Edge-Isoperimetric Inequalities and Ball-Noise Stability: Linear Programming and Probabilistic Approaches

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Abstract

Let Q_n^r be the graph with vertex set $\{-1,1\}^n$ in which two vertices are joined if their Hamming distance is at most r. The edge-isoperimetric problem for Q_n^r is that: For every (n,r,M) such that $1 \leq r \leq n$ and $1 \leq M \leq 2^n$, determine the minimum edge-boundary size of a subset of vertices of Q_n^r with a given size M. In this paper, we apply two different approaches to prove bounds for this problem. The first approach is a linear programming approach and the second is a probabilistic approach. Our bound derived by the first approach generalizes the tight bound for $M=2^{n-1}$ derived by Kahn, Kalai, and Linial in 1989. Moreover, our bound is also tight for $M=2^{n-2}$ and $r\leq \frac{n}{2}-1$. Our bounds derived by the second approach are expressed in terms of the noise stability, and they are shown to be asymptotically tight as $n\to\infty$ when $r=2\lfloor \frac{\beta n}{2}\rfloor+1$ and $M=\lfloor \alpha 2^n\rfloor$ for fixed $\alpha,\beta\in(0,1)$, and is tight up to a factor 2 when $r=2\lfloor \frac{\beta n}{2}\rfloor$ and $M=\lfloor \alpha 2^n\rfloor$. In fact, the edge-isoperimetric problem is equivalent to a ball-noise stability problem which is a variant of the traditional (i.i.d.-) noise stability problem. Our results can be interpreted as bounds for the ball-noise stability problem.

Keywords: Isoperimetric inequalities, noise stability, Fourier analysis, linear programming bound, probabilistic approach, hypercontractivity

1. Introduction

The isoperimetric problem is one of most classic problems, which is to determine the minimum possible boundary-size (i.e., perimeter) of a set with a fixed size (i.e., volume). A famous result for the isoperimetric problem in the n-dimensional Euclidean space states that an n-ball has the smallest surface area per given volume. In last several decades, an analogue of the isoperimetric problem was considered in the discrete setting. Let G = (V, E) be a graph and $A \subseteq V$ a non-empty subset of vertices of G. The edge-boundary ∂A of A is

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the set of all edges of G joining a vertex in A to a vertex in $V \setminus A$. The edge-isoperimetric problem for G asks for the determination of

$$\min\{|\partial A|: A \subseteq V, |A| = M\},\tag{1}$$

for each integer M. When the graph G is set to (the powers of) discrete hypercubes, the corresponding isoperimetric problem attracts a lot of attentions due to its importance to related problems in combinatorics, discrete probability, computer science, social choice theory, and others; see e.g. [15, 7, 5, 3, 1]. For the hypercube² $\{-1, 1\}^n$, the Hamming distance $d_{\rm H}(\mathbf{x}, \mathbf{y}) := |\{i: x_i \neq y_i\}|$ between two vectors \mathbf{x} and \mathbf{y} in $\{-1, 1\}^n$ is defined as the number of coordinates in which they differ. For positive integers n and r such that $r \leq n$, we let Q_n^r denote the r-th power of the n-dimensional discrete hypercube graph, i.e., the graph with vertex-set $\{-1, 1\}^n$ in which two vectors are joined if they are Hamming distance at most r apart. When r = 1, the hypercube graph Q_n^r is denoted as Q_n for brevity. The edge-isoperimetric problem for Q_n^r is hence formulated as follows. For every (n, r, M) such that $r \in [1:n]$ and $M \in [1:2^n]$, determine the minimum boundary-size of a subset (also termed a code) of Q_n^r with a given size M. Throughout this paper, we denote the (normalized) volume as

$$\alpha := \frac{M}{2^n}$$
 and $\beta := \frac{r}{n}$.

The edge-isoperimetric problem is also related to the estimate of distance distribution of a subset in the hypercube Q_n . For a graph G=(V,E) and a non-empty subset $A\subseteq V$, the subgraph induced by A is denoted as G[A], which is the graph whose vertex set is A and whose edge set consists of all of the edges in E that have both endpoints in A. Let e(A) denote the number of edges of G[A]. Indeed, if G is a d-regular graph, then $2e(A) + |\partial A| = d|A|$ for all $A\subseteq V$. Denote $B_r^{(n)}:=\{\mathbf{x}:d_{\mathbf{H}}(\mathbf{x},\mathbf{1})\leq r\}$ (or shortly B_r) as the r-radius ball with center $\mathbf{1}=\{1,1,...,1\}$. Denote the cardinality of $B_r^{(n)}$ as $\binom{n}{\leq r}:=\sum_{i=0}^r\binom{n}{i}$. Similarly, we denote the Hamming sphere with the same radius and center as $S_r^{(n)}:=\{\mathbf{x}:d_{\mathbf{H}}(\mathbf{x},\mathbf{1})=r\}$ (or shortly S_r) and its cardinality as $\binom{n}{r}$. Since Q_r^r is $\binom{n}{\leq r}-1$ -regular, it holds that for $A\subseteq \{-1,1\}^n$ with size M, $2e(A)+|\partial A|=\binom{n}{\leq r}-1$ M. Hence, the edge-isoperimetric problem is equivalent to determining

$$\max\{e(A): A \subseteq \{-1, 1\}^n, |A| = M\}.$$

²Without loss of generality, one can also consider the hypercube as $\{0,1\}^n$, and our results can be easily converted into this case via a simple bijection $x \in \{0,1\} \mapsto (-1)^x$. We choose " $\{-1,1\}^n$ " since the Fourier transform of a function on this set is easier to present.

³Throughout this paper, we denote [a:b] for $a,b \in \mathbb{R}$ such that $a \leq b$ as the set of integers between a and b (i.e., $[a,b] \cap \mathbb{Z}$).

For a non-empty subset $A \subseteq \{-1,1\}^n$, the distance distribution of A is defined as the following probability mass function⁴:

$$P^{(A)}(i) := \frac{1}{|A|^2} |\{(\mathbf{x}, \mathbf{x}') \in A^2 : d_{\mathbf{H}}(\mathbf{x}, \mathbf{x}') = i\}|, \ i \in [0 : n].$$

It is clear that $P^{(A)}(0) = \frac{1}{|A|}$, $\sum_{i=0}^{n} P^{(A)}(i) = 1$, and $P^{(A)}(i) \geq 0$ for $i \in [0:n]$. Furthermore, by definition, if |A| = M then $e(A) = \frac{M^2}{2} \sum_{i=1}^{r} P^{(A)}(i)$. Hence, the edge-isoperimetric problem is also equivalent to determining

$$\max\{\sum_{i=0}^{r} P^{(A)}(i) : A \subseteq \{-1, 1\}^{n}, |A| = M\},\$$

i.e., the estimation of the cumulative distribution function of the distance distribution.

A trivial upper bound for the edge-isoperimetric problem is $\sum_{i=0}^r P^{(A)}(i) \leq 1$, which is attained if M is small enough, more precisely, if $M \leq \binom{n}{\leq b/2}$ (i.e., the optimal sets A are contained in a Hamming ball of radius b/2) [10, 7]. For r=1 (i.e., for Q_n), the edge-isoperimetric problem was solved by Harper [4], Lindsey [13], Bernstein [2] and Hart [6], who showed that lexicographic subsets are optimal in minimizing the edge-boundary size. Here, lexicographic subsets are subsets whose elements are given by initial segments of the lexicographic ordering on $\{-1,1\}^n$. In fact, lexicographic subsets are generalizations of subcubes, and reduce to subcubes when the sizes of them are 2^k for integers k. Furthermore, for $r \geq 2$ and $M = 2^{n-1}$, the edge-isoperimetric problem was solved by Kahn, Kalai, and Linial [7] in 1989, who showed that subcubes are also optimal for this case. However, the problem for $r \geq 2$ and $M \neq 2^{n-1}$ has remained open. In this paper, we make progress on these unsolved cases, more specifically, on the cases of $r \geq 2$ and $M = \alpha 2^n$ with $\alpha \in (0, \frac{1}{2})$. In other words, the case in which M is linear in 2^n is considered in this paper. For this case, we provide two bounds for the edge-isoperimetric problem. In particular, we prove that subcubes are also optimal for r > 2 and $M = 2^{n-2}$.

When M is exponential in n, by using an improved hypercontractivity inequality, Kirshner and Samorodnitsky [9] recently showed that for $M = 2^{nH(\sigma)}$ with $\sigma \in (0, \frac{1}{2})$ and for $i = n\lambda$ with $\lambda \leq 2\sigma(1 - \sigma)$,

$$P^{(A)}(i) \le 2^{n[\sigma H(\frac{\lambda}{2\sigma}) + (1-\sigma)H(\frac{\lambda}{2(1-\sigma)}) - H(\sigma)]},\tag{2}$$

where $H(p) := -p \log_2 p - (1-p) \log_2 (1-p)$ for $p \in [0,1]$ denotes the binary entropy (here, the convention $0 \log_2 0 = 0$ is adopted). By computing the derivative, one can find that given σ , the exponent at the right side of (2) is non-decreasing in λ for $\lambda \leq 2\sigma(1-\sigma)$.

⁴Our definition of the distance distribution is slightly different from the classic one (e.g., in [14]), since in the classic definition, the factor is $\frac{1}{|A|}$, rather than $\frac{1}{|A|^2}$. Our choice is more convenient when the code size is large (e.g., linear in 2^n).

Hence, the inequality in (2) implies that for $M = 2^{nH(\sigma)}$ with $\sigma \in (0, \frac{1}{2})$ and for $r = n\beta$ with $\beta \leq 2\sigma(1-\sigma)$,

$$\sum_{i=0}^{r} P^{(A)}(i) \le (r+1) 2^{n[\sigma H(\frac{\beta}{2\sigma}) + (1-\sigma)H(\frac{\beta}{2(1-\sigma)}) - H(\sigma)]}.$$
 (3)

When $r_n = \lfloor \beta n \rfloor$ for a fixed $\beta \in (0,1)$ and let $n \to \infty$, it holds that

$$\lim_{n \to \infty} -\frac{1}{n} \log_2 \max_{A:|A| \le 2^{nH(\sigma)}} \sum_{i=0}^{r_n} P^{(A)}(i)$$

$$= \begin{cases}
H(\sigma) - \sigma H(\frac{\beta}{2\sigma}) - (1 - \sigma) H(\frac{\beta}{2(1 - \sigma)}), & \beta \le 2\sigma(1 - \sigma) \\
0, & \beta > 2\sigma(1 - \sigma)
\end{cases} . \tag{4}$$

Here, the optimal exponent in (4) is attained by a sequence of Hamming balls with radii (approximately) equal to $n\sigma$. The first clause at the right side of (4) follows from (3), and the second one follows by the following facts: 1. when $\beta = 2\sigma(1-\sigma)$, the right side of (4) vanishes; 2. given a set $A, r \mapsto \sum_{i=0}^{r} P^{(A)}(i)$ is non-decreasing, which implies that the left side of (4) is non-increasing in β ; 3. the left side of (4) is non-negative. It is worth noting that if we replace $\sum_{i=0}^{r_n} P^{(A)}(i)$ at the left side of (4) with $P^{(A)}(r_n)$, then the asymptotic exponent is different from the above. Specifically, this new exponent is zero for $2\sigma(1-\sigma) \le \beta \le \frac{1}{2}$ and it is symmetric with respect to $\beta = \frac{1}{2}$; see details in Remark 29 of [9]. Furthermore, Rashtchian and Raynaud [17] also derived different bounds for the edge-isoperimetric problem for Q_n^r . Their bounds are tight up to a factor of $\exp(\Theta(r))$ (i.e., a factor depending only upon r).

1.1. Ball-Noise Stability: Probabilistic Reformulation of the Edge-Isoperimetric Problem

In this subsection, we reformulate the edge-isoperimetric problem in probabilistic language. Let $\mathbf{X} \sim \mathrm{Unif}\{-1,1\}^n$ and $\mathbf{Y} = \mathbf{X} \circ \mathbf{Z} = (X_i \cdot Z_i)_{1 \leq i \leq n}$ where $\mathbf{Z} \in \{-1,1\}^n$ is independent of \mathbf{X} and \circ denotes the Hadamard product (element-wise product).

Definition 1. For $f: \{-1,1\}^n \to \mathbb{R}$ and $r \in [0:n]$, the sphere-noise stability and ball-noise stability of f at r are respectively

$$\mathbf{SStab}_r[f] := \mathbb{E}[f(\mathbf{X})f(\mathbf{Y})] = \sum_{\mathbf{x}, \mathbf{y} \in f^{-1}(1)} \frac{1\{d_{\mathbf{H}}(\mathbf{x}, \mathbf{y}) = r\}}{2^n \binom{n}{r}}$$
(5)

with $\mathbf{X} \sim \text{Unif}\{-1,1\}^n, \mathbf{Z} \sim \text{Unif}(S_r)$, and

$$\mathbf{BStab}_r[f] := \mathbb{E}[f(\mathbf{X})f(\mathbf{Y})] = \sum_{\mathbf{x}, \mathbf{y} \in f^{-1}(1)} \frac{1\{d_{\mathbf{H}}(\mathbf{x}, \mathbf{y}) \leq r\}}{2^n \binom{n}{\leq r}}$$

with $\mathbf{X} \sim \text{Unif}\{-1,1\}^n, \mathbf{Z} \sim \text{Unif}(B_r).$

Obviously, the joint distribution P_{XY} is symmetric (i.e., $P_{XY} = P_{YX}$) if $X \sim \text{Unif}\{-1,1\}^n$ and $\mathbf{Z} \sim \text{Unif}(S_r)$ or $\mathbf{Z} \sim \text{Unif}(B_r)$. The edge-isoperimetric problem for Q_n^r is equivalent to the following question: For every (n, r, M) such that $1 \le r \le n$ and $1 \le M \le 2^n$, determine

$$\Gamma_{\mathbf{S}}^{(n)}(M,r) := \max_{\substack{f:\{-1,1\}^n \to \{0,1\}\\ \mathbb{P}[f=1]=\alpha}} \mathbf{SStab}_r[f]$$

$$\tag{6}$$

$$\Gamma_{S}^{(n)}(M,r) := \max_{\substack{f:\{-1,1\}^{n} \to \{0,1\} \\ \mathbb{P}[f=1] = \alpha}} \mathbf{SStab}_{r}[f]
\Gamma_{B}^{(n)}(M,r) := \max_{\substack{f:\{-1,1\}^{n} \to \{0,1\} \\ \mathbb{P}[f=1] = \alpha}} \mathbf{BStab}_{r}[f]$$
(6)

and their limits as $n \to \infty$ for fixed $\alpha = \frac{M}{2^n}$ and $\beta = \frac{r}{n}$. Here, the Boolean function $f: \{-1,1\}^n \to \{0,1\}$ in the optimizations can be seen as the indicator of the code A. Our motivation for studying the case in which M is linear in 2^n comes from this probabilistic formulation of the edge-isoperimetric problem, since in this probabilistic formulation, $\alpha = \frac{M}{2n}$ corresponds to the probability of the code A under the uniform measure. For even and odd values of r, the limit behaviour is in fact different for the case of interest here. For $\alpha, \beta \in (0, 1)$, we define

$$\Gamma_{\text{odd,S}}(\alpha,\beta) := \lim_{n \to \infty} \Gamma_{S}^{(n)}(\lfloor \alpha 2^{n} \rfloor, 2\lfloor \frac{\beta n}{2} \rfloor + 1)$$

$$\overline{\Gamma}_{\text{even,S}}(\alpha,\beta) := \limsup_{n \to \infty} \Gamma_{S}^{(n)}(\lfloor \alpha 2^{n} \rfloor, 2\lfloor \frac{\beta n}{2} \rfloor)$$

$$\underline{\Gamma}_{\text{even,S}}(\alpha,\beta) := \liminf_{n \to \infty} \Gamma_{S}^{(n)}(\lfloor \alpha 2^{n} \rfloor, 2\lfloor \frac{\beta n}{2} \rfloor). \tag{8}$$

By replacing sphere noise with ball noise, $\Gamma_{\text{odd},B}(\alpha,\beta)$, $\overline{\Gamma}_{\text{even},B}(\alpha,\beta)$, $\underline{\Gamma}_{\text{even},B}(\alpha,\beta)$ are defined similarly. (The limits in the definitions of $\Gamma_{\text{odd,S}}(\alpha,\beta)$ and $\Gamma_{\text{odd,B}}(\alpha,\beta)$ exist, which will be shown in Theorem 2.) We term the optimization problems in (6) and (7) respectively as the sphere-noise stability and ball-noise stability problems.

