On the growth of Artin–Tits monoids and the partial theta function

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Abstract

We present a new procedure to determine the growth function of a homogeneous Garside monoid, with respect to the finite generating set formed by the atoms. In particular, we present a formula for the growth function of each Artin–Tits monoid of spherical type (hence of each braid monoid) with respect to the standard generators, as the inverse of the determinant of a very simple matrix.

Using this approach, we show that the exponential growth rates of the Artin-Tits monoids of type A_n (positive braid monoids) tend to 3.233636... as n tends to infinity. This number is well-known, as it is the growth rate of the coefficients of the only solution $x_0(y) = -(1+y+2y^2+4y^3+9y^4+\cdots)$ to the classical partial theta function.

We also describe the sequence $1, 1, 2, 4, 9, \ldots$ formed by the coefficients of $-x_0(y)$, by showing that its kth term (the coefficient of y^k) is equal to the number of braids of length k, in the positive braid monoid A_{∞} on an infinite number of strands, whose maximal lexicographic representative starts with the first generator a_1 . This is an unexpected connection between the partial theta function and the theory of braids.

1 Introduction

A Garside monoid is a cancellative monoid where greatest common divisors and least common multiples exist and some finiteness conditions are satisfied. The initial properties of Garside monoids were discovered by F. Garside in his famous paper [15], which was concentrated in the particular case of braids. Garside theory was definitely established in the work of Paris and Dehornoy [12, 11], where in particular the notion of Garside group was defined as the group of fractions of a Garside monoid. Since then, Garside theory has become a remarkable topic in Combinatorial Group Theory, as very important families of groups and monoids are Garside (Artin–Tits groups of spherical type, torus link groups, groups related to solutions of quantum Yang-Baxter equations), and some well-known conjectures have been established with the help of Garside theory, like the $K(\pi, 1)$ conjecture for finite complex reflection arrangements [6]. We recall the basics we need of Garside theory at the beginning of Section 2.

The main objects of study of this paper are the *growth functions* in Garside monoids, particularly in Artin–Tits monoids of spherical type, with respect to the generating set formed by the atoms. Our initial goal was to give a new procedure to compute the growth function of a homogeneous Garside monoid of finite type, which should yield a simple formula in well-known particular cases.

It was already shown by Deligne [13] that the growth function $g_M(t)$ of a monoid M is the inverse of a polynomial (called the *Möbius polynomial* of M) if M is an Artin-Tits monoid of spherical type, and his proof can be generalized to other Garside monoids (Corollary 2.2). Building on this, we develope in Section 3 a new and quite straightforward way of counting the number of elements of given length in a

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Garside monoid of finite type, or in other words, the coefficients of the growth function of the monoid, relative to the atoms.

A major consequence of this new way of counting is the following statement, which includes precise descriptions of the growth functions for all the Artin–Tits monoids of spherical type; these descriptions appear in the text of the paper as Theorem 4.4, Theorem 5.4 and Theorem 5.12, and we present them here in a unified way. Observe that, throughout the paper, we will assume that the number $\binom{k}{2} = \frac{k(k-1)}{2}$ makes sense for every integer $k \geq 0$.

Theorem. Let M an Artin–Tits monoid of spherical type. Then there exists a square matrix \mathcal{M} such that

$$g_M(t) = |\mathcal{M}|^{-1}.$$

Moreover:

- If M is of type A_n , \mathcal{M} is denoted $\mathcal{M}_n^{\mathcal{A}}$, has order n+1, and its entry (i,j) equals $t^{\binom{j-i+1}{2}}$ whenever $j-i+1\geq 0$, and 0 otherwise.
- If M is of type B_n , \mathcal{M} is denoted $\mathcal{M}_n^{\mathcal{B}}$, has order n+1, its entry (i, n+1) equals $t^{(n-i+1)^2}$ for all i, and its entry (i, j) for $j \leq n$ equals $t^{\binom{j-i+1}{2}}$ whenever $j-i+1 \geq 0$, and 0 otherwise.
- If M is of type D_n , \mathcal{M} is denoted $\mathcal{M}_n^{\mathcal{D}}$, has order n, its entry (i,n) equals $2t^{\binom{n-i+1}{2}} t^{(n-i+1)(n-i)}$ for all i, and its entry (i,j) for j < n equals $t^{\binom{j-i+1}{2}}$ whenever $j-i+1 \geq 0$, and 0 otherwise.

The following are concrete examples of the above result, for n = 4 or n = 5:

$$g_{A_4}(t) = \begin{vmatrix} 1 & t & t^3 & t^6 & t^{10} \\ 1 & 1 & t & t^3 & t^6 \\ 0 & 1 & 1 & t & t^3 \\ 0 & 0 & 1 & 1 & t \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}^{-1}$$

$$g_{B_4}(t) = \begin{vmatrix} 1 & t & t^3 & t^6 & t^{16} \\ 1 & 1 & t & t^3 & t^9 \\ 0 & 1 & 1 & t & t^4 \\ 0 & 0 & 1 & 1 & t \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}^{-1}$$

$$g_{D_5}(t) = \begin{vmatrix} 1 & t & t^3 & t^6 & 2t^{10} - t^{20} \\ 1 & 1 & t & t^3 & 2t^6 - t^{12} \\ 0 & 1 & 1 & t & 2t^3 - t^6 \\ 0 & 0 & 1 & 1 & 2t - t^2 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}^{-1}$$

In [8], Bronfman gave a recursive formula to compute (in an efficient way) the Möbius polynomial of M, when $M = A_n$. Using the previous information, we are able to express the same Möbius polynomial as the determinant of a matrix, recovering in particular (Theorem 4.5) Bronfman's recursive formula as the expansion of this determinant along the first row.

Once we obtained these explicit formulae for the growth functions of the Artin–Tits monoids, we drove our attention to a more ambitious goal. Recall that given a monoid M, if we denote by M_k the set of elements in M of length k, the exponential growth rate of M is defined as follows

$$\rho_M = \lim_{k \to \infty} \sqrt[k]{|M_k|}.$$

In the case of an Artin-Tits monoid M of spherical type, since $g_M(t)$ is the inverse of a polynomial, the number ρ_M is inverse of the smallest root of the Möbius polynomial, which is unique, real and positive, as shown by Jugé [19]. It is also known that, in the case of Artin-Tits monoids of type A_n (the braid monoids), the limit of their growth rates as n tends to infinity exists, and lies between 2.5 and 4 (see [35, Theorem 8] and [19, Proposition 7.98]). However, the precise description of this limit remained elusive for years, so we intended to compute it taking account of the information previously obtained in the article.

The strategy to obtain the limit implies the construction of a new bridge between (combinatorial) Monoid Theory and Real Analysis. Surprisingly, the coefficients of the leading root of the partial theta function (see Section 6.1.1), which is defined in a pure analytical framework, can be computed counting braids. Consider the braid monoid A_{∞} on an infinite number of strands, which is the direct limit of the braid monoids $A_1 \subset A_2 \subset A_3 \subset \cdots$ with the natural inclusions. We order the standard generators in the natural way $(a_1 < a_2 < a_3 < \cdots)$, and we consider the words in these generators ordered lexicographically. Every element in A_{∞} has a unique representative which is maximal with respect to this order. We will show the following:

Theorem 6.8. Let $x_0(y)$ be the only solution to the classical partial theta function $\sum_{k=0}^{\infty} y^{\binom{k}{2}} x^k$, and let

 $\xi_0(y) = -x_0(y) = 1 + y + 2y^2 + 4y^3 + 9y^4 + \cdots$ For every $k \ge 0$, the coefficient of y^k in the series $\xi_0(y)$ is equal to the number of braids of length k, in the monoid A_∞ , whose maximal lexicographic representative starts with a_1 .

Then we use the fact that the growth rate of the coefficients $1, 1, 2, 4, 9, \ldots$ of $\xi_0(y)$ is known [33] and, building on some results from [14], we show the following:

Theorem 6.17. Let $\rho = \lim_{n \to \infty} \rho_{A_n}$. Then $\rho = 3.23363...$ is the growth rate of the coefficients of $\xi_0(y)$. That is, ρ is equal to the KLV-constant q_{∞} .

It is overwhelming that this constant, that has a prominent role in different branches of Analysis (see Section 6.1.2), appears here as a purely monoid-theoretic invariant associated to the braid monoid A_n , a fact that opens interesting perspectives of research. Moreover, we should remark that the constant can be determined with arbitrary precision, using for example the results of [20]. The reader interested in this relation between growth of braid monoids and the partial theta function can read Section 2 and Section 3, and jump directly to Section 6.

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2 Growth functions of homogeneous Garside monoids of finite type

In this section we recall some basic facts of Garside theory and we discuss the growth of homogeneous Garside monoids of finite type.

In a Garside monoid M, which is cancellative, its elements form a lattice with respect to the *prefix order*, defined by $x \preccurlyeq y$ if xz = y for some $z \in M$. The lattice property means that for every $a, b \in M$, there exist unique elements $a \land b$ and $a \lor b$, which are the greatest common divisor and the least common multiple, respectively, with respect to \preccurlyeq . It is important to notice that \preccurlyeq is invariant under left multiplication and under left cancellation, that is, $x \preccurlyeq y$ if and only if $cx \preccurlyeq cy$ for every $c, x, y \in M$. This implies that $ca \land cb = c(a \land b)$ and $ca \lor cb = c(a \lor b)$ for every $a, b, c \in M$.

For every element in a Garside monoid M, the number of nontrivial factors that can be used to decompose it as a product of elements in M is bounded above. The maximal number of factors for a given $a \in M$ is denoted ||a||. It follows that every Garside monoid admits a special set of generators, called **atoms**, which are those elements that cannot be decomposed as a product of two nontrivial elements: Every element $a \in M$ can be written as a product of ||a|| atoms. A Garside monoid is said to be of **finite type** if the set of atoms is finite.

Let us fix a Garside monoid M, and its set of atoms \mathcal{A} as a set of generators. Given $a \in M$, its length |a| is defined to be the length of the shortest word (in the atoms) representing a. If all relations in M (written in terms of \mathcal{A}) are homogeneous, then |a| = ||a|| is the word-length of any representative of a as

a word in the atoms. In this paper we shall only consider homogeneous Garside monoids of finite type, which include Artin–Tits monoids of spherical type.

For every integer k, let M_k be the set of elements in M of length k, and let $\alpha_k = |M_k|$ be its cardinal. Notice that $\alpha_0 = 1$, that $\alpha_k = 0$ for k < 0, and that α_k is a positive integer for $k \ge 0$, as we chose a finite set of generators. Let t be an indeterminate. The **growth function** (or growth series, or spherical growth series) of M is defined to be:

$$g_M(t) = \sum_{k>0} \alpha_k t^k$$

It would be more precise to denote this function by $g_{M,\mathcal{A}}(t)$, as it depends on the generating set \mathcal{A} , but in this paper we shall only consider the set of atoms as generating set for M.

There are some well known results concerning the growth function of braid monoids, Artin–Tits monoids of spherical type, and more generally homogeneous Garside monoids of finite type. The first one is that $g_M(t)$ is the inverse of a polynomial (its Möbius polynomial). This was shown by Deligne [13] for Artin–Tits monoids of spherical type with their classical Garside structure. It was rediscovered by Bronfman [8], generalizing it to a wider class of monoids, including homogeneous Garside monoids of finite type, and also rediscovered by Saito [32]. We will show here a simple proof, in the spirit of Bronfman. Given a finite subset $S = \{s_1, \ldots, s_r\} \subset M$, let $\forall S = s_1 \lor \cdots \lor s_r$. If $S = \emptyset$, we set $\forall S = 1 \in M$.

