Combinatorics of Continuants of Continued Fractions with 3 Limits

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Abstract

We give combinatorial descriptions of the terms occurring in continuants of general continued fractions that diverge to three limits. Equating this combinatorics with the usual combinatorial description due to Euler induces nontrivial identities. Special cases and applications to counting sequences are given.

Keywords: Divergent continued fractions, Continuants, Euler-Minding Theorem, Linear recurrences, Integer partitions 2010 MSC: 05A19, 11B39, 40A15

1. Overview

Research on divergent continued fractions usually occurs in the study of analytic continued fractions. Meanwhile, combinatorial aspects of continued fractions are typically studied in in the field of enumerative combinatorics. In this paper we bring the two subjects together and give a combinatorial description of the continuants of a general class of continued fractions that diverge to three limits. This class was previously studied from the analytic point of view by the first author [5]. We are able to relate our combinatorially described polynomials to the classical continuant polynomials going back to Euler. This yields identities that have a flavor similar to the identities between different bases of symmetric polynomials in as much as there is considerable cancellation occurring between the monomials on one side, but not the other.

As usual we write a continued fraction:

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$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}$$

with the more compact notation

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots$$
 (1)

The kth classical numerator A_k , and kth classical denominator B_k , of the continued fraction (1) are the respective numerator and denominator when the finite continued fraction

$$\frac{A_k}{B_k} = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots + \frac{a_k}{b_k}$$

is simplified in the usual way. The polynomials $A_k = A_k(a_1, \ldots, a_k; b_0, \ldots, b_k)$ are also known as *continuants*. Since $B_k = A_{k-1}(a_2, \ldots, a_k; b_1, \ldots, b_k)$, it suffices to consider just the sequence A_k .

1.1. Continuants

A combinatorial description for the terms of polynomials A_k was first given in 1764 by Euler [8] in the case where $a_i = 1$, for $1 \le i \le k$. The case where $b_i = 1$, for $0 \le i \le k$ was considered by Sylvester [18] in 1854. The general case was finally given by Minding [11] in 1869. See also Chrystal [6] and Muir [12].

This description is simplest in the special case when the indeterminates b_i are set equal to unity. There is really no loss of generality due to the simple identity

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_k}{b_k} = b_0 \left(1 + \frac{a_1/b_0b_1}{1} + \frac{a_2/b_1b_2}{1} + \dots + \frac{a_k/b_{k-1}b_k}{1} \right).$$

Euler's combinatorial description [8] is sometimes referred to by the terms *Euler* brackets or *Euler's* rule; see, for example, Davenport [7] or Roberts [14]. In any event the resulting theorem is known as the Euler-Minding Theorem.

Theorem 1 (Euler-Minding Theorem, Sylvester's form). The classical numerators and denominators of

$$1 + \frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \dots$$
 (2)

are given by

$$A_{k} = 1 + \sum_{\substack{k \ge h_{1} >^{2} h_{2} >^{2} \dots >^{2} h_{\ell} \ge 1}} a_{h_{1}} a_{h_{2}} \cdots a_{h_{\ell}},$$
(3)

and

$$B_k = 1 + \sum_{\substack{k \ge h_1 > 2 \\ \ell \ge 1}} a_{h_1} a_{h_2} \cdots a_{h_\ell}, \tag{4}$$

where $i >^2 j$ means i and j have minimal difference 2; $i \ge j + 2$.

Thus the monomials in A_k and B_k are described by sequences h_i of the form

$$k \ge h_1 >^2 h_2 >^2 \dots >^2 h_\ell.$$

We call a sequence satisfying this inequality chain a *minimal difference* 2 *sequence*.

Note that when $k \to \infty$ limits for A_k and B_k exist in the ring of formal power series over the monoid generated by the indeterminates a_i . As we will soon see, this does not necessarily hold for other continued fractions with indeterminate elements.

1.2. Divergent Continued Fractions with Multiple Limits

Apparently, the first theorem on continued fractions that diverge to multiple limits is that of Stern and Stolz [10, 16, 17]:

Theorem 2 (Stern-Stolz). Let the complex sequence $\{b_i\}$ satisfy $\sum |b_i| < \infty$. Then

$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdot$$

diverges. In fact, for $p \in \{0, 1\}$, $\lim_{n \to \infty} A_{2n+p} = C_p \in \mathbb{C}$, and $\lim_{n \to \infty} B_{2n+p} = D_p \in \mathbb{C}$.

The proof of the Stern-Stolz Theorem goes over into the formal power series setting and the conclusion is that limiting formal power series exist for the limits described in the theorem: inspection of the recurrence $A_k = b_k A_{k-1} + A_{k-2}$ shows that it converges for k in the residue classes modulo 2, and the same is true of the sequence B_k , since it satisfies the same recurrence. That the limits are distinct follows from the determinant formula $A_k B_{k-1} - A_{k-1} B_k = (-1)^{k+1}$.

Bowman and McLaughlin [5] established the following result on continued fractions which diverge to three limits as an example of a more general theorem on continued fractions which diverge to any finite number of limits.

Let K be defined to be the following general continued fraction

$$K := b_0 + \frac{-1+a_1}{1+b_1} + \frac{-1+a_2}{1+b_2} + \frac{-1+a_3}{1+b_3} + \cdots$$
(5)

Because we will be interested in giving a combinatorial description for the terms of the continuants of K, we designate its classical numerators and denominators by P_k and Q_k , respectively, to distinguish them from the corresponding polynomials associated with (1). With this notation, the result from [5] of interest is the following theorem.

Theorem 3 (Example 1i from [5]). Let the complex sequences a_i and b_i satisfy $a_i \neq 1$ for $i \geq 1$, and $\sum |a_i| + |b_i| < \infty$. For j = 1, 2, 3,

$$\lim_{n \to \infty} P_{6n+j} = -\lim_{n \to \infty} P_{6n+j+3} = C_j \neq \infty, \tag{6}$$

$$\lim_{n \to \infty} Q_{6n+j} = -\lim_{n \to \infty} Q_{6n+j+3} = D_j \neq \infty.$$
(7)

In fact, for $j \in \{1, 2, 3\}$, K diverges to three limits given by

$$\lim_{\substack{k \to \infty \\ k \equiv j \pmod{3}}} \frac{P_k}{Q_k}$$

Our main result, Theorem 18, which gives a combinatorial description for the terms of the continuants of (5), shows the existence of the limits C_j and D_j as formal power series. (This can also be seen directly from (21) and (23) below.)

1.3. Partition Applications

Putting $a_i = q^i$ in Theorem 1 gives that the Rogers-Ramanujan integer partition identities,

The number of partitions of n into parts with minimal difference two equals the number of partitions of n into parts congruent to 1 or 4 modulo 5.

The number of partitions of n into parts greater than 1 with minimal difference two equals the number of partitions of n into parts congruent to 2 or 3 modulo 5.

are equivalent to the single identity,

$$1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} + \dots \stackrel{\circ}{=} \frac{\left(\prod_{j=1}^{\infty} \frac{1}{(1 - q^{5j+1})(1 - q^{5j+4})}\right)}{\left(\prod_{j=1}^{\infty} \frac{1}{(1 - q^{5j+2})(1 - q^{5j+3})}\right)},$$

where $\stackrel{\circ}{=}$ indicates that the limiting classical numerator and denominator of the continued fraction on the left are equal as formal power series in q to the numerator and denominator on the right.

Thus, a combinatorial description for the terms of the continuants of continued fraction K, in the case where $a_i = 0$, will give a partition interpretation to the limiting classical numerator and denominator (in residue classes modulo 6) of Ramanujan's amazing continued fraction with 3 limits [3, 4]:

$$\lim_{k \to \infty} \frac{1}{1 + \frac{-1}{1 + q}} + \frac{-1}{1 + q^2} + \dots + \frac{-1}{1 + q^{3k+j}} = \omega^2 \left(\frac{\Omega - \omega^{j+1}}{\Omega - \omega^{j-1}}\right) \prod_{m=0}^{\infty} \frac{(1 - q^{3m+2})}{(1 - q^{3m+1})},$$
(8)

where $\omega = e^{2\pi i/3}, j \in \{0, 1, 2\}$, and

$$\Omega = \prod_{p=1}^{\infty} \frac{(1-\omega^2 q^p)}{(1-\omega q^p)}.$$

It follows that when the corresponding products on the right hand have been given interpretations as partition generating functions, one obtains partition identities which are equivalent (via the description of terms for K's continuants) to Ramanujan's three-limit continued fraction. This will be attained in a sequel, and was one of the chief motivations for the present paper.

To state the problem solved in this paper most succinctly, we give a combinatorial description for the terms of the polynomials P_k , defined recursively in the non-commutative indeterminates a_i and b_i by:

$$P_k = (-1 + a_k)P_{k-2} + (1 + b_k)P_{k-1},$$

with initial conditions $P_0 = b_0$ and $P_1 = -1 + b_0 + a_1 + b_1 b_0$.

1.4. Results

This paper studies a number of new and interrelated sequences of polynomials whose terms are described combinatorially. These sequences of polynomials are of two types. The first arise from the classical Euler-Minding Theorem; they exhibit a modulo two or four behavior as a function of their index. The second arise from the sequence P_k ; these exhibit a modulo six behavior. The terms of P_k are characterized by Theorem 18, which is the main result of this paper. Equalities are induced between the two types because the continued fraction (1)can be transformed into (5) by making the change of variables $a_i \mapsto -1 + a_i$ and $b_i \mapsto 1 + b_i$, for $i \ge 1$. This results in non-trivial identities, since the sum in the non-commutative version of the Euler-Minding Theorem (see Section 2.1) now has intensive sieving occurring, while the polynomials on the other side are expressed in terms of their monomials. Important special cases arise when either the variables a_i or b_i vanish. For the continued fraction K, this results in the polynomial sequences C_k , D_k , G_k , and H_k introduced in Section 3. Section 4 examines the resulting polynomial identities and also gives applications to common second order linear recurrence sequences of integers. In a future paper we will apply Corollary 20 of Theorem 18 to find integer partition identities equivalent to (8).

The simplest example of our results is perhaps the following, which comes from Corollaries 22 and 24:

$$-\frac{2\sqrt{3}}{3}\operatorname{Im}\left(e^{k\pi i/3}\right) = \sum_{k \ge \lambda_1 >^2 \lambda_2 >^2 \dots >^2 \lambda_\ell = 1} (-1)^\ell = -\chi_1(k) + \sum_{\lambda \in \mathbf{D}_k} (-1)^{\frac{k-\ell+1}{2}},$$
(9)

where $\chi_1(k)$ is the nonprincipal Dirichlet character modulo 4, and \mathbf{D}_k is the set of finite integer sequences (depending on k) satisfying,

D1 $k \ge \lambda_1 > \lambda_2 > \cdots > \lambda_\ell \ge 2.$

D2 $\lambda_1 \equiv k \pmod{2}$.

D3 $\lambda_j \not\equiv \lambda_{j-1} \pmod{2}$.

D4 $\lambda_{\ell} \equiv 0 \pmod{2}$.

The first expression in (9) indicates a six-fold pattern in the integer sequences given by the sums, although from superficial appearances of the sums, one might expect a two-fold or four-fold pattern. The interpretation of the first equality is beautiful and surprising:

Let \mathbf{C}_k denote the set of increasing sequences of positive integers of minimal difference two, with first term 1 and largest term less than or equal to k. Then the number of elements of \mathbf{C}_k of even length minus the number of elements of odd length is given by the six-periodic integer sequence $0, -1, -1, 0, 1, 1, \ldots$, where the first element of the sequence is indexed by k = 0.

In Section 4.1 we give a simple proof of this result which is independent of the more general theory developed in this paper.