The sphere-noise stability and ball-noise stability problems can be seen as variants of the traditional i.i.d.-noise stability problem. In the traditional noise stability problem, $\mathbf{Z} \sim \mathrm{Rad}^{\otimes n}(\beta)$. Here $\mathrm{Rad}^{\otimes n}(\beta)$ denotes the *n*-product of the biased Rademacher distribution $\operatorname{Rad}(\beta), \beta \in (0, \frac{1}{2})$ with itself, where the biased Rademacher distribution $\operatorname{Rad}(\beta)$ is a distribution having the probability mass function

$$P_Z(z) = \begin{cases} 1 - \beta & z = 1\\ \beta & z = -1 \end{cases}.$$

The noise stability of a function f is defined as

$$\mathbf{Stab}_{\beta}[f] := \mathbb{E}[f(\mathbf{X})f(\mathbf{Y})] = \sum_{\mathbf{x}, \mathbf{y} \in f^{-1}(1)} \frac{\beta^{d_{\mathbf{H}}(\mathbf{x}, \mathbf{y})} (1 - \beta)^{n - d_{\mathbf{H}}(\mathbf{x}, \mathbf{y})}}{2^{n}}, \tag{9}$$

where $\mathbf{X} \sim \text{Unif}\{-1,1\}^n, \mathbf{Z} \sim \text{Rad}^{\otimes n}(\beta)$, and again $\mathbf{Y} = \mathbf{X} \circ \mathbf{Z}$. Similarly to (6)-(8), define

$$\Gamma_{\text{IID}}^{(n)}(M,\beta) := \max_{\substack{f:\{-1,1\}^n \to \{0,1\}\\ \mathbb{P}[f=1] = \alpha}} \mathbf{Stab}_{\beta}[f] \quad \text{ and } \quad \Gamma_{\text{IID}}(\alpha,\beta) := \lim_{n \to \infty} \Gamma_{\text{IID}}^{(n)}(\lfloor \alpha 2^n \rfloor, \beta).$$
 (10)

Obviously, the limit in (10) exists, since $\Gamma_{\text{IID}}^{(n)}(\lfloor \alpha 2^n \rfloor, \beta)$ is non-decreasing in n for given α, β . The edge-isoperimetric problem for Q_n^r and the ball-noise stability problem in (7) are equivalent, as shown in the following proposition.

Proposition 1. For $A \subseteq \{-1,1\}^n$ with size $\alpha 2^n$, $\mathbf{BStab}_r[1_A] = \frac{\alpha^2 2^n}{\binom{n}{s}} \sum_{i=0}^r P^{(A)}(i)$.

Proof.

$$\mathbf{BStab}_{r}[1_{A}] = \mathbb{P}[\mathbf{X} \in A, \mathbf{Y} \in A] = \sum_{\mathbf{x}, \mathbf{y} \in A} \frac{1\{d_{\mathbf{H}}(\mathbf{x}, \mathbf{y}) \leq r\}}{2^{n} \binom{n}{\leq r}}$$
$$= \frac{\alpha^{2} 2^{n}}{\binom{n}{\leq r}} \sum_{\mathbf{x}, \mathbf{y} \in A} \frac{1\{d_{\mathbf{H}}(\mathbf{x}, \mathbf{y}) \leq r\}}{(\alpha 2^{n})^{2}} = \frac{\alpha^{2} 2^{n}}{\binom{n}{\leq r}} \sum_{i=0}^{r} P^{(A)}(i).$$

Combining (4) and Proposition 1 yields that when⁵ $M_n = 2^{n(H(\sigma) + o_n(1))}$ (i.e., $\alpha_n = 2^{n(H(\sigma) - 1 + o_n(1))}$) and $r_n = \lfloor \beta n \rfloor$ for some fixed $\sigma \in (0, \frac{1}{2}), \beta \in (0, 1)$, it holds that

$$\lim_{n \to \infty} -\frac{1}{n} \log_2 \Gamma_{\mathcal{B}}^{(n)}(M_n, r_n)
= \begin{cases} D((1 - \sigma - \frac{\beta}{2}, \frac{\beta}{2}, \frac{\beta}{2}, \sigma - \frac{\beta}{2}) \| (\frac{1-\beta}{2}, \frac{\beta}{2}, \frac{\beta}{2}, \frac{1-\beta}{2})), & \beta \le 2\sigma(1-\sigma) \\ 1 + H(\min\{\beta, \frac{1}{2}\}) - 2H(\sigma), & \beta > 2\sigma(1-\sigma) \end{cases},$$
(11)

where $D(Q||P) := \sum_{x} Q(x) \log_2 \frac{Q(x)}{P(x)}$ denotes the relative entropy between two distributions Q and P.

In the literature, the (i.i.d.) noise stability problem was studied by Benjamini, Kalai, and Schramm [1]. When $\alpha = \frac{1}{2}$ and $n \geq 1$, $\Gamma_{\text{IID}}^{(n)}(2^{n-1},\beta) = \frac{1-\beta}{2}$. This is a consequence of Witsenhausen's results on maximal correlation [18]. When $\alpha = \frac{1}{4}$ and $n \geq 2$, Yu and Tan [21] showed that $\Gamma_{\text{IID}}^{(n)}(2^{n-2},\beta) = (\frac{1-\beta}{2})^2$. In the literature, hypercontractivity inequalities were also used to prove asymptotically tight (up to a factor $(\log \frac{1}{\alpha})^k$ for some k) bounds on $\Gamma_{\text{IID}}(\alpha,\beta)$ as $\alpha \to 0$ for fixed β ; see [15, 8, 21]. Kirshner and Samorodnitsky's improved hypercontractivity inequality in Theorem 8 of [9] implies that when $M_n = 2^{n(H(\sigma) + o_n(1))}$ (i.e., $\alpha = 2^{n(H(\sigma) - 1 + o_n(1))}$) for some $\sigma \in (0, \frac{1}{2})$, the exponent of $\Gamma_{\text{IID}}^{(n)}(M_n, \beta)$ for $\beta \in (0, \frac{1}{2})$ is

$$\lim_{n \to \infty} -\frac{1}{n} \log_2 \Gamma_{\text{IID}}^{(n)}(M_n, \beta) = \min_{Q_{XY}: Q_X = Q_Y = (1 - \sigma, \sigma)} D(Q_{XY} || P_{XY})$$
(12)

$$= \min_{0 \le \theta \le 2\sigma} D((1 - \sigma - \frac{\theta}{2}, \frac{\theta}{2}, \frac{\theta}{2}, \sigma - \frac{\theta}{2}) \| (\frac{1 - \beta}{2}, \frac{\beta}{2}, \frac{\beta}{2}, \frac{1 - \beta}{2})), (13)$$

where the right side of (12) is termed the minimum relative entropy over couplings of $Q_X = Q_Y = (1 - \sigma, \sigma)$ [22, 19], and the unique optimal θ attaining the minimum in (13) is

⁵Throughout this paper, we use $o_n(1)$ to denote a term (or sequence) that vanishes as $n \to \infty$.

 $\theta^* = \frac{\sqrt{1+4(\kappa-1)\sigma(1-\sigma)}-1}{\kappa-1}$ with $\kappa = (\frac{1-\beta}{\beta})^2$. Here, the optimal exponent in (13) is attained by a sequence of Hamming balls (or Hamming spheres). The result in (12) and (13) was generalized to the two set version of noise stability and also generalized to arbitrary distributions on finite alphabets or Polish spaces in [22, 19]. Furthermore, for $\sigma < \frac{1}{2}, \beta \leq 2\sigma(1-\sigma)$, it holds that $\theta^* < \beta$, which implies that for this case, the expression in (13) is strictly smaller than that in (11).

1.2. Main Results

In this paper, we study the discrete edge-isoperimetric problem for Q_n^r with $r \geq 1$. We apply two different techniques to derive bounds for this problem. The first one is Fourier analysis combined with linear programming duality. By such a technique, we prove the following bound which is called linear programming (LP) bound.

Theorem 1 (Linear Programming Bounds). 1. For $\alpha := \frac{M}{2^n} \leq \frac{1}{2}$,

$$\Gamma_{\rm B}^{(n)}(M,r) \le \alpha^2 \left[1 - \frac{\psi_n^+(\alpha,r)}{\binom{n}{\le r}}\right] + \alpha (1 - \alpha) \frac{\binom{n-1}{r}}{\binom{n}{\le r}},\tag{14}$$

where $\psi_n^+(\alpha,r) := [\psi_n(\alpha,r)]^+$ and

$$\psi_{n}(\alpha, r) := \begin{cases}
\max_{\substack{odd \ k \in [n - \tau(n):n] \\ odd \ k \in [n - \tau(n):n] \\ even \ r = n/2 - 1}} \frac{n \binom{n-2}{2k-n}}{2k-n} [2(\frac{1}{\alpha} - 1) - \frac{1}{\alpha} \frac{n+1}{k+1}], & even \ r \leq n/2 - 1 \\
\max_{\substack{k \in [n - \tau(n):n] \cap F \\ odd \ k \in [\frac{n}{2} + 1:n]}} \frac{(n-1)\binom{n-2}{r-1}}{2k-n-1} [2(\frac{1}{\alpha} - 1) - \frac{1}{\alpha} \frac{n}{k}], & odd \ r \leq n/2 - 1 \\
\max_{\substack{k \in [n - \tau(n):n] \cap F \\ odd \ k \in [\frac{n}{2} + 1:n]}} \frac{k\binom{n-2}{r+1}}{(2k-n)} [2(\frac{1}{\alpha} - 1) - \frac{1}{\alpha} \frac{n}{k}], & even \ r > n/2 - 1 \\
\max_{\substack{k \in [\frac{n}{2} + 1:n] \\ odd \ k \in [\frac{n}{2} + 1:n]}} \frac{k\binom{n-2}{r}}{(2k-n)} [2(\frac{1}{\alpha} - 1) - \frac{1}{\alpha} \frac{n}{k}], & odd \ r > n/2 - 1
\end{cases} (15)$$

with

$$\tau(n) := \frac{1}{2} \left(\frac{n}{2} + 2 - \sqrt{\frac{n}{2} + 2} \right) \tag{16}$$

and

$$F := \left\{ even \ k : \ k \ge \frac{n+r+1}{2} \right\}$$

$$\cup \left\{ odd \ k : \ k \ge \max\{\frac{n+1}{2}, \frac{(n-1)r}{n-1-r}\} + \sqrt{\frac{(n+1)r}{2(n-1-r)}} \right\}.$$
(17)

2. When considering the asymptotic case as $n \to \infty$, we have that for fixed $\delta > 0$,

$$\Gamma_{\rm B}^{(n)}(M,r) \le \alpha^2 (1 - (1 - 2\beta)\phi_n(\alpha,r)) + o_n(1)$$
(18)

⁶Throughout this paper, $[x]^+ := \max\{x, 0\}$.

holds for all $\alpha := \frac{M}{2^n} \in [\delta, \frac{1}{2} - \delta], \beta := r/n \in (0, \frac{1}{2} - \delta] \cup [\frac{1}{2}, 1],$ where $o_n(1)$ is only dependent on δ and independent of M, r. Here,

$$\phi_n(\alpha, r) := \begin{cases} \beta \varphi(\alpha) - (\frac{1}{\alpha} - 1), & even \ r \le n/2 - 1\\ \beta \hat{\varphi}(\alpha, \beta) - (\frac{1}{\alpha} - 1), & odd \ r \le n/2 - 1\\ 0, & r > n/2 - 1 \end{cases}$$

where

$$\varphi(\alpha) := \begin{cases} \frac{2(1-\sqrt{\alpha})^2}{\alpha}, & 0 \le \alpha < 1/4\\ \frac{1}{\alpha} - 2, & 1/4 \le \alpha \le 1/2 \end{cases}
\hat{\varphi}(\alpha, \beta) := \begin{cases} \frac{1}{2\hat{\eta} - 1} \left[2(\frac{1}{\alpha} - 1) - \frac{1}{\alpha\hat{\eta}} \right], & 0 \le \alpha < 1/4\\ \frac{1}{\alpha} - 2, & 1/4 \le \alpha \le 1/2 \end{cases}$$

with $\hat{\eta} := \max\left\{\frac{1}{2(1-\sqrt{\alpha})}, \min\left\{\frac{1+\beta}{2}, \frac{\beta}{1-\beta}\right\}\right\}$.

