UNIVERSAL SINGULAR EXPONENTS IN CATALYTIC VARIABLE EQUATIONS

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ABSTRACT. Catalytic equations appear in several combinatorial applications, most notably in the enumeration of lattice path and in the enumeration of planar maps. The main purpose of this paper is to show that the asymptotic estimate for the coefficients of the solutions of (so-called) positive catalytic equations has a universal asymptotic behavior. In particular, this provides a rationale why the number of maps of size n in various planar map classes grows asymptotically like $c \cdot n^{-5/2}\gamma^n$, for suitable positive constants c and γ. Essentially we have to distinguish between linear catalytic equations (where the subexponential growth is $n^{-3/2}$) and non-linear catalytic equations (where we have $n^{-5/2}$ as in planar maps). Furthermore we provide a quite general central limit theorem for parameters that can be encoded by catalytic functional equations, even when they are not positive.

1. INTRODUCTION

A planar map is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane. A map is rooted if an edge e is distinguished and directed. This edge is called the root-edge. The initial vertex v of this (directed) root-edge is then the root-vertex The face to the right of e is called the root-face and is usually taken as the outer face. All maps in this paper are rooted.

The enumeration of rooted maps (up to homeomorphisms) is a classical subject, initiated by Tutte in the 1960's. Tutte (and Brown) introduced the technique now called "the quadratic method" in order to compute the number M_n of rooted maps with n edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!}3^n.$$

This was later extended by Tutte and his school to several classes of planar maps: 2-connected, 3-connected, bipartite, Eulerian, triangulations, quadrangulations, etc. Using the previous formula, Stirling's estimate gives $M_n \sim (2/\sqrt{\pi}) \cdot n^{-5/2}12^n$. In

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all cases where a "natural" condition is imposed on maps, the asymptotic estimates turn out to be of this kind:

$$c \cdot n^{-5/2}\gamma^n.$$

The constants c and γ depend on the class under consideration, but one gets systematically an $n^{-5/2}$ term in the estimate.

This phenomenon is discussed by Banderier et al. [2]: 'This generic asymptotic form is "universal" in so far as it is valid for all known "natural families of maps".' The goal of this paper is to provide to some extent an explanation for this universal phenomenon, based on a detailed analysis of functional equations for generating functions with a catalytic variable. Let us mention that the critical exponent $-5/2$ has been 'explained' previously in at least two different ways. On the one hand, in the classical reference [6] the authors use matrix integrals to compute the generating function of 4-regular planar maps (which are in bijection with general maps) and obtain an asymptotic estimate for its coefficients; see [6, Equation 23] (the exponent is $-7/2$ instead of $-5/2$ because they consider unrooted maps and the growth constant is 48 instead of 12 because of a change of variables). Although some of the arguments in [6] were not fully rigorous, it opened the way to the solution of difficult enumerative problems using matrix integrals and ideas from statistical physics; see for instance [14] and [12].

Another explanation for the critical exponent $-5/2$ comes from bijections between classes of planar maps and 'decorated trees', that is, plane trees with some additional information attached to the nodes or to the edges of a tree. The first such explicit bijection was obtained by Schaeffer between planar maps and binary trees in which a half-edge (a blossom) is added to each internal node (see [19] for a recent survey), which in particular gave the first direct combinatorial interpretation of Tutte's formula for M_n. The bijection is between maps and trees 'up to conjugation', which means that a linear number of decorated trees correspond to the same map. Since the enumeration of rooted trees carries as a rule a subexponential term of the form $n^{-3/2}$ [7], dividing by a linear term gives $n^{-5/2}$. Since then many other classes of maps, such as 2-connected, bipartite, Eulerian, etc. have been shown to be in bijection with certain classes of trees up to conjugation, thus providing further justification for the universal exponent $n^{-5/2}$. Let us mention that some classes of maps, such as outerplanar maps [20], behave like trees and have a $n^{-3/2}$ term instead of $n^{-5/2}$ in the asymptotic estimates.

In order to motivate the statements that follow, let us recall the basic technique for counting planar maps. Let $M_{n,k}$ be the number of maps with n edges and in which the degree of the root-face is equal to k. Let $M(z,u) = \sum_{n,k} M_{n,k} u^k z^n$ be the associated generating function. As shown by Tutte [22], $M(z,u)$ satisfies the quadratic equation

$$M(z,u) = 1 + zu^2 M(z,u)^2 + uz\frac{uM(z,u) - M(z,1)}{u-1}. \tag{1.1}$$

In this context the variable u is usually called a "catalytic variable" (see [5, 24]). It is needed to formulate and solve the equation but is disappears if we are just interested in the generating function $M(z,1) = \sum_n M_n z^n$ related to the numbers $M_n = \sum_k M_{n,k}$ of all rooted planar maps with n edges.

It turns out that

$$M(z,1) = \sum_{n\ge 0} M_n z^n = \frac{18z - 1 + (1-12z)^{3/2}}{54z^2} = 1 + 2z + 9z^2 + 54z^3 + \cdots, \quad (1.2)$$

from which we can deduce the explicit form for the numbers M_n. The remarkable observation here is the singular part $(1-12z)^{3/2}$ that reflects the asymptotic behavior $c \cdot n^{-5/2} 12^n$ of M_n.

A general approach to equations of the form (1.1) was carried out by Bousquet-Mélou and Jehanne [5]. First one rewrites (1.1) into the form

$$P(M(z,u), M_1(z), z, u) = 0,$$

where $P(x_0, x_1, z, u)$ is a polynomial and $M_1(z)$ abbreviates $M(z,0)$ or $M(z,1)$ Next one searches for functions $f(z)$, $y(z)$ and $u(z)$ with[1]

$$\begin{aligned} P(f(z), y(z), z, u(z)) &= 0, \\ P_{x_0}(f(z), y(z), z, u(z)) &= 0, \\ P_u(f(z), y(z), z, u(z)) &= 0. \end{aligned} \quad (1.3)$$

The idea is to bind u and z in the function $G(z,u) = P_{x_0}(M(z,u), M_1(z), z, u)$ so that $G(z, u(z)) = 0$ for a proper function $u(z)$. By taking the derivative of $P(M(z,u), M_1(z), z, u)$ with respect to u one has

$$P_{x_0}(M(z,u), M_1(z), z, u) M_u(z,u) + P_u(M(z,u), M_1(z), z, u). \quad (1.4)$$

Thus, if $G(z,u(z)) = P_{x_0}(M(z,u(z)), M_1(z), z, u(z)) = 0$ then we also have $P_u(M(z,u(z)), M_1(z), z, u(z)) = 0$. This leads to the system (1.3) for $f(z) = M(z,u(z))$, $y(z) = M_1(z)$ and $u(z)$.

At this moment it is not completely clear that this procedure gives the correct solution. To show that this is the case we can argue as follows. Bousquet-Mélou and Jehanne [5] considered in particular equations of the form[2]

$$\boxed{M(z,u) = F_0(u) + zQ\left(M(z,u), \frac{M(z,u) - M(z,0)}{u}, z, u\right)}, \quad (1.5)$$

where $F_0(u)$ and $Q(\alpha_0, \alpha_1, z, u)$ are polynomials, that is

$$P(x_0, x_1, z, u) = F_0(u) + zQ(x_0, (x_0 - x_1)/u, z, u) - x_0,$$

and showed that there is a unique power series solution $M(z,u)$, and that it is also an algebraic function. The equation (1.4) is now (if we multiply by u)

$$\begin{aligned} zuQ_{\alpha_0}&\left(M(z,u), \frac{M(z,u) - M(z,0)}{u}, z, u\right) \\ &+ zQ_{\alpha_1}\left(M(z,u), \frac{M(z,u) - M(z,0)}{u}, z, u\right) - u = 0. \end{aligned} \quad (1.6)$$

Clearly this equation has a power series solutions $u(z)$ with $u(0) = 0$. Thus, the power series $f(z) = M(z,u(z))$, $y(z) = M(z,0)$, $u(z)$ solve the system (1.3).

In Section 3.1 we will rewrite the system (1.3) into an equivalent system (4.4) for $f(z)$, $u(z)$, $w(z) = (f(z) - y(z))/u(z)$. Again we get a power series solution by setting

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[1] Here and in what follows we denote by $P_x = \frac{\partial P}{\partial x}$ the partial derivative of the function P with respect to the variable x.

[2] Actually Bousquet-Mélou and Jehanne [5] considered more general functional equations that contain also higher differences.

$f(z) = M(z,u(z))$ and $w(z) = (M(z,u(z)) - M(z,0))/u(z)$. It will then shown (in Section 3.1) that this system (4.4) has unique power series solutions. Consequently these solutions coincide with $M(z,u(z))$, $u(z)$, $(M(z,u(z)) - M(z,0))/u(z)$. Hence, we get $M(z,0) = f(z) - u(z)w(z)$. Summing up, the general approach in [5] applies to obtain the solution of (1.5).

It should be also mentioned that all the examples that we will discuss can be rewritten into (almost) the form (1.5) (possibly by replacing u by $u+1$). For example, for the equation (1.1) we have

$$F_0(u) = 1 \quad \text{and} \quad Q(\alpha_0, \alpha_1, z, u) = (u+1)^2\alpha_0^2 + (u+1)\alpha_0 + (u+1)\alpha_1.$$

In the context of this paper we always assume that F_0 and Q have non-negative coefficients. This is natural since equation (1.5) can be seen as a translation of a recursive combinatorial description of maps or other combinatorial objects. This also implies that $M(z,u)$ has non-negative coefficients, since the equation (1.5) can be written as an infinite system of equations for the functions $M_j(z) = [u^j]\, M(z,u)$ with non-negative coefficients on the right hand side.

Let us consider the first case, where Q is linear in α_0 and α_1, that is, we can write (1.5) as

$$\boxed{M(z,u) = Q_0(z,u) + zM(z,u)Q_1(z,u) + z\frac{M(z,u) - M(z,0)}{u}Q_2(z,u)}. \tag{1.7}$$

Here we are in the framework of the so-called *kernel method*. We rewrite (1.7) as

$$M(z,u)(u - zuQ_1(z,u) - zQ_2(z,u)) = uQ_0(z,u) - zM(z,0)Q_2(z,u), \tag{1.8}$$

where

$$K(z,u) = u - zuQ_1(z,u) - zQ_2(z,u)$$

is the *kernel*. The idea of the kernel method is to bind u and z so that $K(z,u) = 0$, that is, one considers a function $u = u(z)$ such that $K(z,u(z)) = 0$. Then the left hand side of (1.8) cancels and $M(z,0)$ can be calculated from the right hand side by setting $u = u(z)$. Of course, the kernel equation is precisely the equation (1.6). The kernel method is just a special case of the general procedure of Bousquet-Mélou and Jehanne [5].