Theorem 2.1 ([13, 8, 32]). Let M be a homogeneous Garside monoid of finite type. Let A be its set of atoms, and let $\alpha_i = |M_i|$ be the number of elements of length i, written as a product of atoms. (Notice that $\alpha_i = 0$ if i < 0). Then, for every k > 0, one has:

$$\sum_{S \subset \mathcal{A}} (-1)^{|S|} \alpha_{k-||\vee S||} = 0$$

Proof. Given $a \in M$, let $(aM)_k = \{b \in M; ||b|| = k \text{ and } a \leq b\}$, the set of elements of length k which admit a as a prefix. Let $\mathcal{A} = \{a_1, \ldots, a_m\}$. Every nontrivial element in M must admit some atom as prefix, hence $M_k = (a_1M)_k \cup \cdots \cup (a_mM)_k$. By the inclusion-exclusion principle, it follows that

$$\alpha_k = \sum_{\emptyset \neq S \subset \mathcal{A}} (-1)^{|S|-1} \left| \bigcap_{a_i \in S} (a_i M)_k \right|.$$

As M is cancellative and has homogeneous relations, one has

$$(aM)_k = a M_{k-||a||} = \{ac \in M; ||c|| = k - ||a||\},$$

for every $a \in M$. It follows that counting the elements in $(aM)_k$ is the same as counting the number of elements in M of length k - ||a||. That is, $|(aM)_k| = \alpha_{k-||a||}$.

Notice that given $S = \{x_1, \dots, x_r\} \subset \mathcal{A}$, the common multiples of x_1, \dots, x_r are precisely the multiples of $x_1 \vee \dots \vee x_r$. Hence

$$(x_1M)_k \cap \cdots \cap (x_rM)_k = ((x_1 \vee \cdots \vee x_r)M)_k = ((\vee S)M)_k,$$

hence $|(x_1M)_k \cap \cdots \cap (x_rM)_k| = \alpha_{k-||\vee S||}$. Replacing this in the above expression for α_k , one gets:

$$\alpha_k = \sum_{\emptyset \neq S \subset \mathcal{A}} (-1)^{|S|-1} \alpha_{k-||\vee S||},$$

which is precisely what we wanted to show, as $\alpha_k = \alpha_{k-0} = \alpha_{k-||\vee\emptyset||}$.

Corollary 2.2 ([13, 8, 32]). If M is a homogeneous Garside monoid of finite type, the growth function of M is the inverse of a polynomial. Namely, if $A \subset M$ is the set of atoms,

$$g_M(t) = \left(\sum_{S \subset \mathcal{A}} (-1)^{|S|} t^{||VS||}\right)^{-1}$$

Proof. Let $N = || \vee \mathcal{A} ||$. Notice that, for every k > 0, the formula in Theorem 2.1 is a sum of terms of the form $\pm \alpha_{k-i}$ for some $i \leq N$. Collecting the terms with the same value of i, one obtains:

$$0 = \sum_{S \subset A} (-1)^{|S|} \alpha_{k-||\vee S||} = \sum_{i=0}^{N} c_i \alpha_{k-i}.$$
(1)

It is important to notice that the coefficient c_i does not depend on k: it only depends on the number and size of subsets $S \subset \mathcal{A}$ such that $|| \vee S|| = i$. Since $c_0 = 1$, we have a recurrence relation which determines the sequence $\{\alpha_k\}_{k\geq 0}$:

$$\alpha_k = -\sum_{i=1}^{N} c_i \alpha_{k-i},\tag{2}$$

Each α_k is thus a linear combination of $\alpha_{k-1}, \ldots, \alpha_{k-N}$, where the coefficients $-c_1, \cdots, -c_N$ are fixed (they depend only on the monoid M).

It is easy to check that the coefficients of a power series (in this case $g_M(t)$) satisfy such a recurrence relation if and only if the power series is the inverse of a polynomial, namely:

$$g_M(t) = \left(\sum_{i=0}^N c_i t^i\right)^{-1}.$$

Notice that the polynomial in the above formula is the one obtained from the right hand side of (1) when replacing α_{k-i} with t^i for i = 0, ..., N. Therefore, by the second equality in (1), we obtain:

$$\sum_{S\subset\mathcal{A}}(-1)^{|S|}t^{||\vee S||}=\sum_{i=0}^Nc_it^i.$$

Hence

$$g_M(t) = \left(\sum_{S \subset A} (-1)^{|S|} t^{||VS||}\right)^{-1}.$$

We can then write $g_M(t) = \frac{1}{H_M(t)}$, where $H_M(t) = \sum_{S \subset \mathcal{A}} (-1)^{|S|} t^{||VS||}$, the Möbius polynomial of M, has degree ||VA||. In the case of the braid monoid on n strands, this degree is $\binom{n}{2}$.

We remark that the recurrence relation (2) gives a very fast way to obtain α_k for any given k > 0, starting from $\alpha_0 = 1$ and $\alpha_i = 0$ for i < 0, provided that the coefficients c_i are known. But in order to obtain this recurrence relation (2) explicitly, using the formula in Theorem 2.1, one needs to collect all terms involving each α_{k-i} . This means to run through all subsets of \mathcal{A} . Therefore, computing the recurrence relation (2) using Theorem 2.1 has exponential complexity with respect to the number of atoms.

Nevertheless, when the number of atoms is small, one can compute the growth function without much problem using the formula in Corollary 2.2. For instance, we include here the (already known) growth functions of all Artin–Tits groups of spherical type which are not of type A, D, E.

$$g_{E_6}(t) = \begin{pmatrix} 1 - 6t + 10t^2 - 10t^4 + 5t^5 - 4t^6 + 3t^7 + \\ +4t^{10} - 2t^{11} + t^{12} - t^{15} - 2t^{20} + t^{36} \end{pmatrix}^{-1}$$

$$g_{E_7}(t) = \begin{pmatrix} 1 - 7t + 15t^2 - 5t^3 - 16t^4 + 12t^5 - 3t^6 + 8t^7 - 3t^8 - 3t^9 + \\ +6t^{10} - 5t^{11} + t^{12} - 3t^{15} + t^{16} - 2t^{20} + 2t^{21} + t^{30} + t^{36} - t^{63} \end{pmatrix}^{-1}$$

$$g_{E_8}(t) = \begin{pmatrix} 1 - 8t + 21t^2 - 14t^3 - 21t^4 + 28t^5 - 7t^6 + 12t^7 - 8t^8 - 10t^9 + \\ +10t^{10} - 12t^{11} + 7t^{12} + 2t^{13} - t^{14} - 3t^{15} + 2t^{16} - 2t^{20} + \\ +6t^{21} - t^{22} - t^{23} - t^{28} + t^{30} + t^{36} - t^{37} - t^{42} - t^{63} + t^{120} \end{pmatrix}^{-1}$$

$$g_{F_4}(t) = (1 - 4t + 3t^2 + 2t^3 - t^4 - 2t^9 + t^{24})^{-1}$$

$$g_{H_3}(t) = (1 - 3t + t^2 + t^3 + t^5 - t^{15})^{-1}$$

$$g_{H_4}(t) = (1 - 4t + 3t^2 + 2t^3 - t^4 + t^5 - 2t^6 - t^{15} + t^{60})^{-1}$$

$$g_{I_2(p)}(t) = (1 - 2t + t^p)^{-1}$$

The last case is valid for $p \ge 3$ (even for p = 2, which does not yield an irreducible monoid), although it is usual to denote $I_2(3) = A_2$ and $I_2(4) = B_2$.

Concerning the remaining types, Bronfman [8] gave a procedure to compute the recurrence relation (2), or equivalently the Möbius polynomial $H_M(t)$, in the case of braid monoids (Artin–Tits monoids of type A). Namely, he gave a recurrence relation expressing the polynomial $H_{A_n}(t)$ in terms of $H_{A_1}(t), H_{A_2}(t), \ldots, H_{A_{n-1}}(t)$. Using Bronfman's formula, one has a polynomial time algorithm (with respect to n) to compute the growth function of A_n .

In Theorem 4.5 we will recover Bronfman's recurrence relation as an immediate consequence of our formula for $g_{A_n}(t)$. And we will also obtain new recurrence relations involving Artin–Tits monoids of types B and D. This will be done in the next section.

3 Counting elements in a homogeneous Garside monoid

We will now present a more straightforward way to count the number of elements of given length in the monoid M, that is, the coefficients of the growth function $g_M(t)$, without using the formula in Theorem 2.1.

Recall that M is a homogeneous Garside monoid of finite type, and that M_k is the set of elements of length k (as words in the atoms). Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be the set of atoms. We will study the elements in M_k by choosing a suitable word representing each element. Namely, we define the **lex-representative** of an element of M, as its biggest representative in lexicographical order with $a_1 < a_2 < \cdots < a_n$. Notice that the number of lex-representatives of length k is precisely α_k .

It is clear that $M_0 = \{1\}$. For k > 0, we will be able to count the number of elements in M_k using a stratification of these sets by suitable subsets.

Definition 3.1. For $k \geq 0$ and i = 1, ..., n, we define $L_k^{(i)}$ to be the set of elements in M_k whose lex-representative starts with a_i . That is,

$$L_k^{(i)} = \{b \in M_k; \quad a_i \preccurlyeq b, \quad a_{i+1}, \dots, a_n \not\preccurlyeq b\}.$$

It is then clear that, for k > 0, $M_k = L_k^{(1)} \sqcup L_k^{(2)} \sqcup \cdots \sqcup L_k^{(n)}$.

The number of elements in each $L_k^{(i)}$ is not easy to compute directly. But we can define some related subsets of M_k which will allow us to perform the computation:

Definition 3.2. Given $k \geq 0$ and $1 \leq i \leq j \leq n$, let

$$U_k^{(i,j)} = \{b \in M_k; \quad a_i, \dots, a_j \leq b, \quad a_{j+2}, \dots, a_n \not\leq b\}$$

Notice that if k > 0, every $b \in U_k^{(i,j)}$ admits all atoms a_i, \ldots, a_j as prefixes, which is equivalent to $a_i \vee \cdots \vee a_j \leq b$. As $a_j \leq b$, the lex-representative of b starts either with a_j or with an atom of bigger index. But b does not admit a_t as prefix for $t \geq j+2$. Hence, the lex-representative of b starts with either a_j or a_{j+1} .

Therefore, $b \in U_k^{(i,j)}$ if and only if it has length k, its lex-representative starts with either a_j or a_{j+1} , and $a_i \vee \cdots \vee a_j \leq b$.

Lemma 3.3. For every $k \geq 0$ and i = 1, ..., n one has:

$$\left| L_k^{(i)} \right| = \sum_{j=i}^n (-1)^{j-i} \left| U_k^{(i,j)} \right|$$

Proof. The case k=0 is trivial, as all sets are empty. We will then assume that k>0.

For every element $b \in U_k^{(i,j)}$, its lex-representative starts with either a_j or a_{j+1} . This depends on whether $a_{j+1} \preccurlyeq b$ or not. We can then split $U_k^{(i,j)}$ into two disjoint subsets, if j < n:

$$U_k^{(i,j)} = \{b \in M_k; a_i, \dots, a_j \leq b, a_{j+1}, \dots, a_n \nleq b\}$$

$$\bigsqcup \{b \in M_k; a_i, \dots, a_{j+1} \leq b, a_{j+2}, \dots, a_n \nleq b\}.$$

If j = n we just have:

$$U_k^{(i,n)} = \{ b \in M_k; \ a_i, \dots, a_n \leq b \}.$$

It is important to notice that if j_1 and j_2 are not consecutive, the sets $U_k^{(i,j_1)}$ and $U_k^{(i,j_2)}$ are disjoint. Hence, we can consider the disjoint union $U_k^{(i,i)} \sqcup U_k^{(i,i+2)} \sqcup U_k^{(i,i+4)} \sqcup \cdots$, and also the disjoint union $U_k^{(i,i+1)} \sqcup U_k^{(i,i+3)} \sqcup U_k^{(i,i+5)} \sqcup \cdots$.