For comparison, observe that when $\alpha = 2^{-k}$ for a positive integer k, every (n - k)-dimensional Hamming subcube C_{n-k} (e.g., $\{1\}^k \times \{-1,1\}^{n-k}$) attains the following ball-noise stability:

$$\mathbf{BStab}_r[1_{C_{n-k}}] = \alpha \frac{\binom{n-k}{\leq r}}{\binom{n}{\leq r}}.$$
 (19)

In particular, for fixed k, as $n, r \to \infty$ and $r/n \to \beta$,

$$\mathbf{BStab}_r[1_{C_{n-k}}] \to (\frac{1-\beta}{2})^k. \tag{20}$$

Definition 2. The even part A_{even} of a set $A \subseteq \{-1, 1\}^n$ is defined as $\{\mathbf{x} \in A : d_{\mathbf{H}}(\mathbf{x}, \mathbf{1}) \text{ is even}\}$, i.e., the intersection of A and the set of vectors \mathbf{x} of even Hamming weight $d_{\mathbf{H}}(\mathbf{x}, \mathbf{1})$. The odd part of A is defined as $A_{\text{odd}} = A \setminus A_{\text{even}}$.

For even r, the same ball-noise stability as in (19) can be also achieved by the even part (or odd part) of an (n - k + 1)-dimensional subcube, e.g., $\{\mathbf{x} \in \{1\}^{k-1} \times \{-1,1\}^{n+1-k} : d_{\mathbf{H}}(\mathbf{x}, \mathbf{1}) \text{ is even}\}.$

Comparing Theorem 1 with (19) and (20), we know that the bound in (14) is tight for $\alpha = 1/2$ and $n \ge 1$ as well as for $\alpha = 1/4$, $r \le n/2 - 1$, and $n \ge 2$. For the former case $(\alpha = 1/2 \text{ and } n \ge 1)$, $\psi_n^+(1/2, r) = 0$, which leads to the tight result

$$\Gamma_{\rm B}^{(n)}(2^{n-1},r) = \frac{\binom{n-1}{\leq r}}{2\binom{n}{\leq r}} \text{ and } \Gamma_{\rm B}(1/2,\beta) = \frac{1-\beta}{2}, \ \beta \in (0,1).$$

This recovers a classic result derived by Kahn, Kalai, and Linial [7]. For the latter case $(\alpha = 1/4, r \leq n/2 - 1, \text{ and } n \geq 2), \psi_n(1/4, r) = 2\binom{n-2}{r-1}$, where the optimal k attaining

 $\psi_n(1/4,r)$ in (15) for even $r \le n/2 - 1$ is $k^* = n$ for odd n, and $k^* = n - 1$ for even n, and the optimal k attaining $\psi_n(1/4,r)$ for odd $r \le n/2 - 1$ is $k^* = n$. This leads to that

$$\Gamma_{\rm B}^{(n)}(2^{n-2},r) = \frac{\binom{n-2}{\leq r}}{4\binom{n}{< r}} \text{ and } \Gamma_{\rm B}(1/4,\beta) = (\frac{1-\beta}{2})^2, \ \beta \in (0,1/2).$$

This result is new.

For fixed $\alpha = \frac{M}{2^n} \leq \frac{1}{2}$ and sufficiently large n, Rashtchian and Raynaud's bound in [17] reduces to the following bound:

$$\Gamma_{\rm B}^{(n)}(M,r) \le \frac{2\alpha}{\binom{n}{\le r}} \left[\frac{16en}{r} (n - \log_2 \frac{1}{\alpha}) \right]^{r/2}.$$

When considering the setting of interest in this paper (i.e., the case of fixed $\alpha = \frac{M}{2^n} \in (0, \frac{1}{2}]$, $\beta = \frac{r}{n}$, but $n \to \infty$), Rashtchian and Raynaud's bound becomes $\frac{2\alpha}{\binom{n}{n}} \left[\frac{16e}{\beta}(n - \log_2 \frac{1}{\alpha})\right]^{n\beta/2}$, which tends to ∞ as $n \to \infty$. Hence, their bound is trivial for our setting. We next compare our result with Kirshner and Samorodnitsky's in (2). We first restate their result in the language of the sphere-noise stability. Similarly to Proposition 1, for sphere noise, it holds that $\mathbf{SStab}_r[1_A] = \frac{\alpha^2 2^n}{\binom{n}{r}} P^{(A)}(r)$. Substituting the bound in (2) into this formula yields the following bound: For $M = 2^{nH(\sigma)}$ with $\sigma \in (0, \frac{1}{2})$ and $r = n\beta$ with $\beta \leq 2\sigma(1 - \sigma)$,

$$\Gamma_{S}^{(n)}(M,r) \le \frac{1}{\binom{n}{r}} 2^{n[-\beta \log_2 \beta - (\sigma - \frac{\beta}{2}) \log_2(2\sigma - \beta) - (1 - \sigma - \frac{\beta}{2}) \log_2(2 - 2\sigma - \beta)]}.$$
 (21)

For fixed $0 < \alpha \le \frac{1}{2}$, by solving $2^{nH(\sigma)} = \alpha 2^n$, we obtain $\sigma = \frac{1-t}{2}$ where $t = 2\sqrt{\frac{1}{2n}\ln\frac{1}{\alpha}} + o_n(\frac{1}{\sqrt{n}})$. On the other hand, for fixed $0 < \beta \le \frac{1}{2}$, Stirling's formula implies $\binom{n}{r} = 2^{nH(\beta) - \frac{1}{2}\log_2(2\pi n) + O_n(1)}$. By this formula, the right side of (21) reduces to $2^{-\frac{2}{1-\beta}\log_2\frac{1}{\alpha} + \frac{1}{2}\log_2(2\pi n) + O(1)}$, which obviously also tends to ∞ as $n \to \infty$. In other words, Kirshner and Samorodnitsky's bound becomes trivial as well for our setting. However, it should be noted that Kirshner and Samorodnitsky's bound outperforms our bound when α vanishes exponentially as $n \to \infty$, since their bound is exponentially tight in this setting.

The proof of Theorem 1 is given in Section 2. Here we provide the outline of the proof. In our proof, we first relax the edge-isoperimetric problem to a linear program by employing Fourier analysis. By duality in linear programming, we then rewrite this program as its dual. Finally, we find a feasible solution to the dual program which hence provides a lower bound for the primal program. Such a lower bound also results in a lower bound for the edge-isoperimetric problem.

We next present our second bound, which is proven by a probabilistic approach.

Theorem 2 (Probabilistic Bounds). For $0 < \alpha, \beta \le \frac{1}{2}$,

$$\Gamma_{\text{odd,S}}(\alpha,\beta) = \Gamma_{\text{odd,B}}(\alpha,\beta) = \Gamma_{\text{IID}}(\alpha,\beta),$$
 (22)

$$\Gamma_{\text{IID}}(\alpha, \beta), \frac{1}{2}\Gamma_{\text{IID}}(2\alpha, \beta) \leq \underline{\Gamma}_{\text{even,S}}(\alpha, \beta) \leq \overline{\Gamma}_{\text{even,S}}(\alpha, \beta) \leq 2\Gamma_{\text{IID}}(\alpha, \beta),$$
 (23)

$$\Gamma_{\text{IID}}(\alpha, \beta) \leq \underline{\Gamma}_{\text{even,B}}(\alpha, \beta) \leq \overline{\Gamma}_{\text{even,B}}(\alpha, \beta) \leq 2(1 - \beta)\Gamma_{\text{IID}}(\alpha, \beta).$$
 (24)

We conjecture that for $0 < \alpha, \beta \leq \frac{1}{2}$, $\underline{\Gamma}_{\text{even,S}}(\alpha, \beta) = \overline{\Gamma}_{\text{even,S}}(\alpha, \beta) = \frac{1}{2}\Gamma_{\text{IID}}(2\alpha, \beta)$ and $\underline{\Gamma}_{\text{even,B}}(\alpha, \beta) = \overline{\Gamma}_{\text{even,B}}(\alpha, \beta) = \Gamma_{\text{IID}}(\alpha, \beta)$. This is true if $\overline{\Gamma}_{\text{even,S}}(\alpha, \beta)$ is attained by $\frac{1+\chi_{[1:n]}}{2}f_n$ for some Fourier-weight stable (see Definition 3 in Section 3) sequence of Boolean functions f_n and $\overline{\Gamma}_{\text{even,B}}(\alpha, \beta)$ is attained by some Fourier-weight stable sequence of Boolean functions g_n . For the ball noise case, this conjecture was confirmed positively for $\alpha = 1/2$ by combining Kahn, Kalai, and Linial's result [7] and Witsenhausen's result [18], and for $\alpha = 1/4$ by combining Yu and Tan's result [21] and Theorem 1 above.

The proof of Theorem 2 is given in Section 3. As observed in Proposition 1, the edge-isoperimetric problem for Q_n^r is equivalent to a ball-noise stability problem. By Fourier analysis, we show that this ball-noise stability and the sphere-noise stability are bounded by the traditional i.i.d. noise stability. Hence, we obtain lower bounds for the edge-isoperimetric problem.

Instead of the noise stability problem, a hypercontractivity inequality for the sphere noise was recently investigated by Polyanskiy [16]. Polyanskiy's hypercontractivity inequality differs from the sphere-noise stability problem here in two aspects: 1. The hypercontractivity inequality concerns estimations of $\mathbb{E}[f(\mathbf{X})f(\mathbf{Y})]$ for all possible real-valued or complex-valued functions f, while the noise stability problem restricts f to be a Boolean (i.e., binary-valued) function; 2. the noise model (X, Y) in Polyanskiy's hypercontractivity inequality is formed by adding the sphere noise twice to the uniform random variable X, i.e., $Y = X \circ Z_1 \circ Z_2$ where \mathbf{Z}_1 and \mathbf{Z}_2 are i.i.d. sphere noises, while the noise model (\mathbf{X},\mathbf{Y}) in the sphere-noise stability problem here is formed by adding the sphere noise only once, i.e., $\mathbf{Y} = \mathbf{X} \circ \mathbf{Z}$ where \mathbf{Z} is a sphere noise. Note that for independent Rademacher noises \mathbf{Z}_1 and \mathbf{Z}_2 , the Hadamard product $\mathbf{Z}_1 \circ \mathbf{Z}_2$ is still a Rademacher noise. However, this is not true for sphere noise. Even so, our strategy to prove Theorem 2 is similar to the one used by Polyanskiy [16], and both of them exploit the connections between the i.i.d. Rademacher noise and the sphere-noise (or ball-noise). More precisely, the Rademacher noise $\mathbf{Z} \sim \mathrm{Rad}^{\otimes n}(\beta)$ in fact can be regarded as a "smoothing" version (weighted sum, or 'binomial analogue') of the sphere-noise (compare the expressions in (5) and (9)), and moreover, the corruption effect by the Rademacher noise is mainly determined by the component (the sphere-noise) of radius around $n\beta$ (which is well-known in information theory).

The quantity $\Gamma_{\text{IID}}(\alpha, \beta)$ was widely studied in the literature. Define $\rho := 1 - 2\beta$ and $\Lambda_{\rho}(\alpha)$ as the Gaussian quadrant probability defined by $\Lambda_{\rho}(\alpha) = \mathbb{P}[Z_1 > t, Z_2 > t]$, where Z_1, Z_2 are joint standard Gaussians with correlation $\mathbb{E}[Z_1 Z_2] = \rho$ and t is a real number such that $\mathbb{P}[Z_1 > t] = \alpha$. The small-set expansion theorem on [15, p. 264] states that

$$\Gamma_{\text{IID}}(\alpha, \beta) \le \alpha^{\frac{1}{1-\beta}}.$$
 (25)

On the other hand, for all $0 < \alpha \le \frac{1}{2}$, Hamming balls [15, Exercise 5.32] yield the lower bound

$$\Gamma_{\text{IID}}(\alpha, \beta) \ge \Lambda_{\rho}(\alpha),$$
 (26)

and for $\alpha = 2^{-k}$ with a positive integer k, Hamming subcubes yield the lower bound

$$\Gamma_{\text{IID}}(\alpha, \beta) \ge \left(\frac{1-\beta}{2}\right)^k.$$
 (27)

Combining Theorem 2 with (25)-(27) yields the following theorem.

Theorem 3 (Hypercontractivity Bounds). For $0 < \alpha, \beta \le \frac{1}{2}$,

$$\Lambda_{\rho}(\alpha) \leq \Gamma_{\text{odd,S}}(\alpha, \beta) = \Gamma_{\text{odd,B}}(\alpha, \beta) \leq \alpha^{\frac{1}{1-\beta}},
\frac{1}{2}\Lambda_{\rho}(2\alpha) \leq \underline{\Gamma}_{\text{even,S}}(\alpha, \beta) \leq \overline{\Gamma}_{\text{even,S}}(\alpha, \beta) \leq 2\alpha^{\frac{1}{1-\beta}},
\Lambda_{\rho}(\alpha) \leq \underline{\Gamma}_{\text{even,B}}(\alpha, \beta) \leq \overline{\Gamma}_{\text{even,B}}(\alpha, \beta) \leq 2(1-\beta)\alpha^{\frac{1}{1-\beta}}.$$

Moreover, when $k = \log_2 \frac{1}{\alpha}$ is a positive integer, it also holds that

$$\Gamma_{\text{odd,S}}(\alpha,\beta), \ \underline{\Gamma}_{\text{even,B}}(\alpha,\beta) \ge (\frac{1-\beta}{2})^k \quad and \quad \underline{\Gamma}_{\text{even,S}}(\alpha,\beta) \ge \frac{1}{2}(\frac{1-\beta}{2})^{k-1}.$$
 (28)

Similarly to the ball-noise case, when $\alpha=2^{-k}$ for a positive integer k, Hamming subcubes C_{n-k} attain the sphere-noise stability $\alpha\binom{n-k}{r}/\binom{n}{r}$. However, for even r, the even part (or odd part) of C_{n-k+1} attains the sphere-noise stability $\alpha[\binom{n-k}{r}+\binom{n-k}{r-1}]/\binom{n}{r}$, which is strictly larger than the one by C_{n-k} . In particular, for fixed k, as $n,r\to\infty$ and $r/n\to\beta$, it holds that $\alpha[\binom{n-k}{r}+\binom{n-k}{r-1}]/\binom{n}{r}\to\frac{1}{2}(\frac{1-\beta}{2})^{k-1}$, i.e., the second lower bound in (28). Similarly, for even r, the lower bound $\frac{1}{2}\Lambda_{\rho}(2\alpha)$ is asymptotically achieved by the even part (or odd part) of a Hamming ball with volume 2α .

