $\widehat{\rho}(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in \mathbf{S}_0^*$;*

R_2*: $(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in R_2$ if and only $\widehat{\rho}(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in \mathbf{S}_1$;*

R_3: $(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in R_3$ *if and only* $\widehat{\rho}(\bar{\ell}_{\mathbf{s}}, \bar{\ell}_{\mathbf{t}}) \in \mathbf{T}_0 \setminus \mathbb{F}_q$.

Then the pair $(\mathcal{E}, \{R_i\}_{i=0}^3)$*, where* R_0 *is the identity relation, is a 3-class association scheme.*

The first eigen-matrix of the scheme is

$$P = \begin{pmatrix} 1 & (q-2)(q^2+1)/2 & q(q^2+1)/2 & q(q-2)(q^2+1)/2 \\ 1 & -(q-1)(q-2)/2 & -q(q-1)/2 & q(q-2) \\ 1 & -(q^2-q+2)/2 & q(q+1)/2 & -q \\ 1 & q-1 & 0 & -q \end{pmatrix}; \qquad (4)$$

see [11, Section 7].

Remark 2.3. By identification (1), the set $\mathcal{E}$ may be replaced by the set $\mathcal{X} = \{\mathbf{t} = \{t, t^{q^2}\} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ and the relations R_i, $i = 1, 2, 3$, replaced by

R'_1: $(\mathbf{s}, \mathbf{t}) \in R'_1$ if and only $\widehat{\rho}(\mathbf{s}, \mathbf{t}) \in \mathbf{S}_0^*$;

R'_2: $(\mathbf{s}, \mathbf{t}) \in R'_2$ if and only $\widehat{\rho}(\mathbf{s}, \mathbf{t}) \in \mathbf{S}_1$;

R'_3: $(\mathbf{s}, \mathbf{t}) \in R'_3$ if and only $\widehat{\rho}(\mathbf{s}, \mathbf{t}) \in \mathbf{T}_0 \setminus \mathbb{F}_q$;

here $\widehat{\rho}(\mathbf{s}, \mathbf{t})$ is the quantity defined in (2). Hence, $(\mathcal{X}, \{R'_i\}_{i=0}^3)$ is an association scheme isomorphic to $(\mathcal{E}, \{R_i\}_{i=0}^3)$.

Remark 2.4. Actually, the scheme $(\mathcal{X}, \{R'_i\}_{i=0}^3)$ arises as a fusion of the one given by the following result [11].

Theorem. *Under the identification* ξ*, the action of* $\mathrm{PGL}(2, q^2)$ *on* $\mathcal{E} \times \mathcal{E}$ *gives rise to an association scheme on* $\mathcal{X}$ *with* $q^2/2 - 1$ *classes* $R_{\{\lambda, \lambda^{-1}\}}$, $\lambda \in \mathbb{F}_{q^2} \setminus \{0, 1\}$, *where* $(\mathbf{s}, \mathbf{t}) \in R_{\{\lambda, \lambda^{-1}\}}$ *if and only if* $\{\rho(s, t), \rho(s, t)^{-1}\} = \{\lambda, \lambda^{-1}\}$.

2.2 The Penttila-Williford association schemes

Up to isometries, the vector space $V(4, q^2)$ has precisely one non-degenerate hermitian form, and its Witt index is 2. As usual, $H(3, q^2)$ denotes the unitary polar space of rank 2 defined by it. A *point* (resp. *line*) of $H(3, q^2)$ is a 1-dimensional (resp. 2-dimensional) subspace in $H(3, q^2)$.

Assume q even for the rest of the current section. In $V(4, q^2)$ there is a 4-dimensional $\mathbb{F}_q$-vector space $\widehat{V}$ such that the restriction of the hermitian form on it induces a non-degenerate alternating form $\widehat{b}$ which defines a symplectic polar space $W(3,q)$ of rank 2 of $\mathrm{PG}(\widehat{V})$ [5]. In addition, $\widehat{b}$ is the polar of a non-degenerate quadratic form $\widehat{Q}$ of Witt index 1, whose set of singular point is denoted by $Q^-(3,q)$. By $\widehat{\mathcal{W}}$ (resp. $\widehat{\mathcal{Q}}$) we denote the set of all the totally isotropic (resp. singular) subspaces of $W(3,q)$ (resp. $Q^-(3,q)$) extended over $\mathbb{F}_{q^2}$. As a consequence, for every point of $H(3,q^2)$ not in $\widehat{\mathcal{W}}$ there are exactly q lines of $H(3,q^2)$ disjoint from $\widehat{\mathcal{W}}$ and one in $\widehat{\mathcal{W}}$. Note that $\widehat{\mathcal{W}}$ is an embedding of $W(3,q)$ in $H(3,q^2)$.