Proposition 1. *Suppose that Q_0, Q_1, and Q_2 are polynomials in z and u with non-negative coefficients and let $M(z,u)$ be the power series solution of (1.7). Furthermore let $u(z)$ be the power series solution of the equation*

$$u(z) = zQ_2(z,u(z)) + zu(z)Q_1(z,u(z)), \quad \textit{with } u(0) = 0.$$

Then $M(z,0)$ is given by

$$M(z,0) = \frac{Q_0(z,u(z))}{1 - zQ_1(z,u(z))}.$$

There are three particular *degenerate* cases, where the solution function $M(z,0)$ is a rational function (or even a polynomial). In these cases the asymptotic analysis of M_n is trivial:

- If $Q_0 = R_0(z)$ and $Q_1 = R_1(z)$ depend only on z, then

$$M(z,0) = \frac{R_0(z)}{1 - zR_1(z)}.$$

- If $Q_1 = R_1(z)$ depends only on z and if $Q_2 = T_0(z) + T_1(z)u$ is at most linear in u then

$$u(z) = \frac{zT_0(z)}{1 - zR_1(z) - zT_1(z)} \quad \text{and} \quad M(z,0) = \frac{Q_0(z,u(z))}{1 - zR_1(z)}$$

  are rational functions.
- If Q_2 has u as a factor, then $u(z) = 0$ and we have

$$M(z,0) = \frac{Q_0(z,0)}{1 - zQ_1(z,0)}$$

  is a rational function.

In all other cases $M(z,0)$ has universally a dominant square root singularity as our first main theorem states. We recall that $M(z,0)$ is an algebraic function and has, thus, a Puiseux expansion around its (dominant) singularity.

Theorem 1. *Suppose that Q_0, Q_1, and Q_2 are polynomials in z and u with non-negative coefficients such that none of the three above mentioned cases occurs.*

Let $M(z,u)$ be the power series solution of (1.7) and let $z_0 > 0$ denote the radius of convergence of $M(z,0)$. Then the local Puiseux expansion of $M(z,0)$ around z_0 is given by

$$M(z,0) = a_0 + a_1(1 - z/z_0)^{1/2} + a_2(1 - z/z_0) + \cdots , \tag{1.9}$$

where $a_0 > 0$ and $a_1 < 0$. Furthermore, there exist $b \ge 1$, a non-empty set $J \subseteq \{0,1,\ldots,b-1\}$ of residue classes modulo b and constants $c_j > 0$ such that for $j \in J$

$$M_n = [z^n]M(z,0) = c_j n^{-3/2} z_0^{-n} \left(1 + O\left(\frac{1}{n}\right)\right), \qquad (n \equiv j \bmod b,\ n \to \infty) \tag{1.10}$$

and $M_n = 0$ for $n \equiv j \bmod b$ with $j \notin J$ if Q_1 depends on u or at $M_n = O((z_0(1+\eta))^{-n})$ for some $\eta > 0$ and $n \equiv j \bmod b$ with $j \notin J$ if Q_1 does not depend on u.

This result is quite easy to prove (see Section 3). We just want to mention that there are variations of the above model, for example equations of the form

$$\begin{aligned} M(z,u) =& Q_0(z,u) + zM(z,u)Q_1(z,u) + z\frac{M(z,u) - M(z,0)}{u}Q_2(z,u) \\ &+ uM(z,0)Q_3(z,u), \end{aligned}$$

that can be handled in the same way; see [18]. However, the asymptotics can be slightly different. For example one might have $n^{-1/2}$ instead of $n^{-3/2}$ in the subexponential growth of M_n (namely if $z_0Q_1(z_0,u(z_0)) + u(z_0)Q_3(z_0,u(z_0)) = 1$; if $Q_3 = 0$ then we have $z_0Q_1(z_0,u(z_0)) < 1$).

In the non-linear case the situation is more involved. Here we find the solution function $M(z,0)$ in the following way.

Proposition 2. *Suppose that Q is a polynomial in α_0, α_1, z, u with non-negative coefficients that depends (at least) on α_1, that is, $Q_{\alpha_1} \ne 0$, and let $M(z,u)$ be the power series solution of (1.5). Furthermore we assume that Q is not linear in α_0 and α_1, that is, $Q_{\alpha_0\alpha_0} \ne 0$, or $Q_{\alpha_0\alpha_1} \ne 0$ or $Q_{\alpha_1\alpha_1} \ne 0$.*

Let $f(z), u(z), w(z)$ be the power series solution of the system of equations

$$\begin{aligned}
f(z) &= F_0(u(z)) + zQ(f(z), w(z), z, u(z)),\\
u(z) &= zu(z)Q_{\alpha_0}(f(z), w(z), z, u(z)) + zQ_{\alpha_1}(f(z), w(z), z, u(z)),\\
w(z) &= F_0'(u(z)) + zQ_u(f(z), w(z), z, u(z)) + zw(z)Q_{\alpha_0}(f(z), w(z), z, u(z)).
\end{aligned} \tag{1.11}$$

with $f(0) = F_0(0)$, $u(0) = 0$, $w(0) = F_0'(0)$. Then

$$M(z, 0) = f(z) - w(z)u(z).$$

The meaning of $w(z)$ will become clear later in the proof of the proposition in Section 4. In Theorem 2 we assume that $Q_{\alpha_0 u} \neq 0$, which implies that the system (1.11) is strongly connected. This means that the dependency di-graph of the system is strongly connected as discussed in Section 4.

Again there are some *degenerate* cases. We do not provide a complete list and we just discuss some of them. We also comment on the case $Q_{\alpha_1} = 0$. Given a multivariate function f we replace one of its variables with a dot if f actually does not depend on this variable.

- Suppose that $Q_u = F_0' = 0$, that is, F_0 is constant and Q does not depend on u. Here $w(z) = 0$ and consequently
$$M(z, 0) = f(z),$$
where $f(z)$ is the solution of the equation
$$f(z) = F_0 + zQ(f(z), 0, z, \cdot).$$
Thus, depending on the degree of α_0 in $Q(\alpha_0, 0, z, \cdot)$, the solution function $M(z, 0)$ is either a polynomial, a rational function, or it has a square-root singularity as in (1.9); see [1].
- Next suppose that $Q_u = Q_{\alpha_0} = 0$ but $F_0' \neq 0$. Here we have we are left with the equations
$$f = F_0(u) + zQ(w, z), \quad u = zQ_{\alpha_1}(w, z), \quad w = F_0'(u).$$
Thus, we have to solve the equation $u = zQ_{\alpha_1}(F_0'(u), z)$ to obtain $u = u(z)$ and consequently $w(z) = F_0'(u(z))$ and $f(z) = F_0(u(z)) + zQ(w(z), z)$. Hence, depending on the structure of $zQ_{\alpha_1}(F_0'(u), z)$ we obtain a polynomial, a rational function, or a square-root singularity for
$$\begin{aligned}
M(z, 0) &= f(z) - w(z)u(z)\\
&= zQ(w(z), z) + F_0(u(z)) - u(z)F_0'(u(z)).
\end{aligned}$$
- Finally, if $Q_{\alpha_1} = 0$, then we have an equation of the form
$$M(z, u) = F_0(u) + zQ(M(z, u), z, u).$$
In this case the catalytic variable u is not necessary and we can set it to 0. Hence, depending on the structure of Q we just get a polynomial, a rational function, or a square-root singularity for $M(z, 0)$ (see [1]).

We again recall that $M(z, 0)$ is an algebraic function and has, thus, Puiseux expansions around its singularities.

Theorem 2. *Suppose that Q is a polynomial in α_0, α_1, z, u with non-negative coefficients that depends (at least) on α_1, that is, $Q_{\alpha_1} \neq 0$ and let $M(z, u)$ be the power series solution of (1.5). Furthermore, we assume that Q is not linear in α_0*

and α_1*, that is,* $Q_{\alpha_0\alpha_0} \neq 0$ *or* $Q_{\alpha_0\alpha_1} \neq 0$ *or* $Q_{\alpha_1\alpha_1} \neq 0$*. We assume additionally that* $Q_{\alpha_0 u} \neq 0$.

Let $z_0 > 0$ *denote the radius of convergence of* $M(z,0)$*. Then the local Puiseux expansion of* $M(z,0)$ *around* z_0 *is given by*

$$M(z,0) = a_0 + a_2(1 - z/z_0) + a_3(1 - z/z_0)^{3/2} + O((1 - z/z_0)^2), \tag{1.12}$$

where $a_0 > 0$ *and* $a_3 > 0$.

Furthermore there exists $b \geq 1$ *and a residue class* a *modulo* b *such that*

$$M_n = [z^n]M(z,0) = c\, n^{-5/2} z_0^{-n} \left(1 + O\left(\frac{1}{n}\right)\right), \qquad (n \equiv a \bmod b,\ n \to \infty) \tag{1.13}$$

for some constant $c > 0$*, and* $M_n = 0$ *for* $n \not\equiv a \bmod b$.

The plan of the paper is as follows. In the next section we collect some examples of applying Theorems 1 and 2. We then prove Proposition 1 and Theorem 1 in Section 3, and Proposition 2 and Theorem 2 in Section 4. Finally we provide more information on the solution of catalytic equations. In particular, in Section 5 we formulate a quite general central limit theorem involving an additional parameter.

2. Examples

2.1. **The linear case.** Natural examples for the linear case (Proposition 1 and Theorem 1) come from the enumeration of lattice path. We consider paths on $\mathbb{N}^2$ starting from the coordinate point $(0,0)$ (or from $(0,t)$, $t \in \mathbb{N}$) and allowed to move only to the right (up, straight or down), but forbid going below the x-axis $y = 0$ at each step. Define a step set $\mathcal{S} = \{(a_1,b_1),(a_2,b_2),\cdots,(a_s,b_s)|(a_j,b_j) \in \mathbb{N} \times \mathbb{Z}\}$, and let $f_{n,k}$ be the number of paths ending at point (n,k), where each step is in $\mathcal{S}$. The associated generating function is then defined as

$$F(z,u) = \sum_{n,k\geq 0} f_{n,k} z^n u^k.$$

Example 1. (Motzkin Paths) We start from $(0,0)$ with step set $\mathcal{S} = \{(1,1), (1,0),(1,-1)\}$. The functional equation of its associated generating function is as follows:

$$\begin{aligned} F(z,u) &= 1 + z\left(u + 1 + \frac{1}{u}\right) F(z,u) - \frac{z}{u} F(z,0) \\ &= 1 + z(u+1)F(z,u) + z\frac{F(z,u) - F(z,0)}{u}, \end{aligned}$$

which in the notation of (1.7) corresponds to

$$Q_0(z,u) = 1, \quad Q_1(z,u) = u + 1, \quad \text{and} \quad Q_2(z,u) = 1.$$

We let $u(z)$ be the power series solution of the equation

$$u(z) = zQ_2(z,u(z)) + zu(z)Q_1(z,u(z)) = z + zu(z)(1 + u(z)),$$

that is,

$$u(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}.$$

Then $F(z,0)$ is given by

$$F(z,0) = \frac{Q_0(z,u(z))}{1 - zQ_1(z,u(z))} = \frac{1}{1 - z(1 + u(z))} = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2},$$

and

$$M_n^* = f_{n,0} = [z^n]F(z,0) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!k!(k+1)!} \sim \frac{3\sqrt{3}}{2\sqrt{\pi}} n^{-3/2} 3^n.$$

These numbers are also called "Motzkin numbers".