In fact, it was shown in [15, Exercise 9.24] that given β , $\Lambda_{\rho}(\alpha) = \widetilde{\Theta}(\alpha^{\frac{1}{1-\beta}})$ as $\alpha \to 0$. Hence, the bounds in Theorem 3 are asymptotically tight (up to a factor $(\log \frac{1}{\alpha})^k$ for some k) as $\alpha \to 0$ for fixed β . By comparing the LP bounds in Theorem 1 and the hypercontractivity bounds in Theorem 3, it is easy to see that the LP bounds are tighter when α is close to $\frac{1}{2}$ and the hypercontractivity bounds are tighter when α is close to 0.

2. Proof of Theorem 1

In this section, we apply Fourier analysis combined with linear programming duality to prove Theorem 1. We first introduce Fourier analysis and Krawtchouk polynomials, and also derive new properties of Krawtchouk polynomials. By using Fourier analysis, we then relax the edge-isoperimetric problem to a linear program. By duality in linear programming, we then rewrite this program as its dual. Finally, we find a feasible solution to the dual program which hence provides a lower bound for the primal program. Such a lower bound results in a lower bound for the edge-isoperimetric problem.

2.1. Fourier Analysis and Krawtchouk Polynomials

A subset A is uniquely determined by its characteristics function 1_A . For this Boolean function 1_A , the Fourier expansion and Fourier weights are defined as follows. Consider the Fourier basis $\{\chi_S\}_{S\subseteq[1:n]}$ with $\chi_S(\mathbf{x}) := \prod_{i\in S} x_i$ for $S\subseteq[1:n]$. Then for a function $f:\{-1,1\}^n\to\mathbb{R}$, define its Fourier coefficients as

$$\hat{f}_S := \mathbb{E}_{\mathbf{X} \sim \text{Unif}\{-1,1\}^n} [f(\mathbf{X})\chi_S(\mathbf{X})], \ S \subseteq [1:n].$$

Then the Fourier expansion of the function f (cf. [15, Equation (1.6)]) is $f(\mathbf{x}) = \sum_{S \subseteq [1:n]} \hat{f}_S \chi_S(\mathbf{x})$. The degree-k Fourier weight of f is defined as $\mathbf{W}_k[f] := \sum_{S:|S|=k} \hat{f}_S^2$, $k \in [0:n]$. For brevity, we denote $\mathbf{W}_k[f]$ as \mathbf{W}_k . By definition, it is easily seen that for $f = 1_A$, $\mathbf{W}_0 = \alpha^2$ and $\sum_{k=0}^n \mathbf{W}_k = \alpha$, where $\alpha = |A|/2^n$. For a code $A \subseteq \{-1,1\}^n$, define the scaled degree-k Fourier weight of 1_A as $Q^{(A)}(k) := \frac{1}{\alpha^2} \mathbf{W}_k$. If A is a linear code, then $Q^{(A)}$ is the distance distribution of the dual of code A, and hence is also called the dual distribution of A. For details, please refer to [14]. By definition,

$$Q^{(A)}(0) = 1, \sum_{i=0}^{n} Q^{(A)}(i) = \frac{1}{\alpha}, \text{ and } Q^{(A)}(i) \ge 0 \text{ for } i \in [0:n].$$
 (29)

For each $k \in [0:n]$ and indeterminate x, the Krawtchouk polynomials [14] are defined as⁷

$$K_k^{(n)}(x) := \sum_{j=0}^k (-1)^j \binom{x}{j} \binom{n-x}{k-j},\tag{30}$$

whose generating function satisfies

$$\sum_{k=0}^{\infty} K_k^{(n)}(x) z^k = (1-z)^x (1+z)^{n-x}.$$
 (31)

For brevity and if there is no ambiguity, we denote $K_k^{(n)}$ as K_k . It is well-known that (see,

⁷ Here for a real number x and an integer j, the (generalized) binomial coefficients $\binom{x}{j} := \frac{x(x-1)\cdots(x-j+1)}{j!}$ if j > 0; $\binom{x}{j} := 1$ if j = 0; and $\binom{x}{j} := 0$ if j < 0. Obviously, by definition, $\binom{n}{j} \ge 0$ for nonnegative integer n. In particular, for nonnegative integer n, $\binom{n}{j} = 0$ if j < 0 or n < j. See more properties on MacWilliams and Sloane [14, pp. 13-14].

e.g., [14])

$$K_{0}(i) = 1, \ \forall i \in [0:n];$$

$$K_{k}(0) = \binom{n}{k}, \ K_{k}(1) = \binom{n}{k}(1 - \frac{2k}{n}),$$

$$K_{k}(2) = \binom{n}{k}(1 - \frac{4k(n-k)}{n(n-1)}) \ \forall k \in [0:n];$$

$$\sum_{S \subseteq [1:n]} \chi_{S}(\mathbf{x})\chi_{S}(\mathbf{x}')z^{|S|} = (1-z)^{d_{H}(\mathbf{x},\mathbf{x}')}(1+z)^{n-d_{H}(\mathbf{x},\mathbf{x}')}, \ \forall \mathbf{x}, \mathbf{x}' \in \{-1,1\}^{n};$$

$$K_{k}(d_{H}(\mathbf{x},\mathbf{x}')) = \sum_{S:|S|=k} \chi_{S}(\mathbf{x})\chi_{S}(\mathbf{x}'), \ \forall k \in [0:n], \mathbf{x}, \mathbf{x}' \in \{-1,1\}^{n}.$$
(32)

Taking expectation for both sides of (32), with respect to $(\mathbf{X}, \mathbf{X}') \sim \text{Unif}^{\otimes 2}(A)$, yields the following relationship between $P^{(A)}$ and $Q^{(A)}$:

$$Q^{(A)}(k) = \sum_{i=0}^{n} P^{(A)}(i)K_k(i) \text{ and } P^{(A)}(k) = \frac{1}{2^n} \sum_{i=0}^{n} Q^{(A)}(i)K_k(i).$$
 (33)

These are so-called MacWilliams–Delsarte identities [14].

We now provide the following extremal property of Krawtchouk polynomials. The proof of Lemma 1 is provided in Appendix A.

Lemma 1. For an integer $n \ge 1$, the following hold:

- 1. For $0 \le k, i \le n$, we have $K_k^{(n)}(0) \ge |K_k^{(n)}(i)|$.
- 2. For $0 \le k \le \frac{n-1}{2}$ and $1 \le i \le n-1$, we have $K_k^{(n)}(1) \ge |K_k^{(n)}(i)|$.
- 3. For $0 \le k \le \tau(n)$ (defined in (16)) and $2 \le i \le n-2$, we have $K_k^{(n)}(2) \ge |K_k^{(n)}(i)|$.

Note that the upper threshold $\tau(n)$ in Statement 3 is not sharp. Numerical simulation shows that the upper threshold can be sharpened to a value close to n/2. Proving this seems not easy. However, it can be proven when n is sufficiently large; see the following lemma. This lemma is just a strengthening of [20, Lemma 3] and the proof is similar to that of [20, Lemma 3]. Hence we omit the proof of the following lemma.

Lemma 2. Given a non-negative integer i and $\delta > 0$, for all sufficiently large n,

$$K_k^{(n)}(i) \ge |K_k^{(n)}(x)|, \ \forall k \in [\delta n : (\frac{1}{2} - \delta)n], x \in [i, n - i].$$
 (34)

Combining Lemmas 1 and 2 yields that given $\delta > 0$, for all sufficiently large n,

$$K_k^{(n)}(2) \ge |K_k^{(n)}(j)|, \ \forall k \in [0: (\frac{1}{2} - \delta)n], j \in [2, n - 2].$$
 (35)

2.2. Linear Program and Its Dual

By the MacWilliams–Delsarte identity (33),

$$\sum_{k=0}^{r} P^{(A)}(k) = \frac{1}{2^n} \sum_{i=0}^{n} Q^{(A)}(i) \sum_{k=0}^{r} K_k(i).$$
 (36)

We define

$$\omega_i^{(r)} := \sum_{k=0}^r K_k(i) = K_r^{(n-1)}(i-1)$$
(37)

(for the equality, see [12, Equation (54)]), and for brevity, we also denote $\omega_i^{(r)}$ as ω_i . Then,

$$\sum_{k=0}^{r} P^{(A)}(k) = \frac{1}{2^n} \sum_{i=0}^{n} Q^{(A)}(i)\omega_i.$$
(38)

Note that, in particular, $\omega_0 = \binom{n}{\leq r}$ and $\omega_1 = \binom{n-1}{r}$. From (29), (33), and $P^{(A)}(k) \geq 0$, the following properties of $Q^{(A)}$ hold:

$$Q^{(A)}(i) \ge 0, \ i \in [0:n], \quad Q^{(A)}(0) = 1, \quad \sum_{i=0}^{n} Q^{(A)}(i) = \frac{1}{\alpha},$$
 (39)

and
$$\sum_{i=0}^{n} Q^{(A)}(i)K_k(i) \ge 0, \ k \in [0:n].$$
 (40)

Substituting (39) into (38), we obtain

$$\sum_{k=0}^{r} P^{(A)}(k) = \frac{1}{2^{n}} \left[\omega_{0} + \left(\frac{1}{\alpha} - 1 - \sum_{i=2}^{n} Q^{(A)}(i)\right) \omega_{1} + \sum_{i=2}^{n} Q^{(A)}(i) \omega_{i} \right]$$
$$= \frac{1}{2^{n}} \left[\omega_{0} + \left(\frac{1}{\alpha} - 1\right) \omega_{1} - \sum_{i=2}^{n} Q^{(A)}(i) (\omega_{1} - \omega_{i}) \right].$$

We now consider a relaxed version of the minimization of $\sum_{i=2}^{n} Q^{(A)}(i)(\omega_1 - \omega_i)$ over the dual distance distribution $Q^{(A)}$. Instead of the discrete optimization of $\sum_{i=2}^{n} Q^{(A)}(i)(\omega_1 - \omega_i)$ (since given n, there are only finitely many codes and the corresponding dual distance distributions), we allow $(Q^{(A)}(0), Q^{(A)}(1), ..., Q^{(A)}(n))$ to be any nonnegative vector $(u_0, u_1, ..., u_n)$ such that

$$u_0 = 1, u_i \ge 0, \ i \in [2:n]; \quad \sum_{i=0}^n u_i = \frac{1}{\alpha}; \quad \sum_{i=0}^n u_i K_k(i) \ge 0, \ k \in [0:n].$$

Then in order to lower bound $\sum_{i=2}^{n} Q^{(A)}(i)(\omega_1 - \omega_i)$, we consider the following linear program.

Problem 1 (Primal Problem).

$$\Lambda_n(\alpha, r) := \min_{u_2, u_3, \dots, u_n} \sum_{i=2}^n u_i(\omega_1 - \omega_i)$$

subject to the inequalities

$$u_i \ge 0, \ i \in [2:n];$$

$$\sum_{i=2}^{n} [K_k(1) - K_k(i)] u_i \le K_k(0) + K_k(1) (\frac{1}{\alpha} - 1), \ k \in [1:n].$$

The dual is the following optimization problem.

Problem 2 (Dual Problem).

$$\overline{\Lambda}_n(\alpha, r) := \max_{x_1, x_2, \dots, x_n} -\sum_{k=1}^n [K_k(0) + K_k(1)(\frac{1}{\alpha} - 1)] x_k$$
(41)

subject to the inequalities

$$x_k \ge 0, \ k \in [1:n];$$

$$\sum_{k=1}^n [K_k(1) - K_k(i)] x_k \ge -(\omega_1 - \omega_i), \ i \in [2:n].$$
(42)

By strong duality in linear programming,⁸ $\Lambda_n(\alpha, r) = \overline{\Lambda}_n(\alpha, r)$. Therefore, the following holds.

Theorem 4. For any code A of size M, $\sum_{i=2}^{n} Q^{(A)}(i)(\omega_1 - \omega_i) \geq \overline{\Lambda}_n(\alpha, r)$.

2.3. Linear Programming Bounds

We next provide a lower bound for $\overline{\Lambda}_n(\alpha, r)$.

Theorem 5. For any code A of size M, $\overline{\Lambda}_n(\alpha, r) \ge \psi_n^+(\alpha, r)$, where ψ_n^+ was given in Theorem 1.

The proof of Theorem 5 is provided in Appendix B. In our proof, we constructed different feasible solutions for different cases. For example, for even $r \leq n/2 - 1$ our feasible solution is $\mathbf{x}^* = (0, ..., 0, x_k^*, x_{k+1}^*, 0, ..., 0)$ with $x_k^* = x_{k+1}^* = \frac{n\binom{n-2}{r-1}}{\binom{n}{k}(2k-n)}$, where k is an odd number such that $n - \tau(n) \leq k \leq n$. The feasibility of this solution follows since, on one hand, such a

⁸Obviously, in the primal problem, since $u_i \ge 0$, the primal problem is bounded. On the other hand, the existence of a code A with size $M := \alpha 2^n$ ensures that $u_i = Q^{(A)}(i)$ is a feasible solution. Hence the primal problem has an optimal solution.

solution guarantees that equality holds in (42) for i = 2, n; and on the other hand, after substituting \mathbf{x}^* into (42), it can be found that (42) holds for $i \in [2:n]$ if and only if it holds for i = 2, n. Hence \mathbf{x}^* is feasible. It is easy to see that it leads to the bound in Theorem 5 for even r < n/2 - 1. Other cases are proven similarly.

If we focus on sufficiently large n, then we can obtain a better bound, as shown in the following theorem. The proof of Theorem 6 is almost same as the proof of Theorem 5 except that Lemma 2 (more preciously, the inequality (35)), instead of Lemma 1, is applied.

Theorem 6. For any code A of size M and any $\delta > 0$, for sufficiently large n, the set " $[n-\tau(n):n]$ " in the first two clauses of $\psi_n(\alpha,r)$ in (15) can be replaced with " $[(\frac{1}{2}+\delta)n:n]$ ". In particular, when $n \to \infty$, for fixed $\delta > 0$,

$$\overline{\Lambda}_n(\alpha, r) \ge \kappa_n(\alpha, r) := \begin{cases} \binom{n-2}{r-1} (\varphi(\alpha) + o_n(1)), & even \ r \le n(\frac{1}{2} - \delta) \\ \binom{n-2}{r-1} (\hat{\varphi}(\alpha, \beta) + o_n(1)), & odd \ r \le n(\frac{1}{2} - \delta) \\ \binom{n-2}{r+1} (\varphi(\alpha) + o_n(1)), & even \ r \ge n/2 - 1 \\ \binom{n-2}{r} (\varphi(\alpha) + o_n(1)), & odd \ r \ge n/2 - 1 \end{cases}$$

holds for any $\alpha := \frac{M}{2^n} \in [\delta, \frac{1}{2}], \ \beta := r/n \in (0, \frac{1}{2} - \delta] \cup [\frac{1}{2}, 1], \ where \ all \ the \ terms \ o_n(1)$ are independent of M, r (the ones in first two clauses depend on δ).