The following definition was introduced in [14]. A *relative hemisystem of* $H(3,q^2)$ *with respect to* $W(3,q)$ is a set $\mathcal{H}$ of lines of $H(3,q^2)$ disjoint from $\widehat{\mathcal{W}}$ such that every point of $H(3,q^2)$ not in $\widehat{\mathcal{W}}$ lies on exactly $q/2$ lines of $\mathcal{H}$. For any given line l of $H(3,q^2)$ disjoint from $\widehat{\mathcal{W}}$, let $\mathcal{S}_l$ denote the set of lines of $H(3,q^2)$ which meet both l and $\widehat{\mathcal{W}}$. We stress the fact that $\mathcal{S}_l$ consists of the lines of $\widehat{\mathcal{W}}$ that extend elements of a regular spread of $W(3,q)$[1], and refer to $\mathcal{S}_l$ as the *spread subtended by* l.

Theorem 2.5. [14, Theorem 4] *Let $\mathcal{H}$ be a relative hemisystem of $H(3,q^2)$ with respect to $W(3,q)$. Then a primitive Q-polynomial 3-class association scheme can be constructed on $\mathcal{H}$ by the defining the following relations:*

$\widetilde{R}_1$: *$(l,m) \in \widetilde{R}_1$ if and only $|l \cap m| = 1$;*

$\widetilde{R}_2$: *$(l,m) \in \widetilde{R}_2$ if and only $l \cap m = \emptyset$ and $|\mathcal{S}_l \cap \mathcal{S}_m| = 1$;*

$\widetilde{R}_3$: *$(l,m) \in \widetilde{R}_3$ if and only $l \cap m = \emptyset$ are $|\mathcal{S}_l \cap \mathcal{S}_m| = q+1$.*

Let $\mathrm{PO}^-(\widehat{V})$ be the stabilizer of $\widehat{\mathcal{Q}}$ in the projective unitary group $\mathrm{PGU}(4,q^2)$. By looking at the action of the commutator subgroup $\mathrm{P}\Omega^-(\widehat{V})$ of $\mathrm{PO}^-(\widehat{V})$ on the lines of $H(3,q^2)$, the following result was proved in [14].

Theorem 2.6. $\mathrm{P}\Omega^-(\widehat{V})$ *has two orbits on the lines of $H(3,q^2)$ disjoint from $\widehat{\mathcal{W}}$, and each orbit is a relative hemisystem with respect to $W(3,q)$.*

We consider an association scheme $(\mathcal{H}, \{\widetilde{R}_i\}_{i=0}^3)$ as in Theorem 2.5 by using the hemisystems from Theorem 2.6. As expected, the first eigen-matrix of the scheme is precisely the matrix in (4).

[1] A spread of $W(3,q)$ in $\mathrm{PG}(3,q)$ is a set $\mathcal{S}$ of totally isotropic lines which partition the pointset of $\mathrm{PG}(3,q)$. $\mathcal{S}$ is *regular* if for any three distinct lines of $\mathcal{S}$ there is a set R of $q+1$ lines of $\mathcal{S}$ containing them, with the following property: any line of $\mathrm{PG}(3,q)$ intersecting three lines in R meets all the lines of R.

3 The explicit construction of the relative hemisystem of Penttila-Williford

Let G and H be groups acting on the sets Ω and Δ, respectively. The two actions are said to be *permutationally isomorphic* if there exist a bijection $\theta : \Omega \to \Delta$ and an isomorphism $\chi : G \to H$ such that the following diagram commutes:

$$\begin{tikzcd} G \times \Omega \arrow[r, "\phi"] \arrow[d, "\chi \quad \theta"'] & \Omega \arrow[d, "\theta"] \\ H \times \Delta \arrow[r, "\tilde{\phi}"] & \Delta \end{tikzcd}$$

Here ϕ and $\tilde{\phi}$ are the maps defining the action of G and H on Ω and Δ, respectively.

Let $Q^-(3,q)$ be the orthogonal polar space (of rank 1) defined by $\widehat{Q}$ on the 4-dimensional $\mathbb{F}_q$-vector space $\widehat{V}$ introduced in Section 2.2.
It is known that $(\mathrm{PSL}(2,q^2), \mathrm{PG}(1,q^2))$ and $(\mathrm{P}\Omega^-(\widehat{V}), Q^-(3,q))$ are permutationally isomorphic for all prime powers q. For sake of completeness, we give an explicit description of the above isomorphism which is more suitable for our computation.

In $V(4,q^2) = \{(X_1, X_2, X_3, X_4) : X_i \in \mathbb{F}_{q^2}\}$, let $\widehat{V}$ be the set of all vectors $v = (\alpha, x^q, x, \beta)$ with $\alpha, \beta \in \mathbb{F}_q$, $x \in \mathbb{F}_{q^2}$. With the usual sum and multiplication by scalars from $\mathbb{F}_q$, $\widehat{V}$ is a 4-dimensional vector space over $\mathbb{F}_q$.