Finally, when a decreasing sequence λ_i satisfies condition **D3** above, we say that it is an *alternating parity sequence*. Partitions formed from sequences of such parts have been studied by Andrews [1, 2]. It is easy to show that these kinds of partitions arise naturally from Euler's combinatorial description of the continuants of (1) in the case $a_i = 1$ and $b_i = q^i$. In Section 3 *alternating triality sequences* arise, which are similar, except the congruence conditions on the successive terms are modulo three, instead of two.

2. Preliminaries and Lemmas

2.1. Continued Fractions with Noncommuting indeterminates

The fundamental recurrence formulas for the classical numerators and denominators of continued fractions are used for typical proofs of the Euler-Minding Theorem and they are used to prove Theorem 18. These recurrences state that for $k \ge 1$,

$$A_k = a_k A_{k-2} + b_k A_{k-1}, (10)$$

and

$$B_k = a_k B_{k-2} + b_k B_{k-1}, (11)$$

where $A_{-1} = 1$ and $B_{-1} = 0$. Recurrence formulas with left or right multiplication by noncommuting indeterminates have been considered since at least 1913 [19]. The convention of writing parts of partitions in descending order motivates us to consider recurrences (10) and (11) with noncommuting indeterminates. In this context we speak of the continued fraction (1) as having noncommuting indeterminates; we define the *classical numerators* and *denominators* as the respective sequences of polynomials in noncommutative indeterminates satisfying equations (10) and (11), with initial conditions $A_0 = b_0$, $A_1 = b_1b_0 + a_1$, $B_0 = 1$, and $B_1 = b_1$. Each classical numerator, A_k , and classical denominator B_k is an element of the monoid ring $\mathbb{Z}[\mathcal{M}]$, where \mathcal{M} is the monoid generated by $\{a_{j+1}, b_j\}_{j\geq 0}$ with identity ϵ . The integers are isomorphic to the subring $\mathbb{Z}\epsilon$ of $\mathbb{Z}[\mathcal{M}]$; we abuse 1ϵ , as usual, by writing it simply as 1. The product in \mathcal{M} is denoted by concatenation. Definition 1 provides terminology and notation for $\mathbb{Z}[\mathcal{M}]$ and its elements.

Definition 1. We call the elements P of $\mathbb{Z}[\mathcal{M}]$ polynomials. We write P in the form

$$P = \sum_{m \in \mathcal{M}} c_m m, \tag{12}$$

where $c_m \in \mathbb{Z}$ and all but finitely many c_m are zero. The support of P, denoted by $\operatorname{supp}(P)$, is the set

$$\operatorname{supp}(P) = \{ m \in \mathcal{M} : c_m \neq 0 \}.$$

We write

$$P = \sum_{m \in \text{supp}(P)} c_m m \tag{13}$$

to keep polynomial sums finite. We call the elements of $\operatorname{supp}(P)$ the monomials of P, and for a monomial m of P, we call $c_m m$ a term of P. So here, monomials do not have integer coefficients, while terms do. We call the coefficient of the identity ϵ in (12) (not (13), since it may be that $\epsilon \notin \operatorname{supp}(P)$) the constant of P. Thus the constant of P can be zero.

Since the goal is to give combinatorial descriptions for the terms of classical numerator and denominator polynomials of K, we employ vectors whose components are indices of the elements of the support of these polynomials. In the sequel and throughout, we display the components of an ℓ -dimensional vector λ as $[\lambda_1, \lambda_2, \ldots, \lambda_\ell]$. Definition 2 below defines vectors directly related to the monomials of a given $P \in \mathbb{Z}[\mathcal{M}]$. For the definition, we use the noncommutative product notation inductively defined for $n \geq 1$ by

$$\prod_{j=1}^{n} d_i = d_1 \prod_{j=1}^{n-1} d_{j+1},$$

and the empty product is ϵ as usual.

Definition 2. Let *m* be a monomial of $P \in \mathbb{Z}[\mathcal{M}]$,

$$m = \prod_{j=1}^{\ell} y_j,$$

where $y_j \in \{a_{i+1}, b_i\}_{i \ge 0}$. We denote the *degree* or *length* of the monomial m by $\ell = \ell(m)$; we usually suppress the dependence of ℓ on m.

- (i) The *index of m* is the vector $\boldsymbol{\lambda}(m) = [\lambda_1, \lambda_2, \dots, \lambda_\ell]$, where $y_j = a_u$ implies $\lambda_j = u$ and $y_j = b_u$ implies $\lambda_j = u$.
- (ii) The *a*-index of m is the vector $\boldsymbol{\alpha}(m) = [\alpha_1, \alpha_2, \dots, \alpha_\ell]$, where

$$\alpha_j = \begin{cases} u & \text{if } y_j = a_u, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) The *b*-index of m is the vector $\beta(m) = [\beta_1, \beta_2, \dots, \beta_\ell]$, where

$$\beta_j = \begin{cases} u & \text{if } y_j = b_u \\ 0 & \text{otherwise.} \end{cases}$$

Note that for a monomial m the index of m is the sum of the a-index and b-index: $\lambda(m) = \alpha(m) + \beta(m)$.

Example 1. The monomial $a_6b_4b_3b_2a_1$ has index [6, 4, 3, 2, 1]. It has *a*-index [6, 0, 0, 0, 1] and *b*-index [0, 4, 3, 2, 0]. Monomial $b_5a_4b_2b_0$ has *a*-index [0, 4, 0, 0], *b*-index [5, 0, 2, 0], and index [5, 4, 2, 0].

By a *formal power series* we mean an element of the monoid ring $\mathbb{Z}[[\mathcal{M}]]$, that is, an expression of the form

$$c = \sum_{m \in \mathcal{M}} c_m m,$$

where now we do not require all but finitely many c_m to be 0. Addition and multiplication are defined as usual.

Before studying K we derive the noncommutative description of the terms of the continuants of the general continued fraction (1).

2.2. A Noncommutative Euler-Minding Theorem

Minding [11] seems to have been the first to give the following slightly more general version of Euler's result [8]. See also [13].

Theorem 4 (Euler-Minding Theorem). The classical numerators and denominators of the continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots$$
(14)

in commutative indeterminates $\{a_{j+1}, b_j\}_{j\geq 0}$ are given by

$$A_{k} = b_{k}b_{k-1}\cdots b_{1}b_{0}\left[1 + \sum_{1 \le h_{j} <^{2}h_{j-1} <^{2}\dots <^{2}h_{1} \le k} \frac{a_{h_{1}}a_{h_{2}}\cdots a_{h_{j}}}{b_{h_{1}}b_{h_{1}-1}b_{h_{2}}b_{h_{2}-1}\cdots b_{h_{j}}b_{h_{j}-1}}\right],$$
(15)

$$B_{k} = b_{k}b_{k-1}\cdots b_{1} \left[1 + \sum_{2 \le h_{j} <^{2}h_{j-1} <^{2}\dots <^{2}h_{1} \le k} \frac{a_{h_{1}}a_{h_{2}}\cdots a_{h_{j}}}{b_{h_{1}}b_{h_{1}-1}b_{h_{2}}b_{h_{2}-1}\cdots b_{h_{j}}b_{h_{j}-1}} \right].$$
(16)

Note that this theorem does not immediately give a description for the terms for each continunant since the terms are rational, not monomial. But this is easy to remedy.

Theorem 4 expresses A_k and B_k as rational functions in commuting indeterminates. One obtains the noncommutative version by multiplying through by the *b*-product in front, canceling, and then ordering the terms so that the indices from left to right are decreasing; the construction of the terms in the sum guarantees that the indices are distinct, so no ambiguity between, say $a_i b_i$ and $b_i a_i$ can occur. For the classical numerators, (10) must be satisfied along with the initial conditions $A_0 = b_0$ and $A_1 = b_1 b_0 + a_1$. Induction on (10) gives that A_k is a polynomial in the indeterminates $\{a_{j+1}, b_j\}_{j=0}^k$. Since (10) introduces the new indeterminates a_k and b_k by left multiplication, the indices of the terms of the classical numerators are in descending order. Therefore, the result of expanding each summand of (15) and putting the indices into descending order satisfies (10) with noncommuting indeterminates. Thus,

$$A_{k} = \prod_{t=0}^{k} b_{k-t} + \sum_{\substack{1 \le h_{j} <^{2} h_{j-1} <^{2} \dots <^{2} h_{1} \le k \\ j \ge 1}} \prod_{t=0}^{k-h_{1}-1} b_{k-t} \times \prod_{u=1}^{j} \left(a_{h_{u}} \prod_{v=2}^{h_{u}-h_{u+1}-1} b_{h_{u}-v} \right).$$
(17)

A summand appearing in the second term of (17) has the form

$$b_k b_{k-1} \cdots b_{h_1+1} \times (a_{h_1} b_{h_1-2} b_{h_1-3} \cdots b_{h_2+1}) (a_{h_2} b_{h_2-2} \cdots b_{h_3+1}) \cdots (a_{h_j} b_{h_j-2} \cdots b_0).$$

Observe that the largest index is k and the indices are distinct nonnegative integers. When an *a*-index is equal to some h_j , the next index is $h_j - 2$, since the next index is either *b*-index $h_j - 2$ or *a*-index $h_{j+1} = h_j - 2$. When the index is some *b*-index $h_i - s$, the next index is $h_i - s - 1$, since the next index is either *a*-index $h_{i-1} = h_i - s - 1$ or *b*-index $h_i - s - 1$. Finally, the last index is either zero or one. The last index is a *b*-index zero when $h_j > 1$, and it is the *a*-index 1 when $h_j = 1$.

It is now easy to describe the subset of monomials of \mathcal{M} occurring in the noncommutative Euler-Minding Theorem: let \mathcal{A}_k be the set of monomials with *a*-index α , *b*-index β , and index $\lambda = \alpha + \beta$ satisfying the following properties.

- **A1** $k = \lambda_1 > \lambda_2 > \cdots > \lambda_\ell \ge 0.$
- **A2** If $\lambda_j = \alpha_j$, then $\lambda_{j+1} = \lambda_j 2$.

and

A3 If $\lambda_j = \beta_j$, then $\lambda_{j+1} = \lambda_j - 1$.

A4 Either $\lambda_{\ell} = \beta_{\ell} = 0$ or $\lambda_{\ell} = \alpha_{\ell} = 1$.

It is clear that A1-A4 describe the terms of (17).

For example, $\mathcal{A}_0 = \{b_0\}$, and $\mathcal{A}_1 = \{b_1b_0, a_1\}$. Indeed, the index of any element of \mathcal{A}_0 has $\lambda_1 = 0$ by **A1**. The only possible a and b indices are each [0]. These vectors satisfy **A1–A4**, so $\mathcal{A}_0 = \{b_0\}$. Also $\mathcal{A}_1 = \{b_1b_0, a_1\}$; the index of any element of \mathcal{A}_1 has $\lambda_1 = 1$ by **A1**. So, the possible indices are [1,0] and [1]. By **A2** the vector [1,0] cannot be an a-index. The monomial b_1b_0 with a-index [0,0] and b-index [1,0] satisfies **A1–A4**. Thus, b_1b_0 is in \mathcal{A}_1 . By **A4** the vector [1] is not a b index. The monomial a_1 with a-index [1] and b-index [0] satisfies **A1–A4**. Thus, a_1 is in \mathcal{A}_1 , and $\mathcal{A}_1 = \{b_1b_0, a_1\}$.

It is not hard to show that b_0 is a term of A_k if and only if k is even and that a_1 is a term of A_k if and only if k is odd. Further it can be shown, although we don't take it up here, that $\lim_{k\to\infty} A_{2k}$ and $\lim_{k\to\infty} A_{2k+1}$ exist and are distinct in $\mathbb{Z}[[\mathcal{M}]]$.