Suppose first that n-i is odd. In this case:

$$U_k^{(i,i)} \sqcup U_k^{(i,i+2)} \sqcup \cdots \sqcup U_k^{(i,n-1)} = \bigsqcup_{j=i}^n \{ b \in M_k; \quad a_i, \dots, a_j \leq b, \quad a_{j+1}, \dots, a_n \neq b \}.$$

And also:

$$U_k^{(i,i+1)} \sqcup U_k^{(i,i+3)} \sqcup \cdots \sqcup U_k^{(i,n)} = \bigsqcup_{j=i+1}^n \{b \in M_k; \quad a_i, \dots, a_j \leq b, \quad a_{j+1}, \dots, a_n \not\leq b\}.$$

Therefore, the former disjoint union contains the latter, and we have:

$$\left(U_k^{(i,i)} \sqcup U_k^{(i,i+2)} \sqcup \cdots \sqcup U_k^{(i,n-1)}\right) \setminus \left(U_k^{(i,i+1)} \sqcup U_k^{(i,i+3)} \sqcup \cdots \sqcup U_k^{(i,n)}\right)
= \{b \in M_k; \quad a_i \leq b, \quad a_{i+1}, \ldots, a_n \not\leq b\} = L_k^{(i)},$$

which implies the formula in the statement.

If n-i is even, the argument is analogous. We have:

$$U_k^{(i,i)} \sqcup U_k^{(i,i+2)} \sqcup \cdots \sqcup U_k^{(i,n)} = \bigsqcup_{j=i}^n \{ b \in M_k; \quad a_i, \ldots, a_j \preccurlyeq b, \quad a_{j+1}, \ldots, a_n \not\preccurlyeq b \}.$$

And also:

$$U_k^{(i,i+1)} \sqcup U_k^{(i,i+3)} \sqcup \cdots \sqcup U_k^{(i,n-1)} = \bigsqcup_{j=i+1}^n \{b \in M_k; \quad a_i, \dots, a_j \leq b, \quad a_{j+1}, \dots, a_n \not\leq b\}.$$

Hence, removing the latter union from the former yields $L_k^{(i)}$, and the formula in the statement also holds in this case.

We are mainly interested in Artin–Tits monoids of type A, B and D (also called Artin–Tits monoids of type A_n , B_n and D_n). In those cases, we will be able to describe the sizes of each $L_k^{(i)}$ and each $U_k^{(i,j)}$ in terms of the sizes of the following sets:

Definition 3.4. For $k \geq 0$ and $i = 0, \ldots, n+1$, let

$$M_k^{(i)} = \{ b \in M_k; \quad a_{i+1}, \dots, a_n \not \leq b \}.$$

Notice that $M_0^{(i)} = \{1\}$ for every i. On the other hand, if k > 0, the condition $a_{i+1}, \ldots, a_n \not \leq b$ just means that the lex-representative of b starts with an atom from $\{a_1, \ldots, a_i\}$. Hence, if k > 0:

$$M_k^{(i)} = L_k^{(1)} \sqcup L_k^{(2)} \sqcup \cdots \sqcup L_k^{(i)}.$$

Notice that $M_k^{(0)} = \emptyset$ and that $M_k^{(n+1)} = M_k^{(n)} = M_k$.

It is clear, by definition, that $M_k^{(i-1)}$ is a subset of $M_k^{(i)}$, and that

$$L_k^{(i)} = M_k^{(i)} \setminus M_k^{(i-1)}.$$

Hence:

$$\left|L_k^{(i)}\right| = \left|M_k^{(i)}\right| - \left|M_k^{(i-1)}\right|.$$

In order to avoid cumbersome notation in the following formulae, we will denote $m_{k,i} = \left| M_k^{(i)} \right|$ and $u_{k,i,j} = \left| U_k^{(i,j)} \right|$. Then we have:

Proposition 3.5. For $k \geq 0$ and i = 1, ..., n + 1, one has:

$$m_{k,i} = m_{k,i-1} + \sum_{j=i}^{n} (-1)^{j-i} u_{k,i,j}.$$

Proof. If $i \leq n$, this is just Lemma 3.3, as $m_{k,i} - m_{k,i-1} = \left| L_k^{(i)} \right|$. If i = n+1, the formula reads $m_{k,n+1} = m_{k,n}$, which is true since $M_k^{(n+1)} = M_k^{(n)}$.

If the monoid M is an Artin–Tits monoid of type A_n , B_n or D_n , we will be able to describe each number $u_{k,i,j}$ in terms of some $m_{l,t}$, with l < k. Replacing this in the formula of Proposition 3.5, we will obtain a recurrence relation for the numbers $m_{k,i}$, which is precisely what we need to compute the number $\alpha_k = m_{k,n} = m_{k,n+1}$.

We will then compute a table of the form

$m_{0,1}$	$m_{0,2}$	 $m_{0,n}$	$m_{0,n+1}$
$m_{1,1}$	$m_{1,2}$	 $m_{1,n}$	$m_{1,n+1}$
$m_{2,1}$	$m_{2,2}$	 $m_{2,n}$	$m_{2,n+1}$
$m_{3,1}$	$m_{3,2}$	 $m_{3,n}$	$m_{3,n+1}$
:	:	:	:
$m_{k,1}$	$m_{k,2}$	 $m_{k,n}$	$m_{k,n+1}$
•	•	• •	•

The last two columns of this table will be identical, and they will contain the numbers $m_{k,n} = m_{k,n+1} = \alpha_k$, that is, the number of elements in M_k .

We will be able to compute each number in the above table, as a (signed) sum of at most n of the previous elements. The method for computing each $m_{k,i}$ will produce the new formulae for the Möbius polynomial of the monoid, and a better understanding of its growth rate, in the cases in which M is an Artin-Tits monoid of type A_n , B_n or D_n .

4 Artin–Tits monoid of type A

4.1 Counting elements in the monoid (type A)

Let A_n be an Artin–Tits monoid of type A with n standard generators. That is, the positive braid monoid with n+1 strands. Its standard presentation is the following:

$$A_n = \left\langle a_1, \dots, a_n \mid \begin{array}{c} a_i a_j = a_j a_i, & |i - j| > 1 \\ a_i a_j a_i = a_j a_i a_j, & |i - j| = 1 \end{array} \right\rangle$$

It is well-known [26] that if $1 \le i \le j \le n$, the submonoid of A_n generated by $\{a_i, \ldots, a_j\}$ is again an Artin-Tits monoid of type A_{j-i+1} . Therefore, the least common multiple of the standard generators (in both the submonoid and the monoid) is [27, Lemma 3.1]:

$$a_i \vee \cdots \vee a_j = a_i(a_{i+1}a_i)(a_{i+2}a_{i+1}a_i)\cdots(a_j a_{j-1}\cdots a_i).$$

Hence:

$$||a_i \vee \cdots \vee a_j|| = {j-i+2 \choose 2}.$$

Also, for $t \geq j+2$, the atom a_t commutes with a_i, \ldots, a_j , hence it commutes with the whole element $a_i \vee \cdots \vee a_j$. Therefore:

$$(a_i \vee \cdots \vee a_j) \vee a_t = (a_i \vee \cdots \vee a_j) a_t.$$

Because of these two properties, we can explicitly determine the numbers $u_{k,i,j}$ as follows.

Lemma 4.1. Let M be the Artin–Tits monoid of type A_n , and let $1 \le i \le j \le n$. One has

$$u_{k,i,j} = m_{k-\binom{j-i+2}{2},j+1}$$

Proof. We have

$$U_k^{(i,j)} = \{b \in M_k; \quad a_i, \dots, a_j \leq b, \quad a_{j+2}, \dots, a_n \nleq b\}$$

$$= \{b \in M_k; \quad a_i \vee \dots \vee a_j \leq b, \quad a_{j+2}, \dots, a_n \nleq b\}$$

$$= \{(a_i \vee \dots \vee a_j)c \in M_k; \quad a_{j+2}, \dots, a_n \nleq (a_i \vee \dots \vee a_j)c\}$$

Notice that for every $t \geq j+2$, one has $a_t \leq (a_i \vee \cdots \vee a_j)c$ if and only if $(a_i \vee \cdots \vee a_j) \vee a_t \leq (a_i \vee \cdots \vee a_j)c$, that is $(a_i \vee \cdots \vee a_j) a_t \leq (a_i \vee \cdots \vee a_j)c$, which is equivalent to $a_t \leq c$. Hence

$$U_k^{(i,j)} = \{(a_i \vee \cdots \vee a_j)c \in M_k; \ a_{j+2}, \ldots, a_n \not\preccurlyeq c\}.$$

As M is cancellative and homogeneous, the set of elements $(a_i \vee \cdots \vee a_j)c$ having length k is in bijection with set of elements c having length $k - ||a_i \vee \cdots \vee a_j|| = k - {j-i+2 \choose 2}$. Therefore

$$\left| U_k^{(i,j)} \right| = \left| \left\{ c \in M_{k-\binom{j-i+2}{2}}; \ a_{j+2}, \dots, a_n \not \preccurlyeq c \right\} \right| = \left| M_{k-\binom{j-i+2}{2}}^{(j+1)} \right|$$

Corollary 4.2. In the Artin–Tits monoid A_n , for $k \geq 0$ and i = 1, ..., n + 1, one has:

$$m_{k,i} = m_{k,i-1} + \sum_{j=i}^{n} (-1)^{j-i} m_{k-\binom{j-i+2}{2},j+1}$$

Proof. This is a direct consequence of Proposition 3.5 and Lemma 4.1.

The above recurrence relation allows us to compute, in a very efficient way, a table whose entries are $m_{k,i}$ for i = 1, ..., n + 1 and $k \ge 0$, for the monoid A_n . We start the table with the first row:

$$m_{0,1} = m_{0,2} = \dots = m_{0,n+1} = 1$$

The second row can then be computed, from left to right, using the recurrence relation of Corollary 4.2. That is, we compute $m_{1,1} = m_{1,0} + m_{0,2} = 0 + 1 = 1$, then $m_{1,2} = m_{1,1} + m_{0,3} = 1 + 1 = 2$, then $m_{1,3} = 2 + 1 = 3$ and so on, up to $m_{1,n} = (n-1)+1 = n$ and $m_{1,n+1} = n$. In the same way, we can compute each new row from the previous ones, starting from the leftmost entry, using the recurrence relation of Corollary 4.2. For instance, we can see in Figure 1 the first seven rows of the table corresponding to the monoid A_3 . Notice for instance that $m_{6,1} = m_{6,0} + m_{5,2} - m_{3,3} + m_{0,4} = 0 + 51 - 19 + 1 = 33$, or that $m_{6,2} = m_{6,1} + m_{5,3} - m_{3,4} = 33 + 94 - 19 = 108$.

k i	1	2	3	4
0	1	1	1	1
1	1	2	3	3
2	2	5	8	8
3	4	11	19	19
4	8	24	43	43
5	16	51	94	94
6	33	108	202	202

Figure 1: A table containing $m_{k,i}$ for the Artin–Tits monoid A_3 , for $k \leq 6$.

Now recall that the rightmost column (and also the adjacent column, which is identical) contains precisely the coefficients of the growth function $g_{A_n}(t)$, as $m_{k,n+1} = m_{k,n} = \left| M_k^{(n)} \right| = |M_k| = \alpha_k$. For instance, the table in Figure 1 tells us that there are 202 elements of length 6 in the monoid A_3 . In other words, there are 202 positive braids of length 6 with 4 strands.