As usual we identify $\mathrm{PG}(1,q^2)$ with $\mathbb{F}_{q^2} \cup \{\infty\}$ and we consider the following injective map:

$$\begin{array}{rccc} \theta : & \mathbb{F}_{q^2} \cup \{\infty\} & \longrightarrow & \mathrm{PG}(\widehat{V}) \\ & t & \mapsto & \langle (1, t^q, t, t^{q+1}) \rangle \\ & \infty & \mapsto & \langle (0,0,0,1) \rangle \end{array} . \tag{5}$$

Proposition 3.1. [6] *The image of θ is an orthogonal polar space of rank 1 of $\mathrm{PG}(\widehat{V})$.*

Proof. Let Q be the quadratic form on $V(4,q^2)$ defined by

$$Q(\mathbf{X}) = X_1X_4 - X_2X_3,$$

which has $b(\mathbf{X}, \mathbf{Y}) = X_1Y_4 + X_4Y_1 - X_2Y_3 - X_3Y_2$ as the associated non-degenerate bilinear form. The restriction $\widehat{Q} = Q|_{\widehat{V}}$ is the quadratic form given by

$$\widehat{Q}(v) = \alpha\beta - x^{q+1},$$

which has

$$\widehat{b}(v,v') = \alpha\beta' + \beta\alpha' - xx'^q - x^q x' \tag{6}$$

as the associated bilinear form. Let $v = (\alpha, x^q, x, \beta) \in \mathrm{Rad}(\widehat{V})$, that is $\widehat{b}(v,v') = 0$, for all $v' \in \widehat{V}$. If $\alpha' = \beta' = 0$, a necessary condition for $v \in \mathrm{Rad}(\widehat{V})$ is

$$x^q x' + xx'^q = 0,$$

for all $x' \in \mathbb{F}_{q^2}$. This shows that the polynomial in x' of degree q on the left hand side has at least q^2 roots. Therefore, it must be the zero polynomial, and $x = 0$. We repeat the above argument for $\alpha' = x' = 0$ and for $x' = \beta' = 0$ to show that $v = 0$. This yields that $\widehat{b}$, and hence $\widehat{Q}$, is non-degenerate. Let u be a singular vector for $\widehat{Q}$. Without loss of generality we may take $u = (1,0,0,0)$. Therefore, the subspace $U = \{v \in \widehat{V} : \widehat{b}(v,u) = 0\}$ coincides with $\{(\alpha, x^q, x, 0) : \beta \in \mathbb{F}_q, x \in \mathbb{F}_{q^2}\}$. It is easily seen that $U \cap \ker \widehat{Q} = \{\alpha u : \alpha \in \mathbb{F}_q\}$. Thus, $\widehat{Q}$ is a quadratic form of Witt index 1 giving rise to the orthogonal polar space

$$Q^-(3,q) = \{\langle(1, t^q, t, t^{q+1})\rangle : t \in \mathbb{F}_{q^2}\} \cup \{\langle(0,0,0,1)\rangle\},$$

which is precisely $\mathrm{Im}\,\theta$. $\square$

Let χ be the monomorphism defined by

$$\begin{array}{cccc} \chi : & \mathrm{SL}(2,q^2) & \longrightarrow & \mathrm{SL}(4,q^2) \\ & g & \mapsto & g \otimes g^q \end{array},$$

where $\otimes$ is the Kronecker product and g^q denotes the matrix g with its entries raised to the q-th power. It is straightforward to check that $\chi(g)$ is a $\widehat{Q}$-isometry, for every $g \in \mathrm{SL}(2,q^2)$. Therefore, χ can be regarded as a monomorphism from $\mathrm{PSL}(2,q^2)$ to $\mathrm{PO}^-(\widehat{V})$. It is actually an isomorphism from $\mathrm{PSL}(2,q^2)$ to $\mathrm{P\Omega}^-(\widehat{V})$, as it will be shown below.

Let t_a be the transvection in $\mathrm{SL}(2,q^2)$ with matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, for some $a \in \mathbb{F}_{q^2}^*$. The isometry $\chi(t_a)$ maps (α, x^q, x, β) to $(\alpha, a^q\alpha + x^q, a\alpha + x, a^{q+1}\alpha + ax^q + a^q x + \beta)$. Its restriction on the hyperplane of all $\widehat{b}$-orthogonal vectors to $u = (0,0,0,1)$ is the map

$$\eta_{u,y}(w) = w + \widehat{b}(w,y)u,$$

where $y = (0, -a^q, -a, 0)$. This yields that $\chi(t_a)$ is actually the unique Siegel transformation $\rho_{u,y}$ which extends $\eta_{u,y}$ [15, Theorem 11.18]. By using [15, Theorem 11.19 (ii)] it is possible to show that as a varies in $\mathbb{F}_{q^2}^*$, $\rho_{u,y}$ describes all the Siegel transformations centered at u.