Theorem 5 (Noncommutative Euler-Minding Theorem). The classical numerators of the continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots$$

in noncommutative indeterminates $\{a_{i+1}, b_i\}_{i>0}$ for $k \ge 0$ are given by

$$A_k = \sum_{m \in \mathcal{A}_k} m. \tag{18}$$

Proof. As explained in the paragraph following Theorem 4, ordering the resulting subscripts in descending order and canceling the b_j s in (15) gives (18).

2.3. Lemmas

Let $P_k(a_1, a_2, \ldots, a_k; b_0, b_1, b_2, \ldots, b_k)$ and $Q_k(a_2, \ldots, a_k; b_1, b_2, \ldots, b_k)$ be the kth classical numerators and denominators of the continued fraction

$$K = b_0 + \frac{-1+a_1}{1+b_1} + \frac{-1+a_2}{1+b_2} + \frac{-1+a_3}{1+b_3} + \cdots,$$

where indeterminates $\{a_{j+1}, b_j\}_{j\geq 0}$ are noncommutative. By the fundamental recurrence formulas (10) and (11), the classical numerators and denominators of K satisfy

$$X_k = (-1 + a_k)X_{k-2} + (1 + b_k)X_{k-1},$$

with initial conditions $P_0 = b_0$, $Q_0 = 1$, $P_1 = -1 + b_0 + b_1 b_0 + a_1$, and $Q_1 = 1 + b_1$. The first three classical numerators are:

$$P_0 = b_0,$$

$$P_1 = -1 + b_0 + b_1 b_0 + a_1,$$

$$P_2 = -1 + a_2 b_0 + b_1 b_0 + a_1 - b_2 + b_2 b_0 + b_2 b_1 b_0 + b_2 a_1.$$

The following lemma gives a relationship between the kth classical denominator and k + 1th classical numerator.

Lemma 6.

$$Q_k = -P_{k+1}(0, a_1, a_2, \dots, a_k; 0, 0, b_1, \dots, b_k).$$
⁽¹⁹⁾

Proof. Let x_k denote the right hand side of (19). Then $x_0 = 1$ and $x_1 = 1 + b_1$. Observe that x_k satisfies $x_k = (-1 + a_k)x_{k-2} + (1 + b_k)x_{k-1}$. This is the same recurrence and initial conditions satisfied by Q_k .

Define the sequence of polynomials R_k as follows: set $R_{-1} = 0$ and for $k \ge 0$, let

$$R_k(a_1, a_2, \dots, a_k; b_0, b_1, \dots, b_k) = P_k - R_{k-1}(a_1, a_2, \dots, a_{k-1}; b_0, b_1, \dots, b_{k-1}),$$
(20)

so that

$$P_k = R_k + R_{k-1}.$$
 (21)

The classical recurrence formula for P_k ,

$$P_k = (-1 + a_k)P_{k-2} + (1 + b_k)P_{k-1},$$
(22)

and (21) give a recurrence formula for R_k ,

$$R_k = -R_{k-3} + a_k(R_{k-2} + R_{k-3}) + b_k(R_{k-1} + R_{k-2}).$$
(23)

For consistency, set $a_0 = 0$ and initialize $R_{-3} = 0$, $R_{-2} = 1$, and $R_{-1} = 0$. Interpreting this recurrence formula is the key to our proof of Theorem 18.

For future reference the first seven elements in the sequence $\{R_n\}_{n=0}^{\infty}$ are listed:

$$R_0 = b_0, \tag{24a}$$

$$R_1 = -1 + a_1 + b_1 b_0, (24b)$$

$$R_2 = a_2b_0 - b_2 + b_2a_1 + b_2b_1b_0 + b_2b_0, (24c)$$

$$R_{3} = -b_{0} - a_{3} + a_{3}a_{1} + a_{3}b_{1}b_{0} + a_{3}b_{0} + b_{3}a_{2}b_{0} - b_{3}b_{2} + b_{3}b_{2}a_{1} + b_{3}b_{2}b_{1}b_{0} + b_{3}b_{2}b_{0} - b_{3} + b_{3}a_{1} + b_{3}b_{1}b_{0},$$
(24d)

$$R_4 = 1 - a_1 - b_1b_0 + a_4a_2b_0 - a_4b_2 + a_4b_2a_1 + a_4b_2b_1b_0 + a_4b_2b_0 - a_4 + a_4a_1 + a_4b_1b_0 - b_4b_0 - b_4a_3 + b_4a_3a_1 + b_4a_3b_1b_0 + b_4a_3b_0$$

$$+ h_4h_2a_5h_2 - h_4h_2h_2 + h_4h_2h_2a_1 + h_4h_2h_2h_3 + h_4h_2h_2h_2 - h_4h_3h_2h_3 + h_4h_2h_2h_3 + h_4h_2h_3 + h_4h_3 + h_$$

$$+ 0_4 0_3 a_2 0_0 - 0_4 0_3 0_2 + 0_4 0_3 0_2 a_1 + 0_4 0_3 0_2 0_1 0_0 + 0_4 0_3 0_2 0_0 - 0_4 0_3$$

$$+ b_4 b_3 a_1 + b_4 b_3 b_1 b_0 + b_4 a_2 b_0 - b_4 b_2 + b_4 b_2 a_1 + b_4 b_2 b_1 b_0 + b_4 b_2 b_0, \quad (24e)$$

 $R_5 = -a_2b_0 + b_2 - b_2a_1 - b_2b_1b_0 - b_2b_0 - a_5b_0 - a_5a_3 + a_5a_3a_1$ $+ a_5 a_3 b_1 b_0 + a_5 a_3 b_0 + a_5 b_3 a_2 b_0 - a_5 b_3 b_2 + a_5 b_3 b_2 a_1 + a_5 b_3 b_2 b_1 b_0$ $+ a_5b_3b_2b_0 - a_5b_3 + a_5b_3a_1 + a_5b_3b_1b_0 + a_5a_2b_0 - a_5b_2 + a_5b_2a_1$ $+ a_5b_2b_1b_0 + a_5b_2b_0 + b_5 - b_5a_1 - b_5b_1b_0 + b_5a_4a_2b_0 - b_5a_4b_2$ $+ b_5 a_4 b_2 a_1 + b_5 a_4 b_2 b_1 b_0 + b_5 a_4 b_2 b_0 - b_5 a_4 + b_5 a_4 a_1 + b_5 a_4 b_1 b_0$ $-b_5b_4b_0 - b_5b_4a_3 + b_5b_4a_3a_1 + b_5b_4a_3b_1b_0 + b_5b_4a_3b_0 + b_5b_4b_3a_2b_0$ $-b_5b_4b_3b_2+b_5b_4b_3b_2a_1+b_5b_4b_3b_2b_1b_0+b_5b_4b_3b_2b_0-b_5b_4b_3$ $+ b_5 b_4 b_3 a_1 + b_5 b_4 b_3 b_1 b_0 + b_5 b_4 a_2 b_0 - b_5 b_4 b_2 + b_5 b_4 b_2 a_1 + b_5 b_4 b_2 b_1 b_0$ $+ b_5 b_4 b_2 b_0 - b_5 b_0 - b_5 a_3 + b_5 a_3 a_1 + b_5 a_3 b_1 b_0 + b_5 a_3 b_0 + b_5 b_3 a_2 b_0$ $-b_5b_3b_2+b_5b_3b_2a_1+b_5b_3b_2b_1b_0+b_5b_3b_2b_0-b_5b_3+b_5b_3a_1$ $+ b_5 b_3 b_1 b_0$, (24f)and $R_6 = b_0 + a_3 - a_3a_1 - a_3b_1b_0 - a_3b_0 - b_3a_2b_0 + b_3b_2 - b_3b_2a_1 - b_3b_2b_1b_0$ $-b_3b_2b_0+b_3-b_3a_1-b_3b_1b_0+a_6-a_6a_1-b_1b_0+a_6a_4a_2b_0$ $-a_6a_4b_2 + a_6a_4b_2a_1 + a_6a_4b_2b_1b_0 + a_6a_4b_2b_0 - a_6a_4 + a_4a_1$ $+ a_6 a_4 b_1 b_0 - a_6 b_4 b_0 - a_6 b_4 a_3 + a_6 b_4 a_3 a_1 + a_6 b_4 a_3 b_1 b_0 + a_6 b_4 a_3 b_0$ $+ a_{6}b_{4}b_{3}a_{2}b_{0} - a_{6}b_{4}b_{3}b_{2} + a_{6}b_{4}b_{3}b_{2}a_{1} + a_{6}b_{4}b_{3}b_{2}b_{1}b_{0} + a_{6}b_{4}b_{3}b_{2}b_{0}$ $-a_{6}b_{4}b_{3} + a_{6}b_{4}b_{3}a_{1} + a_{6}b_{4}b_{3}b_{1}b_{0} + a_{6}b_{4}a_{2}b_{0} - a_{6}b_{4}b_{2} + a_{6}b_{4}b_{2}a_{1}$ $+ a_6 b_4 b_2 b_1 b_0 + a_6 b_4 b_2 b_0 - a_6 b_0 - a_6 a_3 + a_6 a_3 a_1 + a_6 a_3 b_1 b_0 + a_6 a_3 b_0$ $+ a_6 b_3 a_2 b_0 - a_6 b_3 b_2 + a_6 b_3 b_2 a_1 + a_6 b_3 b_2 b_1 b_0 + a_6 b_3 b_2 b_0 - a_6 b_3$ $+ a_6 b_3 a_1 + a_6 b_3 b_1 b_0 - b_6 a_2 b_0 + b_6 b_2 - b_6 b_2 a_1 - b_6 b_2 b_1 b_0 - b_6 b_2 b_0$ $-b_6a_5b_0 - b_6a_5a_3 + b_6a_5a_3a_1 + b_6a_5a_3b_1b_0 + b_6a_5a_3b_0 + b_6a_5b_3a_2b_0$ $-b_6a_5b_3b_2 + b_6a_5b_3b_2a_1 + b_6a_5b_3b_2b_1b_0 + b_6a_5b_3b_2b_0 - b_6a_5b_3b_2b_0$ $+ b_6a_5b_3a_1 + b_6a_5b_3b_1b_0 + b_6a_5a_2b_0 - b_6a_5b_2 + b_6a_5b_2a_1 + b_6a_5b_2b_1b_0$ $+ b_6 a_5 b_2 b_0 + b_6 b_5 - b_6 b_5 a_1 - b_6 b_5 b_1 b_0 + b_6 b_5 a_4 a_2 b_0 - b_6 b_5 a_4 b_2$ $+ b_6 b_5 a_4 b_2 a_1 + b_6 b_5 a_4 b_2 b_1 b_0 + b_6 b_5 a_4 b_2 b_0 - b_6 b_5 a_4 + b_6 b_5 a_4 a_1$ $+ b_6 b_5 a_4 b_1 b_0 - b_6 b_5 b_4 b_0 - b_6 b_5 b_4 a_3 + b_6 b_5 b_4 a_3 a_1$ $+ b_6 b_5 b_4 a_3 b_1 b_0 + b_6 b_5 b_4 a_3 b_0 + b_6 b_5 b_4 b_3 a_2 b_0 - b_6 b_5 b_4 b_3 b_2$ $+ b_6 b_5 b_4 b_3 b_2 a_1 + b_6 b_5 b_4 b_3 b_2 b_1 b_0 + b_6 b_5 b_4 b_3 b_2 b_0 - b_6 b_5 b_4 b_3$ $+ b_6 b_5 b_4 b_3 a_1 + b_6 b_5 b_4 b_3 b_1 b_0 + b_6 b_5 b_4 a_2 b_0 - b_6 b_5 b_4 b_2 + b_6 b_5 b_4 b_2 a_1$

 $+ b_6 b_5 b_4 b_2 b_1 b_0 + b_6 b_5 b_4 b_2 b_0 - b_6 b_5 b_0 - b_6 b_5 a_3 + b_6 b_5 a_3 a_1$