4.2 A new formula for the growth function (type A)

From the recurrence relation given in Corollary 4.2 to compute the table of $m_{k,i}$'s, we will be able to provide a new formula for the growth function of the Artin–Tits monoid A_n . We will also do the same for types B_n and D_n in Section 5.

Notice that every new entry of the table is obtained from the previous ones by a linear combination with coefficients 0 or ± 1 . Moreover, in order to compute the kth row of the table, one just needs to use values from the previous $\binom{n+1}{2}$ rows.

More precisely, for $k \ge 1$ let \mathbf{v}_{k-1} be a column vector whose entries correspond to the entries of rows k-1 to $k-\binom{n+1}{2}$ in the table (we consider $m_{t,i}=0$ if t<0). Namely

$$\mathbf{v}_{k-1} = \left(m_{k-1,1} \cdots m_{k-1,n+1} \ m_{k-2,1} \cdots m_{k-2,n+1} \cdots m_{k-\binom{n+1}{2},1} \cdots m_{k-\binom{n+1}{2},n+1} \right)^t$$

For instance, if $M = A_3$, we can check from the table in Figure 1 that:

$$\mathbf{v}_2 = (2\ 5\ 8\ 8\ |\ 1\ 2\ 3\ 3\ |\ 1\ 1\ 1\ 1\ |\ 0\ 0\ 0\ 0\ |\ 0\ 0\ 0\ 0\ 0\ 0)^t$$

And so on. Just to give an example with nonzero entries, we have:

$$\mathbf{v}_6 = (33\ 108\ 202\ 202\ |\ 16\ 51\ 94\ 94\ |\ 8\ 24\ 43\ 43\ |\ 4\ 11\ 19\ 19\ |\ 2\ 5\ 8\ 8\ |\ 1\ 2\ 3\ 3)^t$$

From the above arguments, every entry of \mathbf{v}_k is a linear combination of the entries of \mathbf{v}_{k-1} , for every k > 0. And the coefficients of the linear combination do not depend on k. Hence, we have:

Lemma 4.3. There is a square matrix A with $\binom{n+1}{2}(n+1)$ rows, whose entries belong to $\{0,1,-1\}$ such that for every $k \geq 1$

$$A\mathbf{v}_{k-1} = \mathbf{v}_k$$

Proof. Just to give an idea of how the matrix \mathcal{A} looks like, here is \mathcal{A} when n=3:

	$ \left(\begin{array}{cccccc} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) $	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	0 0 -1 0 0 0 -1 -1 0 0 -1 -1 0 0 -1 -1	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$
	I	0	0	0	0	0
$\mathcal{A} = $	0	I	O	O	O	0
	0	0	I	0	0	0
	0	0	0	I	0	0
,	0	0	0	0	I	0

Here I is the 4×4 identity matrix, and \mathcal{O} is the 4×4 zero matrix.

In general, the matrix A can be defined as follows. Let L be the $(n+1) \times (n+1)$ lower triangular matrix whose entries are $\ell_{i,j} = 1$ if $i \leq j$ and 0 otherwise. We will shift the columns of L in the following way: Given a matrix P, define sh(P) to be the matrix obtained from P by removing its rightmost column and adjoining a zero-column to the left. In other words, sh(P) is obtained from P by shifting its columns one position to the right and inserting zeroes in the first column. When n = 3 we can repeatedly shift the matrix L to obtain:

$$L = \begin{pmatrix} \begin{smallmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad sh(L) = \begin{pmatrix} \begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad sh^{2}(L) = \begin{pmatrix} \begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad sh^{3}(L) = \begin{pmatrix} \begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then \mathcal{A} is defined as a matrix made of blocks of size $(n+1) \times (n+1)$. The block in position (i,j) with $1 \leq i,j \leq \binom{n+1}{2}$ is

$$\Gamma_{i,j} = \begin{cases} (-1)^t s h^{t-1}(L) & \text{if } i = 1 \text{ and } j = {t \choose 2} \text{ for some } t. \\ \\ I_{(n+1)\times(n+1)} & \text{if } i = j+1. \\ \\ \mathcal{O}_{(n+1)\times(n+1)} & \text{otherwise.} \end{cases}$$

Let us show that $A\mathbf{v}_{k-1} = \mathbf{v}_k$ for every $k \geq 1$. First, it is clear from the definition of A that the (n+1+j)th entry of $A\mathbf{v}_{k-1}$ equals the jth entry of \mathbf{v}_{k-1} , which is precisely the (n+1+j)th entry of \mathbf{v}_k . Hence we just need to prove the equality for the first n+1 entries of $A\mathbf{v}_{k-1}$ and \mathbf{v}_k .

We will first see that the first entry of $A\mathbf{v}_{k-1}$ is $m_{k,1}$. Notice that the nonzero entries of the first row of A form an alternate sequence of 1's and -1's placed at columns

$$\left[\binom{2}{2} - 1 \right] (n+1) + 2, \quad \left[\binom{3}{2} - 1 \right] (n+1) + 3, \quad \dots \quad \left[\binom{n+1}{2} - 1 \right] (n+1) + (n+1).$$

The entries occupying these positions in the vector \mathbf{v}_{k-1} are precisely

$$m_{k-\binom{2}{2},2}, \quad m_{k-\binom{3}{2},3}, \quad \dots \quad m_{k-\binom{n+1}{2},n+1}.$$

This implies that the first entry of $\mathcal{A}\mathbf{v}_{k-1}$ is $\sum_{j=1}^{n} (-1)^{j-1} m_{k-\binom{j+1}{2},j+1}$ which by Corollary 4.2 (as $m_{k,0}=0$)

is precisely equal to $m_{k,1}$.

Now suppose, by induction, that $2 \le i \le n$ and that the (i-1)st entry of $\mathcal{A}\mathbf{v}_{k-1}$ is $m_{k,i-1}$. We will show that the *i*th entry of $\mathcal{A}\mathbf{v}_{k-1}$ is $m_{k,i}$ and this will finish the proof.

Notice that the *i*th row of \mathcal{A} is the sum of the (i-1)st row plus a vector whose nonzero entries form an alternate sequence of 1's and -1's placed at columns

$$\left[\binom{2}{2} - 1 \right] (n+1) + (i+1), \quad \left[\binom{3}{2} - 1 \right] (n+1) + (i+2), \quad \dots \quad \left[\binom{n-i+2}{2} - 1 \right] (n+1) + (n+1).$$

(In the case i = n + 1 all entries of this vector are zero, as the rows n and n + 1 of \mathcal{A} are equal.)

The entries occupying these positions in \mathbf{v}_{k-1} are

$$m_{k-\binom{2}{2},i+1}, \quad m_{k-\binom{3}{2},i+2}, \quad \dots \quad m_{k-\binom{n-i+2}{2},n+1}.$$

Therefore the *i*th entry of $A\mathbf{v}_{k-1}$ equals

$$((i-1)\text{st row of }\mathcal{A})\mathbf{v}_{k-1} + \sum_{i=i}^{n} (-1)^{j-i} m_{k-\binom{j-i+2}{2},j+1}$$

which by induction hypothesis and by Corollary 4.2 equals

$$m_{k,i-1} + \sum_{j=i}^{n} (-1)^{j-i} m_{k-\binom{j-i+2}{2},j+1} = m_{k,i}$$

as we wanted to show.

We can now prove our new formula for the growth function of A_n , in a similar way in which growth functions of automatic groups are computed from transition matrices of finite state automata.

Theorem 4.4. Let M be the Artin–Tits monoid of type A_n , that is the positive braid monoid on n+1 strands. Let $\mathcal{M}_n^{\mathcal{A}}$ be the square matrix of order n+1 whose entry (i,j) equals $t^{\binom{j-i+1}{2}}$ whenever $j-i+1 \geq 0$, and 0 otherwise. Then

$$g_M(t) = |\mathcal{M}_n^{\mathcal{A}}|^{-1}$$

Proof. It follows immediately from Lemma 4.3 that $\mathcal{A}^k \mathbf{v}_0 = \mathbf{v}_k$ for all $k \geq 0$. Now notice that the (n+1)st entry of \mathbf{v}_k is precisely $m_{k,n+1} = \alpha_k$. Hence, if we define the row vector $\mathbf{v} = (0 \cdots 0 \ 1 \ 0 \cdots 0)$, where the number 1 occupies the (n+1)st position, we obtain

$$\alpha_k = \mathbf{v} \mathcal{A}^k \mathbf{v}_0$$

for $k \geq 0$. Therefore

$$g_{A_n}(t) = \sum_{k \ge 0} \alpha_k t^k = \sum_{k \ge 0} \left(\mathbf{v} \mathcal{A}^k \mathbf{v}_0 \right) t^k = \mathbf{v} \left(\sum_{k \ge 0} A^k t^k \right) \mathbf{v}_0 = \mathbf{v} \left(I - \mathcal{A} t \right)^{-1} \mathbf{v}_0$$

Let \mathcal{A}' be the adjugate matrix of $I - \mathcal{A}t$. We know that $(I - \mathcal{A}t)^{-1} = \frac{\mathcal{A}'}{|I - \mathcal{A}t|}$, so

$$g_{A_n}(t) = \frac{\mathbf{v}\mathcal{A}'\mathbf{v}_0}{|I - \mathcal{A}t|}$$

We will now show that $\mathbf{v} \mathcal{A}' \mathbf{v}_0 = 1$. To achieve this goal, notice that both $\mathbf{v} \mathcal{A}' \mathbf{v}_0$ and $|I - \mathcal{A}t|$ are polynomials in $\mathbb{Z}[t]$. We will show that $|I - \mathcal{A}t|$ has degree $\binom{n+1}{2}$, and it follows directly from Corollary 2.2 that $g_{A_n}(t)$ is the inverse of a polynomial of degree $\binom{n+1}{2}$. Hence $\mathbf{v} \mathcal{A}' \mathbf{v}_0$ is a constant. From $g_{A_n}(0) = \alpha_0 = 1$ it follows that the constant is 1. Therefore:

$$g_{A_n}(t) = \frac{1}{|I - \mathcal{A}t|}$$

Now let us apply column operations to the matrix I - At which preserve its determinant. It is easier to explain these operations in terms of blocks. Keep in mind the case n = 3 as a clarifying example:

$$|I - \mathcal{A}t| = \begin{vmatrix} I - sh(L)t & \mathcal{O} & sh^2(L)t & \mathcal{O} & \mathcal{O} & -sh^3(L)t \\ -It & I & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & -It & I & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & -It & I & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & -It & I & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & -It & I \end{vmatrix}$$

If we add to each (block) column the column on its right multiplied by t, starting from the right hand side, we obtain

$$|I - \mathcal{A}t| = \begin{vmatrix} I - sh(L)t + sh^2(L)t^3 - sh^3(L)t^6 & \star & \star & \star & \star & \star \\ \mathcal{O} & I & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & I & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & I & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & I & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}$$

The symbol \star stands for an entry whose value is not important for us, as it does not affect the determinant. Hence

$$|I - \mathcal{A}t| = |I - sh(L)t + sh^{2}(L)t^{3} - sh^{3}(L)t^{6}| = \begin{vmatrix} 1 & -t & t^{3} & -t^{6} \\ 0 & 1 - t & -t + t^{3} & t^{3} - t^{6} \\ 0 & -t & 1 - t + t^{3} & -t + t^{3} - t^{6} \\ 0 & -t & -t + t^{3} & 1 - t + t^{3} - t^{6} \end{vmatrix}$$