Every transvection g is conjugate in $\mathrm{SL}(2,q^2)$ to a transvection of type t_a. This implies that $\chi(g)$ is also a Siegel transformation [15, Theorem 11.19 (iii)]. Therefore, χ gives rise to a bijection from the set of all transvections in $\mathrm{SL}(2,q^2)$ to all Siegel transformations of $\widehat{V}$. Since transvections generate $\mathrm{SL}(2,q^2)$, and Siegel transformations generate $\Omega^-(\widehat{V})$, we achieve $\chi(\mathrm{PSL}(2,q^2)) \leq \mathrm{P}\Omega^-(\widehat{V})$. As $|\mathrm{PSL}(2,q^2)| = |\mathrm{P}\Omega^-(\widehat{V})|$, χ is actually the desired isomorphism. It is a matter of fact that the diagram

$$\begin{array}{ccccc}
\mathrm{PSL}(2,q^2) & \times & \mathbb{F}_{q^2}\cup\{\infty\} & \xrightarrow{\ \phi\ } & \mathbb{F}_{q^2}\cup\{\infty\} \\
\chi\big\downarrow & & \theta\big\downarrow & & \theta\big\downarrow \\
\mathrm{P}\Omega^-(\widehat{V}) & \times & Q^-(3,q) & \xrightarrow{\ \tilde{\phi}\ } & Q^-(3,q)
\end{array}$$

commutes.

For the rest of this section, assume q is even. The bilinear form $\widehat{b}$ defined by (6) is a (non-degenerate) alternating form on $\widehat{V}$. Let h be the non-degenerate hermitian form on $V(4,q^2)$ given by

$$h(\mathbf{X},\mathbf{Y}) = X_1Y_4^q + X_2Y_2^q + X_3Y_3^q + X_4Y_1^q,$$

with associated unitary polar space $H(3,q^2)$. It is evident that $h|_{\widehat{V}} = \widehat{b}$. Therefore, the symplectic polar space $W(3,q)$ defined by $\widehat{b}$, as well as the orthogonal polar space $Q^-(3,q)$, can be embedded in $H(3,q^2)$ by extending the scalars, so getting $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{Q}}$ introduced in Section 2.2. This also implies that $\mathrm{P}\Omega^-(\widehat{V})$ is a subgroup of the projective symplectic group $\mathrm{PSp}(\widehat{V})$ which is in turn a subgroup of the projective unitary group $\mathrm{PGU}(4,q^2)$.

The semilinear involutorial transformation τ of $V(4,q^2)$ given by

$$\begin{array}{cccc}
\tau: & V(4,q^2) & \longrightarrow & V(4,q^2) \\
 & (X_1,X_2,X_3,X_4) & \mapsto & (X_1^q,X_3^q,X_2^q,X_4^q).
\end{array}$$

fixes $H(3,q^2)$ and acts as the identity on $\widehat{\mathcal{W}}$.

If we embed $V(4,q^2)$ in $V(4,q^4)$ by extending the scalars, then $\mathrm{PG}(3,q^4)$ embeds $\mathrm{PG}(\widehat{V})$. Therefore, θ defined by (5) can be naturally thought as the restriction of the following map:

$$\begin{array}{cccc}
\theta: & \mathbb{F}_{q^4}\cup\{\infty\} & \longrightarrow & \mathrm{PG}(3,q^4) \\
 & t & \longmapsto & \langle(1,t^q,t,t^{q+1})\rangle \\
 & \infty & \longmapsto & \langle(0,0,0,1)\rangle.
\end{array}$$

Note that, for any $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, $\theta(t)$ is not the span of a vector of $V(4, q^2)$. Moreover,

$$\theta(t^{q^2}) = \langle(1, t^{q^3}, t^{q^2}, t^{q^3+q^2})\rangle = \theta(t)^{\tau^2} \neq \theta(t).$$

For each $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, we associate the pair $\mathbf{t} = \{t, t^{q^2}\}$ with the line $M_{\mathbf{t}}$ of $\mathrm{PG}(3, q^4)$ spanned by $\theta(t)$ and $\theta(t^{q^2})$, which is distinct from $M_{\mathbf{t}}^{\tau}$.

Lemma 3.2. *For each pair* $\mathbf{t}$*,* $M_{\mathbf{t}} \cap V(4, q^2)$ *is a line of* $H(3, q^2)$*, say* $m_{\mathbf{t}}$*, which is disjoint from* $\widehat{\mathcal{W}}$*.*

Proof. A straightforward computation shows that the vectors in $M_{\mathbf{t}} \cap V(4, q^2)$ are precisely $\mathbf{X}_\lambda = (\lambda + \lambda^{q^2}, \lambda t^q + \lambda^{q^2} t^{q^3}, \lambda t + \lambda^{q^2} t^{q^2}, \lambda t^{q+1} + \lambda^{q^2} t^{q^3+q^2})$, for all $\lambda \in \mathbb{F}_{q^4}$, and they form a line $m_{\mathbf{t}}$ of $\mathrm{PG}(3, q^2)$. Since $h(\mathbf{X}_\lambda, \mathbf{X}_\lambda) = 0$ for all $\lambda \in \mathbb{F}_{q^4}$, $m_{\mathbf{t}}$ is a line of $H(3, q^2)$. Finally, in order to prove that $m_{\mathbf{t}}$ is disjoint from $\widehat{\mathcal{W}}$, consider the following system:

$$\begin{aligned} a\,\alpha &= \lambda + \lambda^{q^2} \\ a\,x^q &= \lambda t^q + \lambda^{q^2} t^{q^3} \\ a\,x &= \lambda t + \lambda^{q^2} t^{q^2} \\ a\,\beta &= \lambda t^{q+1} + \lambda^{q^2} t^{q^3+q^2} \end{aligned}\quad, \tag{7}$$

with $\alpha, \beta \in \mathbb{F}_q$, $x, a \in \mathbb{F}_{q^2}$, $\lambda \in \mathbb{F}_{q^4}$, $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. The existence of a solution for (7), or rather the existence of $a \in \mathbb{F}_{q^2}$, makes the system inconsistent. This concludes the proof. $\square$