 $+ b_6 b_5 a_3 b_1 b_0 + b_6 b_5 a_3 b_0 + b_6 b_5 b_3 a_2 b_0 - b_6 b_5 b_3 b_2 + b_6 b_5 b_3 b_2 a_1 b_0 + b_6 b_5 b_3 b_2 b_0 + b_6 b_5 b_3 b_0 + b_6 b_5 b_0 + b_6 b_0 + b_$

 $+ b_6 b_5 b_3 b_2 b_1 b_0 + b_6 b_5 b_3 b_2 b_0 - b_6 b_5 b_3 + b_6 b_5 b_3 a_1 + b_6 b_5 b_3 b_1 b_0 + b_6 b_5 b_3 b_1 b_0 + b_6 b_5 b_3 b_2 b_1 b_0 + b_6 b_5 b_3 b_2 b_1 b_0 + b_6 b_5 b_3 b_2 b_0 - b_6 b_5 b_3 b_1 b_0 + b_6 b_5 b_3 b_0 + b_6 b_5 b_0 + b_6 b_0 + b_6$

 $-b_6a_1 - b_6b_1b_0 + b_6a_4a_2b_0 - b_6a_4b_2 + b_6a_4b_2a_1 + b_6a_4b_2b_1b_0$

 $+ b_6 a_4 b_2 b_0 - b_6 a_4 + b_6 a_4 a_1 + b_6 a_4 b_1 b_0 - b_6 b_4 b_0 - b_6 b_4 a_3 + b_6 b_4 a_3 a_1$

 $+ \, b_6 b_4 b_3 b_2 b_1 b_0 + b_6 b_4 b_3 b_2 b_0 - b_6 b_4 b_3 + b_6 b_4 b_3 a_1 + b_6 b_4 b_3 b_1 b_0$

$$+ b_6 b_4 a_2 b_0 - b_6 b_4 b_2 + b_6 b_4 b_2 a_1 + b_6 b_4 b_2 b_1 b_0 + b_6 b_4 b_2 b_0.$$
(24g)

Lemma 7. For $k \ge 2$, the polynomials R_k , R_{k-1} , and R_{k-2} have pairwise disjoint supports; there is no cancellation of terms in the sum $R_k + R_{k-1} + R_{k-2}$.

Proof. This follows easily by induction on recurrence formula (23). \Box

Corollary 8. For $k \ge 0$, let r_k count the number of terms of R_k . The sequence of integers $\{r_k\}_{k=0}^{\infty}$ satisfies the recurrence formula

$$\begin{cases} r_0 = 1, \quad r_1 = 3, \quad r_2 = 5\\ r_k = r_{k-1} + 2r_{k-2} + 2r_{k-3}, \end{cases}$$

and has generating function

$$\sum_{k \ge 0} r_k x^k = \frac{1+2x}{1-x-2x^2-2x^3}.$$

Proof. This is immediate from Lemma 7 and (23). The calculation of the generating function follows by the usual method. \Box

Lemma 9. Let T be a term of R_k . For j > 0:

- 1. The degree of T in each variable $a_1, a_2, \ldots, a_k, b_0, b_1, \ldots, b_k$ is at most one.
- 2. If a_j is a factor of T, then b_j is not a factor of T.

Proof. By induction these statements are true for the terms of P_k by (22). The result for R_k then follows from (21).

Let $\rho(k)$ be the periodic sequence:

$$\rho(k) = \begin{cases} -1 & \text{if } k \equiv 1 \pmod{6} \\ 1 & \text{if } k \equiv 4 \pmod{6} \\ 0 & \text{otherwise} \end{cases}.$$

Observe that the constant of R_k equals $\rho(k)$ for k = 0, 1, ..., 5. Further observe that the coefficient of each term in R_k is ± 1 , for k = 0, 1, ..., 5. More generally the following lemma holds.

Lemma 10. The constant of each polynomial R_k is $\rho(k)$. Further, the coefficient of any term T of R_k is ± 1 .

Proof. Let $\text{Const}(R_i) = R_i(0, 0, \dots, 0; 0, 0, \dots, 0)$ be the constant of R_i . By (23), $\text{Const}(R_k) = -\text{Const}(R_{k-3})$. That the constant term of R_k is $\rho(k)$ follows by induction. In (24), the coefficients of R_0 , R_1 , and R_2 are ± 1 . The lemma now follows by Lemma 7 and (23).

Proposition 13 will show the following definition characterizes $\operatorname{supp}(R_k) \setminus \{\epsilon\}$.

Definition 3. For $k \ge 0$, define \mathcal{R}_k to be the set of monomials whose index λ , *a*-index α , and *b*-index β satisfy the following properties:

- **R1** $k \geq \lambda_1 > \lambda_2 > \dots \lambda_\ell \geq 0.$
- **R2** $\lambda_1 \equiv k \pmod{3}$.

R3 If $\lambda_j = \alpha_j$, then $\lambda_j \not\equiv \lambda_{j+1} + 1 \pmod{3}$.

R4 If $\lambda_j = \beta_j$, then $\lambda_j \not\equiv \lambda_{j+1} \pmod{3}$.

R5 If $\lambda_{\ell} = \alpha_{\ell}$, then $\lambda_{\ell} \not\equiv 2 \pmod{3}$.

R6 If $\lambda_{\ell} = \beta_{\ell}$, then $\lambda_{\ell} \not\equiv 1 \pmod{3}$.

Note that property **R1** implies that monomials in \mathcal{R}_k satisfy the conditions of Lemma 9. Example 2 below shows the sets $\{b_0\}$, $\{b_1b_0, a_1\}$, and $\{b_2b_1b_0, b_2a_1, a_2b_0, b_2b_0, b_2\}$ are \mathcal{R}_0 , \mathcal{R}_1 , and \mathcal{R}_2 , respectively.

Example 2. Property **R1** implies that all elements of \mathcal{R}_0 have an index with $\lambda_1 = \lambda_\ell = 0$. Thus, any monomial in \mathcal{R}_0 has index, *a*-index, and *b*-index each equal to [0]. This index, *a*-index, and *b*-index satisfy **R1–R6**, thus $\mathcal{R}_0 = \{b_0\}$.

Properties **R1** and **R2** imply that all elements of \mathcal{R}_1 have an index with $\lambda_1 = 1$. Possible indices are [1,0] and [1]. When $\lambda = [1,0]$, the *a*-index [1,0] and *b*-index [0,0] do not satisfy **R3**, so $a_1b_0 \notin \mathcal{R}_1$. However, the monomial with *a*-index [0,0] and *b*-index [1,0] satisfies **R1–R6**. Thus $b_1b_0 \in \mathcal{R}_1$. The monomials index [1] with *a*-index [1] and *b*-index [0] satisfies **R1–R6**, thus $a_1 \in \mathcal{R}_1$. The monomials index [1] with *a*-index [1] with *a*-index [0] and *b*-index [1] does not satisfy **R6**, so $b_1 \notin \mathcal{R}_1$. Thus $\mathcal{R}_1 = \{b_1b_0, a_1\}$.

Properties **R1** and **R2** imply that all elements of \mathcal{R}_2 have an index with $\lambda_1 = 2$. Possible monomial indices are [2, 1, 0], [2, 1], [2, 0], and [2]. For a monomial in \mathcal{R}_2 with index [2, 1, 0], $\alpha_1 \neq 2$ and $\alpha_1 \neq 1$ by **R3**. Thus, $a_2b_1b_0$, $a_2a_1b_0$, $b_2a_1b_0 \notin \mathcal{R}_2$. However the monomial with index [2, 1, 0], a-index [0, 0, 0] and b-index [2, 1, 0] does satisfy **R1–R6**. Thus, $b_2b_1b_0 \in \mathcal{R}_2$. For a monomial in \mathcal{R}_2 with index [2, 1], $\alpha_1 \neq 2$, so **R3** implies that $a_2b_1, a_2a_1 \notin \mathcal{R}_2$. For a monomial with index [2, 0], the monomials with $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ equal to ([2, 0], [0, 0]) or ([0, 0], [2, 0]) satisfy **R1–R6**. Thus $a_2b_0, b_2b_0 \in \mathcal{R}_2$. For index [2, 0], the monomial with $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = ([0, 1], [2, 0])$ satisfies **R1–R6**. Thus $b_2a_1 \notin \mathcal{R}_2$. For index [2, 0], the monomial with $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = ([0, 1], [2, 0])$ satisfies **R1–R6**. Thus $b_2a_1 \notin \mathcal{R}_2$. For index [2, 0], the monomial with $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = ([0, 1], [2, 0])$ satisfies **R1–R6**. Thus $b_2a_1 \notin \mathcal{R}_2$. For index [2, 0], the monomial with $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = ([0, 1], [2, 0])$ satisfies **R1–R6**. Thus $b_2a_1 \notin \mathcal{R}_2$. For index [2, 0], the monomial with $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = ([0, 1], [2])$ satisfies **R1–R6**. Thus $b_2 \in \mathcal{R}_2$. For index [2], **R5** implies $\alpha_1 \neq 2$. Thus, $a_2 \notin \mathcal{R}_2$. The monomial with $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = ([0], [2])$ satisfies **R1–R6**. Thus $b_2 \in \mathcal{R}_2$. Finally, $\mathcal{R}_2 = \{b_2b_1b_0, b_2a_1, a_2b_0, b_2b_0, b_2\}$.

The following remark gives conditions for when monomials a_k or b_k are in \mathcal{R}_k .

Remark 1. For k > 0, the monomial a_k with $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = ([k], [0])$ is an element of \mathcal{R}_k if and only if $k \equiv 0, 1 \pmod{3}$ by **R5**. Similarly by **R6**, the monomial b_k with $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = ([0], [k])$ is an element of \mathcal{R}_k if and only if $k \equiv 0, 2 \pmod{3}$. Thus for $i \ge 0$, $\{a_{3i+1}\} = \mathcal{R}_{3i+1} \cap \{a_{3i+1}, b_{3i+1}\}, \{b_{3i+2}\} = \mathcal{R}_{3i+2} \cap \{a_{3i+2}, b_{3i+2}\},$ and $\{a_{3i+3}, b_{3i+3}\} \subset \mathcal{R}_{3i+3}$.

Lemma 11. The sequence $r_k - |\rho(k)|$ counts the number of elements in \mathcal{R}_k .

Proof. Our proof uses induction on k. From Example 2, the sets \mathcal{R}_0 , \mathcal{R}_1 , and \mathcal{R}_2 have 1, 2, and 5 elements, respectively. We verify $r_0 - |\rho(0)| = 1 - 0 = 1$, $r_1 - |\rho(1)| = 3 - 1 = 2$, and $r_2 - |\rho(2)| = 5 - 0 = 5$.

Make the induction hypothesis that \mathcal{R}_{k-3} , \mathcal{R}_{k-2} , and \mathcal{R}_{k-1} have $r_{k-3} - |\rho(k-3)|$, $r_{k-2} - |\rho(k-2)|$, and $r_{k-1} - |\rho(k-1)|$ elements, respectively. Let $\overline{\mathcal{R}}_j$ be the monomials of \mathcal{R}_j after substitutions $a_{j+1} \mapsto \overline{a_{j+1}}$ and $b_j \mapsto \overline{b_j}$ for $j \ge 0$. Here the overline denotes a different copy of the indeterminates.