Example 2. We start from $(0, k_0)$ with step set $\mathcal{S} = \{(2,0),(1,-1)\}$. Here the functional equation is given by

$$\begin{aligned} F(z,u) &= u^{k_0} + (z^2 + \frac{z}{u})F(z,u) - \frac{z}{u}F(z,0) \\ &= u^{k_0} + z^2 F(z,u) + z\frac{F(z,u)-F(z,0)}{u}, \end{aligned}$$

which corresponds to

$$Q_0(z,u) = u^{k_0}, \quad Q_1(z,u) = z \quad \text{and} \quad Q_2(z,u) = 1.$$

This is actually a *degenerate* case since Q_1 and Q_2 depend only on z. Here $u(z)$ is a rational function

$$u(z) = \frac{zQ_2(z,\cdot)}{1 - zQ_1(z,\cdot)} = \frac{z}{1-z^2},$$

as well as

$$F(z,0) = \frac{Q_0(z,u(z))}{1 - zQ_1(z,u(z))} = \frac{u(z)^{k_0}}{1-z^2} = \frac{z^{k_0}}{(1-z^2)^{k_0+1}} = \sum_{\ell \ge 0} \binom{\ell + k_0}{k_0} z^{k_0 + 2\ell}.$$

Thus, we have for $n \ge k_0$, $n \equiv k_0 \bmod 2$,

$$f_{n,0} = [z^n]F(z,0) \sim \frac{n^{k_0}}{2^{k_0} k_0!}.$$

Example 3. We start again from $(0,0)$ but now with step set $\mathcal{S} = \{(2,0),$ $(1,1),(1,0)\}$, and we also assume that the step $(1,0)$ is forbidden on the x-axis $y = 0$. The functional equation in this case is

$$\begin{aligned} F(z,u) &= 1 + z(z+u+1)F(z,u) - zF(z,0) \\ &= 1 + z(z+u)F(z,u) + zu\frac{F(z,u)-F(z,0)}{u}, \end{aligned}$$

that is, we have

$$Q_0(z,u) = 1, \quad Q_1(z,u) = z + u \quad \text{and} \quad Q_2(z,u) = u.$$

Here Q_2 has u as a factor so that we are again in a *degenerate* case. We have $u(z) = 0$ and consequently

$$F(z,0) = \frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \cdots.$$

Hence we have $f_{n,0} = [z^n]F(z,0) = 1$ if n is even and 0 else.

2.2. **The non-linear case.** We collect here some examples from the enumeration of planar maps. The starting point is the classical example of all planar maps.

Example 4. Let $M(z,u)$ be the generating function of planar maps with n edges and in which the degree of the root-face is equal k. We have already mentioned that $M(z,u)$ satisfies the non-linear catalytic equation (1.1). In order to apply Proposition 2 and Theorem 2 we use the substitution $u \to u+1$ and obtain

$$M(z,u+1) = 1 + z(u+1)\Bigg((u+1)M(z,u+1)^2 + M(z,u+1) \\ + \frac{M(z,u+1) - M(z,1+0)}{u}\Bigg),$$

that is, we have $F_0(u) = 1$, and $Q(\alpha_0,\alpha_1,z,u) = (u+1)^2\alpha_0^2 + (u+1)\alpha_0 + (u+1)\alpha_1$. Here $Q_{\alpha_1} = u+1 \neq 0$, $Q_{\alpha_0,u} \neq 0$, and $Q_{\alpha_0,\alpha_0} \neq 0$, so that Theorem 2 fully applies. Of course this is in accordance with

$$M(z,1) = \sum_{n\geq 0} M_n z^n = \frac{18z - 1 + (1-12z)^{3/2}}{54z^2},$$

and

$$M_n = [z^n]M(z,1) \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n.$$

Example 5. Let $E(z,u)$ be the generating function of bipartite planar maps which satisfies the catalytic equation (see [10])

$$E(z,u) = 1 + zu^2E(z,u)^2 + u^2z\frac{E(z,u) - E(z,1)}{u^2-1}.$$

Here we use the substitution $u = \sqrt{1+v}$ and obtain

$$E(z,\sqrt{1+v}) = 1 + z(v+1)E(z,\sqrt{1+v})^2 + (v+1)z\frac{E(z,\sqrt{1+v}) - E(z,1)}{v},$$

which is of a type where Theorem 2 fully applies:

$$F(v) = 1, \quad Q(\alpha_0,\alpha_1,z,v) = \alpha_0^2(v+1) + \alpha_1(v+1).$$

Thus, we obtain

$$E_n = [z^n]\,E(z,1) \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 8^n.$$

Example 6. Let $B(z,u)$ be the generating function of 2-connected planar maps. It satisfies (see [4, 10])

$$B(z,u) = z^2u + zuB(z,u) + u(z + B(z,u))\frac{B(z,u) - B(z,1)}{u-1}.$$

After substituting u by $u+1$ we obtain

$$B(z,u+1) = z^2(u+1) + z(u+1)B(z,u+1) \\ + (u+1)(z + B(z,u+1))\frac{B(z,u+1) - B(z,1)}{u},$$

which is not exactly of the form (1.5). Nevertheless the same methods as in the proof of Theorem 2 apply – we just have to observe that the analogue of the system of equations (4.4) has proper positive power series solutions – and we obtain the same result:

$$B_n = [z^n]\, B(z,1) \sim \frac{2\sqrt{3}}{27\sqrt{\pi}} n^{-5/2} \left(\frac{27}{4}\right)^n.$$

Example 7. Let $T(z,u)$ be the generating function for planar triangulations, which satisfies (see [10, 21])

$$T(z,u) = (1 - uT(z,u)) + (z+u)T(z,u)^2 + z(1-uT(z,u))\frac{T(z,u)-T(z,0)}{u}.$$

Here $T_{n,k} = [z^n u^k]\, T(z,u)$ denotes the number of near-triangulations, that is, all finite faces are triangles, with n internal vertices and $k+3$ external vertices. In order to get rid of the negative sign we use the substitution $\widetilde{T}(z,u) = T(z,u)/(1 - uT(z,u))$ and we obtain

$$\widetilde{T}(z,u) = 1 + u\widetilde{T}(z,u) + z(1+\widetilde{T}(z,u))\frac{\widetilde{T}(z,u)-\widetilde{T}(z,0)}{u}.$$

Again this is not precisely of the form (1.5) but our methods apply once more. Note that $\widetilde{T}(z,0) = T(z,0)$. We finally get for the number $T_{n,0}$ or triangulations

$$T_{n,0} = [z^n]\, T(z,0) \sim \frac{8\sqrt{6}}{27\sqrt{\pi}} n^{-5/2} \left(\frac{256}{27}\right)^n.$$

3. Proofs of Proposition 1 and Theorem 1

3.1. **Proof of Proposition 1.** As already mentioned in the Introduction, we rewrite (1.7) as

$$M(z,u)\,(u - zuQ_1(z,u) - zQ_2(z,u)) = uQ_0(z,u) - zM(z,0)Q_2(z,u).$$

It is clear that if $u = u(z)$ satisfies

$$u = zQ_2(z,u) + zuQ_1(z,u), \tag{3.1}$$

which is the same as the equation (1.6). The *kernel* $K(z,u) = u - zuQ_1(z,u) - zQ_2(z,u)$ is then identically zero, which implies that $M(z,0)$ is given by $M(z,0) = u(z)Q_0(z,u(z))/(zQ_2(z,u(z)))$. Since $zQ_2(z,u(z)) = u(z)(1-zQ_1(z,u(z)))$, we also have

$$M(z,0) = \frac{Q_0(z,u(z))}{1 - zQ_1(z,u(z))}, \tag{3.2}$$

as claimed.

We recall that the equation (3.1) has always a unique power series solution $u = u(z)$. We add some comments on this fact. On a formal level this is immediately clear by comparing coefficients and rewriting (3.1) as a recurrence for the coefficients of $u(z)$; for instance, this implies that we always have $u(0) = 0$. Analytically we can argue in various ways. We can apply the implicit function theorem (for analytic functions) and obtain a convergent power series solution $u(z)$, provided that z is sufficiently small in modulus. Alternatively we can consider (3.1) as a fixed point equation, which is a contraction if z and u are sufficiently small. This means that the recurrence $u_0(z) = 0$, $u_{k+1}(z) = zQ_2(z,u_k(z)) + zu_k(z)Q_1(z,u_k(z))$, $k \ge 0$, has an analytic limit $u(z)$ (if z is sufficiently small in modulus). Note that this approach also implies that the coefficients of $u(z)$ are non-negative.

3.2. **Proof of Theorem 1.** Since $u(z)$ has non-negative coefficients the dominant singularity is positive and equals the radius of convergence of $u(z)$.

We do not comment on the *degenerate cases* that are discussed after Theorem 1, since the generating functions involved are only rational functions. In the non-degenerate case equation (3.1) is a non-linear positive polynomial equation for $u(z)$, that is, the right hand side of (3.1) a polynomial in z, u with non-negative coefficients. These kind of functional equations are very well studied (see. for example, [1] and [7, Theorem 2.19]) It follows that $u(z)$ has a square-root singularity at the radius of convergence z_0:

$$u(z) = u_0 + u_1(1 - z/z_0)^{1/2} + u_2(1 - z/z_0) + u_3(1 - z/z_0)^{3/2} + \cdots , \qquad (3.3)$$

where $z_0 > 0$ and $u_0 > 0$ are (uniquely) given by the system of equations

$$\begin{aligned} u_0 &= z_0Q_2(z_0, u_0) + z_0u_0Q_1(z_0, u_0), \\ 1 &= z_0Q_{2,u}(z_0, u_0) + z_0Q_1(z_0, u_0) + z_0u_0Q_{1,u}(z_0, u_0). \end{aligned}$$

It is important to note that $u_1 \neq 0$. By inserting the local expansion (3.3) into the equation (3.1) and by comparing coefficients we have (compare also with [7, Theorem 2.19])

$$u_1 = \pm\sqrt{\frac{2Q_2(z_0, u_0) + 2u_0Q_1(z_0, u_0) + 2z_0u_0Q_{1,z}(z_0, u_0)}{Q_{2,uu}(z_0, u_0) + 2Q_{1,u}(z_0, u_0) + u_0Q_{1,uu}(z_0, u_0)}}.$$

Note also that the positive sign cannot occur. It would imply asymptotically negative coefficients for $u(z)$. Therefore we actually have $u_1 < 0$.