Proposition 3.3. *The sets* $\{m_{\mathbf{t}} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ *and* $\{m_{\mathbf{t}}^{\tau} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ *are precisely the two orbits of* $\mathrm{P}\Omega^-(\widehat{V})$ *on the lines of* $H(3, q^2)$ *disjoint from* $\widehat{\mathcal{W}}$*.*

Proof. From the proof of [14, Theorem 5], $\mathrm{P}\Omega^-(\widehat{V})$ has two orbits on the lines of $H(3, q^2)$ disjoint from $\widehat{\mathcal{W}}$, and these two orbits are interchanged by τ. We recall that $m_{\mathbf{t}}$ is uniquely defined by the line $M_{\mathbf{t}}$ of $\mathrm{PG}(3, q^4)$, which is spanned by $\theta(t)$ and $\theta(t^{q^2})$. Hence, it suffices to prove that $\{M_{\mathbf{t}} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ is an orbit of $\mathrm{P}\Omega^-(\widehat{V})$.

Let $\omega \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ such that $\omega^{q^2} = \omega + 1$ and $\omega^2 + \omega = \delta$, with $\delta \in \mathbb{F}_{q^2} \setminus \mathbf{T}_0$, $\delta \neq 1$. For all $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, write $t = x + y\omega$, with $x, y \in \mathbb{F}_{q^2}$, $y \neq 0$.

As a group acting on the projective line $\mathrm{PG}(1, q^2)$ assimilated to the set $\mathbb{F}_{q^2} \cup \{\infty\}$, $\mathrm{PSL}(2, q^2)$ may be identified with the group of linear fractional transformations

$$z \mapsto \frac{az + b}{cz + d},$$

where $ad-bc$ is a non-zero square in $\mathbb{F}_{q^2}$ [15]. For any given $t = x + y\omega \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, let $g \in \mathrm{PSL}(2, q^2)$ with matrix $[\,1, 0;\, x, y\,]$. Then, $\chi(g) = g \otimes g^q$ maps $\Big(\theta(\omega), \theta(\omega^{q^2})\Big)$ to $\Big(\theta(t), \theta(t^{q^2})\Big)$ by taking into account $\omega^{q^2} = \omega+1$. This implies that $\chi(g) \in \mathrm{P}\Omega(\widehat{V})$ maps the line $M_{\{\omega,\omega^{q^2}\}}$ to $M_{\mathbf{t}}$. $\square$

Corollary 3.4. *The sets $\{m_{\mathbf{t}} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ and $\{m_{\mathbf{t}}^{\tau} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ are the Penttila-Williford relative hemisystems.*

4 The proof of the Main Theorem

Define $T = \{\{t, t^{q^2}\} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$, and put $\mathcal{X} = T$. By Remark 2.3, we need to find a bijection between the set $\mathcal{X}$ and the relative hemisystem $\mathcal{H} = \{m_{\mathbf{t}} : \mathbf{t} \in T\}$ preserving the relations defined on them.

From the arguments in Section 3, we may associate the pair $\mathbf{t} = \{t, t^{q^2}\} \in \mathcal{X}$ with the line $m_{\mathbf{t}} \in \mathcal{H}$. Moreover, Corollary 3.4 gives $|\mathcal{H}| = (q^4 - q^2)/2 = |\mathcal{X}|$, and this contributes to make the mapping $\varphi : \mathcal{X} \to \mathcal{H}, \mathbf{t} \mapsto m_{\mathbf{t}}$ a bijection. In order to show that φ, in fact, preserves the relations, we will move into a different geometric setting. More precisely, we will use the following dual representation of $H(3, q^2)$. Via the Klein correspondence κ, the lines of $\mathrm{PG}(3, q^2)$ are mapped to the points of an orthogonal polar space $Q^+(5, q^2)$ of rank 3 of $\mathrm{PG}(5, q^2)$, which is the so-called the *Klein quadric*. In particular, the lines of $H(3, q^2)$ are mapped to the points of an orthogonal polar space $Q^-(5, q)$ of rank 2 in a $\mathrm{PG}(5, q)$ embedded in $\mathrm{PG}(5, q^2)$. When q is even, κ maps the lines of any symplectic polar space of rank 2 embedded in $H(3, q^2)$ to the points of an orthogonal polar space of rank 2, which is the intersection of $Q^-(5, q)$ with a hyperplane of $\mathrm{PG}(5, q)$. The reader is referred to [10] for more details on the Klein correspondence.