We define a bijection $\psi : \overline{\mathcal{R}}_{k-3} \cup \mathcal{R}_{k-3} \cup \overline{\mathcal{R}}_{k-2} \cup \mathcal{R}_{k-2} \cup \mathcal{R}_{k-1} \to \mathcal{R}_k \setminus \{a_k, b_k\}$ as follows. ψ left multiplies elements of $\overline{\mathcal{R}}_{k-3} \cup \overline{\mathcal{R}}_{k-2}$ by $\overline{a_k}$ and then removes all overlines, ψ left multiplies elements of $\mathcal{R}_{k-2} \cup \mathcal{R}_{k-1}$ by b_k , and ψ leaves each element of \mathcal{R}_{k-3} fixed. ψ^{-1} is described as follows. When a_k is a factor of a monomial in $\mathcal{R}_k \setminus \{a_k, b_k\}, \psi^{-1}$ removes the factor a_k and overlines the remaining indeterminate factors. The result of this is in $\overline{\mathcal{R}}_{k-3}$ or $\overline{\mathcal{R}}_{k-2}$ due to property **R3**. Similarly, when b_k is a factor of a monomial in $\mathcal{R}_k \setminus \{a_k, b_k\}, \psi^{-1}$ removes the factor b_k , and the result is in either \mathcal{R}_{k-2} or \mathcal{R}_{k-1} by property **R4**. Otherwise, ψ^{-1} leaves a monomial of $\mathcal{R}_k \setminus \{a_k, b_k\}$ fixed.

Since there is a bijection between $\mathcal{R}_k \setminus \{a_k, b_k\}$ and the pairwise disjoint union $\overline{\mathcal{R}}_{k-3} \cup \mathcal{R}_{k-3} \cup \overline{\mathcal{R}}_{k-2} \cup \mathcal{R}_{k-2} \cup \mathcal{R}_{k-1}$, the number of elements in $\mathcal{R}_k \setminus \{a_k, b_k\}$ is

$$r_{k-1} - |\rho(k-1)| + 2r_{k-2} - 2|\rho(k-2)| + 2r_{k-3} - 2|\rho(k-3)|.$$

By the recurrence formula for r_k in Corollary 8, the above equals

$$r_k - |\rho(k-1)| - 2|\rho(k-2)| - 2|\rho(k-3)|.$$
(25)

Remark 1 gives that the number of elements in $\mathcal{R}_k \cap \{a_k, b_k\}$ is one when $k \equiv 1, 2 \pmod{3}$ and two when $k \equiv 0 \pmod{3}$. Since $|\rho(k)| = 1$ if $k \equiv 1 \pmod{3}$ and is zero otherwise, the total number of elements in $\mathcal{R}_k \cap \{a_k, b_k\}$ is expressible as $|\rho(k)| + |\rho(k-1)| + 2|\rho(k-2)|$ or $|\rho(k-3)| + |\rho(k-1)| + 2|\rho(k-2)|$. Adding this to the number of elements of $\mathcal{R}_k \setminus \{a_k, b_k\}$ found in (25) gives that the number of monomials in \mathcal{R}_k is $r_k - |\rho(k-3)| = r_k - |\rho(k)|$.

Corollary 12. Let $s_k = |\mathcal{R}_k|$. Then s_k satisfies the linear recurrence $s_k = s_{k-1} + 2s_{k-2} + 3s_{k-3} - s_{k-4} - 2s_{k-5} - 2s_{k-6}$ with initial conditions, $s_0 = 1$, $s_1 = 2$, $s_2 = 5$, $s_3 = 13$, $s_4 = 28$, and $s_5 = 65$.

Proof. From the fact that s_k and $|\rho(k)|$ satisfy linear recurrences of order three, with constant coefficients, it follows that s_k can satisfy a similar recurrence of order at most 9. Standard linear algebra gives the recurrence for s_k .

Proposition 13. For $k \ge 0$, supp $(R_k - \rho(k)) = \mathcal{R}_k$.

Proof. By Lemma 11 supp $(R_k - \rho(k))$ and \mathcal{R}_k have the same number of elements, $r_k - |\rho(k)|$. Therefore, it is enough to show that the support of $R_k - \rho(k)$ is a subset of \mathcal{R}_k . Our proof of this uses induction on k. The support of $R_0 - \rho(0)$ is $\{b_0\}$, the support of $R_1 - \rho(1)$ is $\{a_1, b_1b_0\}$, and the support of $R_2 - \rho(2)$ is $\{a_2b_0, b_2, b_2a_1, b_2b_1b_0, b_2b_0\}$. These sets are identical to the corresponding sets \mathcal{R}_0 , \mathcal{R}_1 , and \mathcal{R}_2 found in Example 2.

Let *m* be a monomial of R_k with degree ℓ , *a*-index α , *b*-index β , and index λ . Suppose that each monomial *m'* in the support of R_i is also in \mathcal{R}_i for $i = 1, 2, \ldots, k-1$. By (23) and Lemma 7, *m* is either in the support of $-R_{k-3}-\rho(k)$, $a_k R_{k-2}$, $a_k R_{k-3}$, $b_k R_{k-1}$, or $b_k R_{k-2}$. We verify that the index, *a*-index, and *b*-index of *m* satisfy the conditions **R1**–**R6** in each of these cases.

Suppose *m* is a monomial of $-R_{k-3} - \rho(k)$. Then *m* is a monomial of $R_{k-3} - \rho(k-3)$, since $-\rho(k) = \rho(k-3)$ and the supports of polynomials -P and *P* are the same. By the induction hypothesis, *m* is in \mathcal{R}_{k-3} . From property **R2** for \mathcal{R}_{k-3} , the first component of the index of *m* satisfies $\lambda_1 \equiv k - 3 \pmod{3}$, so $\lambda_1 \equiv k \pmod{3}$ and *m* satisfies **R2**. The other properties **R1**, **R3-R6** clearly follow from the respective properties of \mathcal{R}_{k-3} .

Suppose *m* is a monomial of $a_k R_{k-3+p}$ where p = 0, 1. If $\ell = 1$, then $m = a_k$ and $\rho(k-3+p)$ is nonzero. Thus, $k-3+p \equiv 1 \pmod{3}$ and $k \equiv 1, 0 \pmod{3}$, and **R5** holds. The monomial a_k has index $\lambda = [k] + [0]$. Properties **R1** and **R2** of \mathcal{R}_k hold for a_k . Properties **R4** and **R6** hold for a_k since no $\lambda_j = \beta_j$. Property **R3** holds since $\ell = 1$. Next, if $\ell > 1$, then by the inductive hypothesis $m = a_k m'$, where $m' \in \mathcal{R}_{k-3+p}$. The index of *m* is $\lambda = [k, \alpha'] + [0, \beta']$, where $\lambda' = \alpha' + \beta'$ is the index of *m'*. Clearly *m* satisfies **R1** and **R2** for \mathcal{R}_k . From **R2** for $\mathcal{R}_{k-3+p}, \lambda'_1 \equiv k-3+p \pmod{3}$, thus $\lambda_1 = \alpha_1 = k \neq k-3+p+1 \pmod{3}$ and **R3** holds for j = 1. Property **R3** holds for j > 1 since $m' \in \mathcal{R}_{k-3+p}$. Properties **R4–R6** are satisfied by *m* from the respective properties of $m' \in \mathcal{R}_{k-3+p}$.

Suppose *m* is a monomial of $b_k R_{k-2+p}$ where p = 0, 1. If $\ell = 1$, then $m = b_k$ and $\rho(k-2+p)$ is nonzero. Thus, $k-2+p \equiv 1 \pmod{3}$ and $k \equiv 0, 2 \pmod{3}$, and **R6** holds. The monomial b_k has index $\lambda = [0] + [k]$. Properties **R1** and **R2** of \mathcal{R}_k hold for b_k . Properties **R3** and **R5** of \mathcal{R}_k hold for b_k since no $\lambda_j = \alpha_j$. Property **R4** holds since $\ell = 1$. Next if $\ell > 1$, then by the inductive hypothesis $m = b_k m'$, where $m' \in \mathcal{R}_{k-2+p}$. The index of *m* is $\lambda = [0, \alpha'] + [k, \beta']$, where $\lambda' = \alpha' + \beta'$ is the index of *m'*. Clearly *m* satisfies **R1** and **R2** for \mathcal{R}_k . From **R2** for \mathcal{R}_{k-2+p} , $\lambda' \equiv k-2+p \pmod{3}$, thus $\lambda_1 = \beta_1 = k \neq k-2+p \pmod{3}$ and **R4** holds for j = 1. Property **R4** holds for j > 1 since $m' \in \mathcal{R}_{k-2+p}$.

We now turn our attention to the coefficients of R_k . By Lemma 10, each monomial $m \in \text{supp}(R_k)$ has coefficient $c_m = \pm 1$. The determination of the sign depends on the following definition.

Definition 4. We call a set of three consecutive integers an *adjacent triple*. For a monomial m of \mathcal{R}_k with index λ , the integers in the set

$$\{-1,0,\ldots,k\}\setminus\{\lambda_1,\lambda_2,\ldots,\lambda_\ell\}$$

are called the *omitted subscripts of m*. For a monomial $m \in \mathcal{R}_k$ with index λ , define the function $g_k(m)$ to be the maximum number of pairwise disjoint adjacent triples whose union is a *subset* of the omitted subscripts of m.

The coefficient c_m of $m \in \text{supp}(R_k)$ is determined by the parity of $g_k(m)$. Specifically, $c_m = (-1)^{g_k(m)}$. We show this in Lemma 16. The coefficients of three monomials are computed in Example 3.

Example 3. First, consider the monomial b_2 in supp (R_5) with index [2]. Monomial b_2 has omitted subscripts $\{5, 4, 3, 1, 0, -1\}$. This set is the union of $g_5(b_2) = 2$ disjoint triples: $\{5, 4, 3\}$ and $\{1, 0, -1\}$. Thus, the coefficient of b_2 is $(-1)^2 = 1$.

Second, consider the monomial $b_6b_4a_3$ in $\operatorname{supp}(R_6)$ with index [6, 4, 3]. It has omitted subscripts $\{5, 2, 1, 0, -1\}$. The omitted subscripts contain two adjacent triples, $\{2, 1, 0\}$ and $\{1, 0, -1\}$. Since these adjacent triples are not disjoint, $g_6(b_6b_4a_3) = 1$, and the coefficient of $b_6b_4a_3$ is $(-1)^1 = -1$.

Third, consider the monomial $a_6b_4b_2b_1b_0$ in $\operatorname{supp}(R_6)$. This monomial has index [6, 4, 2, 1, 0] and has omitted subscripts $\{5, 3, -1\}$. The omitted subscripts give $g_6(a_6b_4b_2b_1b_0) = 0$ pairwise disjoint adjacent triple subsets. Thus the coefficient of $a_6b_4b_2b_1b_0$ is $(-1)^0 = 1$.

Observe that the coefficient of b_2 in R_8 should be the opposite of its coefficient in R_5 , since the omitted subscripts in the former case contains an additional adjacent triple $\{8, 7, 6\}$. More generally, Lemma 14 describes how the coefficient of a monomial of R_k is based upon recurrence formula (23).

Lemma 14. For each monomial m of $R_k - \rho(k)$ with index $\lambda = \alpha + \beta \in \mathcal{R}_k$ of length $\ell > 1$, let m' be the monomial with index $\lambda' = \alpha' + \beta' = [\alpha_2, \alpha_3, \dots, \alpha_\ell] + [\beta_2, \beta_3, \dots, \beta_\ell]$. Define $\operatorname{sgn}_k(m)$ to be the coefficient of m (or sign of m) in the polynomial R_k . Then for $k \geq 3$ and p = 1, 2, 3,

$$\operatorname{sgn}_k(m) = \begin{cases} \operatorname{sgn}_{k-p}(m') & \text{if } \lambda_1 = k \text{ and } \lambda_2 \equiv k - p \pmod{3}, \\ -\operatorname{sgn}_{k-3}(m) & \text{if } \lambda_1 \neq k. \end{cases}$$

Proof. Let $\operatorname{sgn}_k(m)m$ be a term of $R_k - \rho(k)$. There are five cases corresponding to the five summands when the right hand side of (23) is expanded. If $\operatorname{sgn}_k(m)m$ is a term of $b_k R_{k-1}$, then $\operatorname{sgn}_k(m) = \operatorname{sgn}_{k-1}(m')$. If $\operatorname{sgn}_k(m)m$ is a term of $a_k R_{k-2}$ or $b_k R_{k-2}$, then $\operatorname{sgn}_k(m) = \operatorname{sgn}_{k-2}(m')$. If $\operatorname{sgn}_k(m)m$ is a term of $a_k R_{k-3}$, then $\operatorname{sgn}_k(m) = \operatorname{sgn}_{k-3}(m')$. If $\operatorname{sgn}_k(m)m$ is a term of $-R_{k-3}$, then $\operatorname{sgn}_k(m) = -\operatorname{sgn}_{k-3}(m)$.