By (3.2) and the property that $z_0Q_1(z_0, u_0) < 1$ it follows that $M(z, 0)$ has a corresponding square-root singularity:

$$y(z) = y_0 + y_1(1 - z/z_0)^{1/2} + y_2(1 - z/z_0) + y_3(1 - z/z_0)^{3/2} + \cdots ,$$

where $y_0 = M(z_0, 0) > 0$ and $y_1 < 0$.

More precisely, $u(z)$ can be represented as $u(z) = z^aU(z^b)$, where $a \geq 0$, $b \geq 1$, and $U(z)$ has also a square-root singularity at $z = z_0^{1/b}$, that is the only singularity on the circle $|z| \leq z_0^{1/b}$, compare with [1, Lemma 8]. From (3.1) it follows that $zQ_1(z, u(z))$ can be represented as $zQ_1(z, u(z)) = \tilde{Q}_1(z^b, U(z^b))$, where $\tilde{Q}_1(z, U)$ is a polynomial with non-negative coefficients. Next we represent $Q_0(z, u(z))$ as

$$Q_0(z, u(z)) = \sum_{j=0}^{b-1} z^j \overline{Q}_j(z^b, U(z^b)),$$

where $\overline{Q}_j(z, U)$ are polynomials with non-negative coefficients.

First suppose that $Q_1(z, u)$ depends on u and let J be the set of $j \in \{0, 1, \ldots, b-1\}$ for which $\overline{Q}_j \neq 0$. Hence we get

$$M(z, 0) = \sum_{j\in J} z^j \frac{\overline{Q}_j(z^b, U(z^b))}{1 - \tilde{Q}_1(z^b, U(z^b))}.$$

Clearly the functions

$$\frac{\overline{Q}_j(z, U(z))}{1 - \tilde{Q}_1(z, U(z))}$$

have a square-root singularity at $z = z_0^{1/b}$, that is the only singularity on the circle $|z| \le z_0^{1/b}$. Thus, it follows that, as $k \to \infty$,

$$[z^k]\frac{\overline{Q}_j(z,U(z))}{1-\tilde{Q}_1(z,U(z))} = \overline{c}_j z_0^{-k/b} k^{-3/2}\left(1+O\left(\frac{1}{k}\right)\right) \tag{3.4}$$

for certain positive constants $\overline{c}_j > 0$ (see [7, Corollary 2.15]). Clearly this completes the proof of Theorem 1 by considering residue classes modulo b in this case.

If $Q_1(z,u) = R_1(z)$ does not depend on u then $Q_0(z,u)$ has to depend on u. Similarly as above we have $R_1(z) = \tilde{R}_1(z^b)$ and consequently

$$M(z,0) = \sum_{j=0}^{b-1} z^j \frac{\overline{Q}_j(z^b,U(z^b))}{1-\tilde{R}_1(z^b)}.$$

Here we define J as the set of $j \in \{0,1,\ldots,b-1\}$ for which $\overline{Q}_j(z,u)$ depends on u. Hence, for $j \in J$ we get again a square-root singularity of $\overline{Q}_j(z,U(z))/(1-\tilde{R}_1(z))$ at $z = z_0^{1/b}$ and deduce an asymptotic relation as in (3.4). If $j \ne J$ then $\overline{Q}_j(z,U(z))/(1-\tilde{R}_1(z))$ is just a rational function in z (the function $U(z)$ does not appear) that has radius of convergence $> z_0^{1/b}$. This completes the proof of Theorem 1.

4. Proof of Proposition 2 and Theorem 2

4.1. **Proof of Proposition 2.** As mentioned in the Introduction, general catalytic equations can be solved with the help of the method of Bousquet-Mélou and Jehanne [5]. In our present case we set

$$P(x_0,x_1,z,u) = F_0(u) + zQ(x_0,(x_0-x_1)/u,z,u) - x_0 \tag{4.1}$$

and are looking for solutions of the equation

$$P(M(z,u),M(z,0),z,u) = 0.$$

The next step is to find functions $x_0 = f(z)$, $x_1 = y(z)$, and $u = u(z)$, that are power series in z, such that the system of equations (1.3) is satisfied, that is, we have $P = 0$, $P_{x_0} = 0$, and $P_u = 0$ (where we set $x_0 = f(z)$, $x_1 = y(z)$, and $u = u(z)$). We recall that this procedure leads to the unique power series solution $M(z,u)$ (see [5]), in particular we have $f(z) = M(z,u(z))$ and $y(z) = M(z,0)$ if the system (1.3) has unique power series solutions.

In our situation the system of equations (1.3) rewrites to

$$\begin{aligned}
f(z) &= F_0(u(z)) + zQ(f(z),(f(z)-y(z))/u(z),z,u(z)),\\
1 &= zQ_{\alpha_0}(f(z),(f(z)-y(z))/u(z),z,u(z))\\
&\quad + \frac{z}{u(z)}Q_{\alpha_1}(f(z),(f(z)-y(z))/u(z),z,u(z)), \qquad (4.2)\\
0 &= F_0'(u(z)) + zQ_u(f(z),(f(z)-y(z))/u(z),z,u(z))\\
&\quad - z\frac{f(z)-y(z)}{u(z)^2}Q_{\alpha_1}(f(z),(f(z)-y(z))/u(z),z,u(z)). \qquad (4.3)
\end{aligned}$$

We already mentioned we slightly modify this system by setting $w = w(z) = (f(z)-y(z))/u(z)$. We also multiply the second equation by $u(z)$ and replace $zQ_{\alpha_1}/u(z)$

by $1 - zQ_{\alpha_0}$ in the third equation. This leads to the equivalent system

$$\begin{aligned} f(z) &= F_0(u(z)) + zQ(f(z), w(z), z, u(z)), \\ u(z) &= zu(z)Q_{\alpha_0}(f(z), w(z), z, u(z)) + zQ_{\alpha_1}(f(z), w(z), z, u(z)), \\ w(z) &= F_0'(u(z)) + zQ_u(f(z), w(z), z, u(z)) + zw(z)Q_{\alpha_0}(f(z), w(z), z, u(z)), \end{aligned} \tag{4.4}$$

which is precisely (1.11). This is a (so-called) positive polynomial system of equations for the unknown functions $f(z)$, $w(z)$, and $u(z)$; recall that the coefficients of F_0 and Q are non-negative.

As in the case of one equation it is easy to show that the system (4.4) has unique power series solutions. Clearly we have to have $f(0) = F_0(0)$, $w(0) = F_0'(0)$, $u(0) = 0$. Then we can apply the implicit function theorem (compare also with Section 1). We can solve the system (4.4) iteratively and observe that all the coefficients of the power series $f(z)$, $u(z)$, $w(z)$ are non-negative.

Hence, $M(z, 0) = f(z) - u(z)w(z)$. This completes the proof of Proposition 2.

4.2. **Proof of Theorem 2.** Positive polynomial systems of equations are discussed in detail in [1]. In particular if the system is strongly connected then we know that there is a common dominant singularity z_0 and $f(z)$, $w(z)$, and $u(z)$ have a square root singularity at z_0 of the form (1.9):

$$\begin{aligned} f(z) &= f_0 + f_1 Z + f_2 Z^2 + f_3 Z^3 + \cdots, \\ u(z) &= u_0 + u_1 Z + u_2 Z^2 + u_3 Z^3 + \cdots, \\ w(z) &= w_0 + w_1 Z + w_2 Z^2 + w_3 Z^3 + \cdots, \end{aligned} \tag{4.5}$$

with $Z = \sqrt{1 - z/z_0}$ and where $f_1 < 0$, $u_1 < 0$ and $w_1 < 0$. The situation is analogue to a single non-linear positive equation. Thus, it follows that $M(z, 0) = f(z) - w(z)u(z)$ has the same kind of singularity:

$$y(z) = y_0 + y_1 Z + y_2 Z^2 + y_3 Z^3 + \cdots. \tag{4.6}$$

Hence, in order to prove Theorem 2 we have to show the following properties:

(1) If $Q_{\alpha_0 u} \neq 0$ then the system (4.4) is strongly connected.
(2) We have $y_1 = 0$ in the expansion (4.6).
(3) We have $y_3 > 0$ in the expansion (4.6).

With these properties the singular structure of $M(z, 0)$ at z_0 is precisely of the form (1.12). Furthermore, the asymptotics of M_n follows in the following way. Since the system (4.4) is non-linear and strongly connected we know from [1] that there exists $a_1, a_2, a_3 \geq 0$ and $b \geq 1$ such that $f(z) = z^{a_1} F(z^b)$, $w(z) = z^{a_2} W(z^b)$, $u(z) = z^{a_3} U(z^b)$, where F, W, and U have square-root singularities at $z = z_0^{1/b}$ but no other singularities on the circle $|z| \leq z_0^{1/b}$. It also follows that $M(z, 0) = z^{a_1} F(z^b) - z^{a_2 + a_3} W(z^b) U(z^b)$. If $a_1 \not\equiv a_2 + a_3 \bmod b$ then $M(z, 0)$ would have negative coefficients M_n for $n \equiv a_2 + a_3 \bmod b$ which is impossible. Thus, $a_1 \equiv a_2 + a_3 \bmod b$ and we have positive coefficients for $n \equiv a_1 \bmod b$ (if n is sufficiently large) and zero coefficients else. From $M(z, 0) = z^{a_1} \widetilde{M}(z^b)$, where $\widetilde{M}(z)$ has $z = z_0^{1/b}$ as a singularity of type (1.12) and no other singularities on the circle $|z| \leq z_0^{1/b}$, we obtain the asymptotics (1.13).

Finally we comment on the computation of z_0. Let $\mathbf{J} = \mathbf{J}(f, w, u, z)$ denote the Jacobian matrix (with derivatives with respect to f, w, u) of the right hand side of

(4.4). Then we consider the extended system of equations

$$\begin{aligned} f_0 &= F_0(u_0) + z_0 Q(f_0, w_0, z_0, u_0), \\ u_0 &= z_0 u_0 Q_{\alpha_0}(f_0, w_0, z_0, u_0) + z_0 Q_{\alpha_1}(f_0, w_0, z_0, u_0), \\ w_0 &= F_0'(u_0) + z_0 Q_u(f_0, w_0, z_0, u_0) + z_0 w_0 Q_{\alpha_0}(f_0, w_0, z_0, u_0), \\ 0 &= \det(\mathbf{I} - \mathbf{J}(f_0, w_0, u_0, z_0)), \end{aligned} \tag{4.7}$$

and search for the unique positive solution (f_0, w_0, u_0, z_0) such that the spectral radius of $\mathbf{J}(f_0, w_0, u_0, z_0)$ equals 1. This gives the correct value z_0.[3]

4.2.1. ***Strong connectedness***. Let $y_j = F_j(z, y_1, \ldots, y_d)$ a d-dimensional system of equations. The dependency di-graph of such a system consists of vertices $\{y_1, \ldots, y_d\}$ and there is an oriented edge from y_i to y_i if F_j depends on y_i, that is, $F_{j,y_i} \neq 0$. We say that the system is strongly connected if the dependency di-graph is strongly connected (see [1, 7]). In our present situation our vertex set is $\{f, u, w\}$. By assumption we have $Q_{\alpha_1} \neq 0$. Thus, there is always an edge from w to f.