Theorems 5 and 6 implies the following linear programming bound on $\sum_{k=0}^{r} P^{(A)}(k)$.

Theorem 7 (Linear Programming Bound). 1. For $\alpha = \frac{M}{2^n} \leq \frac{1}{2}$,

$$\sum_{k=0}^{r} P^{(A)}(k) \le \frac{1}{2^{n}} \left[\omega_0 + \left(\frac{1}{\alpha} - 1\right)\omega_1 - \psi_n^+(\alpha, r)\right]. \tag{43}$$

2. When considering the asymptotic case as $n \to \infty$, $\psi_n^+(\alpha, r)$ in (43) can be replaced by $\kappa_n(\alpha, r)$.

In the perspective of ball-noise stability, Theorem 7 can be rewritten as the bounds in Theorem 1. In particular, for the asymptotic case, it follows by the fact that for fixed $\delta > 0$, $\frac{\binom{n}{r}}{\binom{n}{\leq r}} \to \frac{1-2\beta}{1-\beta}$ uniformly for all $\beta := r/n \in [\delta, \frac{1}{2} - \delta]$; see Lemma 4 in the next section. This completes the proof of Theorem 1.

3. Proof of Theorem 2

In this section, we apply a probabilistic approach to prove Theorem 2. A similar approach was also used by Polyanskiy to study the hypercontractivity phenomenon under the sphere noise [16].

Similar to the i.i.d.-noise stability, the sphere- or ball-noise stability of a function can be also expressed in terms of Fourier weights of this function. For a random noise $\mathbf{Z} \in \{-1,1\}^n$, let \mathbb{T} be a noise operator such that for a function $f: \{-1,1\}^n \to \mathbb{R}$, $[\mathbb{T}f](\mathbf{x}) = \mathbb{E}[f(\mathbf{x} \circ \mathbf{Z})]$.

Then it is easy to verify that $\mathbb{T}\chi_S = (2\mathbb{P}[\chi_S(\mathbf{Z}) = 1] - 1)\chi_S$. Hence, for any function $f: \{-1, 1\}^n \to \mathbb{R}$,

$$\mathbb{T}f = \mathbb{T}\sum_{S \subseteq [1:n]} \hat{f}_S \chi_S = \sum_{S \subseteq [1:n]} \hat{f}_S \mathbb{T}\chi_S = \sum_{S \subseteq [1:n]} (2\mathbb{P}[\chi_S(\mathbf{Z}) = 1] - 1)\hat{f}_S \chi_S, \tag{44}$$

which further implies that

$$\mathbb{E}[f(\mathbf{X})f(\mathbf{Y})] = \langle f, \mathbb{T}f \rangle = \sum_{S \subseteq [1:n]} (2\mathbb{P}[\chi_S(\mathbf{Z}) = 1] - 1)\hat{f}_S^2$$

where $\mathbf{Y} = \mathbf{X} \circ \mathbf{Z}$. In particular, if $\mathbf{Z} \sim \text{Unif}(B_r)$, i.e., \mathbb{T} corresponds to the ball-noise operator \mathbb{B}_r , then

$$\mathbb{P}[\chi_S(\mathbf{Z}) = 1] = \mathbb{P}[\prod_{i \in S} Z_i = 1] = \mathbb{P}[\prod_{i=1}^{|S|} Z_i = 1],$$

where the last equality follows since the distribution of **Z** is invariant under the permutation operation. Hence, for ball-noise operator \mathbb{B}_r ,

$$\mathbf{BStab}_r[f] = \langle f, \mathbb{B}_r f \rangle = \sum_{k=0}^n (2\mathbb{P}[\prod_{i=1}^k Z_i = 1] - 1) \mathbf{W}_k. \tag{45}$$

For sphere noise operator \mathbb{S}_r , $\mathbf{SStab}_r[f]$ admits the same expression above, but in which $\mathbf{Z} \sim \mathrm{Unif}(S_r)$. We next provide the exact and asymptotic expressions for the coefficients $2\mathbb{P}[\prod_{i=1}^k Z_i = 1] - 1$ for sphere noise and ball noise.

Lemma 3. 1. For $\mathbb{Z} \sim \text{Unif}(S_r)$,

$$2\mathbb{P}[\prod_{i=1}^{k} Z_i = 1] - 1 = \frac{K_r(k)}{\binom{n}{r}}.$$
(46)

For fixed k and $\delta > 0$, as $n \to \infty$,

$$2\mathbb{P}[\prod_{i=1}^{k} Z_i = 1] - 1 \to (1 - 2\beta)^k \text{ and } \max_{j \in [k:n-k]} |2\mathbb{P}[\prod_{i=1}^{j} Z_i = 1] - 1| \to (1 - 2\beta)^k$$
 (47)

uniformly for all $\beta := r/n \in [\delta, 1/2 - \delta]$. 2. For $\mathbf{Z} \sim \text{Unif}(B_r)$,

$$2\mathbb{P}[\prod_{i=1}^{k} Z_i = 1] - 1 = \frac{K_r^{(n-1)}(k-1)}{\binom{n}{\leq r}}.$$
(48)

For fixed k and $\delta > 0$, as $n \to \infty$,

$$2\mathbb{P}[\prod_{i=1}^{k} Z_i = 1] - 1 \to (1 - 2\beta)^k \text{ and } \max_{j \in [k:n-k+1]} |2\mathbb{P}[\prod_{i=1}^{j} Z_i = 1] - 1| \to (1 - 2\beta)^k$$

uniformly for all $\beta := r/n \in [\delta, 1/2 - \delta]$.

Proof. We first prove Statement 1. Assume $\mathbf{Z} \sim \mathrm{Unif}(S_r)$. By (32) with $\mathbf{x}' \leftarrow \mathbf{1}$ and $\mathbf{x} \leftarrow \mathbf{Z}$, we have

$$\mathbb{E}[K_k(d_{\mathcal{H}}(\mathbf{Z}, \mathbf{1}))] = \mathbb{E}[\sum_{S:|S|=k} \chi_S(\mathbf{Z})\chi_S(\mathbf{1})]. \tag{49}$$

The right side of (49) is equal to $\sum_{S:|S|=k} \mathbb{E}[\chi_S(\mathbf{Z})] = \binom{n}{k} (2\mathbb{P}[\prod_{i=1}^k Z_i = 1] - 1)$. The left side of (49) is equal to $K_k(r) = \frac{\binom{n}{k}}{\binom{n}{r}} K_r(k)$, since $\binom{n}{i} K_k(i) = \binom{n}{k} K_i(k)$ holds for any nonnegative integers i, k. Hence, we obtain (46).

Lemma 4. 1. For fixed $k \geq i \geq 0$, $\frac{\binom{n-k}{r-i}}{\binom{n}{r}} \rightarrow \beta^i (1-\beta)^{k-i}$ uniformly for all $\beta := r/n \in [0,1]$.

- 2. For fixed $\delta > 0$, $\frac{\binom{n}{\leq r}}{\binom{n}{r}} \to \sum_{j=0}^{r} (\frac{\beta}{1-\beta})^j$ uniformly for all $\beta := r/n \in [0, \frac{1}{2} \delta]$.
- 3. For fixed $k \geq i \geq 0$ and $\delta > 0$, $\frac{\binom{n-k}{\leq r-i}}{\binom{n}{\leq r}} \to \beta^i (1-\beta)^{k-i}$ uniformly for all $\beta := r/n \in [\delta, \frac{1}{2} \delta]$.

The proof of Lemma 4 is provided in Appendix C.

When $\mathbf{Z} \sim \mathrm{Unif}(S_r)$, by definition, $\mathbb{P}[\prod_{i=1}^k Z_i = 1] = \frac{\sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \binom{n-k}{r-2j}}{\binom{n}{r}}$. By Statement 1 of Lemma 4, for fixed k,

$$\mathbb{P}[\prod_{i=1}^{k} Z_i = 1] \to \sum_{j=0}^{\lfloor k/2 \rfloor} {k \choose 2j} \beta^{2j} (1-\beta)^{k-2j} = \frac{1 + (1-2\beta)^k}{2}$$
 (50)

uniformly for all $\beta := r/n \in [0, 1]$. The equality above follows by the following interpretation of the middle term in (50). Consider $Y^{(k)} = \sum_{i=1}^k X_i$ with i.i.d. $X_i \sim \text{Bern}(\beta)$ (Bernoulli distribution on $\{0, 1\}$ with 1 having mass β) and denote $p_k := \mathbb{P}[Y^{(k)}]$ is even] and $q_k := \mathbb{P}[Y^{(k)}]$ is odd]. Then, the middle term in (50) is just p_k . Obviously, p_k , q_k satisfy the following recursive relations: $p_k = (1 - \beta)p_{k-1} + \beta q_{k-1}$ and $q_k = (1 - \beta)q_{k-1} + \beta p_{k-1}$, which imply $p_k - q_k = (1 - 2\beta)(p_{k-1} - q_{k-1}) = \dots = (1 - 2\beta)^k$. Combining this with $p_k + q_k = 1$ yields the equality in (50). Obviously, (50) is just restatement of the first convergence result in (47).

The second convergence result in (47) follows from the first one in (47), (46), and Lemma 2. This completes the proof of Statement 1.

Statement 2 follows similarly, and hence the proof is omitted here.

Substituting (48) into (45), we obtain

$$\mathbf{BStab}_{r}[f] = \frac{1}{\binom{n}{< r}} \sum_{k=0}^{n} \mathbf{W}_{k} K_{r}^{(n-1)}(k-1).$$

Similarly,

$$\mathbf{SStab}_r[f] = \frac{1}{\binom{n}{r}} \sum_{k=0}^n \mathbf{W}_k K_r^{(n)}(k). \tag{51}$$

In contrast, note that [15, Theorem 2.49] for i.i.d. noise,

$$\mathbf{Stab}_{\beta}[f] = \sum_{k=0}^{n} \mathbf{W}_{k} (1 - 2\beta)^{k}. \tag{52}$$

By Lemma 3, we obtain several relationships between $\mathbf{SStab}_r[f_n]$, $\mathbf{BStab}_r[f_n]$, and $\mathbf{Stab}_{\beta}[f_n]$ for a sequence of functions $\{f_n\}$. Before introducing these relationships, we first introduce a new concept on the tail behavior of Fourier weights.

Definition 3. A sequence of functions $\{f_n : \{-1,1\}^n \to \mathbb{R}\}_{n=1}^{\infty}$ is called Fourier-weight stable if $\lim_{n\to\infty} \sum_{k=n-k_0}^n \mathbf{W}_k[f_n] = 0, \forall k_0 \geq 0$ (or equivalently, $\lim_{k_0\to\infty} \lim_{n\to\infty} \sum_{k=n-k_0}^n \mathbf{W}_k[f_n] = 0$).

Theorem 8. 1. Let $\{f_n\}$ be a sequence of nonnegative (not necessarily Boolean) functions with bounded L^2 -norm, i.e., $\limsup_{n\to\infty} \mathbb{E}[f_n^2(\mathbf{X})] < \infty$. Let $\delta > 0$. Given n, let $r \in \mathbb{N}$ such that $\beta := r/n \in [\delta, 1/2 - \delta]$. Then, for even r,

$$\mathbf{Stab}_{\beta}[f_n] + o_n(1) \le \mathbf{SStab}_r[f_n] \le 2\mathbf{Stab}_{\beta}[f_n] + o_n(1), \tag{53}$$

$$\mathbf{Stab}_{\beta}[f_n] + o_n(1) \le \mathbf{BStab}_r[f_n] \le 2(1 - \beta)\mathbf{Stab}_{\beta}[f_n] + o_n(1), \tag{54}$$

and for odd r,

$$0 \le \mathbf{SStab}_r[f_n] \le \mathbf{Stab}_\beta[f_n] + o_n(1), \tag{55}$$

$$2\beta \mathbf{Stab}_{\beta}[f_n] + o_n(1) \le \mathbf{BStab}_r[f_n] \le \mathbf{Stab}_{\beta}[f_n] + o_n(1). \tag{56}$$

Here, the terms $o_n(1)$ in (53)-(56) are independent of r (given n), but dependent of δ .

- 2. Any sequence of nonnegative functions f_n supported on a subset of $\{\mathbf{x}: d_H(\mathbf{x}, \mathbf{1}) \text{ is even}\}\$ (or $\{\mathbf{x}: d_H(\mathbf{x}, \mathbf{1}) \text{ is odd}\}\$) attains the upper bounds in (53) and (54) for even r, and the lower bounds in (55) and (56) for odd r.
- 3. Any Fourier-weight stable sequence of nonnegative functions f_n (e.g., Hamming subcubes or Hamming balls) attains the lower bounds in (53) and (54) for even r, and the upper bounds in (55) and (56) for odd r.

Proof. We first consider the sphere noise case. By Lemma 3, for fixed k,

$$2\mathbb{P}\left[\prod_{i=1}^{k} Z_i = 1\right] - 1 = (1 - 2\beta)^k + o_n(1),$$

$$2\mathbb{P}\left[\prod_{i=1}^{n-k} Z_i = 1\right] - 1 = \frac{K_r(n-k)}{\binom{n}{r}} = \frac{(-1)^r}{\binom{n}{r}} K_r(k) = (-1)^r (1 - 2\beta)^k + o_n(1),$$

and for fixed k_0 ,

$$\max_{k \in [k_0+1: n-k_0-1]} |2\mathbb{P}[\prod_{i=1}^k Z_i = 1] - 1| = (1-2\beta)^{k_0+1} + o_n(1).$$

By these equalities, we have that for fixed k_0 ,

$$\mathbf{SStab}_r[f_n] = \sum_{k=0}^n \mathbf{W}_k(2\mathbb{P}[\prod_{i=1}^k Z_i = 1] - 1) \leq a(k_0) + (-1)^r b(k_0) \pm c(k_0) + o_n(1).$$

where

$$a(k_0) = \sum_{k=0}^{k_0} \mathbf{W}_k (1 - 2\beta)^k, \ b(k_0) = \sum_{k=0}^{k_0} \mathbf{W}_{n-k} (1 - 2\beta)^k, \ c(k_0) = (\sum_{k=k_0+1}^{n-k_0-1} \mathbf{W}_k) (1 - 2\beta)^{k_0+1}.$$
(57)

For odd r,

$$0 \le \mathbf{SStab}_r[f_n] = a(k_0) - b(k_0) + c(k_0) + o_n(1). \tag{58}$$

Equation (58) implies (55) since $a(k_0) \leq \sum_{k=0}^{n} \mathbf{W}_k (1 - 2\beta)^k = \mathbf{Stab}_{\beta}[f_n]$ and $c(k_0) \leq \limsup_{n \to \infty} \mathbb{E}[f_n^2] (1 - 2\beta)^{k_0 + 1} \to 0$ as $k_0 \to \infty$.