Now we substract to each row the previous one, starting from the bottom, and we obtain

$$|I - \mathcal{A}t| = \begin{vmatrix} 1 & -t & t^3 & -t^6 \\ -1 & 1 & -t & t^3 \\ 0 & -1 & 1 & -t \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

Finally we point out the alternating signs, all of which can be turned positive by changing the sign of the even rows and then changing the sign of the even columns. As the number of sign changes is even, the final result is

$$|I - \mathcal{A}t| = \left| egin{array}{cccc} 1 & t & t^3 & t^6 \ 1 & 1 & t & t^3 \ 0 & 1 & 1 & t \ 0 & 0 & 1 & 1 \end{array}
ight| = |\mathcal{M}_3^{\mathcal{A}}|$$

In the case of arbitrary n, exactly the same operations lead to the matrix $\mathcal{M}_n^{\mathcal{A}}$ defined in the statement. It remains to show that $|I - \mathcal{A}t|$ is a polynomial of degree $\binom{n+1}{2}$. To achieve this, one just needs to expand the determinant $|\mathcal{M}_n^{\mathcal{A}}|$ along the first row:

$$|\mathcal{M}_n^{\mathcal{A}}| = 1|\mathcal{M}_{n-1}^{\mathcal{A}}| - t|\mathcal{M}_{n-2}^{\mathcal{A}}| + t^3|\mathcal{M}_{n-3}^{\mathcal{A}}| - \dots + (-1)^{n-1}t^{\binom{n}{2}}|\mathcal{M}_0^{\mathcal{A}}| + (-1)^nt^{\binom{n+1}{2}}.$$

By induction in n, the ith summand $(i \le n)$ has degree $\binom{i}{2} + \binom{n-i+1}{2}$, which is smaller than $\binom{n+1}{2}$, hence the last summand contains the leading term.

The expansion of the determinant of \mathcal{M}_n^A along the first row yields precisely the previously known result by Bronfman [8], relating the Möbius polynomials of Artin–Tits monoids of type A_n , for distinct values of n:

Theorem 4.5. [8] If the growth function of the Artin–Tits monoid of type A_n is

$$g_{A_n}(t) = \frac{1}{H_n(t)}$$

and we denote $H_{-1}(t) = H_0(t) = 1$, then for $n \ge 1$ one has:

$$H_n(t) = \sum_{i=1}^{n+1} (-1)^{i-1} t^{\binom{i}{2}} H_{n-i}(t)$$

Proof. By Theorem 4.4 one has $H_n(t) = |\mathcal{M}_n^{\mathcal{A}}|$ for all $n \geq 1$. The formula in the statement is just the expansion of $|\mathcal{M}_n^{\mathcal{A}}|$ along the first row.

To finish the section, we exhibit some examples of the matrices and the growth functions, for small values of n.

Examples:

$$g_{A_1}(t) = \begin{vmatrix} 1 & t \\ 1 & 1 \end{vmatrix}^{-1}, \qquad g_{A_2}(t) = \begin{vmatrix} 1 & t & t^3 \\ 1 & 1 & t \\ 0 & 1 & 1 \end{vmatrix}^{-1},$$

$$g_{A_3}(t) = \begin{vmatrix} 1 & t & t^3 & t^6 \\ 1 & 1 & t & t^3 \\ 0 & 1 & 1 & t \\ 0 & 0 & 1 & 1 \end{vmatrix}^{-1}, \qquad g_{A_4}(t) = \begin{vmatrix} 1 & t & t^3 & t^6 & t^{10} \\ 1 & 1 & t & t^3 & t^6 \\ 0 & 1 & 1 & t & t^3 \\ 0 & 0 & 1 & 1 & t \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}^{-1}.$$

In other words,

$$g_{A_1}(t) = \frac{1}{1-t}, \quad g_{A_2}(t) = \frac{1}{1-2t+t^3}, \quad g_{A_3}(t) = \frac{1}{1-3t+t^2+2t^3-t^6},$$
$$g_{A_4}(t) = \frac{1}{1-4t+3t^2+3t^3-2t^4-2t^6+t^{10}}.$$

5 Artin-Tits monoids of type B and D

In the previous section we treated the case of Artin–Tits monoids of type A. In this section we will see that the same techniques can be applied to Artin–Tits monoids of type B and D. As the proofs are almost identical, we shall basically provide just the parts in which they differ.

5.1 Counting elements in the monoid (type B)

Let B_n be the Artin–Tits monoid of type B with n standard generators. Its standard presentation is the following:

$$B_n = \left\langle a_1, \dots, a_n \middle| \begin{array}{c} a_i a_j = a_j a_i, & |i - j| > 1 \\ a_i a_j a_i = a_j a_i a_j, & |i - j| = 1, i, j \neq n \\ a_{n-1} a_n a_{n-1} a_n = a_n a_{n-1} a_n a_{n-1} \end{array} \right\rangle$$

In this case (see [26]), for $0 \le i \le j \le n$, the consecutive generators a_i, \ldots, a_j either generate an Artin–Tits group of type A_{j-i+1} (if j < n), or generate an Artin–Tits group of type B_{j-i+1} (if j = n). In the latter case, the least common multiple of the generators is [27, Lemma 4.1]:

$$a_i \vee \cdots \vee a_n = (a_i \cdots a_{n-1} a_n a_{n-1} \cdots a_i)(a_{i+1} \cdots a_{n-1} a_n a_{n-1} \cdots a_{i+1}) \cdots (a_{n-1} a_n a_{n-1}) a_n.$$

Therefore, one has

$$||a_i \vee \dots \vee a_j|| = \begin{cases} \binom{j-i+2}{2} & \text{if } j < n \\ (n-i+1)^2 & \text{if } j = n \end{cases}$$

Also, as it happened in A_n , if $t \ge j + 2$ the atom a_t commutes with a_i, \ldots, a_j , so one has

$$(a_i \vee \cdots \vee a_j) \vee a_t = (a_i \vee \cdots \vee a_j)a_t.$$

Therefore we obtain the following result.

Lemma 5.1. Let M be the Artin-Tits monoid of type B_n . For $1 \le i \le j \le n$ one has

$$u_{k,i,j} = \begin{cases} m_{k-\binom{j-i+2}{2},j+1} & \text{if } j < n, \\ \\ m_{k-(n-i+1)^2,n+1} & \text{if } j = n. \end{cases}$$

Proof. Same proof as Lemma 4.1.

Corollary 5.2. In the Artin–Tits monoid B_n , for $k \geq 0$ and i = 1, ..., n + 1, one has:

$$m_{k,i} = m_{k,i-1} + \left(\sum_{j=i}^{n-1} (-1)^{j-i} m_{k-\binom{j-i+2}{2},j+1}\right) + (-1)^{n-i} m_{k-(n-i+1)^2,n+1}$$

Proof. This is a direct consequence of Proposition 3.5 and Lemma 5.1.

Using this recurrence relation we can also compute, in the case of B_n , a table containing the numbers $m_{k,i}$, where each row is computed from left to right, using the data in the previous ones. In Figure 2 we can see the first seven rows of the table corresponding to B_3 . We recall that the rightmost column contains the number of elements of length k in B_3 .

5.2 A new formula for the growth function (type B)

We keep proceeding as in the case of Artin–Tits monoids of type A, although this time the defined vectors have length $n^2(n+1)$. Namely, we define for $k \ge 1$, the column vector

$$\mathbf{v}_{k-1} = \left(m_{k-1,1} \cdots m_{k-1,n+1} \ m_{k-2,1} \cdots m_{k-2,n+1} \cdots m_{k-n^2,1} \cdots m_{k-n^2,n+1} \right)^t$$

where the $m_{i,j}$'s are defined for the Artin–Tits monoid B_n .

k i	1	2	3	4
0	1	1	1	1
1	1	2	3	3
2	2	5	8	8
3	4	12	20	20
4	9	28	48	48
5	20	65	113	113
6	45	150	263	263

Figure 2: A table containing $m_{k,i}$ for the Artin–Tits monoid B_3 , for $k \leq 6$.

Lemma 5.3. There is a square matrix \mathcal{B} with $n^2(n+1)$ rows, whose entries belong to $\{0,1,-1\}$ such that for every $k \geq 1$

$$\mathcal{B}\mathbf{v}_{k-1} = \mathbf{v}_k$$

Proof. As in the previous section, we can indicate how the matrix \mathcal{B} looks like, by showing it explicitly when n=3. As in the case of type A, let us denote (for n=3):

$$sh(L) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad sh^2(L) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad sh^3(L) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This time the last column of these matrices behaves differently, so given a matrix T, we will denote T' the matrix obtained from T by replacing its last column by a column of zeroes, and T'' the matrix obtained from T by keeping its last column and replacing all other entries by zeroes. Of course, T' + T'' = T.

As an example, we have:

$$sh^2(L) = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right) \quad sh^2(L)' = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right) \quad sh^2(L)'' = \left(\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$$

then the matrix \mathcal{B} is the following block matrix:

where I is the 4×4 identity matrix and \mathcal{O} is the 4×4 zero matrix.

For every $n \ge 2$, the $(n+1) \times (n+1)$ blocks of the matrix \mathcal{B} are defined in the following way. The block in position (i,j) with $1 \le i,j \le n^2$ is

$$B_{i,j} = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where

$$\Gamma_1 = \begin{cases} (-1)^t s h^{t-1}(L)' & \text{if } i = 1 \text{ and } j = {t \choose 2} \text{ for some } 2 < t \le n \\ \mathcal{O}_{(n+1)\times(n+1)} & \text{otherwise.} \end{cases}$$

$$\Gamma_2 = \begin{cases} (-1)^t s h^{t-1}(L)'' & \text{if } i = 1 \text{ and } j = t^2 \text{ for some } 2 \le t \le n \\ \mathcal{O}_{(n+1)\times(n+1)} & \text{otherwise.} \end{cases}$$

$$\Gamma_3 = \begin{cases} I_{(n+1)\times(n+1)} & \text{if } i = j+1 \\ \mathcal{O}_{(n+1)\times(n+1)} & \text{otherwise.} \end{cases}$$

The proof which shows that $\mathcal{B}\mathbf{v}_{k-1} = \mathbf{v}_k$ for every $k \geq 1$ is analogous to that of type A, this time using the formula in Corollary 5.2.

Theorem 5.4. Let M be the Artin-Tits monoid of type B_n . Let $\mathcal{M}_n^{\mathcal{B}}$ be the square matrix of order n+1 whose entry (i, n+1) equals $t^{(n-i+1)^2}$ for all i, and whose entry (i, j) for $j \leq n$ equals $t^{\binom{j-i+1}{2}}$ whenever $j-i+1 \geq 0$, and 0 otherwise. Then

$$g_M(t) = |\mathcal{M}_n^{\mathcal{B}}|^{-1}$$

Proof. Using the same arguments as in the proof of Theorem 4.4, one has:

$$g_{B_n}(t) = \frac{\mathbf{v}\mathcal{B}'\mathbf{v}_0}{|I - \mathcal{B}t|}$$

Later we will see that $|I - \mathcal{B}t|$ is a polynomial of degree n^2 , and $g_{B_n}(t)$ is the inverse of a polynomial of degree n^2 by Corollary 2.2. Hence $\mathbf{v}\mathcal{B}'\mathbf{v}_0$ is a constant, namely 1 (as $g_{B_n}(0) = \alpha_0 = 1$).