Assume q even. In $V(6, q^2)$ consider the 6-dimensional $\mathbb{F}_q$-subspace $\widetilde{V} = \{(x, x^q, y, y^q, z, z^q) : x, y, z \in \mathbb{F}_{q^2}\}$. Let $\mathrm{PG}(\widetilde{V})$ be the projective geometry defined by $\widetilde{V}$.

We consider the Klein quadric $Q^+(5, q^2)$ defined by the (non-degenerate) quadratic form $Q(\mathbf{X}) = X_1X_6+X_2X_5+X_3X_4$ on $V(6, q^2)$. For any given $w = (x, x^q, y, y^q, z, z^q) \in \widetilde{V}$,

$$\widetilde{Q}(w) = Q|_{\widetilde{V}}(w) = xz^q + x^q z + y^{q+1}.$$

From [12, Proposition 2.4], $\widetilde{Q}$ is a non-degenerate quadratic form of Witt index 2 on $\widetilde{V}$ with associated alternating form

$$\widetilde{b}(w, w') = xz'^q + x^q z' + yy'^q + y^q y' + zx'^q + z^q x'.$$

Therefore, $\widetilde{Q}$ gives rise to an orthogonal polar space $Q^-(5,q)$ of $\mathrm{PG}(\widetilde{V})$ embedded in $Q^+(5,q^2)$.

For any subspace X of $\widetilde{V}$, set

$$X^\perp = \{w \in \widetilde{V} : \widetilde{b}(w,u) = 0, \text{ for all } u \in X\}.$$

Let $Q(4,q)$ be the polar space whose points are the κ-image of the lines of $\widehat{\mathcal{W}}$, and Γ be the hyperplane of $\mathrm{PG}(\widetilde{V})$ containing $Q(4,q)$. For a complete description of Γ observe that the pairs

$$\{(1,0,0,0),(0,x,x^q,0)\}, \quad \{(0,0,0,1),(0,x,x^q,0)\}, \quad \{(1,1,1,1),(x+x^q,x,x^q,0)\},$$

with $x \in \mathbb{F}_{q^2}$, span lines of $\widehat{\mathcal{W}}$ which give three skew lines of $Q(4,q)$ under κ generating Γ. It follows that $\Gamma = \{(x,x^q,c,c,z,z^q) : x,z \in \mathbb{F}_{q^2}, c \in \mathbb{F}_q\}$.

Under κ, the line $m_\mathbf{t}$ of $\mathcal{H}$ is mapped to the point $P_\mathbf{t} = \langle w_\mathbf{t}\rangle$ of $\mathrm{PG}(\widetilde{V})$, where

$$w_\mathbf{t} = (t^q+t^{q^3}, t+t^{q^2}, t^{1+q}+t^{q^2+q^3}, t^{1+q^3}+t^{q+q^2}, t^{1+q+q^3}+t^{q+q^2+q^3}, t^{1+q+q^2}+t^{1+q^2+q^3}).$$

Note that $P_\mathbf{t}$ is in $Q^-(5,q)$, but not in $Q(4,q)$. Let $P'_\mathbf{t} = \kappa(m_\mathbf{t}^\tau)$. Since $m_\mathbf{t}$ and $m_\mathbf{t}^\tau$ are disjoint lines of $H(3,q^2)$, the line $L_\mathbf{t}$ spanned by $P_\mathbf{t}$ and $P'_\mathbf{t}$ intersects $Q^-(5,q)$ just at $P_\mathbf{t}$ and $P'_\mathbf{t}$. On the other hand, $m_\mathbf{t}$ and $m_\mathbf{t}^\tau$ subtend the same spread $\mathcal{S}_\mathbf{t} = \mathcal{S}_{m_\mathbf{t}}$ in $\widehat{\mathcal{W}}$. The κ-image of $\mathcal{S}_\mathbf{t}$ is an orthogonal polar space of rank 1 contained in $Q(4,q)$ [10], and it turns out this is precisely $Q(4,q) \cap L_\mathbf{t}^\perp$. Consequently, $L_\mathbf{t}^\perp$ is in Γ and $\Gamma^\perp = \langle(0,0,1,1,0,0)\rangle = \langle w_0\rangle$ is a point of $L_\mathbf{t}$, for all $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. The symbol $\widetilde{\mathcal{O}}_\mathbf{t}$ will be used to indicate $Q(4,q) \cap L_\mathbf{t}^\perp$.