Lemma 15.

$$g_k(m) = \left\lfloor \frac{k - \lambda_1}{3} \right\rfloor + \left\lfloor \frac{\lambda_\ell + 1}{3} \right\rfloor + \sum_{j=2}^{\ell} \left\lfloor \frac{\lambda_{j-1} - \lambda_j - 1}{3} \right\rfloor,$$

the last sum being zero when $\ell = 1$.

Proof. The maximum number of disjoint three adjacent triples strictly between integers λ_{j-1} and λ_j is

$$\left\lfloor \frac{\lambda_{j-1} - \lambda_j - 1}{3} \right\rfloor.$$

The lemma follows from summing and using the conventions $\lambda_0 = k + 1$ and $\lambda_{\ell+1} = -2$.

Lemma 16. For a monomial m of R_k ,

$$\operatorname{sgn}_k(m) = (-1)^{g_k(m)}.$$

Proof. We proceed by induction on k. The initial cases are given by (24). Let m be in the support of $R_k - \rho(k)$ with index $\lambda = \alpha + \beta$. Suppose that each monomial m^* in the support of $R_j - \rho(j)$ has coefficient $\operatorname{sgn}_j(m^*) = (-1)^{g_j(m^*)}$, for $j = 0, 1, 2, \ldots, k - 1$. From Lemma 14 and the induction hypothesis,

$$\operatorname{sgn}_{k}(m) = \begin{cases} (-1)^{g_{k-1}(m')} & \text{if } \lambda_{1} = k \text{ and } \lambda_{2} \equiv k - 1 \pmod{3}, \\ (-1)^{g_{k-2}(m')} & \text{if } \lambda_{1} = k \text{ and } \lambda_{2} \equiv k - 2 \pmod{3}, \\ (-1)^{g_{k-3}(m')} & \text{if } \lambda_{1} = k \text{ and } \lambda_{2} \equiv k - 3 \pmod{3}, \\ (-1)^{1+g_{k-3}(m)} & \text{if } \lambda_{1} \neq k, \end{cases}$$
(26)

where m' has index $\lambda' = [\alpha_2, \alpha_3, \dots, \alpha_\ell] + [\beta_2, \dots, \beta_\ell]$. Each of the cases in (26) gives $(-1)^{g_k(m)}$. Indeed, in each case where $\lambda_1 = k$,

$$\{-1, 0, 1, 2, 3, \dots, k\} \setminus \{\lambda_1, \lambda_2, \dots, \lambda_\ell\}$$

= $\{-1, 0, 1, 2, 3, \dots, k-1\} \setminus \{\lambda_2, \lambda_3, \dots, \lambda_\ell\}.$

So when $\lambda_1 = k$, the maximum number of disjoint adjacent triples of these equal sets $g_k(m)$ and $g_{k-1}(m')$, respectively, are equal.

We use that $g_k(m) = g_{k-1}(m')$ for the first three cases where $\lambda_1 = k$ and $\lambda_2 \equiv k - p \pmod{3}$ for p = 1, 2, 3. Let d be the integer such that $\lambda_2 = k - p - 3d$. By Lemma 15,

$$g_{k-1}(m') = \left\lfloor \frac{\lambda_{\ell} + 1}{3} \right\rfloor + \left\lfloor \frac{k - 1 - \lambda_2}{3} \right\rfloor + \sum_{j=3}^{\ell} \left\lfloor \frac{\lambda_{j-1} - \lambda_j - 1}{3} \right\rfloor.$$
 (27)

Substituting k - p - 3d for λ_2 in the second summand of (27) yields,

$$\left\lfloor \frac{k-1-\lambda_2}{3} \right\rfloor = \left\lfloor \frac{3d+p-1}{3} \right\rfloor = d = \left\lfloor \frac{3d}{3} \right\rfloor = \left\lfloor \frac{k-p-\lambda_2}{3} \right\rfloor.$$

Thus we can replace the second summand of (27):

$$g_{k-1}(m') = \left\lfloor \frac{\lambda_{\ell} + 1}{3} \right\rfloor + \left\lfloor \frac{k - p - \lambda_2}{3} \right\rfloor + \sum_{j=3}^{\ell} \left\lfloor \frac{\lambda_{j-1} - \lambda_j - 1}{3} \right\rfloor = g_{k-p}(m').$$

For the last case when $\lambda_1 \neq k$, property **R2** gives that $\lambda_1 \neq k, k-1, k-2$. Thus, the adjacent triple k, k-1, k-2 is in

$$\{-1,0,1,2,3,\ldots,k\}\setminus\{\lambda_1,\lambda_2,\lambda_3\ldots,\lambda_\ell\},\$$

and $g_k(m)$ is one more than the number of adjacent triples in

$$\{-1,0,1,2,3,\ldots,k-3\}\setminus\{\lambda_1,\lambda_2,\lambda_3\ldots,\lambda_\ell\}.$$

Thus $1 + g_{k-3}(m) = g_k(m)$ and $\operatorname{sgn}_k(m) = (-1)^{g_k(m)}$ by (26).

3. Main Results

The following proposition gives a combinatorial description for the terms of the polynomials R_k .

Proposition 17. For $k \ge 0$,

$$R_k(a_1, a_2, \dots, a_k, b_0, b_1, \dots, b_k) = \rho(k) + \sum_{m \in \mathcal{R}_k} (-1)^{g_k(m)} m.$$
(28)

Proof. Applying Lemmas 10, 13, and 16 gives (28).

Theorem 18 below results from piecing together Lemma 6, (21), and Proposition 17. Lemma 6 states $Q_k = \phi(-P_k)$, where ϕ is the substitution that maps a_1 , b_0 , and b_1 to 0, and for j > 1, substitutes a_{j-1} for a_j , and substitutes b_{j-1} for b_j . The linearity of ϕ and (21) gives $Q_k = -\phi(R_{k+1}) - \phi(R_k)$.

Define $\sigma(k)$ to be

$$\sigma(k) = \frac{2\sqrt{3}}{3} \sin\left(\frac{k\pi}{3}\right) = \frac{2\sqrt{3}}{3} \operatorname{Im}\left(e^{k\pi i/3}\right),\tag{29}$$

and note that $\sigma(k)$ is the six-periodic sequence which begins 0, 1, 1, 0, -1, -1, ..., for $k \ge 0$ that satisfies $\sigma(k) = -\sigma(k-3)$.

Theorem 18. The kth classical numerator and denominator P_k and Q_k of

$$b_0 + \frac{-1+a_1}{1+b_1} + \frac{-1+a_2}{1+b_2} + \frac{-1+a_3}{1+b_3} + \cdots$$

are

$$P_{k} = -\sigma(k) + \sum_{m \in \mathcal{R}_{k-1}} (-1)^{g_{k-1}(m)} m + \sum_{m \in \mathcal{R}_{k}} (-1)^{g_{k}(m)} m,$$
(30)

and

$$Q_k = \sigma(k+1) - \sum_{\substack{m \in \mathcal{R}_k \\ \lambda_\ell > 1}} (-1)^{g_k(m)} \phi(m) - \sum_{\substack{m \in \mathcal{R}_{k+1} \\ \lambda_\ell > 1}} (-1)^{g_{k+1}(m)} \phi(m).$$
(31)

Proof. First observe that $\sigma(k) = -\rho(k-1) - \rho(k)$. The proof follows immediately from (21), (28), and (19). The condition $\lambda_{\ell} > 1$ in (31) simplifies the summations by removing all summands with $\phi(m) = 0$, where $\phi(m)$ is as defined after the proof of Proposition 17.

The next two corollaries are specializations of Theorem 18. The first of these is the case when $b_j = 0$ for $j \ge 0$. Making this substitution into (30) causes all monomials whose b index has nonzero components and those elements with $\beta_{\ell} = \alpha_{\ell} = 0$ to vanish from the summation in (30). This implies that the index equals the a index for these monomials. Thus for $k \ge 1$ the support of the classical numerators is the subset \mathcal{U}_k of \mathcal{R}_k whose b index is zero and whose index satisfies the following properties:

- **U1** $k \geq \lambda_1 > \lambda_2 > \dots \lambda_\ell \geq 1.$
- **U2** $\lambda_1 \equiv k \pmod{3}$.
- **U3** $\lambda_j \not\equiv \lambda_{j+1} + 1 \pmod{3}$.
- U4 $\lambda_{\ell} \not\equiv 2 \pmod{3}$.

Corollary 19. The kth classical numerator and denominator C_k and D_k of

$$\frac{-1+a_1}{1} + \frac{-1+a_2}{1} + \frac{-1+a_3}{1} + \cdots$$

are

$$C_k = -\sigma(k) + \sum_{m \in \mathcal{U}_{k-1}} (-1)^{g_{k-1}(m)} m + \sum_{m \in \mathcal{U}_k} (-1)^{g_k(m)} m,$$
(32)

and

$$D_{k} = \sigma(k+1) - \sum_{\substack{m \in \mathcal{U}_{k} \\ \lambda_{\ell} > 1}} (-1)^{g_{k}(m)} \phi(m) - \sum_{\substack{m \in \mathcal{U}_{k+1} \\ \lambda_{\ell} > 1}} (-1)^{g_{k+1}(m)} \phi(m).$$
(33)

The second case of Theorem 18 is $b_0 = 0$ and $a_j = 0$ for $j \ge 1$. The monomials whose a index has nonzero components as well as those elements with $\beta_{\ell} = \alpha_{\ell} = 0$ vanish from the summation in (30). Hence for $k \ge 1$ the support of the classical numerators is the subset \mathcal{V}_k of \mathcal{R}_k whose a index is zero and whose index satisfies the following properties:

- **V1** $k \ge \lambda_1 > \lambda_2 > \dots \lambda_\ell \ge 2.$
- **V2** $\lambda_1 \equiv k \pmod{3}$.
- **V3** $\lambda_j \not\equiv \lambda_{j+1} \pmod{3}$.
- **V4** $\lambda_{\ell} \not\equiv 1 \pmod{3}$.

Corollary 20. The kth classical numerator and denominator G_k and H_k of

$$\frac{-1}{1+b_1} + \frac{-1}{1+b_2} + \frac{-1}{1+b_3} + \cdots$$

are

$$G_k = -\sigma(k) + \sum_{m \in \mathcal{V}_{k-1}} (-1)^{g_{k-1}(m)} m + \sum_{m \in \mathcal{V}_k} (-1)^{g_k(m)} m,$$
(34)

and

$$H_{k} = \sigma(k+1) - \sum_{\substack{m \in \mathcal{V}_{k} \\ \lambda_{\ell} > 1}} (-1)^{g_{k}(m)} \phi(m) - \sum_{\substack{m \in \mathcal{V}_{k+1} \\ \lambda_{\ell} > 1}} (-1)^{g_{k+1}(m)} \phi(m).$$
(35)

Corollary 20 was first given as Theorem 33 of [15].

We refer to a finite decreasing sequence λ_i satisfying $\lambda_j \not\equiv \lambda_{j+1} + t \pmod{3}$, for a fixed $t \in \{0, 1, 2\}$, as an *alternating triality sequence*. Only the cases t = 0, 1 occur in this paper.