We now consider even r. For this case,

$$\mathbf{SStab}_r[f_n] \le a(k_0) + b(k_0) \pm c(k_0) + o_n(1). \tag{59}$$

Since $\operatorname{\mathbf{Stab}}_{\beta}[f_n] = \sum_{k=0}^n \mathbf{W}_k (1-2\beta)^k \le a(k_0) + b(k_0) + c(k_0)$ and $c(k_0) \to 0$ as $k_0 \to \infty$. We have $\operatorname{\mathbf{SStab}}_{\beta}[f_n] \ge \operatorname{\mathbf{Stab}}_{\beta}[f_n] + o_n(1)$.

On the other hand, for even r, (58) also implies that for a sequence of functions $\{f_n\}$,

$$b(k_0) \le a(k_0) + c(k_0) + o_n(1), \tag{60}$$

where the β 's in definitions of a, b, c in (57) are reset to $(r-1)/n = \beta - 1/n$ so that the radius r-1 is odd. Note that 1/n vanishes as $n \to \infty$, and hence, this asymptotically vanishing term can be merged into $o_n(1)$, which means that (60) still holds if a, b, c remain unchanged as in (57) (in other words, the β 's there are r/n). Substituting (60) into (59) yields that for even r,

$$\mathbf{SStab}_r[f_n] \le 2a(k_0) + 2c(k_0) + o_n(1) \le 2\mathbf{Stab}_{\beta}[f_n] + 2c(k_0) + o_n(1).$$

Since $k_0 > 0$ is arbitrary and $c(k_0) \to 0$ as $k_0 \to \infty$, we obtain (53).

For ball noise case, inequalities (54) and (56) can be proven similarly. The proof is omitted. Furthermore, if f is supported on a subset of $\{\mathbf{x}: d_{\mathbf{H}}(\mathbf{x}, \mathbf{1}) \text{ is even}\}$, then by definition, the Fourier coefficients of f satisfy that $\hat{f}_S = \hat{f}_{S^c}$ for any $S \subseteq [1:n]$. Hence, Statement 2 can be easily verified. In addition, Statement 3 can be easily verified as well. \square

We next turn back to prove Theorem 2.

Proof of Theorem 2. The upper bound in (23) and the upper and lower bounds in (24), as well as $\Gamma_{\text{odd},S}(\alpha,\beta)$, $\Gamma_{\text{odd},B}(\alpha,\beta) \leq \Gamma_{\text{IID}}(\alpha,\beta)$ and $\underline{\Gamma}_{\text{even},S}(\alpha,\beta) \geq \Gamma_{\text{IID}}(\alpha,\beta)$ follow directly from Theorem 8. It remains to prove $\Gamma_{\text{odd},S}(\alpha,\beta)$, $\Gamma_{\text{odd},B}(\alpha,\beta) \geq \Gamma_{\text{IID}}(\alpha,\beta)$ and $\underline{\Gamma}_{\text{even},S}(\alpha,\beta) \geq \frac{1}{2}\Gamma_{\text{IID}}(2\alpha,\beta)$.

We first prove $\Gamma_{\text{odd,S}}(\alpha, \beta) \ge \Gamma_{\text{IID}}(\alpha, \beta)$. Let $A \subseteq \{-1, 1\}^n$ be a subset of size M. Now we construct a new subset $B_k = A \times \{-1, 1\}^k$. Obviously, $B_k \subseteq \{-1, 1\}^{n+k}$ and $|B_k| = 2^k M$. Next, we prove that for fixed n, A,

$$\lim_{k \to \infty} \mathbf{SStab}_{2\lfloor \frac{(n+k)\beta}{2} \rfloor + 1} [1_{B_k}] \ge \mathbf{Stab}_{\beta} [1_A]. \tag{61}$$

For any $\mathbf{x} \in B_k$, we can write $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ where $\mathbf{x}_1 \in A$ and $\mathbf{x}_2 \in \{-1, 1\}^k$. Then we have

$$d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) = d_{\mathrm{H}}(\mathbf{x}_{1}, \mathbf{y}_{1}) + d_{\mathrm{H}}(\mathbf{x}_{2}, \mathbf{y}_{2}). \tag{62}$$

Denote $r_k := 2\lfloor \frac{(n+k)\beta}{2} \rfloor + 1$. Using (62) we obtain that under sphere noise,

$$\mathbf{SStab}_{r_k}[1_{B_k}] = \mathbb{P}[\mathbf{X} \in B_k, \mathbf{Y} \in B_k]$$

$$= \frac{\#\{(\mathbf{x}, \mathbf{y}) \in B_k^2 : d_{\mathbf{H}}(\mathbf{x}, \mathbf{y}) = r_k\}}{2^{n+k} \binom{n+k}{r_k}}$$

$$= \sum_{i=0}^n \#\{(\mathbf{x}_1, \mathbf{y}_1) \in A^2 : d_{\mathbf{H}}(\mathbf{x}_1, \mathbf{y}_1) = i\} \frac{\#\{(\mathbf{x}_2, \mathbf{y}_2) : d_{\mathbf{H}}(\mathbf{x}_2, \mathbf{y}_2) = r_k - i\}}{2^{n+k} \binom{n+k}{r_k}}$$

$$= \sum_{i=0}^n \#\{(\mathbf{x}_1, \mathbf{y}_1) \in A^2 : d_{\mathbf{H}}(\mathbf{x}_1, \mathbf{y}_1) = i\} \frac{\binom{k}{r_k - i}}{2^n \binom{n+k}{r_k}}$$

$$\to \frac{1}{2^n} \sum_{i=0}^n \#\{(\mathbf{x}_1, \mathbf{y}_1) \in A^2 : d_{\mathbf{H}}(\mathbf{x}_1, \mathbf{y}_1) = i\} \beta^i (1 - \beta)^{n-i} \text{ as } k \to \infty \qquad (63)$$

$$= \mathbf{Stab}_{\beta}[1_A],$$

where (63) follows by Lemma 4. Therefore, (61) holds, which implies $\Gamma_{\text{odd,S}}(\alpha, \beta) \geq \Gamma_{\text{IID}}(\alpha, \beta)$. Similarly, one can prove $\Gamma_{\text{odd,B}}(\alpha, \beta) \geq \Gamma_{\text{IID}}(\alpha, \beta)$.

We next prove $\underline{\Gamma}_{\text{even,S}}(\alpha,\beta) \geq \frac{1}{2}\Gamma_{\text{IID}}(2\alpha,\beta)$. We have shown that $\Gamma_{\text{odd,S}}(\alpha,\beta) = \Gamma_{\text{IID}}(\alpha,\beta)$ holds. We now claim that $\Gamma_{\text{odd,S}}(\alpha,\beta)$ is attained by a Fourier-weight stable sequence of Boolean functions, and moreover, this sequence also attains $\Gamma_{\text{IID}}(\alpha,\beta)$. We now prove it. For any optimal sequence of Boolean functions $\{f_n\}$ attaining $\Gamma_{\text{odd,S}}(\alpha,\beta)$, it holds that $\mathbf{Stab}_{\beta}[f_n] \geq a(k_0)$, where $a(k_0)$ was defined in (57). Combining this inequality with (58) yields that

$$\mathbf{SStab}_r[f_n] \le \mathbf{Stab}_{\beta}[f_n] - b(k_0) + c(k_0) + o_n(1).$$
 (64)

Taking limits as $n \to \infty$, we obtain

$$\Gamma_{\text{odd,S}}(\alpha,\beta) \leq \liminf_{n \to \infty} \mathbf{Stab}_{\beta}[f_n] - b(k_0) + c(k_0) \leq \Gamma_{\text{IID}}(\alpha,\beta) - \limsup_{n \to \infty} b(k_0) + (1-2\beta)^{k_0+1}.$$

Then, taking limits as $k_0 \to \infty$ and by using the equality $\Gamma_{\text{odd,S}}(\alpha,\beta) = \Gamma_{\text{IID}}(\alpha,\beta)$, we have that $\lim_{k_0 \to \infty} \lim \sup_{n \to \infty} b(k_0) = 0$, or equivalently, $\lim \sup_{n \to \infty} b(k_0) = 0$, $\forall k_0 \ge 0$, which implies that $\lim_{n \to \infty} \sum_{k=n-k_0}^{n} \mathbf{W}_k[f_n] = 0$, $\forall k_0 \ge 0$, i.e., $\{f_n\}$ is Fourier-weight stable.

Moreover, we also have that $\liminf_{n\to\infty} \mathbf{Stab}_{\beta}[f_n]$ is equal to $\Gamma_{\text{IID}}(\alpha,\beta)$. Hence, $\Gamma_{\text{IID}}(\alpha,\beta)$ is attained by $\{f_n\}$ as well, i.e., the claim is true.

We also need the following decomposition of a Boolean function. Any Boolean f can be written as $f = f_{\text{even}} + f_{\text{odd}}$, where $f_{\text{even}} = \frac{1+\chi_{[1:n]}}{2}f$ and $f_{\text{odd}} = \frac{1-\chi_{[1:n]}}{2}f$ are Boolean functions respectively supported on vectors \mathbf{x} of even and odd Hamming weights $d_{\text{H}}(\mathbf{x}, \mathbf{1})$. In fact, if A is the support of f, then the supports of f_{even} and f_{odd} are respectively the even part and odd part of A; see Definition 2. For functions f_{even} , f_{odd} , their Fourier coefficients satisfy that

$$\hat{f}_{\text{even},S} = \mathbb{E}_{\mathbf{X} \sim \text{Unif}\{-1,1\}^n} [f(\mathbf{X}) \frac{1 + \chi_{[1:n]}(\mathbf{X})}{2} \chi_S(\mathbf{X})]$$
$$= \mathbb{E}_{\mathbf{X} \sim \text{Unif}\{-1,1\}^n} [f(\mathbf{X}) \frac{\chi_S(\mathbf{X}) + \chi_{S^c}(\mathbf{X})}{2}] = \frac{\hat{f}_S + \hat{f}_{S^c}}{2}$$

and $\hat{f}_{\text{odd},S} = \frac{\hat{f}_S - \hat{f}_{S^c}}{2}$. Define the Fourier weights $\mathbf{W}_{\text{even},k} := \sum_{S:|S|=k} \hat{f}_{\text{even},S}^2$ and $\mathbf{W}_{\text{odd},k} := \sum_{S:|S|=k} \hat{f}_{\text{odd},S}^2$. From (44), under sphere noise,

$$\mathbb{E}[f_{\text{even}}(\mathbf{X})f_{\text{odd}}(\mathbf{Y})] = \langle f_{\text{even}}, \mathbb{S}_r f_{\text{odd}} \rangle = \sum_{S \subseteq [1:n]} \hat{f}_{\text{even},S} \hat{f}_{\text{odd},S} \frac{K_r(|S|)}{\binom{n}{r}}$$

$$= \sum_{S:|S| < n/2} \hat{f}_{\text{even},S} \hat{f}_{\text{odd},S} \frac{K_r(|S|)}{\binom{n}{r}} - \sum_{S:|S| < n/2} \hat{f}_{\text{even},S} \hat{f}_{\text{odd},S} \frac{K_r(|S|)}{\binom{n}{r}} = 0, \quad (65)$$

where $\mathbf{Y} = \mathbf{X} \circ \mathbf{Z}$ with $\mathbf{Z} \sim \text{Unif}(S_r)$, and in (65) $K_r(|S^c|) = K_r(|S|)$ for even r is applied. Given (α, β) , denote $\{f_n\}$ as an optimal Fourier-weight stable sequence of Boolean functions with support size $\lfloor 2\alpha 2^n \rfloor$ that attains $\Gamma_{\text{IID}}(2\alpha, \beta)$. For brevity, we omit the subscript n of f_n . Then for $r = 2\lfloor \frac{\beta n}{2} \rfloor$,

$$\Gamma_{\text{IID}}(2\alpha, \beta) = \mathbf{Stab}_{\beta}[f] = \mathbf{SStab}_{r}[f] + o_{n}(1)$$

$$= \mathbb{E}[[f_{\text{even}} + f_{\text{odd}}](\mathbf{X})[f_{\text{even}} + f_{\text{odd}}](\mathbf{Y})] + o_{n}(1)$$

$$= \mathbb{E}[f_{\text{even}}(\mathbf{X})f_{\text{even}}(\mathbf{Y})] + \mathbb{E}[f_{\text{odd}}(\mathbf{X})f_{\text{odd}}(\mathbf{Y})] + o_{n}(1)$$

$$\leq \underline{\Gamma}_{\text{even S}}(\alpha_{\text{even}}, \beta) + \underline{\Gamma}_{\text{even S}}(\alpha_{\text{odd}}, \beta) + o_{n}(1),$$
(67)

where (66) follows by Statement 3 of Theorem 8, and $\alpha_{\text{even}} = \mathbb{E}[f_{\text{even}}(\mathbf{X})]$, $\alpha_{\text{odd}} = \mathbb{E}[f_{\text{odd}}(\mathbf{X})]$. On the other hand, since f is Fourier-weight stable, we have

$$\mathbf{W}_{\text{even},0} = (\frac{\hat{f}_{\emptyset} + \hat{f}_{[1:n]}}{2})^2 \le (\frac{\sqrt{\mathbf{W}_0} + \sqrt{\mathbf{W}_n}}{2})^2 \to \frac{\mathbf{W}_0}{4}, \text{ as } n \to \infty,$$

and similarly $\mathbf{W}_{\text{odd},0} \to \frac{\mathbf{W}_0}{4}$ as $n \to \infty$. This means

$$\alpha_{\text{even}} \to \alpha \text{ and } \alpha_{\text{odd}} \to \alpha.$$
 (68)