For any given distinct pairs $\mathbf{s}$ and $\mathbf{t}$, let $\Pi_{\mathbf{s},\mathbf{t}}$ be the plane of $\mathrm{PG}(\widetilde{V})$ spanned by $\Gamma^\perp$, $P_\mathbf{s}$ and $P_\mathbf{t}$. The restriction of $\widetilde{Q}$ and $\widetilde{b}$ on $\Pi_{\mathbf{s},\mathbf{t}}$ will be denoted by $\widetilde{Q}_{\mathbf{s},\mathbf{t}}$ and $\widetilde{b}_{\mathbf{s},\mathbf{t}}$, respectively. Identifying a triple $(a,b,c) \in \mathbb{F}_q^3$ with the vector $v = aw_\mathbf{s} + bw_0 + cw_\mathbf{t} \in \Pi_{\mathbf{s},\mathbf{t}}$, we obtain that the action of $\widetilde{b}_{\mathbf{s},\mathbf{t}}$ induced on $\mathbb{F}_q^3$ is given by the matrix

$$B = \begin{pmatrix} \widetilde{b}(w_\mathbf{s},w_\mathbf{s}) & \widetilde{b}(w_\mathbf{s},w_0) & \widetilde{b}(w_\mathbf{s},w_\mathbf{t}) \\ \widetilde{b}(w_0,w_\mathbf{s}) & \widetilde{b}(w_0,w_0) & \widetilde{b}(w_0,w_\mathbf{t}) \\ \widetilde{b}(w_\mathbf{t},w_\mathbf{s}) & \widetilde{b}(w_\mathbf{t},w_0) & \widetilde{b}(w_\mathbf{t},w_\mathbf{t}) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Tr}(s^{q+1}) & \widetilde{b}(w_\mathbf{s},w_\mathbf{t}) \\ \mathrm{Tr}(s^{q+1}) & 0 & \mathrm{Tr}(t^{q+1}) \\ \widetilde{b}(w_\mathbf{s},w_\mathbf{t}) & \mathrm{Tr}(t^{q+1}) & 0 \end{pmatrix};$$

here Tr is the trace map from $\mathbb{F}_{q^4}$ on $\mathbb{F}_q$. A straightforward calculation shows that $\Pi_{\mathbf{s},\mathbf{t}}$ is degenerate as $\mathrm{Rad}(\Pi_{\mathbf{s},\mathbf{t}}) = \langle v_{\mathbf{s},\mathbf{t}}\rangle$, where

$$v_{\mathbf{s},\mathbf{t}} = \mathrm{Tr}(t^{q+1})w_\mathbf{s} + \widetilde{b}(w_\mathbf{s},w_\mathbf{t})w_0 + \mathrm{Tr}(s^{q+1})w_\mathbf{t}.$$

It is easily seen that

$$\begin{aligned}\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) &= \widetilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) \left(\widetilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) + \mathrm{Tr}(s^{q+1})\mathrm{Tr}(t^{q+1})\right) \\ &= \widetilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}})\, \widetilde{b}(w_{\mathbf{s}}, w'_{\mathbf{t}}),\end{aligned} \tag{8}$$

where

$w'_{\mathbf{t}} = (t^q + t^{q^3}, t + t^{q^2}, t^{q+q^2} + t^{1+q^3}, t^{1+q} + t^{q^2+q^3}, t^{1+q+q^3} + t^{q+q^2+q^3}, t^{1+q+q^2} + t^{1+q^2+q^3})$.

Note that $P'_{\mathbf{t}} = \kappa(m^{\tau}_{\mathbf{t}}) = \langle w'_{\mathbf{t}} \rangle$.

Now two cases are possible according as $v_{\mathbf{s},\mathbf{t}}$ is singular or not.

If $\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) = 0$, then $\widetilde{Q}_{\mathbf{s},\mathbf{t}}$ is degenerate, and $\mathcal{C}_{\mathbf{s},\mathbf{t}} = \Pi_{\mathbf{s},\mathbf{t}} \cap Q^-(5, q)$ consists of two distinct lines through $\langle v_{\mathbf{s},\mathbf{t}} \rangle$, as $P_{\mathbf{s}}$, $P'_{\mathbf{s}}$, $P_{\mathbf{t}}$ and $P'_{\mathbf{t}}$ are distinct points no three of them collinear. This yields that $L^{\perp}_{\mathbf{s}}$ meets $L^{\perp}_{\mathbf{t}}$ in the plane $\Pi^{\perp}_{\mathbf{s},\mathbf{t}}$ of Γ, with $\widetilde{Q}|_{\Pi^{\perp}_{\mathbf{s},\mathbf{t}}}$ degenerate, and $\widetilde{\mathcal{O}}_{\mathbf{s}} \cap \widetilde{\mathcal{O}}_{\mathbf{t}} = \Pi^{\perp}_{\mathbf{s},\mathbf{t}} \cap Q^-(5, q) = \langle v_{\mathbf{s},\mathbf{t}} \rangle$. By taking into account (8), there are two possibilities of obtaining zero for $\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}})$: either $\widetilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) = 0$ or $\widetilde{b}(w_{\mathbf{s}}, w'_{\mathbf{t}}) = 0$.