Suppose first the $Q_{\alpha_0\alpha_0} \neq 0$. Then by the second equation there is an edge from f to u. By assumption we always have $Q_{\alpha_0 u} \neq 0$ which implies that there is an edge from u to w. This implies a circle $w \to f \to u \to w$ and consequently strongly connectedness.

Second suppose that $Q_{\alpha_0\alpha_1} \neq 0$ or $Q_{\alpha_1\alpha_1} \neq 0$. In this case there is certainly an edge from w to u. Furthermore, since $Q_{\alpha_0 u} \neq 0$ there is an edge from u to w and another one from f to w. This again leads to a strongly connected di-graph and completes the proof of the first assertion.

4.2.2. ***The condition*** $y_1 = 0$. In order to prove that y_1 vanishes we recall first the approach by Bousquet-Mélou and Jehanne [5]. Starting with the function $P(x_0, x_1, z, u)$ that is given by (4.1) we have to solve the system

$$\begin{aligned} P(f(z), y(z), z, u(z)) &= 0, \\ P_{x_0}(f(z), y(z), z, u(z)) &= 0, \\ P_u(f(z), y(z), z, u(z)) &= 0 \end{aligned} \tag{4.8}$$

that is precisely the system (4.2). The system (4.2) transfers into the system (4.4) by the substitution $w(z) = (f(z) - y(z))/u(z)$ and by an elementary linear operation. Thus, as long as $u(z) \neq 0$ the functional determinant of the system (4.2) (with respect to the unknown functions $f(z), w(z), u(z)$) equals zero if and only if the functional determinant of the system (4.4) (with respect to the unknown functions $f(z), y(z), u(z)$) equals zero. For the system (4.4) we know that this actually happens for $(f, w, u, z) = (f(z_0), w(z_0), u(z_0), z_0)$. Consequently the functional determinant of the system (4.8) (that is the same as (4.2)) has to be zero, evaluated at $(f(z_0), y(z_0), z_0, u(z_0))$. By (4.8) $P_{x_0} = P_u = 0$ at $(f(z_0), y(z_0), z_0, u(z_0))$. Hence,

[3] If we just consider the system of these four equations then the solution is not necessarily unique, even if we are searching for positive solutions. The key property is that 1 is the largest eigenvalue of $\mathbf{J}(f_0, w_0, u_0, z_0)$. This ensures uniqueness and allows us to compute the (common) radius of convergence z_0. This was not explicitly stated in [7, Theorem 2.33], however this was then clarified in [3] and later in [1].

we get (at $(f(z_0), y(z_0), z_0, u(z_0))$)

$$\det \begin{pmatrix} P_{x_0} & P_{x_1} & P_u \\ P_{x_0x_0} & P_{x_0x_1} & P_{x_0u} \\ P_{x_0u} & P_{x_1u} & P_{uu} \end{pmatrix} = -P_{x_1}\left(P_{x_0x_0}P_{uu} - P_{x_0u}^2\right) = 0.$$

By assumption Q_{α_1} is a non-zero polynomial with non-negative coefficients. Hence $Q_{\alpha_1} \neq 0$ at $(f(z_0), y(z_0), z_0, u(z_0))$ which implies that $P_{x_1} = -zQ_{\alpha_1}/u \neq 0$ at this point. Thus, we obtain the relation $P_{x_0x_0}P_{uu} = P_{x_0u}^2$ (at $(f(z_0), y(z_0), z_0, u(z_0))$).

We now discuss the analytic function P at the point $(f_0, y_0, z_0, u_0) = (f(z_0), y(z_0), z_0, u(z_0))$ in more detail. We already know that $P_{x_0} = 0$ (at this point). However, by differentiating (4.1) it follows that

$$P_{x_0x_0} = z_0 Q_{\alpha_0\alpha_0} + 2\frac{z_0}{u_0}Q_{\alpha_0\alpha_1} + \frac{z_0}{u_0^2}Q_{\alpha_1\alpha_1} > 0.$$

Hence by the Weierstrass preparation theorem[4] [16] it follows that P can be locally written in a unique way as

$$P(x_0, x_1, z, u) = K(x_0, x_1, z, u)\left((x_0 - G(x_1, z, u))^2 - H(x_1, z, u)\right),$$

where K, G and H are analytic function with the properties that $K(f_0, y_0, z_0, u_0) \neq 0$, $G(y_0, z_0, u_0) = f_0$, and $H(y_0, z_0, u_0) = 0$.

Since $P = 0$ if and only if $(f-G)^2 = H$ and

$$\begin{aligned} P_{x_0} &= K_{x_0}\left((f-G)^2 - H\right) + 2K(f-G), \\ P_u &= K_u\left((f-G)^2 - H\right) + K\left(-2(f-G)G_u - H_u\right) \end{aligned}$$

it follows from (4.8) that

$$H(y(z), z, u(z)) = 0 \quad \text{and} \quad H_u(y(z), z, u(z)) = 0$$

for z close to z_0. We note that this is precisely a system of equations that appears in the context of the quadratic method (see [5, 10]).

Next we will show how the singular condition $P_{x_0x_0}P_{uu} = P_{x_0u}^2$ at (f_0, y_0, z_0, u_0) translates into $H_{uu}(y_0, z_0, u_0) = 0$. Since

$$\begin{aligned} P_{x_0x_0} =& K_{x_0x_0}\left((f-G)^2 - H\right) + 4K_{x_0}(f-G) + 2K, \\ P_{uu} =& K_{uu}\left((f-G)^2 - H\right) + 2K_u\left(-2(f-G)G_u - H_u\right) \\ &+ K\left(2G_u^2 - 2(f-G)G_{uu} - H_{uu}\right), \\ P_{x_0u} =& K_{x_0u}\left((f-G)^2 - H\right) + 2K_u(f-G) + K_{x_0}\left(-2(f-G)G_u - H_u\right) \\ &+ K\left(-2G_u - 2(f-G)G_{x_0u}\right) \end{aligned}$$

it follows that we have at (f_0, y_0, z_0, u_0)

$$\begin{aligned} P_{x_0x_0} &= 2K, \\ P_{uu} &= (2G_u^2 - H_{uu})K, \\ P_{x_0u} &= -2G_uK. \end{aligned}$$

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[4]The Weierstrass preparation theorem says that every non-zero function $F(z_1, \ldots, z_d)$ with $F(0, \ldots, 0) = 0$ that is analytic at $(0, \ldots, 0)$ has a unique factorization $F(z_1, \ldots, z_d) = K(z_1, \ldots, z_d)W(z_1; z_2, \ldots, z_d)$ into analytic factors, where $K(0, \ldots, 0) \neq 0$ and $W(z_1; z_2, \ldots, z_d) = z_1^d + z_1^{d-1}g_1(z_2, \ldots, z_d) + \cdots + g_d(z_2, \ldots, z_d)$ is a so-called Weierstrass polynomial, that is, all g_j are analytic and satisfy $g_j(0, \ldots, 0) = 0$.

Consequently the condition $P_{x_0x_0}P_{uu} = P_{x_0u}^2$ at (f_0, y_0, z_0, u_0) implies $H_{uu}(y_0, z_0, u_0) = 0$.

In a similar (but much easier way) it also follows that that $P_{x_1} = -KH_{x_1}$ at (f_0, y_0, z_0, u_0). This also implies that $H_{x_1} \neq 0$ at this point since $P_{x_1} \neq 0$ (by assumption $Q_{\alpha_1} \neq 0$).

Nest we recall that $u(z)$ and $y(z) = f(z) - u(z)w(z)$ have singular (and convergent) expansions of the form

$$\begin{aligned}
u(z) &= u_0 + u_1 Z + u_2 Z^2 + u_3 Z^3 + \cdots , \\
y(z) &= y_0 + y_1 Z + y_2 Z^2 + y_3 Z^3 + \cdots ,
\end{aligned}$$

where $Z = \sqrt{1 - z/z_0}$ and $u_1 < 0$. By using the Taylor expansion of H at (y_0, z_0, u_0) and the property $H(y(z), z, u(z)) = 0$ it follows that

$$\begin{aligned}
0 =& H_{x_1}\left(y_1 Z + y_2 Z^2 + y_3 Z^3 + \cdots\right) - z_0 H_z Z^2 + \frac{1}{2} H_{x_1x_1}\left(y_1^2 Z^2 + 2y_1y_2 Z^3 + \cdots\right) \\
&+ H_{x_1u}\left(y_1u_1 Z^2 + (y_1u_2 + y_2u_1) Z^3 + \cdots\right) - z_0 H_{zu}(u_1 Z^3 + \cdots) \qquad (4.9) \\
&- z_0 H_{x_1z}(y_1 Z^3 + \cdots) + \frac{1}{6} H_{uuu}(u_1^3 Z^3 + \cdots) + \frac{1}{2} H_{x_1uu}(u_1^2 y_1 Z^3 + \cdots) \\
&+ \frac{1}{2} H_{x_1x_1u}(u_1 y_1^2 Z^3 + \cdots) + \frac{1}{6} H_{x_1x_1x_1}(y_1^3 Z^3 + \cdots) + O(Z^4).
\end{aligned}$$

By comparing coefficients of Z this implies

$$\begin{aligned}
0 =& H_{x_1} y_1, \\
0 =& H_{x_1} y_2 - z_0 H_z + \frac{1}{2} H_{x_1x_1} y_1^2 + H_{x_1u} y_1 u_1, \qquad (4.10) \\
0 =& H_{x_1} y_3 + H_{x_1x_1} y_1 y_2 + H_{x_1u}(y_1u_2 + y_2u_1) - z_0 H_{zu} u_1 - z_0 H_{yz} y_1 + \frac{1}{6} H_{uuu} u_1^3 \\
&+ \frac{1}{2} H_{x_1uu} u_1^2 y_1 + \frac{1}{2} H_{x_1x_1u} u_1 y_1^2 + \frac{1}{6} H_{x_1x_1x_1} y_1^3,
\end{aligned}$$

where the derivatives of H are evaluated at (y_0, z_0, u_0). In particular, since $H_{x_1} \neq 0$ (at (y_0, z_0, u_0)) it follows that $y_1 = 0$, which completes the proof of the second property.