Therefore:

$$g_{B_n}(t) = \frac{1}{|I - \mathcal{B}t|}$$

In the case n=3 we have:

$$|I - Sh(L)t| = \begin{vmatrix} I - sh(L)t & \mathcal{O} & sh^2(L)'t & sh^2(L)''t & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & -sh^3(L)''t \\ -It & I & \mathcal{O} \\ \mathcal{O} & -It & I & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & -It & I & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & -It & I & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & -It & I & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & -It & I & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & -It & I & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & -It & I & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It & I \\ \mathcal{O} & -It \\ \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal$$

As we did for type A, if we add to each (block) column the column on its right multiplied by t, starting from the right hand side, we obtain

where

$$T = I + \left(-sh(L)'t + sh^2(L)'t^3\right) + \left(-sh(L)''t + sh^2(L)''t^4 - sh^3(L)''t^9\right)$$

Hence

$$|I - \mathcal{B}t| = |T| = \begin{vmatrix} 1 & -t & t^3 & -t^9 \\ 0 & 1 - t & -t + t^3 & t^4 - t^9 \\ 0 & -t & 1 - t + t^3 & -t + t^4 - t^9 \\ 0 & -t & -t + t^3 & 1 - t + t^4 - t^9 \end{vmatrix}$$

Notice that the exponents in the last column are of the form k^2 , while the exponents on the other columns are of the form $\binom{k}{2}$.

Substracting to each row the previous one, starting from the bottom, we obtain

$$|I - \mathcal{B}t| = \left| egin{array}{cccc} 1 & -t & t^3 & -t^9 \ -1 & 1 & -t & t^4 \ 0 & -1 & 1 & -t \ 0 & 0 & -1 & 1 \end{array}
ight| = \left| egin{array}{cccc} 1 & t & t^3 & t^9 \ 1 & 1 & t & t^4 \ 0 & 1 & 1 & t \ 0 & 0 & 1 & 1 \end{array}
ight| = |\mathcal{M}_3^{\mathcal{B}}|$$

In the case of arbitrary n, exactly the same operations lead to the matrix $\mathcal{M}_n^{\mathcal{B}}$ defined in the statement of Theorem 5.4. Hence $|I - \mathcal{B}t| = |\mathcal{M}_n^{\mathcal{B}}|$.

Notice that $M_n^{\mathcal{B}}$ equals $M_n^{\mathcal{A}}$ except for the last column. Hence, if we expand the determinant of $\mathcal{M}_n^{\mathcal{B}}$ along the last column, we see that we can express $|\mathcal{M}_n^{\mathcal{B}}|$ in terms of $|\mathcal{M}_t^{\mathcal{A}}|$ for t < n. Namely:

$$|\mathcal{M}_{n}^{\mathcal{B}}| = 1|\mathcal{M}_{n-1}^{\mathcal{A}}| - t|\mathcal{M}_{n-2}^{\mathcal{A}}| + t^{4}|\mathcal{M}_{n-3}^{\mathcal{A}}| - \dots + (-1)^{n-1}t^{(n-1)^{2}}|\mathcal{M}_{0}^{\mathcal{A}}| + (-1)^{n}t^{n^{2}}.$$

As $|\mathcal{M}_t^{\mathcal{A}}|$ is a polynomial of degree $\binom{t}{2}$, it follows that $|\mathcal{M}_n^{\mathcal{B}}|$ is a polynomial of degree n^2 . This finishes the proof.

The above expansion of the determinant of $\mathcal{M}_n^{\mathcal{B}}$ along the last column, yields a formula relating the Möbious polynomials of B_n to the Möbious polynomials of A_m for m < n.

Corollary 5.5. If the growth functions of the Artin-Tits monoids of types A_n and B_n are

$$g_{A_n} = \frac{1}{H_n(t)}, \qquad g_{B_n} = \frac{1}{F_n(t)}$$

and we denote $H_{-1}(t) = H_0(t) = 1$, then for $n \ge 1$ one has:

$$F_n(t) = \sum_{i=0}^{n} (-1)^i t^{i^2} H_{n-1-i}(t)$$

Proof. By Theorem 5.4 one has $F_n(t) = |\mathcal{M}_n^{\mathcal{B}}|$ for all $n \geq 1$. The formula in the statement is just the expansion of $|\mathcal{M}_n^{\mathcal{B}}|$ along the last column.

But we can also expand the determinant $|\mathcal{M}_n^{\mathcal{B}}|$ along the first row, yielding a formula relating the growth function of B_n to the growth functions of B_m for m < n.

Corollary 5.6. If the growth functions of the Artin-Tits monoid of type B_n is

$$g_{B_n} = \frac{1}{F_n(t)}$$

and we denote $F_{-1}(t) = F_0(t) = 1$, then for $n \ge 1$ one has:

$$F_n(t) = \left(\sum_{i=1}^n (-1)^{i-1} t^{\binom{i}{2}} F_{n-i}(t)\right) + (-1)^n t^{n^2}$$

Proof. By Theorem 5.4 one has $F_n(t) = |\mathcal{M}_n^{\mathcal{B}}|$ for all $n \geq 1$. Now expand this determinant along the first row.

As we did with the monoid of type A, we finish the section with some examples.

Examples:

$$g_{B_1}(t) = \begin{vmatrix} 1 & t \\ 1 & 1 \end{vmatrix}^{-1}, \qquad g_{B_2}(t) = \begin{vmatrix} 1 & t & t^4 \\ 1 & 1 & t \\ 0 & 1 & 1 \end{vmatrix}^{-1},$$

$$g_{B_3}(t) = \begin{vmatrix} 1 & t & t^3 & t^9 \\ 1 & 1 & t & t^4 \\ 0 & 1 & 1 & t \\ 0 & 0 & 1 & 1 \end{vmatrix}^{-1}, \qquad g_{B_4}(t) = \begin{vmatrix} 1 & t & t^3 & t^6 & t^{16} \\ 1 & 1 & t & t^3 & t^9 \\ 0 & 1 & 1 & t & t^4 \\ 0 & 0 & 1 & 1 & t \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}^{-1}.$$

In other words,

$$g_{B_1}(t) = \frac{1}{1-t}, \quad g_{B_2}(t) = \frac{1}{1-2t+t^4}, \quad g_{B_3}(t) = \frac{1}{1-3t+t^2+t^3+t^4-t^9},$$
$$g_{B_4}(t) = \frac{1}{1-4t+3t^2+2t^3-t^5-t^6-t^9+t^{16}}.$$

5.3 Counting elements in the monoid (type D)

The standard presentation of the Artin-Tits monoid of type D with $n \geq 4$ generators is the following:

$$D_n = \left\langle a_1, \dots, a_n \middle| \begin{array}{l} a_i a_j = a_j a_i, & |i-j| > 1, \ \{i, j\} \neq \{n-2, n\} \\ a_i a_j a_i = a_j a_i a_j, & |i-j| = 1, \ \{i, j\} \neq \{n-1, n\} \\ a_{n-2} a_n a_{n-2} = a_n a_{n-2} a_n \\ a_{n-1} a_n = a_n a_{n-1} \end{array} \right\rangle$$

In this monoid (see [26]), the submonoid generated by the consecutive standard generators a_i, \ldots, a_j is an Artin-Tits monoid of type D if j = n and $i \le n - 3$, it is \mathbb{Z}^2 if j = n and i = n - 1, and it is an Artin-Tits monoid of type A otherwise. In the first case, the least common multiple of the generators is [27, Lemma 5.1]:

$$a_i \vee \dots \vee a_n = \prod_{j=i}^{n-1} (a_j a_{j+1} \cdots a_{n-2})(a_{n-1} a_n)(a_{n-2} a_{n-3} \cdots a_j).$$

This word has length (n-i+1)(n-i). We remark that if i=n-2 and j=n, the least common multiple of a_{n-2}, a_{n-1}, a_n (which generate a group of type A_3) is an element of length 6, and this number is also equal to (n-i+1)(n-i). And if i=n-1 and j=n, the generators a_{n-1} and a_n commute, so their least common multiple is $a_{n-1}a_n$ which has length 2, that is, equal to (n-i+1)(n-i). Therefore:

$$||a_i \vee \dots \vee a_j|| = \begin{cases} (n-i+1)(n-i) & \text{if } j = n \text{ and } i < n \\ {j-i+2 \choose 2} & \text{otherwise.} \end{cases}$$

On the other hand, if $t \geq j+2$, the atom a_t commutes with a_i, \ldots, a_j , except when t=n and j=n-2. In this latter case, the generators $a_i, a_{i+1}, \ldots, a_{n-2}, a_n$ behave as consecutive generators in a monoid of type A. Hence, when $t \geq j+2$:

$$(a_i \vee \dots \vee a_j) \vee a_t = \begin{cases} (a_i \vee \dots \vee a_j) a_t & \text{if } j < n-2 \\ (a_i \vee \dots \vee a_j) a_t a_j a_{j-1} \cdots a_i & \text{if } j = n-2 \text{ (hence } t = n) \end{cases}$$

Therefore, in D_n the situation is the following.

Lemma 5.7. Let M be the Artin–Tits monoid of type D_n $(n \ge 4)$. For $1 \le i \le j \le n$ one has

$$u_{k,i,j} = \begin{cases} m_{k-(n-i+1)(n-i),n+1} & \text{if } j = n \text{ and } i < n \\ m_{k-\binom{n-i}{2},n} - m_{k-\binom{n-i+1}{2},n+1} & \text{if } j = n-2, \\ m_{k-\binom{j-i+2}{2},j+1} & \text{otherwise.} \end{cases}$$

Proof. If j = n and i < n we have

$$u_{k,i,j} = |\{b \in M_k; \quad a_i \vee \dots \vee a_n \preccurlyeq b\}| = |M_{k-(n-i+1)(n-i)}| = m_{k-(n-i+1)(n-i),n+1}.$$

Suppose now that j = n - 2. In this case:

$$U_k^{(i,n-2)} = \{b \in M_k; \quad a_i \vee \dots \vee a_{n-2} \leq b, \quad a_n \not\leq b\}$$
$$= \{(a_i \vee \dots \vee a_{n-2})c \in M_k; \quad a_n \not\leq (a_i \vee \dots \vee a_{n-1})c\}$$

But $a_n \preceq (a_i \lor \cdots \lor a_{n-2})c$ if and only if $(a_i \lor \cdots \lor a_{n-2}) \lor a_n \preceq (a_i \lor \cdots \lor a_{n-2})c$, which occurs if and only if $(a_i \lor \cdots \lor a_{n-2}) a_n a_{n-2} a_{n-3} \cdots a_i \preceq (a_i \lor \cdots \lor a_{n-2})c$, which is equivalent to $a_n a_{n-2} a_{n-3} \cdots a_i \preceq c$. Hence

$$U_k^{(i,n-2)} = \{ (a_i \vee \dots \vee a_{n-2}) c \in M_k; \ a_n a_{n-2} a_{n-3} \dots a_i \not \preccurlyeq c \}.$$

Therefore

$$u_{k,i,n-2} = \left| U_k^{(i,n-2)} \right| = \left| \{ c \in M_{k-\binom{n-i}{2}}; \ a_n a_{n-2} a_{n-3} \cdots a_i \not \leqslant c \} \right|.$$

Hence, we must remove from the elements in $M_{k-\binom{n-i}{2}}$ those having $a_n a_{n-2} a_{n-3} \cdots a_i$ as prefix. As D_n is a cancellative monoid, there are exactly $\left| M_{k-\binom{n-i}{2}-(n-i)} \right| = \left| M_{k-\binom{n-i+1}{2}} \right|$ elements to remove. Therefore

$$u_{k,i,n-2} = \left| M_{k-\binom{n-i}{2}} \right| - \left| M_{k-\binom{n-i+1}{2}} \right| = m_{k-\binom{n-i}{2},n} - m_{k-\binom{n-i+1}{2},n+1}$$

as we wanted to show. The reason of using either n or n+1 as second subindex will become clear later. Finally, in all the remaining cases (either i=j=n or $j\notin\{n-2,n\}$), we have:

$$U_k^{(i,j)} = \{b \in M_k; \quad a_i \vee \dots \vee a_j \leq b, \quad a_{j+2}, \dots, a_n \not\leq b\} = m_{k-\binom{j-i+2}{2}, j+1}$$

in the same way as in Lemma 4.1.