If $\widetilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) = 0$, then $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are collinear in $Q^-(5, q)$, or equivalently, the lines $m_{\mathbf{s}}$ and $m_{\mathbf{t}}$ are concurrent, that is $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_1$ (see Theorem 2.5). On the other hand, by taking into account (3),

$$\begin{aligned}0 &= \widetilde{b}(w_{\mathbf{s}}, w_{\mathbf{t}}) \\ &= (s^{q^2} + s)^q (t^{q^2} + t)^q (s + t^{q^2})(s^{q^2} + t) + (s^{q^2} + s)(t^{q^2} + t)(s + t^{q^2})^q (s^{q^2} + t)^q\end{aligned}$$

if and only if

$$\nu = \frac{1}{\rho(s, t) + 1} = \frac{(s + t^{q^2})(s^{q^2} + t)}{(s^{q^2} + s)(t^{q^2} + t)} \in \mathbb{F}_q.$$

When $\nu \in \mathbb{F}_q$, $\widehat{\rho}(s, t) \in \mathbf{S}^*_0$ by Lemma 2.1, that is $(\mathbf{s}, \mathbf{t}) \in R'_1$ (see Remark 2.3). On the other hand, if $\widehat{\rho}(s, t) = \nu^2 + \nu \in \mathbf{S}^*_0$, then there exists $z \in \mathbb{F}_q$ such that $(z + \nu)^2 + (z + \nu) = 0$, which implies either $z = \nu$ or $z + 1 = \nu$. In both cases $\nu \in \mathbb{F}_q$. Therefore, $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_1$ if and only if $(\mathbf{s}, \mathbf{t}) \in R'_1$.

If $\widetilde{b}(w_{\mathbf{s}}, w'_{\mathbf{t}}) = 0$, then $P_{\mathbf{s}}$ and $P'_{\mathbf{t}}$ are collinear in $Q^-(5, q)$, and this leads to the non-collinearity of $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$. This means that $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_2$ on one side, and $(\mathbf{s}, \mathbf{t}) \in R'_2$ on the other one. In fact,

$$\begin{aligned}0 &= \widetilde{b}(w_{\mathbf{s}}, w'_{\mathbf{t}}) \\ &= (s^{q^2} + s)^q (t^{q^2} + t)^q (s + t^{q^2})(s^{q^2} + t) + (s^{q^2} + s)(t^{q^2} + t)(s + t^{q^2})^q (s^{q^2} + t)^q + \\ &\quad + (s^{q^2} + s)^{q+1} (t^{q^2} + t)^{q+1}\end{aligned}$$

if and only if

$$\nu^q + \nu = \left(\frac{1}{\rho(s,t)+1}\right)^q + \frac{1}{\rho(s,t)+1} = 1.$$

When $\nu^q + \nu = 1$, then $\nu \notin \mathbb{F}_q$. This implies that the equation $Z^2 + Z = \widehat{\rho}(s,t)$ has no solutions in $\mathbb{F}_q$, that is $\widehat{\rho}(s,t) \in \mathbf{S}_1$, i.e. $(\mathbf{s},\mathbf{t}) \in R_2'$. On the other hand, $\widehat{\rho}(s,t) = \nu^2+\nu \in \mathbf{S}_1 \subset \mathbb{F}_q$ implies $\nu \notin \mathbb{F}_q$. As $\widehat{\rho}(s,t) \in \mathbb{F}_q$, then $(\nu^q+\nu)^2+(\nu^q+\nu) = 0$ holds, whence $\nu^q + \nu = 1$.

Finally, if $\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) \neq 0$, then $\widetilde{Q}_{\mathbf{s},\mathbf{t}}$ is non-degenerate and $\langle v_{\mathbf{s},\mathbf{t}} \rangle$ is the nucleus of the (non-degenerate) conic $\mathcal{C}_{\mathbf{s},\mathbf{t}}$. Therefore, $\widetilde{\mathcal{O}}_{\mathbf{t}}$ and $\widetilde{\mathcal{O}}_{\mathbf{s}}$ meet in $q+1$ points of $\Pi_{\mathbf{s},\mathbf{t}}^{\perp} \cap Q(4,q)$. Then, $\mathcal{S}_{m_{\mathbf{t}}} = \kappa^{-1}(\widetilde{\mathcal{O}}_{\mathbf{t}})$ and $\mathcal{S}_{m_{\mathbf{s}}} = \kappa^{-1}(\widetilde{\mathcal{O}}_{\mathbf{s}})$ meet in exactly $q+1$ lines in $\widehat{\mathcal{W}}$, that is $(m_{\mathbf{t}}, m_{\mathbf{s}}) \in \widetilde{R}_3$. It is clear that $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_3$ if and only if $(\mathbf{s},\mathbf{t}) \in R_3'$ by exclusion.

Summing up, for each $i = 0, \ldots, 3$, we have

$$(\mathbf{s},\mathbf{t}) \in R_i' \quad \textit{if and only if} \quad (m_{\mathbf{s}}, m_{\mathbf{t}}) = \varphi(\mathbf{s},\mathbf{t}) \in \widetilde{R}_i,$$

i.e. φ induces a bijection between R_i' and $\widetilde{R}_i$, thus achieving our aim.

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