4.2.3. ***The condition*** $y_3 > 0$. The idea of the proof is to relate it with an elimination procedure that leads to the singular behavior of $u(z)$. We note that it is sufficient to prof $y_3 \neq 0$. It is impossible that $y_3 < 0$. If so, we would get asymptotically negative coefficients for $y(z) = M(z, 0)$.

First we have to obtain a proper representation for y_3. By taking also into account the second and third relations from (4.10) and by using $y_1 = 0$ we get

$$\begin{aligned}
y_2 &= \frac{z_0 H_z}{H_{x_1}}, \\
y_3 &= \frac{u_1}{H_{x_1}}\left(z_0 H_{zu} - H_{x_1u} y_2 - \frac{1}{6} H_{uuu} u_1^2\right),
\end{aligned}$$

where the derivatives of H are evaluated at (y_0, z_0, u_0). Next we expand H_u locally (as we did it for H in (4.9)), compare coefficients, and obtain by using the property $H_{uu} = 0$ the additional relation

$$H_{x_1u} y_2 - z_0 H_{zu} + \frac{1}{2} H_{uuu} u_1^2 = 0. \qquad (4.11)$$

(Again all derivatives of H are evaluated at (y_0, z_0, u_0).) Hence, y_3 can be also represented as

$$y_3 = \frac{2u_1 z_0}{3H_{x_1}^2} \left(H_{x_1} H_{zu} - H_z H_{x_1 u} \right).$$

We already know that $u_1 \neq 0$ and $H_{x_1} \neq 0$. Thus it remains to show that

$$H_{x_1} H_{zu} - H_z H_{x_1 u} \neq 0, \tag{4.12}$$

where we evaluate at (y_0, z_0, u_0). This will show that $y_3 \neq 0$.

Next we show that (4.12) holds if and only if the functional determinant

$$\begin{aligned} \Delta = \begin{vmatrix} P_{x_0} & P_{x_1} & P_z \\ P_{x_0 x_0} & P_{x_0 x_1} & P_{x_0 z} \\ P_{x_0 u} & P_{x_1 u} & P_{uz} \end{vmatrix} =& - P_{x_1} \left(P_{x_0 x_0} P_{uz} - P_{x_0 z} P_{x_0 u} \right) \\ &+ P_z \left(P_{x_0 x_0} P_{u x_1} - P_{x_0 x_1} P_{u x_0} \right) \neq 0, \end{aligned} \tag{4.13}$$

where we evaluate at (f_0, y_0, z_0, u_0), so that $P_{x_0} = 0$.

Since

$$\begin{aligned} P_{x_1} =& K_{x_1} \left((f-G)^2 - H \right) + K \left(-2(f-G) G_{x_1} - H_{x_1} \right), \\ P_z =& K_z \left((f-G)^2 - H \right) + K \left(-2(f-G) G_z - H_z \right), \\ P_{x_0 x_1} =& K_{x_0 x_1} \left((f-G)^2 - H \right) + 2K_{x_1}(f-G) + K_{x_0} \left(-2(f-G) G_{x_1} - H_{x_1} \right) \\ &- 2K G_{x_1}, \\ P_{x_0 z} =& K_{x_0 z} \left((f-G)^2 - H \right) + 2K_z(f-G) + K_{x_0} \left(-2(f-G) G_z - H_z \right) - 2K G_z, \\ P_{x_1 u} =& K_{x_1 u} \left((f-G)^2 - H \right) + K_{x_1} \left(-2(f-G) G_u - H_u \right) \\ &+ K_u \left(-2(f-G) G_{x_1} - H_{x_1} \right) + K \left(2G_{x_1} G_u - 2(f-G) G_{x_1 u} - H_{x_1 u} \right), \\ P_{zu} =& K_{zu} \left((f-G)^2 - H \right) + K_z \left(-2(f-G) G_u - H_u \right) \\ &+ K_u \left(-2(f-G) G_z - H_z \right) + K \left(2G_z G_u - 2(f-G) G_{zu} - H_{zu} \right), \end{aligned}$$

it follows that we have (at (f_0, y_0, z_0, u_0))

$$\begin{aligned} P_{x_1} =& - H_{x_1} K, \\ P_z =& - H_z K, \\ P_{x_0 x_1} =& - K_{x_0} H_{x_1} - 2G_{x_1} K \\ P_{x_0 z} =& - K_{x_0} H_z - 2G_z K, \\ P_{x_1 u} =& - K_u H_{x_1} + K \left(2G_{x_1} G_u - H_{x_1 u} \right), \\ P_{zu} =& - K_u H_z + K \left(2G_z G_u - H_{zu} \right), \end{aligned} \tag{4.14}$$

for $(f,y,z,u)=(f_0,y_0,z_0,u_0)$. Thus, we have

$$\begin{aligned}
\Delta =& - P_{x_1}\left(P_{x_0x_0}P_{zu} - P_{x_0z}P_{x_0u}\right) + P_z\left(P_{x_0x_0}P_{x_1u} - P_{x_0x_1}P_{x_0u}\right)\\
=& H_{x_1}K\Big(2K\big(-K_uH_z + K\left(2G_zG_u - H_{zu}\right)\big) - 2G_uK\left(K_{x_0}H_z + 2G_zK\right)\Big)\\
&- H_zK\Big(2K\big(-K_uH_{x_1} + K\left(2G_{x_1}G_u - H_{x_1u}\right)\big) - 2G_uK\left(K_{x_0}H_{x_1} + 2G_{x_1}K\right)\Big)\\
=& H_{x_1}K\Big(2K\big(-K_uH_z - KH_{zu}\big) - 2G_uKK_{x_0}H_z\Big)\\
&- H_zK\Big(2K\big(-K_uH_{x_1} - KH_{x_1u}\big) - 2G_uKK_{x_0}H_{x_1}\Big)\\
=& -2K^2(K_u + G_uK_{x_0})(H_{x_1}H_z - H_zH_{x_1}) + 2K^3(H_{x_1}H_{zu} - H_zH_{x_1u})\\
=& 2K^3\left(H_{x_1}H_{zu} - H_zH_{x_1u}\right).
\end{aligned}$$

Together with the fact $K(f_0,y_0,z_0,u_0)\neq 0$, we proved that (4.12) holds if and only if (4.13) holds.

We recall that the system (4.8) is up to the substitution $w=(f-y)/u$ and up to an elementary operation the same as the system (4.4). More precisely the system (4.4) can be rewritten as

$$\begin{aligned}
P(f,f-uw,z,u) &= 0,\\
u\,P_{x_0}(f,f-uw,z,u) &= 0,\\
P_u(f,f-uw,z,u) + w\,P_{x_0}(f,f-uw,z,u) &= 0.
\end{aligned}\tag{4.15}$$

We just put all terms to the left hand side of the equations.

The system (4.4) is a positive strongly connected polynomial system of equations. In order to handle such systems we apply an elimination procedure. In our context we consider the first and the third equation (of (4.4) or equivalently of (4.15)) and solve them as functions $f=\overline{f}(z,u)$, $w=\overline{w}(z,u)$ in two independent variables z,u. These functions are the unique power series solutions of the first and third equation, and they have non-negative coefficients (see [1] for a thorough discussion). They also satisfy $\overline{f}(z_0,u_0)=f_0$ and $\overline{w}(z_0,u_0)=w_0=f_0-u_0y_0$. Due to the strong connectedness and the positivity properties it follows that the corresponding functional determinant (evaluated at (f_0,w_0,z_0,u_0))

$$\begin{aligned}
\Delta_1 &= \begin{vmatrix} P_{x_0}+P_{x_1} & -u\,P_{x_1}\\ P_{ux_0}+P_{ux_1}+w\,P_{x_0x_0}+w\,Px0x1 & -u\,P_{ux_1}-uw\,P_{x_0x_1}+P_{x_0}\end{vmatrix}\\
&= u\,P_{x_1}\left(P_{ux_0}+w\,P_{x_0x_0}\right)
\end{aligned}$$

is non-zero so that by the implicit function theorem the functions $f=\overline{f}(z,u)$, $w=\overline{w}(z,u)$ are analytic in a neighborhood of (z_0,u_0). Actually this can directly checked, too. We have $u_0>0$, $P_{x_1}\neq 0$, and

$$P_{ux_0}+w\,P_{x_0x_0} = zQ_{\alpha_0u} + \frac{z}{u}Q_{\alpha_1u} + zwQ_{\alpha_0\alpha_0} + \frac{zw}{u}Q_{\alpha_0\alpha_1} > 0,$$

(if we evaluate at (f_0,w_0,z_0,u_0)).

In order to solve the whole system we finally substitute these two functions into the second equation and get a final relation between u and z:

$$\overline{P}(z,u) := u\,P_{x_0}(\overline{f}(z,u),\overline{f}(z,u)-u\overline{w}(z,u),z,u) = 0,\tag{4.16}$$

or more explicitly (and rewritten into a positive equation):

$$u = zuQ_{\alpha_0}(\overline{f}(z,u),\overline{w}(z,u),z,u) + zQ_{\alpha_1}(\overline{f}(z,u),\overline{w}(z,u),z,u).$$

For this equation we have the following properties:

$$\overline{P}(z,u(z))=0,\quad \overline{P}(z_0,u_0)=0,\quad \overline{P}_u(z_0,u_0)=0$$

and

$$\overline{P}_z(z_0,u_0)>0,\quad \overline{P}_{uu}(z_0,u_0)>0.$$

The last two properties are immediate due to the positivity assumptions. (All together these properties imply that $u(z)$ has a squareroot singularity (3.3) with $u_1\neq 0$, compare also with [7, Theorem 2.19].)

In a final step we show that

$$\overline{P}_z(z_0,u_0)=u_0^2\frac{\Delta}{\Delta_1}. \tag{4.17}$$

Since we know that $\overline{P}_z(z_0,u_0)\neq 0$, $u_0\neq 0$, and $\Delta_1\neq 0$ it with then follow that $\Delta\neq 0$.

By taking the derivative of (4.16) with respect to z we get

$$\overline{P}_z=u\left((P_{x_0x_0}+P_{x_0x_1})\overline{f}_z-uP_{x_0x_0}\overline{w}_z+P_{x_0z}\right).$$

The derivatives $\overline{f}_z$, $\overline{w}_z$ can be calculated from (4.15) by implicit differentiation. In particular for $(z,u)=(z_0,u_0)$ (where we can use the property $P_{x_0}=0$) we obtain

$$\begin{aligned} f_z(z_0,u_0)&=\frac{u_0}{\Delta_1}\left(P_z(P_{x_1u}+w_0P_{x_0x_1})-P_{x_1}(P_{zu}+w_0P_{x_0z})\right),\\ w_z(z_0,u_0)&=\frac{1}{\Delta_1}\left(P_z(P_{x_0u}+P_{x_1u}+w_0P_{x_0x_0}+w_0P_{x_0x_1})-P_{x_1}(P_{zu}+w_0P_{x_0z})\right).\end{aligned}$$

This directly leads to (4.17) and completes the proof of the property $y_3\neq 0$.