Furthermore, we claim that $\underline{\Gamma}_{\text{even,S}}(\alpha,\beta)$ is continuous in α . Let $0 \leq \alpha_1 < \alpha_2 \leq 1$. We now prove this claim. Let 1_{A_n} be an optimal Boolean function attaining $\Gamma_{\text{S}}^{(n)}(\lfloor \alpha_2 2^n \rfloor, 2\lfloor \frac{\beta n}{2} \rfloor)$,

where A_n is the support of this optimal Boolean function. Let B_n be an arbitrary subset of A_n such that $\mathbb{P}[\mathbf{X} \in B_n] = \alpha_1$. Then,

$$\Gamma_{\mathbf{S}}^{(n)}(\lfloor \alpha_{2} 2^{n} \rfloor, 2\lfloor \frac{\beta n}{2} \rfloor) = \mathbb{P}[\mathbf{X} \in A_{n}, \mathbf{Y} \in A_{n}]$$

$$= \mathbb{P}[\mathbf{X} \in B_{n}, \mathbf{Y} \in B_{n}] + \mathbb{P}[\mathbf{X} \in A_{n} \backslash B_{n}, \mathbf{Y} \in B_{n}]$$

$$+ \mathbb{P}[\mathbf{X} \in B_{n}, \mathbf{Y} \in A_{n} \backslash B_{n}] + \mathbb{P}[\mathbf{X} \in A_{n} \backslash B_{n}, \mathbf{Y} \in A_{n} \backslash B_{n}]$$

$$\leq \mathbb{P}[\mathbf{X} \in B_{n}, \mathbf{Y} \in B_{n}] + 3\mathbb{P}[\mathbf{X} \in A_{n} \backslash B_{n}]$$

$$\leq \Gamma_{\mathbf{S}}^{(n)}(\lfloor \alpha_{1} 2^{n} \rfloor, 2\lfloor \frac{\beta n}{2} \rfloor) + 3\frac{\lfloor \alpha_{2} 2^{n} \rfloor - \lfloor \alpha_{1} 2^{n} \rfloor}{2^{n}}.$$

Taking $\liminf_{n\to\infty}$, we obtain

$$\underline{\Gamma}_{\text{even,S}}(\alpha_1, \beta) \leq \underline{\Gamma}_{\text{even,S}}(\alpha_2, \beta) \leq \underline{\Gamma}_{\text{even,S}}(\alpha_1, \beta) + 3(\alpha_2 - \alpha_1),$$

which implies the continuity of $\underline{\Gamma}_{\text{even},S}(\alpha,\beta)$ in α . (In fact, by the same argument, other quantities such as $\Gamma_{\text{IID}}(\alpha, \beta)$, $\overline{\Gamma}_{\text{even,S}}(\alpha, \beta)$, $\overline{\Gamma}_{\text{even,B}}(\alpha, \beta)$ etc. are also continuous in α .)

Finally, combining (67) and (68) and applying the continuity of $\underline{\Gamma}_{\text{even,S}}(\alpha,\beta)$ yields $\underline{\Gamma}_{\text{even,S}}(\alpha,\beta) \ge \frac{1}{2}\Gamma_{\text{IID}}(2\alpha,\beta).$

Appendix A. Proof of Lemma 1

Here we prove Lemma 1 by using the generating function method. Statement 1: By the equality $\sum_k K_k^{(n)}(i)z^k = (1-z)^i(1+z)^{n-i}$ where \sum_k means the summation over all integers (in fact, it can be replaced by $\sum_{k=0}^n$ for this equality), we have that

$$\sum_{k} \left[K_k^{(n)}(0) - K_k^{(n)}(i) \right] z^k = (1+z)^{n-i} \left[(1+z)^i - (1-z)^i \right]$$
(A.1)

$$=2(1+z)^{n-i}\left[\sum_{\text{odd }j} \binom{i}{j} z^j\right]. \tag{A.2}$$

Here in fact, the variable j under the summation in (A.2) can be additionally restricted to belong to [0:i]. Similarly, we have

$$\sum_{k} \left[K_k^{(n)}(0) + K_k^{(n)}(i) \right] z^k = 2(1+z)^{n-i} \left[\sum_{\text{even } j} {i \choose j} z^j \right]. \tag{A.3}$$

Since all coefficients in (A.2) and (A.3) are nonnegative, we have $K_k^{(n)}(0) \ge |K_k^{(n)}(i)|$. Statement 2: We first prove Statement 2 for odd i, i.e., the following claim.

Claim 1. $K_k^{(n)}(1) \ge |K_k^{(n)}(i)|$ holds for $0 \le k \le \frac{n-1}{2}$ and odd i such that $1 \le i \le n-1$.

Similarly to (A.1)-(A.2), we obtain for $1 \le i \le n-1$,

$$\sum_{k} [K_{k}^{(n)}(1) - K_{k}^{(n)}(i)] z^{k} = 2(1-z) \left[\sum_{j} {n-i \choose j} z^{j} \right] \left[\sum_{\text{odd } j} {i-1 \choose j} z^{j} \right]$$

$$= 2(1-z) \sum_{k} \left[\sum_{\text{odd } j} {n-i \choose k-j} {i-1 \choose j} \right] z^{k}$$

$$= 2 \sum_{k} a_{i,k} z^{k},$$

where

$$a_{i,k} := \sum_{\text{odd } i} \binom{i-1}{j} \left[\binom{n-i}{k-j} - \binom{n-i}{k-j-1} \right].$$

By the formula $\binom{m}{l} = \binom{m-1}{l} + \binom{m-1}{l-1}$, we can rewrite $a_{i,k} = \sum_{\text{odd } j} \binom{i-1}{j} (b_{k-j} - b_{k-j-2})$, where $b_l := \binom{n-i-1}{l}$. We next prove $a_{i,k} \geq 0$ for $k \leq \frac{n}{2}$ and odd $i \in [1:n-1]$. Observe that for odd i,

$$a_{i,k} = \sum_{\text{odd } j} {\binom{i-1}{j} - \binom{i-1}{j-2}} b_{k-j}$$

$$= \sum_{\text{odd } j < \frac{i+1}{2}} {\binom{i-1}{j} - \binom{i-1}{j-2}} b_{k-j} + {\binom{i-1}{i+1-j} - \binom{i-1}{i-1-j}} b_{k-(i+1-j)}$$

$$= \sum_{\text{odd } j < \frac{i+1}{2}} {\binom{i-1}{j} - \binom{i-1}{j-2}} [b_{k-j} - b_{k-(i+1-j)}].$$

We now require some basic properties of binomial coefficients. For a nonnegative integer m, the function $g: j \in \mathbb{Z} \mapsto \binom{m}{j}$ satisfies following properties.

- 1. g is symmetric with respect to $\frac{m}{2}$, i.e., g(j) = g(m-j).
- 2. g is nondecreasing for $j \leq \frac{m+1}{2}$ and nonincreasing for $j \geq \frac{m+1}{2}$.
- 3. $g(j_1) \leq g(j_2)$ for all integers j_1, j_2 such that $j_1 \leq j_2$ and $j_1 + j_2 \leq m$.

The first two properties follow by definition, and the third one follows by the first two since $g(j_1) \leq g(j_2)$ if $j_2 \leq \frac{m+1}{2}$, and $g(j_1) \leq g(m-j_2) = g(j_2)$ if $j_2 > \frac{m+1}{2}$. By the second property above, we have $\binom{i-1}{j} \geq \binom{i-1}{j-2}$ since $j < \frac{i+1}{2}$ (or equivalently,

By the second property above, we have $\binom{i-1}{j} \geq \binom{i-1}{j-2}$ since $j < \frac{i+1}{2}$ (or equivalently, $j \leq \frac{i}{2}$). By the last one, we have $b_{k-(i+1-j)} \leq b_{k-j}$, since $k-(i+1-j) \leq k-j$ and $k-j+k-(i+1-j)=2k-i-1 \leq n-i-1$. Hence $a_{i,k} \geq 0$, which implies $K_k^{(n)}(1) \geq K_k^{(n)}(i)$ for odd $i \in [1:n-1]$. Similarly, one can show that $K_k^{(n)}(1) \geq -K_k^{(n)}(i)$ for odd $i \in [1:n-1]$, just by replacing all the summations above over odd j with the corresponding ones over even j. Hence, Claim 1 holds.

We next use Claim 1 to prove Statement 2 for odd n and all $i \in [1:n-1]$. From Claim 1, $K_k^{(n)}(1) \ge |K_k^{(n)}(i)|$ for odd $i \in [1:n-1]$. On the other hand, $K_k^{(n)}(i) = (-1)^k K_k^{(n)}(n-i), 0 \le i, k \le n$ Krasikov [11, (11)]. Hence, $K_k^{(n)}(1) \ge |K_k^{(n)}(n-i)|$ for odd $i \in [1:n-1]$. If, additionally, n is odd, then n-i takes all even numbers in [1:n-1]. Hence, for odd n, $K_k^{(n)}(1) \ge |K_k^{(n)}(i)|$ holds for all $i \in [1:n-1]$.

We lastly prove Statement 2 for even n and all $i \in [1:n-1]$. By the conclusion for odd n above, $K_k^{(n-1)}(1) \geq |K_k^{(n-1)}(i)|$ holds for $0 \leq k \leq \frac{n-2}{2}$ and $1 \leq i \leq n-2$, and $K_{k-1}^{(n-1)}(1) \geq |K_{k-1}^{(n-1)}(i)|$ holds for $1 \leq k \leq \frac{n}{2}$ and $1 \leq i \leq n-2$. By the property that $K_k^{(n)}(i) = K_k^{(n-1)}(i) + K_{k-1}^{(n-1)}(i), 1 \leq k \leq n, 0 \leq i \leq n$ Levenshtein [12, (47)], we have that for $1 \leq k \leq \frac{n-2}{2}$ and $1 \leq i \leq n-2$,

$$K_k^{(n)}(1) = K_k^{(n-1)}(1) + K_{k-1}^{(n-1)}(1) \ge |K_k^{(n-1)}(i)| + |K_{k-1}^{(n-1)}(i)|$$

$$\ge |K_k^{(n-1)}(i) + K_{k-1}^{(n-1)}(i)| = |K_k^{(n)}(i)|.$$

Hence, it remains to verify the case that $k=0,\frac{n-1}{2},$ or i=n-1. First, note that $\frac{n-1}{2}$ is not an integer, and hence, k cannot equal it. We next verify the case that k=0 or i=n-1. By definition, for k=0, $K_0^{(n)}(i)=1$, and hence, $K_0^{(n)}(1)\geq |K_0^{(n)}(i)|$ holds obviously. For i=n-1, by definition, $K_k^{(n)}(1)=\binom{n}{k}(1-\frac{2k}{n})\geq 0$ for $k\leq n/2$, and $K_k^{(n)}(n-1)=(-1)^kK_k^{(n)}(1)$. Obviously, $K_k^{(n)}(1)=|K_k^{(n)}(n-1)|$ holds. Hence, Statement 2 holds for even n. Combining two points above (the cases of odd n and even n), Statement 2 holds for all $n\geq 1$.

Statement 3: For $2 \le i \le n-2$,

$$\sum_{k=0}^{n} \left[K_k^{(n)}(2) - K_k^{(n)}(i) \right] z^k = (1-z)^2 (1+z)^{n-i} \left[(1+z)^{i-2} - (1-z)^{i-2} \right]. \tag{A.4}$$

Observe that $(1+z)^{i-2} - (1-z)^{i-2} = \sum_{\text{odd } j \in [0:i-2]} {i-2 \choose j} z^j$ and

$$(1-z)^{2}(1+z)^{n-i} = (1-z)^{2} \left[\sum_{j=0}^{n-i} \binom{n-i}{j} z^{j} \right]$$

$$= \sum_{j=0}^{n-i} \left[\binom{n-i}{j} + \binom{n-i}{j-2} - 2\binom{n-i}{j-1} \right] z^{j}. \tag{A.5}$$

It is easy to verify that $\binom{n-i}{j}+\binom{n-i}{j-2}-2\binom{n-i}{j-1}\geq 0$ if $j\leq \frac{n-i+2-\sqrt{n-i+2}}{2}$. It means that $K_k^{(n)}(2)\geq K_k^{(n)}(i)$ when $k\leq \frac{n-i+2-\sqrt{n-i+2}}{2}$, since the terms z^j in (A.5) with j>k have no contribution to the term z^k in the final expansion in (A.4).

On the other hand,

$$\begin{split} (1-z)^2(1+z)^2[(1+z)^{i-2}-(1-z)^{i-2}] &= 2(1-z^2)^2[\sum_{\substack{\text{odd } j \in [0:i-2]}} \binom{i-2}{j} z^j] \\ &= 2\sum_{\substack{\text{odd } j \in [0:i-2]}} [\binom{i-2}{j} + \binom{i-2}{j-4} - 2\binom{i-2}{j-2}] z^j. \end{split}$$

It is easy to verify that $\binom{i-2}{j} + \binom{i-2}{j-4} - 2\binom{i-2}{j-2} \ge 0$ if $j \le \frac{i+2-\sqrt{i+2}}{2}$. Hence $K_k^{(n)}(2) \ge K_k^{(n)}(i)$ also holds when $k \le \frac{i+2-\sqrt{i+2}}{2}$.

Combining the two points above, we have that for $2 \le i \le n-2$, $K_k^{(n)}(2) \ge K_k^{(n)}(i)$ whenever

$$k \le \max\{\frac{n-i+2-\sqrt{n-i+2}}{2}, \frac{i+2-\sqrt{i+2}}{2}\}.$$

Taking minimization over $2 \le i \le n-2$ to find the worst case, we have that $K_k^{(n)}(2) \ge K_k^{(n)}(i)$ holds for $2 \le i \le n-2$, when

$$k \le \min_{2 \le i \le n-2} \max\{\frac{n-i+2-\sqrt{n-i+2}}{2}, \frac{i+2-\sqrt{i+2}}{2}\} = \frac{1}{2}(\frac{n}{2}+2-\sqrt{\frac{n}{2}+2}). \quad (A.6)$$

Similarly, one can show $K_k^{(n)}(2) \ge -K_k^{(n)}(i)$ if k satisfies (A.6).

Appendix B. Proof of Theorem 5

One can easily observe that $\mathbf{x} = \mathbf{0}$ is a feasible solution to Problem 2, since by Lemma 1, $K_r^{(n-1)}(0) \geq K_r^{(n-1)}(i-1), \forall i \in [2:n], \forall r \in [0:n-1]$. This solution leads to the lower bound $\overline{\Lambda}_n(\alpha,r) \geq 0$. In the following, we construct another two kinds of feasible solutions for different cases: the 1-sparse solution which contains only one non-zero component, and the 2-sparse solution which contains two non-zero components. By using these feasible solutions, we will show that $\overline{\Lambda}_n(\alpha,r) \geq \psi_n(\alpha,r)$. We partition all the possible cases into four parts, according to whether $r \leq n/2 - 1$ and whether r is even.