Corollary 5.8. In the Artin–Tits monoid D_n $(n \ge 4)$, for $k \ge 0$ and i = 1, ..., n-2, one has:

$$m_{k,i} = m_{k,i-1} + \left(\sum_{j=i}^{n-3} (-1)^{j-i} m_{k-\binom{j-i+2}{2},j+1}\right) + (-1)^{n-2-i} \left(m_{k-\binom{n-i}{2},n} - 2 m_{k-\binom{n-i+1}{2},n+1} + m_{k-(n-i+1)(n-i),n+1}\right),$$

and also:

$$m_{k,n-1} = m_{k,n-2} + m_{k-1,n} - m_{k-2,n+1}$$

$$m_{k,n} = m_{k,n-1} + m_{k-1,n+1},$$

$$m_{k,n+1} = m_{k,n}.$$

Proof. This is a direct consequence of Proposition 3.5 and Lemma 5.7.

The first formula in Corollary 5.8 has an interesting property: It involves no expression of the form $m_{t,n-1}$ for any t. Hence, the formulae can be simplified if we remove the column corresponding to i = n-1 from the table containing the $m_{k,i}$'s. For that purpose, we define:

Definition 5.9. Let M be the Artin–Tits monoid of type D_n . For $k \geq 0$ we define:

$$d_{k,i} = \begin{cases} m_{k,i} & \text{if } i \in \{1, \dots, n-2\}, \\ m_{k,i+1} & \text{if } i \in \{n-1, n\}. \end{cases}$$

We have then defined the numbers $d_{k,1}, \ldots, d_{k,n}$, such that $d_{k,n-1} = d_{k,n} = |M_k| = \alpha_k$. With this definition, the recurrence relation in Corollary 5.8 can be rewritten as follows:

Corollary 5.10. In the Artin–Tits monoid D_n , for $k \geq 0$ and i = 1, ..., n one has:

$$d_{k,i} = d_{k,i-1} + \left(\sum_{j=i}^{n-1} (-1)^{j-i} d_{k-\binom{j-i+2}{2},j+1}\right) + (-1)^{n-i-1} \left(d_{k-\binom{n-i+1}{2},n} - d_{k-(n-i+1)(n-i),n}\right)$$

Proof. For i = 1, ..., n - 2, this is a direct consequence of Corollary 5.8.

For i = n - 1, we have (using Corollary 5.8 and the fact that $m_{k-1,n} = m_{k-1,n+1}$):

$$\begin{array}{lcl} d_{k,n-1} = m_{k,n} & = & m_{k,n-1} + m_{k-1,n+1} \\ & = & \left(m_{k,n-2} + m_{k-1,n} - m_{k-2,n+1} \right) + m_{k-1,n+1} \\ & = & m_{k,n-2} + 2 \, m_{k-1,n+1} - m_{k-2,n+1} \\ & = & d_{k,n-2} + 2 \, d_{k-1,n} - d_{k-2,n} \end{array}$$

which satisfies the statement.

Finally, if i = n:

$$d_{k,n} = m_{k,n+1} = m_{k,n} = d_{k,n-1} = d_{k,n-1} - (d_{k,n} - d_{k,n})$$

so the result also holds in this case.

Thanks to Corollary 5.10 we can compute a table containing the numbers $d_{k,i}$ for $k \geq 0$ and i = 1, ..., n. Notice that the rightmost column contains the numbers $d_{k,n}$ for $k \geq 0$, which correspond to the number of elements of length k, with respect to the standard generators of D_n .

k i	1	2	3	4
0	1	1	1	1
1	1	2	4	4
2	2	6	13	13
3	5	16	38	38
4	12	42	105	105
5	29	108	280	280
6	72	277	732	732

Figure 3: A table containing the numbers $d_{k,i}$ for the Artin-Tits monoid D_4 , for $k \leq 6$.

5.4 A new formula for the growth function (type D)

The linear recurrence obtained in the case of Artin–Tits monoids of type D forces us to define vectors with $n^2(n-1)$ entries. Namely, we define for $k \ge 1$ the column vector

$$\mathbf{v}_{k-1} = \left(d_{k-1,1} \cdots d_{k-1,n} \ d_{k-2,1} \cdots d_{k-2,n} \cdots d_{k-n(n-1),1} \cdots d_{k-n(n-1),n}\right)^t$$

Lemma 5.11. There is a square matrix \mathcal{D} with $n^2(n-1)$ rows, whose entries belong to $\{0,1,-1\}$ such that for every $k \geq 1$

$$\mathcal{D}\mathbf{v}_{k-1} = \mathbf{v}_k$$

Proof. This proof is analogous to those of type A and B. One can see the matrix \mathcal{D} as a squared block matrix, with n(n-1) rows, each block made of $n \times n$ matrices. In the case n=4, the first row of blocks is the following:

0	1	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0-1
0	1	1	0	0	0	0	0	0	0	-1	-2	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0 - 1
0	1	1	2	0	0	0 -	-1	0	0	-1	-2	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0 - 1
0	1	1	2	0	0	0 -	-1	0	0	-1	-2	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0-1

And the remaining rows are made of identity and zero matrices, with the identities placed at the subdiagonal.

Recall the definition of $sh^k(L)$ for k > 0. As for type B, the last column of these matrices behaves differently than in the case of type A, so given a matrix T, we denote T' the matrix obtained from T by replacing its last column by a column of zeroes, and T'' the matrix obtained from T by keeping its last column and replacing all other entries by zeroes. Then, for every $n \ge 2$, the $n \times n$ blocks of the matrix \mathcal{D} are defined in the following way.

The block in position (i, j) with $1 \le i, j \le n(n-1)$ is

$$D_{i,j} = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$$

where

$$\Gamma_1 = \begin{cases} (-1)^t s h^{t-1}(L)' & \text{if } i = 1 \text{ and } j = {t \choose 2} \text{ for some } 2 \le t < n \\ \mathcal{O}_{n \times n} & \text{otherwise.} \end{cases}$$

$$\Gamma_2 = \begin{cases} (-1)^t 2 s h^{t-1}(L)'' & \text{if } i = 1 \text{ and } j = {t \choose 2} \text{ for some } 2 \le t \le n \\ \mathcal{O}_{n \times n} & \text{otherwise.} \end{cases}$$

$$\Gamma_3 = \begin{cases} (-1)^{t-1} s h^{t-1}(L)'' & \text{if } i = 1 \text{ and } j = t(t-1) \text{ for some } 2 \le t \le n \\ \mathcal{O}_{n \times n} & \text{otherwise.} \end{cases}$$
otherwise.

$$\Gamma_4 = \begin{cases} I_{n \times n} & \text{if } i = j + 1 \\ \mathcal{O}_{n \times n} & \text{otherwise.} \end{cases}$$

This description follows directly from the formula in Corollary 5.10.

Theorem 5.12. Let M be the Artin-Tits monoid of type D_n , for $n \geq 4$. Let $\mathcal{M}_n^{\mathcal{D}}$ be the square matrix of order n whose entry (i,n) equals $2t^{\binom{n-i+1}{2}} - t^{(n-i+1)(n-i)}$ for all i, and whose entry (i,j) for j < n equals $t^{\binom{j-i+1}{2}}$ whenever $j-i+1 \geq 0$, and 0 otherwise. Then

$$g_M(t) = |\mathcal{M}_n^{\mathcal{D}}|^{-1}$$

Proof. Following the lines of Theorem 4.4 and Theorem 5.4, one has:

$$g_{D_n}(t) = \frac{\mathbf{v}\mathcal{D}'\mathbf{v}_0}{|I - \mathcal{D}t|}$$

Later we will see that $|I - \mathcal{D}t|$ is a polynomial of degree n(n-1), and $g_{D_n}(t)$ is the inverse of a polynomial of degree n(n-1) by Corollary 2.2. Hence $\mathbf{v}\mathcal{D}'\mathbf{v}_0$ is a constant, namely 1.

Therefore:

$$g_{D_n}(t) = \frac{1}{|I - \mathcal{D}t|}$$

As we did for types A and B, if we add to each (block) column the column on its right multiplied by t, starting from the right hand side, we obtain a matrix which for n = 4 looks like the following:

Hence

$$|I - \mathcal{D}t| = |T| = \begin{vmatrix} 1 & -t & t^3 & -2t^6 + t^{12} \\ 0 & 1 - t & -t + t^3 & 2t^3 - 3t^6 + t^{12} \\ 0 & -t & 1 - t + t^3 & -2t + t^2 + 2t^3 - 3t^6 + t^{12} \\ 0 & -t & -t + t^3 & 1 - 2t + t^2 + 2t^3 - 3t^6 + t^{12} \end{vmatrix}$$

Substracting to each row the previous one, starting from the bottom, we obtain

$$|I - \mathcal{D}t| = \begin{vmatrix} 1 & -t & t^3 & -2t^6 + t^{12} \\ -1 & 1 & -t & 2t^3 - t^6 \\ 0 & -1 & 1 & -2t + t^2 \\ 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & t & t^3 & 2t^6 - t^{12} \\ 1 & 1 & t & 2t^3 - t^6 \\ 0 & 1 & 1 & 2t - t^2 \\ 0 & 0 & 1 & 1 \end{vmatrix} = |\mathcal{M}_4^{\mathcal{D}}|$$

In the case of arbitrary n, exactly the same operations lead to the matrix $\mathcal{M}_n^{\mathcal{D}}$ defined in the statement of Theorem 5.12. Hence $|I - \mathcal{D}t| = |\mathcal{M}_n^{\mathcal{D}}|$.

Notice that, as for type B, the matrix $\mathcal{M}_n^{\mathcal{D}}$ equals $\mathcal{M}_{n-1}^{\mathcal{A}}$ except for the last column. Hence, if we expand the determinant of $\mathcal{M}_n^{\mathcal{D}}$ along the last column, we see that we can express $|\mathcal{M}_n^{\mathcal{D}}|$ in terms of $|\mathcal{M}_t^{\mathcal{A}}|$ for t < n-1. Namely:

$$|\mathcal{M}_{n}^{\mathcal{D}}| = 1|\mathcal{M}_{n-2}^{\mathcal{A}}| - (2t - t^{2})|\mathcal{M}_{n-3}^{\mathcal{A}}| + \dots + (-1)^{n}(2t^{\binom{n-1}{2}} - t^{(n-1)(n-2)})|\mathcal{M}_{0}^{\mathcal{A}}| + (-1)^{n-1}(2t^{\binom{n}{2}} - t^{n(n-1)}).$$

As $|\mathcal{M}_t^{\mathcal{A}}|$ is a polynomial of degree $\binom{t}{2}$, it follows that $|\mathcal{M}_n^{\mathcal{D}}|$ is a polynomial of degree n(n-1). This finishes the proof.

From this formula it is easy to relate the growth functions of the monoid D_n with the growth functions of monoids of type A_n .

Corollary 5.13. If the growth functions of the Artin-Tits monoids of types A_n and D_n are

$$g_{A_n} = \frac{1}{H_n(t)}, \qquad g_{D_n} = \frac{1}{G_n(t)}$$

and we denote $H_{-1}(t) = H_0(t) = 1$, then for n > 1 one has:

$$G_n(t) = \sum_{i=1}^{n} (-1)^{i-1} \left(2t^{\binom{i}{2}} - t^{i(i-1)} \right) H_{n-i-1}(t)$$

Proof. By Theorem 5.12 one has $G_n(t) = |\mathcal{M}_n^{\mathcal{D}}|$ for all n > 1. The formula in the statement is just the expansion of $|\mathcal{M}_n^{\mathcal{D}}|$ along the last column.