5. Central Limit Theorems for Additional Parameters

Let $M(z,x,u)$ denote the generating function of rooted planar maps, where the variable z corresponds to the number of edges, x to the number of vertices and u to the root face valency. Then by the usual combinatorial decomposition of maps we have

$$M(z,x,u)=x+zu^2M(z,x,u)^2+zu\frac{M(z,x,1)-uM(z,x,u)}{1-u}.$$

Thus, for every positive x this is a catalytic equation of the form (1.5) so that Proposition 2 and Theorem 2 apply. In particular we obtain an expansion and asymptotics of the form

$$M(z,x,1)=a_0(x)+a_2(x)\left(1-\frac{z}{\rho(x)}\right)+a_3(x)\left(1-\frac{z}{\rho(x)}\right)^{3/2}+\cdots, \tag{5.1}$$

where $z=\rho(x)$ satisfies the equation

$$\begin{aligned} 768\,x^4z^4-1536\,x^3z^4-512\,x^3z^3+2304\,x^2z^4+768\,x^2z^3-1536\,xz^4\\ +96\,x^2z^2+768\,xz^3+768\,z^4-96\,xz^2-512\,z^3+96\,z^2-1=0\end{aligned}$$

with $\rho(1)=\frac{1}{12}$ and where $a_0(1)=\frac{4}{3}$, $a_2(1)=-\frac{4}{3}$, $a_3(1)=\frac{8}{3}$, and consequently

$$[z^n]\,M(z,x,1)=c(x)n^{-5/2}\rho(x)^{-n}\left(1+O\left(\frac{1}{n}\right)\right).$$

Actually all the functions $\rho(x)$, $c(x)$, and $a_j(x)$ are not only defined for positive x but extend to analytic functions around the positive real axis, and by inspection of the proof even the asymptotics can be extended to non-real x that are close to the positive real axis.

Let X_n denote the random variable equal to the number of vertices in a random planar rooted map with n edges, where each map of size n is considered to be equally likely. Then the probability generating function $\mathbb{E}[x^{X_n}]$ can be written as

$$\mathbb{E}[x^{X_n}] = \frac{[z^n]\, M(z,x,1)}{[z^n]\, M(z,1,1)} = \frac{c(x)}{c}\left(\frac{\rho(1)}{\rho(x)}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

At this stage we can apply standard tools (see [7, Chapter 2]) to obtain a central limit theorem for X_n of the form $(X_n - \mu n)/\sqrt{\sigma^2 n} \to \mathcal{N}(0,1)$, where $\mu = -\rho'(1)/\rho(1)$ and $\sigma^2 = \mu + \mu^2 - \rho''(1)/\rho(1)$. Since $\rho'(1) = -\frac{1}{24}$ and $\rho''(1) = \frac{19}{384}$ we immediately obtain $\mu = \frac{1}{2}$ and $\sigma^2 = \frac{5}{32}$. We also have $\mathbb{E}[X_n] = \mu n + O(1)$ and $\mathbb{V}\mathrm{ar}[X_n] = \sigma^2 n + O(1)$. In this special case Euler's relation and duality can be used to obtain (the even more precise representation) $\mathbb{E}[X_n] = n/2 + 1$.[5]

Actually we can easily generalize Proposition 2 and Theorem 2 in order to obtain the following central limit theorem.

Theorem 3. *Suppose that Q is a polynomial in $\alpha_0, \alpha_1, z, x, u$ with non-negative coefficients that depends (at least) on α_1, that is, $Q_{\alpha_1} \neq 0$, and $F_0(x,u)$ is another polynomial with non-negative coefficients. Let $M(z,x,u)$ be the power series solution of the equation*

$$M(z,x,u) = F_0(x,u) + zQ\left(M(z,x,u), \frac{M(z,x,u) - M(z,x,0)}{u}, z, x, u\right).$$

Furthermore assume that Q is not linear in α_0 and α_1, that is, $Q_{\alpha_0\alpha_0} \neq 0$, or $Q_{\alpha_0\alpha_1} \neq 0$ or $Q_{\alpha_1\alpha_1} \neq 0$ and additionally that $Q_{\alpha_0 u} \neq 0$.

Furthermore we denote by a and b integers with the property that $[z^n]\, M(z,1,0) > 0$ for $n \equiv a \bmod b$ and $n \geq n_0$, whereas $[z^n]\, M(z,1,0) = 0$ for $n \not\equiv a \bmod b$ (see Theorem 2).

Let X_n with $n \equiv a \bmod b$, $n \geq n_0$, be a sequence of random variables defined by

$$\mathbb{E}[x^{X_n}] = \frac{[z^n]\, M(z,x,0)}{[z^n]\, M(z,1,0)}.$$

For positive x, let $\rho(x) > 0$ denote the radius of convergence of $z \mapsto M(z,x,0)$. Then $\rho(x)$ can be extended to an analytic function around the positive real axis and we have with

$$\mu = -\frac{\rho'(1)}{\rho(1)}, \quad \sigma^2 = \mu + \mu^2 - \frac{\rho''(1)}{\rho(1)}$$

the following asymptotic moment properties:

$$\mathbb{E}[X_n] = \mu n + O(1) \quad \text{and} \quad \mathbb{V}\mathrm{ar}[X_n] = \sigma^2 n + O(1),$$

for $n \equiv a \bmod b$. Furthermore, if $\sigma^2 \neq 0$ then we also have a central limit theorem of the form

$$\frac{X_n - \mathbb{E}[X_n]}{\sqrt{\mathbb{V}\mathrm{ar}[X_n]}} \to \mathcal{N}(0,1) \qquad (n \equiv a \bmod b).$$

[5]This central limit theorem seems to be a folklore result. However, to the best of our knowledge it was first explicitly mentioned by the second author at the Alea-meeting 2010 in Luminy: `https://www-apr.lip6.fr/alea2010/` . However, it also follows from Schaeffer's bijection between pointed quadrangulations and well-labelled trees [19].

Proof. It is easy to show that, for every positive x, we can apply Proposition 2 and Theorem 2 and obtain (for $n \equiv a \bmod b$)

$$[z^n]\, M(z,x,1) = c(x)\, n^{-5/2} \rho(x)^{-n} \left(1 + O\left(\frac{1}{n}\right)\right) \tag{5.2}$$

for some positive valued function $c(x)$. Note that the error term comes from the remainder terms

$$a_4(x) \left(1 - \frac{z}{\rho(x)}\right)^2 + O\left(\left(1 - \frac{z}{\rho(x)}\right)^{5/2}\right)$$

in the singular expansion (5.1) of $M(z,x,1)$. The term $a_4(x)\,(1 - z/\rho(x))^2$ has no asymptotic contribution, wheres the other term gives rise to the error term $O(n^{-7/2}\rho(x)^{-n})$; see also [7]. This proves (5.2). Furthermore, since F_0 and Q are polynomials it follows that $\rho(x)$ is an algebraic function since it is determined by the algebraic system of equations (4.7), where we just have to add the algebraic dependence on x.

Actually it can be shown that $\rho(x)$ has no singular point for $x > 0$. As explained in the proof of Proposition 2, we can reduce the solution of the catalytic equation to a system of three positive polynomial equations. Such a system can be reduced to a single equation $u(x,z) = F(x,z,u(x,z))$ in one unknown function $u = u(x,z)$, where $u = F(x,z,u)$ has a power series expansion with non-negative coefficients (see [7]). Note that we certainly have $F_z \ne 0$ and $F_{uu} \ne 0$. The system of equations that determines the values $z = \rho(x)$ and $u = u(x,\rho(x))$, where the solution function $z \mapsto u(x,z)$ gets singular, is given by

$$u = F(x,z,u), \quad 1 = F_u(x,z,u).$$

The functional determinant of this system, when we solve it for $z = \rho(x)$ and $u = u(x,\rho(x))$, is given by

$$F_z F_{uu} - (F_u - 1) F_{uz} = F_z F_{uu} \ne 0.$$

By the implicit function theorem $z = \rho(x)$, as well as $u = u(x,\rho(x))$ are analytic. Moreover $\rho'(x) = -F_x/F_z < 0$, since $F_x > 0$ and $F_z > 0$.

By the methods of [7] it also follows that the singular expansion (4.5), where Z has to replaced by $\sqrt{1 - z/\rho(x)}$ and all coefficient functions f_j, u_j, w_j depend on x, can be extended to complex x that are sufficiently close to the positive real axis. Accordingly the asymptotic expansion (5.2) holds uniformly if x varies in a compact subset of the complex plane, where $\rho(x)$ is well defined.

As mentioned during the discussion of the example ahead of Theorem 3 an expansion of the form (5.2) is sufficient to prove the asymptotic expansion for $\mathbb{E}[X_n]$, $\mathbb{V}\mathrm{ar}[X_n]$, as well the central limit theorem (see again see [7, Chapter 2]). $\square$

We note the crucial point in the proof of Theorem 3 was to prove a singular expansion of the form (5.1) that holds in a complex neighborhood of $x = 1$. We finally add a theorem for catalytic equations, where we do not necessarily have a polynomial equation with non-negative coefficients. Again, we assume that there is an additional variable x, where we non necessarily assume that the defining catalytic equations contains only non-negative coefficients. This kind of approach was first applied in [9], where the number of faces of given valency in random planar maps was discussed; see below. (It was first stated without a proof in [11]).

Theorem 4. *Suppose that $M(z,x,u)$ and $M_1(z,x)$ are the solutions of the catalytic equation $P(M(z,x,u),M_1(z,x),z,x,u)=0$, where the function $P(x_0,x_1,z,x,u)$ is analytic and $M_1(z,1)$ has a singularity at $z=z_0$ of form*

$$M_1(z,1)=y_0+y_2\left(1-\frac{z}{z_0}\right)+y_3\left(1-\frac{z}{z_0}\right)^{3/2}+\cdots, \tag{5.3}$$

with $y_3\neq 0$ such that for $x_0=M(z_0,1,u_0)$, $x_1=M_1(z_0,1)$, $z=z_0$, $x=1$, and $u=u_0$ we have

$$P=0,\quad P_u=0,\quad P_{x_0}=0,\quad P_{x_1}\neq 0,\quad P_{x_0x_0}P_{uu}=P_{x_0u}^2. \tag{5.4}$$

Furthermore, let $z=\rho(x)$, $u=u_0(x)$, $x_0=x_0(x)$, $x_1=x_1(x)$ for x close to 1 be defined by $\rho(1)=z_0$, $u_0(1)=u_0$, $x_0(1)=M(z_0,1,u_0)$, $x_1(1)=M_1(z_0,1)$ and by the system

$$P=0,\quad P_u=0,\quad P_{x_0}=0,\quad P_{x_0x_0}P_{uu}=P_{x_0u}^2. \tag{5.5}$$

Then for x close to 1 the function $M_1(z,x)$ has a local singular representation of the form

$$M_1(z,x)=a_0(x)+a_2(x)\left(1-\frac{z}{\rho(x)}\right)+a_3(x)\left(1-\frac{z}{\rho(x)}\right)^{3/2}+\cdots \tag{5.6}$$

where the functions $a_j(x)$ are analytic at $x=1$ and satisfy $a_j(1)=y_j$.