4. Applications to Polynomial Identities and Integer Sequences

4.1. Relating Theorems 5 and 18

1

What do Theorems 5 with 18 imply when taken together?

To relate these theorems, the following change of variables is used. Let $\delta : \mathbb{Z}[\mathcal{M}] \to \mathbb{Z}[\mathcal{M}]$ be the homomorphism induced by $\delta(a_i) = -1 + a_i$ and $\delta(b_i) = 1 + b_i$. When applied to \mathcal{A}_k , each monomial of length l gives rise to 2^l monomials with different signs, since the monomials in \mathcal{A}_k are of degree one in each of their indeterminates. Thus Theorems 5 and 18 give,

$$\sum_{m \in \mathcal{A}_k} \delta(m) = -\sigma(k) + \sum_{m \in \mathcal{R}_{k-1}} (-1)^{g_{k-1}(m)} m + \sum_{m \in \mathcal{R}_k} (-1)^{g_k(m)} m.$$
(36)

Notice the left side of (36) has intense cancellation, while the right side has none. This identity becomes more explicit in the special cases corresponding to Corollaries 19 and 20. First we provide a corollary of Theorem 5 that can be equated to Corollary 19. Define the set of minimal difference two sequences \mathbf{C}_k by

$$\mathbf{C}_k = \{ \boldsymbol{\lambda} : k \ge \lambda_1 >^2 \lambda_2 >^2 \dots >^2 \lambda_\ell = 1 \}$$

It is easy to see that $|\mathbf{C}_k|$ equals the *k*th Fibonacci number, for an element of \mathbf{C}_k is either an element of \mathbf{C}_{k-1} , or is obtained by adjoining the integer *k* to an element of \mathbf{C}_{k-2} . The initial values $F_0 = |\mathbf{C}_0| = 0$ and $F_1 = |\mathbf{C}_1| = 1$ give the conclusion.

Corollary 21. The classical numerators of the continued fraction

$$\frac{-1+a_1}{1} + \frac{-1+a_2}{1} + \frac{-1+a_3}{1} + \dots$$
(37)

in noncommutative indeterminates $\{a_{j+1}\}_{j\geq 0}$ for $k\geq 0$ are given by

$$C_k = \sum_{\boldsymbol{\lambda} \in \mathbf{C}_k} (-1 + a_{\lambda_1})(-1 + a_{\lambda_2}) \cdots (-1 + a_{\lambda_\ell}).$$
(38)

Proof. By Theorem 4 (with $b_0 = 0$ and $b_i = 1$, for i > 0), the kth classical numerator of

$$\frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \cdots$$

equals

$$\sum_{\boldsymbol{\lambda}\in\mathbf{C}_k}a_{\lambda_1}a_{\lambda_2}\cdots a_{\lambda_\ell}$$

after writing subscripts in descending order. Substituting sequence $\{-1+a_j\}_{j\geq 1}$ for $\{a_j\}_{j\geq 1}$ yields (38).

Equating Corollaries 19 and 21 gives Corollary 22 below.

Corollary 22.

$$\sum_{\lambda \in \mathbf{C}_k} \prod_{j=1}^{\ell} (-1 + a_{\lambda_j}) = -\sigma(k) + \sum_{m \in \mathcal{U}_{k-1}} (-1)^{g_{k-1}(m)} m + \sum_{m \in \mathcal{U}_k} (-1)^{g_k(m)} m,$$
(39)

$$\sum_{\lambda \in \mathbf{C}_k} (-1)^\ell = -\sigma(k), \tag{40}$$

and

$$-\sigma(k) + \sum_{m \in \mathcal{U}_{k-1}} (-1)^{g_{k-1}(m)} + \sum_{m \in \mathcal{U}_k} (-1)^{g_k(m)} = 0$$
(41)

Example 4 below demonstrates these identities in the k = 5 case. Before the example we give the simple proof of (40) mentioned in Section 1.4.

Proof. Let e_n and o_n denote the number of elements of \mathbf{C}_n of even and odd lengths, respectively. Since every element of \mathbf{C}_n is either an element of \mathbf{C}_{n-1} , or is obtained by adjoining the integer n to an element of \mathbf{C}_{n-2} , it is clear that $o_n = o_{n-1} + e_{n-2}$, and $e_n = e_{n-1} + o_{n-2}$. Putting $x_n = e_n - o_n$, and subtracting the first of the two equations from the second, gives that $x_n = x_{n-1} - x_{n-2}$. Clearly $x_{n-1} = x_{n-2} - x_{n-3}$. Substituting the second of these two equations into the first gives $x_n = -x_{n-3}$, which is the same recurrence satisfied by $-\sigma(k)$. For $k = 0, 1, 2, -\sigma(k) = 0, -1, -1$, and the left-hand side of (40) also equals 0, -1, -1, since $C_0 = \emptyset$ and C_1 and C_2 each contain only the sequence $\{1\}$.

Example 4. Let k = 5 in (39). It is found that $\sigma(5) = -1$, $\mathcal{U}_4 = \{[4, 1], [4], [1]\}$ and $\mathcal{U}_5 = \{[5, 3, 1], [5, 3]\}$, so that right-hand side of (39) is:

$$1 + a_4 a_1 - a_4 - a_1 + a_5 a_3 a_1 - a_5 a_3. \tag{42}$$

The left-hand side of (39) for k = 5 can be computed by summing the contributions from each sequence in \mathbf{C}_k and making several cancellations. First, we find the contribution due to $\{5,3,1\} \in \mathbf{C}_5$:

$$(a_5-1)(a_3-1)(a_1-1) = a_5a_3a_1 - (a_5a_3 + a_5a_1 + a_3a_1) + (a_5 + a_3 + a_1) - 1.$$
(43)

Similarly, the sequence $\{5, 1\}$ contributes

$$(a_5 - 1)(a_1 - 1) = a_5a_1 - a_5 - a_1 + 1.$$
(44)

The sequence $\{4, 1\}$ contributes

$$(a_4 - 1)(a_1 - 1) = a_4a_1 - a_4 - a_1 + 1.$$
(45)

The sequence $\{3,1\}$ contributes

$$a_3a_1 - a_3 - a_1 + 1. \tag{46}$$

Finally, the sequence $\{1\}$ contributes

$$a_1 - 1.$$
 (47)

Summing (43)-(47) and canceling terms recovers (42).

The sum on the left-hand side of (40) in the k = 5 case is over the sequences $\{5,3,1\}, \{5,1\}, \{4,1\}, \{3,1\}$ and $\{1\}$. The sum can be computed according to the lengths of these sequences as $(-1)^3 + 3(-1)^2 + (-1)^1 = 1$, which indeed is equal to $-\sigma(5)$. Also observe that the substitution $a_j \mapsto 1$ in (42) yields zero, giving (41).

To relate Theorem 5 to Corollary 20, define \mathbf{D}_k to be the set of alternating parity sequences satisfying:

D1 $k \ge \lambda_1 > \lambda_2 > \cdots > \lambda_\ell \ge 2.$

D2 $\lambda_1 \equiv k \pmod{2}$.

D3 $\lambda_j \not\equiv \lambda_{j-1} \pmod{2}$.

D4 $\lambda_{\ell} \equiv 0 \pmod{2}$.

Theorem 5 now reduces to:

Corollary 23. The classical numerators of the continued fraction

$$\frac{-1}{1+b_1} + \frac{-1}{1+b_2} + \frac{-1}{1+b_3} + \cdots$$

in noncommutative indeterminates $\{b_j\}_{j\geq 0}$ for $k\geq 0$ are given by

$$G_k = -\chi_1(k) + \sum_{\lambda \in \mathbf{D}_k} (-1)^{\frac{k-\ell+1}{2}} (1+b_{\lambda_1})(1+b_{\lambda_2}) \cdots (1+b_{\lambda_\ell}).$$
(48)

.

Proof. By Theorem 4, the kth classical numerator of

$$\frac{-1}{b_1} + \frac{-1}{b_2} + \frac{-1}{b_3} + \cdots$$

in commuting variables is

$$b_2 \cdots b_k \left[1 + \sum_{3 \le h_1 < {}^2h_2 < {}^2 \cdots < {}^2h_j \le k} \frac{(-1)^{j+1}}{b_{h_1-1}b_{h_1}b_{h_2-1}b_{h_2} \cdots b_{h_j-1}b_{h_j}} \right].$$

When expanded, the degree of a summand is $\ell = k - 1 - 2j$. So $j = \frac{k - \ell - 1}{2}$. Note that after cancellation the largest subscript has the same parity as k and the smallest subscript is even. When k is even, $b_2 \cdots b_k$ is not canceled when distributed and the constant is zero. When k is odd, the constant is $(-1)^{1+(k-1)/2}$. Thus the constant is $-\chi_1(k)$, the nonprincipal Dirichlet character modulo 4. Rearranging subscripts in descending order gives that in noncommuting variables, the classical numerator is

$$-\chi_1(k) + \sum_{\lambda \in \mathbf{D}_k} (-1)^{\frac{k-\ell+1}{2}} (1+b_{\lambda_1})(1+b_{\lambda_2}) \cdots (1+b_{\lambda_\ell}).$$

Equating Corollaries 20 and 23 yields the following corollary. Corollary 24.

$$-\chi_{1}(k) + \sum_{\lambda \in \mathbf{D}_{k}} (-1)^{\frac{k-\ell+1}{2}} \prod_{j=1}^{\ell} (1+b_{\lambda_{j}}) = -\sigma(k) + \sum_{m \in \mathcal{V}_{k-1}} (-1)^{g_{k-1}(m)} m + \sum_{m \in \mathcal{V}_{k}} (-1)^{g_{k}(m)} m,$$
$$-\chi_{1}(k) + \sum_{\lambda \in \mathbf{D}_{k}} (-1)^{\frac{k-\ell+1}{2}} = -\sigma(k),$$

and

$$-\chi_1(k) + \sigma(k) = \sum_{m \in \mathcal{V}_{k-1}} (-1)^{g_{k-1}(m)+\ell} + \sum_{m \in \mathcal{V}_k} (-1)^{g_k(m)+\ell}$$

4.2. Applications to some linear recurrence sequences

Corollaries 19 and 20 lead to new formulas for Fibonacci and Pell numbers. In this section, we use notation P_k for the kth Pell number, not the kth classical numerator of K as before. Thus, in this section $P_k = 2P_{k-1} + P_{k-2}$, with initial conditions $P_0 = 0$ and $P_1 = 1$. Despite possible interest, we do not take up the corresponding results that follow from Corollaries 21 and 23 here, nor do we investigate the consequences for other or more general integer sequences.

The definition of $\sigma(k)$ and the following corollary imply the well-known fact that the kth Fibonacci number, F_k , is even if and only if $k \equiv 0 \pmod{3}$.

Corollary 25.

$$F_k = -\sigma(k) + \sum_{\substack{m \in \mathcal{U}_{k-1} \\ \lambda_\ell > 0}} (-1)^{g_{k-1}(m)} 2^\ell + \sum_{\substack{m \in \mathcal{U}_k \\ \lambda_\ell > 0}} (-1)^{g_k(m)} 2^\ell,$$
(49)

and

$$F_{k} = \sum_{\substack{m \in \mathcal{U}_{k-1} \\ \lambda_{\ell} = 1}} (-1)^{g_{k-1}(m)} 2^{\ell-1} + \sum_{\substack{m \in \mathcal{U}_{k} \\ \lambda_{\ell} = 1}} (-1)^{g_{k}(m)} 2^{\ell-1},$$
(50)

where F_k is the kth Fibonacci number.