• Even $r \le n/2 - 1$

For this case, we first construct a 2-sparse feasible solution. Let k be an odd number such that $n-\tau(n) \le k \le n-1$. Consider the vector $\mathbf{x}^* := (0,...,0,x_k^*,x_{k+1}^*,0,...,0)$ with the k-th and (k+1)-th components (x_k^*,x_{k+1}^*) satisfying

$$[K_k(2) - K_k(1)]x_k^* + [K_{k+1}(2) - K_{k+1}(1)]x_{k+1}^* + K_r^{(n-1)}(1) - K_r^{(n-1)}(0) = 0$$
(B.1)

$$[K_k(n) - K_k(1)]x_k^* + [K_{k+1}(n) - K_{k+1}(1)]x_{k+1}^* + K_r^{(n-1)}(n-1) - K_r^{(n-1)}(0) = 0.$$
 (B.2)

That is, if we define

$$\varphi(i) := [K_k(i) - K_k(1)]x_k^* + [K_{k+1}(i) - K_{k+1}(1)]x_{k+1}^* + K_r^{(n-1)}(i-1) - K_r^{(n-1)}(0),$$

then $\varphi(2) = \varphi(n) = 0$. Solving the equations (B.1) and (B.2), we obtain

$$x_k^* = x_{k+1}^* = \frac{n\binom{n-2}{r-1}}{\binom{n}{k}(2k-n)}.$$
 (B.3)

We next prove that \mathbf{x}^* is a feasible solution to Problem 2. That is, for all $i \in [2:n]$, $\varphi(i) \leq 0$. By the choice of \mathbf{x}^* , we have $\varphi(2) = \varphi(n) = 0$. Hence we only need to show $\varphi(i) \leq 0$ for all $i \in [3:n-1]$. We next prove this.

By the property $K_k(i) = (-1)^i K_{n-k}(i), 0 \le i, k \le n$ [11, (13)], we have

$$\varphi(i) = [(-1)^i K_{n-k}(i) + K_{n-k}(1)] x_k^* + [(-1)^i K_{n-k-1}(i) + K_{n-k-1}(1)] x_{k+1}^* + K_r^{(n-1)}(i-1) - K_r^{(n-1)}(0).$$

By Lemma 1, for $i \in [3: n-2]$ and $n-\tau(n) \le k \le n-1$,

$$\varphi(i) \leq [|K_{n-k}(i)| + K_{n-k}(1)]x_k^* + [|K_{n-k-1}(i)| + K_{n-k-1}(1)]x_{k+1}^* + K_r^{(n-1)}(i-1) - K_r^{(n-1)}(0)$$

$$\leq [K_{n-k}(2) + K_{n-k}(1)]x_k^* + [K_{n-k-1}(2) + K_{n-k-1}(1)]x_{k+1}^* + K_r^{(n-1)}(1) - K_r^{(n-1)}(0)$$

$$= \varphi(2) = 0.$$

Hence, it remains to verify that $\varphi(n-1) \leq 0$.

By the property $K_k(i) = (-1)^k K_k(n-i), 0 \le i, k \le n$ [11, (11)], we have

$$\varphi(n-1) = [(-1)^{n-1}K_{n-k}(n-1) + K_{n-k}(1)]x_k^* + [(-1)^{n-1}K_{n-k-1}(n-1) + K_{n-k-1}(1)]x_{k+1}^* + K_r^{(n-1)}(n-2) - K_r^{(n-1)}(0)$$

$$= [(-1)^{n-1+n-k}K_{n-k}(1) + K_{n-k}(1)]x_k^* + [(-1)^{n-1+n-k-1}K_{n-k-1}(1) + K_{n-k-1}(1)]x_{k+1}^* + K_r^{(n-1)}(1) - K_r^{(n-1)}(0)$$

$$= 2K_{n-k}(1)x_k^* + K_r^{(n-1)}(1) - K_r^{(n-1)}(0)$$

$$= 2(\frac{2k}{n} - 1)\frac{n\binom{n-2}{r-1}}{(2k-n)} + \binom{n-1}{r}(1 - \frac{2r}{n-1}) - \binom{n-1}{r} = 0.$$

Until now, we have shown that \mathbf{x}^* is a feasible solution to Problem 2. This immediately yields the following bound on Problem 1:

$$\overline{\Lambda}_{n}(\alpha, r) \geq -\binom{n}{k} \left[1 + \left(1 - \frac{2k}{n}\right) \left(\frac{1}{\alpha} - 1\right)\right] x_{k}^{*} - \binom{n}{k+1} \left[1 + \left(1 - \frac{2(k+1)}{n}\right) \left(\frac{1}{\alpha} - 1\right)\right] x_{k+1}^{*} \\
= \frac{n\binom{n-2}{r-1}}{2k-n} \left[2\left(\frac{1}{\alpha} - 1\right) - \frac{1}{\alpha} \frac{n+1}{k+1}\right].$$
(B.4)

We next construct a 1-sparse feasible solution for odd n. Let k = n and $\mathbf{x}^* := (0, ..., 0, x_n^*)$, where $x_n^* = \binom{n-2}{r-1}$ which coincides with (B.3) with k = n. For this case, all the derivations above for the 2-sparse feasible solution still hold since $K_{n+1}(i) = 0$ for all $0 \le i \le n$.

In conclusion, for the lower bound in (B.4), k is allowed to be chosen as an odd number in $[n - \tau(n) : n]$. We maximize this lower bound over all such k's and obtain the desired bound.

• Odd r < n/2 - 1

Let $k \in [n - \tau(n) : n] \cap F$ be an integer where F is defined in (17). Consider the vector $\mathbf{x}^* := (0, ..., 0, x_k^*, 0, ..., 0)$ with the k-th component x_k^* satisfying

$$[K_k(2) - K_k(1)]x_k^* + K_r^{(n-1)}(1) - K_r^{(n-1)}(0) = 0.$$

That is, $x_k^* = \frac{n(n-1)\binom{n-2}{r-1}}{\binom{n}{k}k(2k-n-1)}$. For this case, we re-define

$$\varphi(i) := [K_k(i) - K_k(1)]x_k^* + K_r^{(n-1)}(i-1) - K_r^{(n-1)}(0)$$

= $[(-1)^i K_{n-k}(i) + K_{n-k}(1)]x_k^* + K_r^{(n-1)}(i-1) - K_r^{(n-1)}(0),$

which satisfies $\varphi(2) = 0$. We next show $\varphi(i) \leq 0$ for all $i \in [3:n]$.

Similarly to the case of even $r \le n/2 - 1$, for $i \in [3:n-2]$ and $n - \tau(n) \le k \le n$, one can easily verify that $\varphi(i) \le \varphi(2) = 0$. We next verify that $\varphi(n-1), \varphi(n) \le 0$.

$$\varphi(n-1) = [(-1)^{n-1}K_{n-k}(n-1) + K_{n-k}(1)]x_k^* + K_r^{(n-1)}(n-2) - K_r^{(n-1)}(0)$$
$$= [(-1)^{k+1}K_{n-k}(1) + K_{n-k}(1)]x_k^* - K_r^{(n-1)}(1) - K_r^{(n-1)}(0).$$

If k is even, then obviously, $\varphi(n-1) \leq 0$. Otherwise,

$$\varphi(n-1) = 2K_{n-k}(1)x_k^* - K_r^{(n-1)}(1) - K_r^{(n-1)}(0)$$
$$= 2\binom{n-2}{r-1}\left[\left(\frac{2k}{n}-1\right)\frac{n(n-1)}{k(2k-n-1)} - \frac{n-1}{r} + 1\right]$$

which is non-positive if $k > \frac{n+1}{2}$ and

$$k \ge \frac{2(n-1) + s(n+1) + \sqrt{(2(n-1) - s(n+1))^2 + 8s(n-1)}}{4s}$$
 (B.5)

with $s = \frac{n-1}{r} - 1$. By the inequality $\sqrt{a^2 + b^2} \le a + b$ for $a, b \ge 0$, (B.5) is satisfied if

$$k \geq \max\{\frac{n+1}{2}, \frac{n-1}{s}\} + \sqrt{\frac{n+1}{2s}} = \max\{\frac{n+1}{2}, \frac{(n-1)r}{n-1-r}\} + \sqrt{\frac{(n+1)r}{2(n-1-r)}}.$$

Similarly to the case of even $r \leq n/2 - 1$, for $\varphi(n)$, we have

$$\varphi(n) = [(-1)^n K_{n-k}(n) + K_{n-k}(1)] x_k^* + K_r^{(n-1)}(n-1) - K_r^{(n-1)}(0)$$
$$= [(-1)^k K_{n-k}(0) + K_{n-k}(1)] x_k^* - 2K_r^{(n-1)}(0).$$

If k is odd, then obviously, $\varphi(n) \leq 0$. Otherwise,

$$\varphi(n) = [K_{n-k}(0) + K_{n-k}(1)]x_k^* - 2K_r^{(n-1)}(0)$$
$$= 2(n-1)\binom{n-2}{r-1}(\frac{1}{2k-n-1} - \frac{1}{r})$$

which is non-positive if $k \ge \frac{n+r+1}{2}$.

Therefore, the solution constructed above is feasible if $k \in [n - \tau(n) : n] \cap F$. This immediately yields the following bound on Problem 1:

$$\overline{\Lambda}_n(\alpha, r) \ge \frac{(n-1)\binom{n-2}{r-1}}{2k-n-1} \left[2\left(\frac{1}{\alpha} - 1\right) - \frac{1}{\alpha} \frac{n}{k} \right].$$

Maximizing this lower bound over all $k \in [n - \tau(n) : n] \cap F$ yields the desired lower bound.

• Odd r > n/2 - 1

For odd r > n/2 - 1, consider the vector $\mathbf{x}^* := (0, ..., 0, x_k^*, 0, ..., 0)$ with the k-th component x_k^* satisfying

$$2K_{n-k}(1)x_k^* + K_{n-1-r}^{(n-1)}(1) - K_{n-1-r}^{(n-1)}(0) = 0,$$

i.e., $x_k^* = \frac{n(n-1)\binom{n-2}{r-1}}{\binom{n}{k}k(2k-n-1)}$, where $k \geq \frac{n}{2} + 1$ is odd. Re-define

$$\varphi(i) := [K_k(i) - K_k(1)] x_k^* + K_r^{(n-1)}(i-1) - K_r^{(n-1)}(0)$$

$$= [(-1)^{i+n-k} K_{n-k}(n-i) + K_{n-k}(1)] x_k^* + (-1)^{i+n-r} K_{n-1-r}^{(n-1)}(n-i) - K_{n-1-r}^{(n-1)}(0),$$

which satisfies $\varphi(n-1)=0$. Then for odd k, it holds that for all $i\in[2:n]$,

$$\varphi(i) \le [K_{n-k}(1) + K_{n-k}(1)]x_k^* + K_{n-1-r}^{(n-1)}(1) - K_{n-1-r}^{(n-1)}(0)$$

= $\varphi(n-1) = 0$.

Hence, the vector \mathbf{x}^* is feasible and leads to the desired bound for this case.

• Even r > n/2 - 1

For even r > n/2 - 1, by Lemma 1, it holds that $K_{r+1}(1) \ge |K_{r+1}(i)|$ for all $i \in [2:n]$. This inequality implies that $-(\omega_1^{(r)} - \omega_i^{(r)}) \le -(\omega_1^{(r+1)} - \omega_i^{(r+1)})$ where $\omega_i^{(r)}$ is defined in (37). Hence, inequality $\overline{\Lambda}_n(\alpha, r) \ge \overline{\Lambda}_n(\alpha, r+1)$ holds, which, combined with the lower bound for the case "odd r > n/2 - 1", implies the desired lower bound for even r > n/2 - 1.

Appendix C. Proof of Lemma 4

Statement 1 is obvious. Statement 3 follows from Statements 1 and 2. Hence it suffices to prove Statement 2. Next we do this. On one hand,

$$\frac{\binom{n}{\leq r}}{\binom{n}{r}} = \frac{\sum_{k=0}^{r} \binom{n}{r-k}}{\binom{n}{r}} = \sum_{k=0}^{r} \frac{r...(r-k+1)}{(n-r+k)...(n-r+1)}$$

$$= \sum_{k=0}^{r} \frac{\beta...(\beta - \frac{k-1}{n})}{(1-\beta + \frac{k}{n})...(1-\beta + \frac{1}{n})} \leq \sum_{k=0}^{r} (\frac{\beta}{1-\beta})^{k}. \tag{C.1}$$

On the other hand, for a fixed N and for any $r/n \in [0, \frac{1}{2} - \delta]$,

$$\frac{\binom{n}{\leq r}}{\binom{n}{r}} - \sum_{k=0}^{r} \left(\frac{\beta}{1-\beta}\right)^{k} \ge \sum_{k=0}^{\min\{r,N\}} \frac{\beta ... (\beta - \frac{k-1}{n})}{(1-\beta + \frac{k}{n})...(1-\beta + \frac{1}{n})} - \sum_{k=0}^{r} \left(\frac{\beta}{1-\beta}\right)^{k}$$

$$\ge \sum_{k=0}^{\min\{r,N\}} \left[\left(\frac{\beta - \frac{k}{n}}{1-\beta + \frac{k}{n}}\right)^{k} - \left(\frac{\beta}{1-\beta}\right)^{k} \right] - \sum_{k=\min\{r,N\}+1}^{r} \left(\frac{\beta}{1-\beta}\right)^{k}$$

Therefore, for fixed N,

$$\liminf_{n\to\infty}\inf_{r/n\in[0,\frac12-\delta]}\{\frac{\binom{n}{\leq r}}{\binom{n}{r}}-\sum_{k=0}^r(\frac{\beta}{1-\beta})^k\}\geq \liminf_{n\to\infty}\inf_{\beta\in[0,\frac12-\delta]}-\sum_{k=\min\{r,N\}+1}^r(\frac{\beta}{1-\beta})^k\geq -\frac{\gamma^{N+1}}{1-\gamma}.$$

where $\gamma := \frac{\frac{1}{2} - \delta}{\frac{1}{2} + \delta} < 1$. Since N is arbitrary,

$$\liminf_{n \to \infty} \inf_{r/n \in [0, \frac{1}{2} - \delta]} \left\{ \frac{\binom{n}{\leq r}}{\binom{n}{r}} - \sum_{k=0}^{r} \left(\frac{\beta}{1 - \beta}\right)^k \right\} \ge 0. \tag{C.2}$$

Combining (C.1) and (C.2) yields that given $\delta > 0$, $\frac{\binom{n}{\leq r}}{\binom{n}{r}} \to \sum_{k=0}^{r} (\frac{\beta}{1-\beta})^k$ uniformly for all $\beta := r/n \in [0, \frac{1}{2} - \delta]$.

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