Proof. As in the proof of Theorem 2 we can replace the (catalytic) equation $P(M(z,x,u),M_1(z,x),z,x,u)=0$ by

$$(M(z,x,u)-G(M_1(z,x),z,x,u))^2=H(M_1(z,x),z,x,u)$$

around $z=z_0$, $x=1$, $u=u_0$. In particular we have

$$H=0,\quad H_u=0,\quad H_{uu}=0,\quad H_{x_1}\neq 0$$

for $x_0=M(z_0,1,u_0)$, $x_1=M_1(z_0,1)$, $z=z_0$, $x=1$, and $u=u_0$.

In the next step we set $x=1$ and apply the methods from [10, Lemma 2] that ensure that there exist precisely two (local) solutions $u(z)$ and $y(z)$ of the system of equations

$$H(y(z),z,1,u(z))=0,\quad H_u(y(z),z,1,u(z))=0$$

with $y(z_0)=M_1(z_0,1)$ and $u(z_0)=u_0$ and with local expansions

$$u(z)=u_0\pm u_1Z+u_2Z^2\pm u_3Z^3+\cdots,\qquad y(z)=y_0+y_2Z^2\pm y_3Z^3+\cdots,$$

where $Z=\sqrt{1-z/z_0}$ (and the signs are either all positive or all negative). By assumption, one of these two solutions has to be equal to $M_1(z,1)$ which implies that $y_3\neq 0$.

By the methods of the proof of Theorem 2 we also have

$$y_3=\frac{2u_1z_0}{3H_{x_1}^2}\left(H_{x_1}H_{zu}-H_zH_{x_1u}\right).$$

Recall that $H_{x_1}\neq 0$. Thus it follows that $z_0\neq 0$, $u_1\neq 0$, and $H_{x_1}H_{zu}-H_zH_{x_1u}\neq 0$. Furthermore, since $y_2=z_0H_z/H_{x_1}$ it follows from the relation (4.11) that

$$H_{uuu}=\frac{2z_0}{u_1^2}\left(H_{x_1}H_{zu}-H_zH_{x_1u}\right)\neq 0.$$

We want to do a similar analysis for x close to 1. For this purpose we have to check whether the conditions (5.4) can be extended to x different from 1, that is, whether is is possible to solve the system (5.5) for $z=\rho(x)$, $u=u_0(x)$, $x_0=x_0(x)$,

$x_1 = x_1(x)$ (if x is close to 1). Note that the condition $P_{x_1} \neq 0$ certainly extends to a neighborhood. By the same procedure as in the proof of Theorem 2 it follows that the system (5.5) is equivalent to the system

$$H = 0, \quad H_u = 0, \quad H_{uu} = 0 \tag{5.7}$$

for $x_1 = x_1(x)$, $z = \rho(x)$, $u = u_0(x)$, Note that $x_1(x)$, $\rho(x)$, $u_0(x)$ are the same functions as above; and the function $x_0(x)$ can be recovered by $x_0(x) = M(\rho(x), x, u_0(x))$. Now the functional determinant of the system (5.7) is given by

$$\begin{vmatrix} H_{x_1} & H_{ux_1} & H_{uux_1} \\ H_z & H_{uz} & H_{uuz} \\ H_u & H_{uu} & H_{uuu} \end{vmatrix} = H_{uuu}\left(H_{x_1}H_{uz} - H_z H_{ux_1}\right)$$

which is non-zero at $x_1 = M_1(z_0, 1)$, $z = z_0$, $x = 1$, and $u = u_0$. Hence by the implicit function theorem the system (5.7) has an analytic (and unique) local solution $x_1 = x_1(x)$, $z = \rho(x)$, $u = u_0(x)$ with $x_1(1) = M_1(z_0, 1)$, $\rho(1) = z_0$, $u_0(1) = u_0$.

Summing up, we can apply the same techniques as in [10, Lemma 2] that are now valid uniformly in a small (complex) neighborhood of $x = 1$ and leads to an expansion of the form (5.6). □

Expansions of the form (5.3) or (5.6), respectively, are in particular useful if $z = z_0$ or $z = \rho(x)$ is the only singularity on the slit disc

$$\{z \in \mathbb{C} : |z| < z_0 + \varepsilon\} \setminus [z_0, \infty) \quad \text{or} \quad \{z \in \mathbb{C} : |z| < |\rho(x)| + \varepsilon\} \setminus [z_0, \infty)$$

for some $\varepsilon > 0$. In this case it follows directly that, as $n \to \infty$,

$$[z^n]\, M_1(z, 1) = \frac{3y_3}{4\sqrt{\pi}} z_0^{-n} n^{-5/2} \left(1 + O\left(\frac{1}{n}\right)\right)$$

or

$$[z^n]\, M_1(z, x) = \frac{3a_3(x)}{4\sqrt{\pi}} \rho(x)^{-n} n^{-5/2} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Thus, if

$$M_1(z, x) = \sum_{n \ge 0} \mathbb{E}[x^{X_n}] \cdot [z^n]\, M_1(z, 1) \cdot z^n$$

encodes the distribution of a sequence of random variables X_n it follows that

$$\mathbb{E}[x^{X_n}] = \frac{[z^n]\, M_1(z, x)}{[z^n]\, M_1(z, 1)} = \frac{a_3(x)}{y_3} \left(\frac{z_0}{\rho(x)}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

and we obtain a central limit theorem by standard tools (see [7] and the above discussion).

Example 8. Let $k \ge 2$ be a fixed integer and let $M(z, x, u)$ be the ordinary generating function enumerating rooted planar maps, where z corresponds to the number of edges, x to the number of non-root faces of degree k, and u to the root-face degree. In [9, Lemma 2] is was shown that $M(z, x, u)$ satisfies the equation

$$\begin{aligned} M(z, x, u)\left(1 - z(x-1)u^{-k+2}\right) =& 1 + zu^2 M(z, x, u) + zu \frac{uM(z, x, u) - M(z, x, 1)}{u - 1} \\ & - z(x-1)u^{-k+2} G(z, x, M(z, x, 1), u), \end{aligned}$$

where $G(z, x, y, u)$ is a polynomial of degree $k - 2$ in u with coefficients that are analytic functions in (z, x, y) for $|z| \le 1/10$, $|x - 1| \le 2^{1-k}$, and $|y| \le 2$. It should be noted that the function G is not explicitly given but is (one of) the solution(s) of in infinite system of equations that can be solved with the help of Banach's fixed point theorem.

Clearly, $M(z,1,u)$ is just the usual planar map counting generating function for which we know that $M(z,1,1)$ is explicitly given by (1.2) so that all assumptions of Theorem 4 are satisfied. Alternatively we could have used Theorem 2 to obtain the local expansion of $M(z,1,1)$. Furthermore a central limit theorem follows, where X_n is just the number of non-root faces of valency k in a random planar map with n edges. (This is also one of the main results of [9].)

Example 9. We say that a face is a pure k-gon ($k \ge 2$) if it is incident exactly to k different edges and k different vertices. let $P(z,x,u)$ be the ordinary generating function enumerating rooted planar maps, where z corresponds to the number of edges, x to the number of non-root faces that are pure k-gons, and u to the root-face degree. Similarly to the previous case it can be shown (see [23]) that $P(z,x,u)$ satisfies an equation of the form

$$\begin{aligned} P(z,x,u) =& 1 + zu^2 P(z,x,u) + zu\frac{uP(z,x,u) - P(z,x,1)}{u-1} \\ & - z(x-1)u^{-k+2}\tilde{G}(z,x,P(z,x,1),u), \end{aligned}$$

where $\tilde{G}(z,x,y,u)$ is a polynomial of degree $k-2$ in u with coefficients that are analytic functions in (z,x,y) for $|z| \le 1/10$, $|x-1| \le 2^{1-k}$, and $|y| \le 2$.

Again, if we set $x = 1$ we recover $M(z,u) = P(z,1,u)$ so that all assumptions of Theorem 4 are satisfied. Hence, for fixed $k \ge 2$, the number of pure k-gons in a random planar map satisfies a central limit theorem.

Example 10. A planar map is simple if is has no loops and no multiple edges. The corresponding generating function $S(z,u)$ (where z corresponds to the number of edges and u to the root face valency) satisfies the catalytic equation (see [23])

$$\begin{aligned} S(z,u) =& 1 + zu^2 S(z,u)^2 + zu\frac{uS(z,u) - S(z,1)}{u-1} \\ & - zuS(z,u)S(z,1) - (S(z,u)-1)(S(z,1)-1) \end{aligned}$$

and the solution $S(z,1)$ is explicitly given by

$$S(z,1) = \frac{1 + 20z - 8z^2 + (1-8z)^{3/2}}{2(z+1)^3}.$$

Let $k \ge 2$ be a fixed integer and let $S(z,x,u)$ be the ordinary generating function enumerating simple rooted planar maps, where z corresponds to the number of edges, x to the number of non-root faces of degree k, and u to the root-face degree.

In [23] is was shown that $S(z,x,u)$ satisfies the equation

$$\begin{aligned}S(z,x,u) =& 1 + zu^2 S(z,x,u) + zu\frac{uS(z,x,u)-S(z,x,1)}{u-1}\\ &- zuS(z,x,u)S(z,x,1) - (S(z,x,u)-1)(S(z,x,1)-1)\\ &+ (x-1)\Bigg(zu^{-k+2}S(z,x,u)G_1(z,x,S(z,x,1),u)\\ &\qquad - zuS(z,x,u)G_2(z,x,S(z,x,1))\\ &\qquad - (S(z,x,u)-1)G_3(z,x,S(z,x,1))\Bigg),\end{aligned}$$

where $G_1(z,x,y,u)$ is a polynomial of degree $k-2$ in u with coefficients that are analytic functions in (z,x,y) for $|z| \le 2/25$, $|x-1| \le 2^{-k-5}$, and $|y-1| \le 2/5$. Similarly the functions $G_2(z,x,y)$ and $G_3(z,x,y)$ are analytic functions in (z,x,y) for $|z| \le 2/25$, $|x-1| \le 2^{-k-5}$, and $|y-1| \le 2/5$.

Again all assumptions of Theorem 4 are satisfied. Hence, for fixed $k \ge 2$, the number of faces of valency k in a random simple planar map satisfies a central limit theorem.

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