Proof. The substitution $a_i = 1$, $b_0 = 0$, and $b_j = 1$ in (1) and the recurrence formulas (10) and (11) gives that $A_k = F_k$ and $B_k = F_{k+1}$. The same classical numerators and denominators arise from the substitution $a_i = 2$ in Corollary 19. Then (32) and (33) gives (49) along with

$$F_{k+1} = \sigma(k+1) - \sum_{\substack{m \in \mathcal{U}_k \\ \lambda_\ell > 1}} (-1)^{g_k(m)} 2^\ell - \sum_{\substack{m \in \mathcal{U}_{k+1} \\ \lambda_\ell > 1}} (-1)^{g_{k+1}(m)} 2^\ell.$$

Shifting $k \mapsto k-1$ in this identity and adding it to (49) yields (50).

It is also possible to compute F_k using Corollary 20. The classical numerators of the continued fraction

$$\frac{-1}{-1} + \frac{-1}{1} + \frac{-1}{-1} + \frac{-1}{1} + \dots$$
(51)

are $\tau(k)F_k$, where

$$\tau(k) = \begin{cases} -1 & \text{if } k \equiv 1, 2 \pmod{4} \\ 1 & \text{if } k \equiv 3, 4 \pmod{4}. \end{cases}$$

The substitutions $b_{2j-1} = -2$ and $b_{2j} = 0$ in the continued fraction in Corollary 20 give (51).

Corollary 26.

$$\tau(k)F_k = -\sigma(k) + \sum_{m \in \mathcal{V}_{k-1}^{\text{odd}}} (-1)^{g_{k-1}(m)} (-2)^{\ell} + \sum_{m \in \mathcal{V}_k^{\text{odd}}} (-1)^{g_k(m)} (-2)^{\ell},$$

where $\mathcal{V}_k^{\text{odd}}$ is the subset of \mathcal{V}_k whose monomials have indices which are all odd.

Note that when k is even, property **V2** implies $\mathcal{V}_k^{\text{odd}} = \emptyset$. Since $\tau(2k-1) = (-1)^k$ and $\tau(2k) = (-1)^k$,

$$(-1)^k F_{2k-1} = -\sigma(2k-1) + \sum_{m \in \mathcal{V}_{2k-1}^{\text{odd}}} (-1)^{g_{2k-1}(m)} (-2)^\ell,$$

and

$$(-1)^k F_{2k} = -\sigma(2k) + \sum_{m \in \mathcal{V}_{2k-1}^{\text{odd}}} (-1)^{g_{2k-1}(m)} (-2)^{\ell}.$$

Turning to the Pell numbers, the kth classical numerator of the continued fraction

$$\frac{1/4}{1} + \frac{1/4}{1} + \frac{1/4}{1} + \frac{1/4}{1} + \cdots$$
 (52)

is $P_k/2^{k+1}$. Substituting $a_i = 5/4$ into the continued fraction in Corollary 19 yields (52).

Corollary 27.

$$P_{k} = -2^{k+1}\sigma(k) + \sum_{m \in \mathcal{U}_{k-1}} (-1)^{g_{k-1}(m)} 5^{\ell} 2^{k+1-2\ell} + \sum_{m \in \mathcal{U}_{k}} (-1)^{g_{k}(m)} 5^{\ell} 2^{k+1-2\ell},$$
(53)

and

$$P_k = 2^{k+1} \left(-\sigma(k) + \sum_{m \in \mathcal{U}_{k-1}} (-1)^{g_{k-1}(m)} (5/4)^{\ell} + \sum_{m \in \mathcal{U}_k} (-1)^{g_k(m)} (5/4)^{\ell} \right).$$

This gives an interpretation of the fact that Pell number

$$P_k \equiv v(k) \pmod{5},\tag{54}$$

where v(k) is the 12 periodic sequence $0, 1, 2, 0, 2, 4, 0, 4, 3, 0, 3, 1, \ldots$ starting from k = 0. Observe that multiplying both sides of (53) by 2^{k-1} gives

$$2^{k-1}P_k \equiv -2^{2k}\sigma(k) \pmod{5},$$

since in the sums $2k - 2\ell \ge 0$. Because $gcd(2^{k-1}, 5) = 1$,

$$P_k \equiv -2^{k+1}\sigma(k) \pmod{5}.$$

The periodicity of $2^k \pmod{5}$ and $-\sigma(k) \pmod{5}$ now yield (54).

Next, the kth classical numerator of the continued fraction

$$\frac{-1}{-2} + \frac{-1}{2} + \frac{-1}{-2} + \frac{-1}{2} + \dots$$
(55)

is $\tau(k)P_k$. We can apply Corollary 20 by making substitutions $b_{2k-1} = -3$ and $b_{2k} = 1$.

Corollary 28.

$$\tau(k)P_k = -\sigma(k) + \sum_{m \in \mathcal{V}_{k-1}} (-1)^{g_{k-1}(m)} (-3)^{\ell_{\text{odd}}} + \sum_{m \in \mathcal{V}_k} (-1)^{g_k(m)} (-3)^{\ell_{\text{odd}}},$$

where $\ell_{odd}(m)$ counts the odd indices of m.

We conclude with an application of Theorem 18. Let n_k and p_k be the number of terms of R_k that have negative sign and positive sign, respectively. From the recurrence formula (23), these sequences satisfy

$$n_k = p_{k-3} + n_{k-3} + 2n_{k-2} + n_{k-1},$$

and

$$p_k = n_{k-3} + p_{k-3} + 2p_{k-2} + p_{k-1}$$

Let $J_{k+1} = p_k - n_k$. Then J_k satisfies the recurrence formula $J_k = J_{k-1} + 2J_{k-2}$, with initial conditions $J_1 = J_2 = 1$; these are the Jacobsthal numbers. One property of the Jacobosthal numbers is that $J_k + J_{k-1} = 2^{k-1}$ for $k \ge 0$. Thus from (30),

$$-\sigma(k) + \sum_{m \in \mathcal{R}_{k-1}} (-1)^{g_{k-1}(m)} + \sum_{m \in \mathcal{R}_k} (-1)^{g_k(m)} = 2^k.$$
(56)

5. Tables

The following two figures show the relations between the different polynomials associated with K encountered in this paper.

$R_n(a_1, a_2, \ldots$	$(a_n; b_0, b_1, \ldots, b_n)$
Support counted by	Monomials in \mathcal{R}_n counted by
$\int r_0 = 1, \ r_1 = 3, \ r_2 = 5,$	$s_0 = 1, \ s_1 = 2, \ s_2 = 5,$
$\int r_n = r_{n-1} + 2r_{n-2} + 2r_{n-3}$	$s_3 = 13, \ s_4 = 28, \ s_5 = 65,$
with generating function	$s_n = s_{n-1} + 2s_{n-2} + 3s_{n-3}$
1 + 2x	$-s_{n-4}-2s_{n-5}-2s_{n-6}$
$1 - x - 2x^2 - 2x^3$.	with generating function
	$1 - x + x^2 + x^3$
	$\frac{1-x-2x^2-3x^3+x^4+2x^5+2x^6}{1-x-2x^6}.$

$$- \begin{array}{c} R_n(a_1, a_2, \dots, a_k; 0, 0, \dots, 0) \\ \text{Support counted by} \\ u_0 = 0, \ u_1 = 2, \ u_2 = 0, \\ u_n = u_{n-2} + 2u_{n-3} \\ \text{with generating function} \\ \frac{2x}{1 - x^2 - 2x^3} \\ \end{array} \qquad \begin{array}{c} s_0 = 0, \ s_1 = 1, \ s_2 = 0, \\ s_3 = 2, \ s_4 = 3, \ s_5 = 2, \\ s_n = s_{n-2} + 3s_{n-3} - s_{n-5} \\ -2s_{n-6} \\ \text{with generating function} \\ \frac{x + x^3}{1 - x^2 - 3x^3 + x^5 + 2x^6} \\ \end{array}$$

$R_n(0,0,\ldots)$	$(0,0,0,b_1,\ldots,b_n)$
Tribonaccis	$s_0 = 0, s_1 = 0, s_2 = 1.$
$\int T_0 = 0, \ T_1 = 1, \ T_2 = 1,$	$s_{3} = 2, s_{4} = 3, s_{5} = 7,$
 $ \left(T_n = T_{n-1} + T_{n-2} + T_{n-3} \right) $ with momentum function	$\begin{cases} s_n = s_{n-1} + s_{n-2} + 2s_{n-3} \end{cases}$
x	$(-s_{n-4}-s_{n-5}-s_{n-6})$
$\overline{1-x-x^2-x^3}$	with generating function
	$x^2 + x^3$
	$1 - x - x^2 - 2x^3 + x^4 + x^5 + x^6$

Figure 1: Counting sequences related to polynomials R_n and their special cases. For Tribonaccis, see [9].

$P_n(a_1, a_2,$	$(a_n; b_0, b_1, \dots, b_n)$
Support counted by	Monomials in $\mathcal{R}_n \cup \mathcal{R}_{n-1}$ counted by
$\int p_0 = 1, \ p_1 = 4, \ p_2 = 8,$	$s_0 = 1, \ s_1 = 3, \ s_2 = 7,$
$p_n = p_{n-1} + 2p_{n-2} + 2p_{n-3}$	$s_3 = 18, \ s_4 = 41, \ s_5 = 93,$
with generating function	$s_n = s_{n-1} + 2s_{n-2} + 3s_{n-3}$
$\frac{1+3x+2x^2}{2}$	$-s_{n-4}-2s_{n-5}-2s_{n-6}$
$1 - x - 2x^2 - 2x^3$	with generating function
	$1 + 2x + 2x^2 + 2x^3 + x^4$
	$\overline{1-x-2x^2-3x^3+x^4+2x^5+2x^6}$.

$P_n(a_1, a_2, a_3)$	$, \ldots, a_k; 0, 0, \ldots, 0)$
Support counted by	Monomials in $\mathcal{U}_n \cup \mathcal{U}_{n-1}$ counted by
$\int p_0 = 0, \ p_1 = 2, \ p_2 = 2,$	$s_0 = 0, \ s_1 = 1, \ s_2 = 1,$
$p_n = p_{n-2} + 2p_{n-3}$	$s_3 = 2, \ s_4 = 5, \ s_5 = 5,$
with generating function	$s_n = s_{n-2} + 3s_{n-3} - s_{n-5}$
$\frac{2x+2x^2}{1-2x^2}$	$(-2s_{n-6})$
$1 - x^2 - 2x^3$	with generating function
	$x + x^2 + x^3$
	$\overline{1 - x^2 - 3x^3 + x^5 + 2x^6}.$
	$P_n(a_1, a_2, a_3)$ Support counted by $\begin{cases} p_0 = 0, \ p_1 = 2, \ p_2 = 2, \\ p_n = p_{n-2} + 2p_{n-3} \end{cases}$ with generating function $\frac{2x + 2x^2}{1 - x^2 - 2x^3}.$

$P_n(0, 0,$	$(\ldots, 0; 0, b_1, \ldots, b_n)$
Support counted by	Monomials in $\mathcal{V}_n \cup \mathcal{V}_{n-1}$ counted by
Tribonacci-type sequence	$s_0 = 0, \ s_1 = 0, \ s_2 = 1,$
$\int p_0 = 0, \ p_1 = 1, \ p_2 = 2,$	$s_3 = 3, s_4 = 5, s_5 = 10,$
 $ \int p_n = p_{n-1} + p_{n-2} + p_{n-3} $	$s_n = s_{n-1} + s_{n-2} + 2s_{n-3}$
with generating function	$-s_{n-4}-s_{n-5}-s_{n-6}$
$x + x^2$	with generating function
$\overline{1-x-x^2-x^3}.$	$x^2 + 2x^3 + x^4$
	$1 - x - x^2 - 2x^3 + x^4 + x^5 + x^6$

Figure 2: Counting sequences related to classical numerators P_n and their special cases. Note that the generating functions are a product of (1 + x) and the generating functions of the polynomials in the previous figure.

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