But we can also expand the determinant $|\mathcal{M}_n^{\mathcal{D}}|$ along the first row, yielding a formula relating the growth function of D_n to the growth functions of D_m for m < n. To simplify the statement, although the monoid D_n is defined for $n \ge 4$, we will naturally assume that $D_2 = \mathbb{Z} \times \mathbb{Z}$ and $D_3 = A_3$.

Corollary 5.14. If the growth functions of the Artin-Tits monoid of type D_n , for $n \geq 2$, is

$$g_{D_n} = \frac{1}{G_n(t)}$$

and we denote $G_1(t) = 1$, then for $n \ge 2$ one has:

$$G_n(t) = \left(\sum_{i=1}^{n-1} (-1)^{i-1} t^{\binom{i}{2}} G_{n-i}(t)\right) + (-1)^{n-1} \left(2t^{\binom{n}{2}} - t^{n(n-1)}\right)$$

Proof. By Theorem 5.12 one has $G_n(t) = |\mathcal{M}_n^{\mathcal{D}}|$ for all $n \geq 4$. Now expand this determinant along the first row.

As in the previous cases, here are some examples to illustrate the computations.

Examples:

$$g_{D_2}(t) = \begin{vmatrix} 1 & 2t - t^2 \\ 1 & 1 \end{vmatrix}^{-1}, \qquad g_{D_3}(t) = \begin{vmatrix} 1 & t & 2t^3 - t^6 \\ 1 & 1 & 2t - t^2 \\ 0 & 1 & 1 \end{vmatrix}^{-1},$$

$$g_{D_4}(t) = \begin{vmatrix} 1 & t & t^3 & 2t^6 - t^{12} \\ 1 & 1 & t & 2t^3 - t^6 \\ 0 & 1 & 1 & 2t - t^2 \\ 0 & 0 & 1 & 1 \end{vmatrix}^{-1}, \qquad g_{D_5}(t) = \begin{vmatrix} 1 & t & t^3 & t^6 & 2t^{10} - t^{20} \\ 1 & 1 & t & t^3 & 2t^6 - t^{12} \\ 0 & 1 & 1 & t & 2t^3 - t^6 \\ 0 & 0 & 1 & 1 & 2t - t^2 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}^{-1}.$$

In other words,

$$g_{D_2}(t) = \frac{1}{1 - 2t + t^2}, \quad g_{D_3}(t) = \frac{1}{1 - 3t + t^2 + 2t^3 - t^6}, \quad g_{D_4}(t) = \frac{1}{1 - 4t + 3t^2 + 2t^3 - 3t^6 + t^{12}},$$

$$g_{D_5}(t) = \frac{1}{1 - 5t + 6t^2 + 2t^3 - 4t^4 + t^5 - 4t^6 + t^7 + 2t^{10} + t^{12} - t^{20}}.$$

6 Growth rates

One of the big advantages of the way we use for counting the number of elements in a monoid, is that one can easily extract some conclusions on its growth rate. We recall that the exponential growth rate of a monoid, with respect of a given set of generators is equal to

$$\rho_M = \lim_{k \to \infty} \sqrt[k]{|M_k|}$$

In the case of Artin-Tits monoids of type A_n , B_n and D_n , this growth rate has been extensively studied by several authors [32, 35, 19]. In [19], it is shown that the Möbius polynomial for M has a unique root of smallest modulus, which is a real number. Its inverse is precisely ρ_M .

If one considers the family of monoids $\{A_n\}_{n\geq 1}$, it is a natural question, posed by several authors, to find the limit:

$$\rho = \lim_{n \to \infty} \rho_{A_n}.$$

It was shown in [35, Theorem 8] that $2 \le \rho_{A_n} \le 4$ for all n, so we have $2 \le \rho \le 4$. This was improved in [19, Proposition 7.98], where it is shown that the sequence ρ_{A_n} is non-decreasing, and that its limit ρ satisfies $2.5 < \rho < 4$.

We will now see that we can determine ρ with arbitrary precision. For this sake, we will first describe two concepts of analytical nature: the partial theta function and the KLV-constant q_{∞} .

6.1 Topics from Real Analysis

6.1.1 The partial theta function

We start by recalling the formal power series:

$$f(x,y) = \sum_{k=0}^{\infty} y^{\binom{k}{2}} x^k.$$

In Sokal's paper [33] this series is called the *partial theta function*, a name that comes from the relation with the classical Jacobi theta function. It is noteworthy that this name is sometimes also used for the more general series

$$f(x,y) = \sum_{k=0}^{\infty} y^{Ak^2 + Bk} x^k,$$

corresponding the previous case to A = 1/2, B = -1/2. We will follow here Sokal's nomenclature.

Having already appeared in Ramanujan's work [2], the partial theta function is an instance of a more general family of series which also contains the deformed exponential function (see for example [25]). It is also a particular example of the three-variable Rogers-Ramanujan function [31], and should not be confused with the different -but related- "false theta functions" defined by Rogers in [30]. The study of the partial theta function has become a fruitful field of research, as relations have been discovered with the so-called section-hyperbolic polynomials [24], Dirichlet series via asymptotic expansions [5], Garrett-Ismail-Stanton type identities [36, 16] or q-hypergeometric series, particularly mock modular forms [7]; a good guide for the theory is the year-long work of Vladimir Kostov (see for example [21, 22, 23]).

To our knowledge, however, no relationship had been found so far between partial theta functions and algebraic invariants, in particular coming for Group/Monoid Theory. To describe such a relation, we need to introduce the concept of leading root. Following Sokal, given a formal series f(x, y) with coefficients in a commutative unital ring R, there is a unique formal series $x_0(y)$ in R[y] such that $f(x_0(y), y) = 0$.

This series is called the *leading root* of f(x, y), and in the case of the partial theta function defined above, it has the shape:

$$x_0(y) = -(1 + y + 2y^2 + 4y^3 + 9y^4 + \cdots)$$

The sequence of coefficients of $-x_0(y)$, which is

$$1, 1, 2, 4, 9, 21, 52, 133, \dots$$

is proved to be an increasing sequence of positive integers [33], and its first 7,000 terms were computed by the author.

In Proposition 6.6 below we will state that this sequence is in particular, and very surprisingly, a combinatorial invariant of Artin–Tits monoids. A different combinatorial (non-algebraic) approach to these coefficients, based on stack polyominos, can be found in unpublished work of Prellberg [29].

6.1.2 The KLV-constant q_{∞}

In the last section of Sokal's paper [33] the exponential growth of the previous sequence of coefficients is also established, as a consequence of Pringhseim theorem. This number is the inverse of the first real singularity of the leading root of the partial theta function, and turns to be a constant whose value is 3.2336366652... As this number was first effectively computed by Katkova-Lobova-Vishnyakova in [20] and seems not to have a standard name, we will call it here the KLV-constant and denote it by q_{∞} , as these authors did.

The KLV-constant appeared in the context of a long-standing problem, very easy to formulate. Consider the series

$$g_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}},$$

for a>1. The goal is to find the smallest a such that $g_a(z)$ has only real roots. This problem was first undertaken by Hardy [18], who proved that $a^2\geq 9$ was a sufficient condition. Afterwards, different authors attacked the problem and lowered this bound, as for example Pólya-Szegő [28] or Craven-Csordas [10], who reached a bound of 3.42. The final solution finally appeared in 2003 in [20, Theorem 4], as the mentioned authors proved that $a^2\geq q_\infty$ is a necessary and sufficient condition for the series $g_a(z)$ to have only real roots. Moreover, their computation provided a way of approximating q_∞ with arbitrary precision.

We will show in the next section that $q_{\infty} = \rho$. This result permits in particular a completely unexpected description of the KLV-constant in terms of growths of monoids.

6.2 Growth rates of braid monoids and the partial theta function

Recall from Corollary 4.2 that, for the Artin-Tits monoid of type A_n , the numbers $m_{k,i}$ satisfy the following recurrence relation:

$$m_{k,i} = m_{k,i-1} + \sum_{j=i}^{n} (-1)^{j-i} m_{k-\binom{j-i+2}{2},j+1},$$

where $m_{0,1} = 1$.

Using this recurrence relation, we compute the table containing the numbers $m_{k,i}$ for $k \geq 0$ and i = 1, ..., n + 1, as in Figure 1.

k i	1	2	3	4	5	6
0	1	1	1	1	1	1
1	1	2	3	4	5	5
2	2	5	9	14	19	19
3	4	12	25	43	62	62
4	9	30	68	125	187	187
5	21	75	181	349	536	536
6	51	190	478	952	1488	1488
7	126	484	1254	2555	4043	4043
8	317	1241	3279	6786	10829	10829

$egin{array}{c} i \ k \end{array}$	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1	1	2	3	4	5	6	6
2	2	5	9	14	20	26	26
3	4	12	25	44	69	95	95
4	9	30	69	132	221	316	316
5	21	76	188	383	673	989	989
6	52	197	512	1091	1985	2974	2974
7	132	517	1393	3068	5726	8700	8700
8	343	1373	3794	8557	16268	24968	24968

Figure 4: Tables containing $m_{k,i}$ for the Artin-Tits monoids A_5 and A_6 , for $k \leq 8$.

We will now compare the tables corresponding to distinct values of n. To give an example, in Figure 4 we can see the first 9 rows of the tables corresponding to A_5 and A_6 .

In Figure 4 we have boldfaced the elements $m_{k,i}$ such that $k+i \leq n+1$. We will call them *stabilized* entries. From the recurrence relation in Corollary 4.2, it is easy to see that each stabilized entry in the table for A_n is computed using exactly the same values as the corresponding element in the table for A_{n+1} . Hence, if $m_{k,i}$ is a stabilized entry for some A_n , the value $m_{k,i}$ will be the same in the table corresponding to A_{n_1} , for every $n_1 > n$. In other words, the table stabilizes when n tends to infinity: The value of each $m_{k,i}$ will become constant. By abuse of notation, we will denote $m_{k,i}$ the value of $m_{k,i}$ for $n \geq k+i-1$, that is, when its value has stabilized.

The value of the stabilized $m_{k,i}$ can be computed as follows:

$$m_{k,i} = m_{k,i-1} + \sum_{j=i}^{\infty} (-1)^{j-i} m_{k-\binom{j-i+2}{2},j+1}$$
 (3)

Notice that this sum is always finite, as the value $k - {j-i+2 \choose 2}$ must be non-negative in order to produce a nonzero summand.

There is a nice way to understand the *limit table*, which contains the numbers which are already stabilized.

Notice that if $n_1 < n_2$, there is a natural inclusion $A_{n_1} \subset A_{n_2}$. The direct limit of the monoids $\{A_n\}_{n\geq 1}$ with respect to these natural inclusions is known as A_{∞} , the braid monoid on an infinite number of strands. It has the same presentation as the usual braid monoid A_n , but with an infinite number of generators:

$$A_{\infty} = \left\langle a_1, a_2, \dots \middle| \begin{array}{c} a_i a_j = a_j a_i, & |i - j| > 1 \\ a_i a_j a_i = a_j a_i a_j, & |i - j| = 1 \end{array} \right\rangle$$

If $M=A_{\infty}$ we can define, for $k\geq 0$ and $i\geq 1$, the number $m'_{k,i}=\left|M_k^{(i)}\right|$ as the number of elements in M having length k, whose lex-representative starts with a letter from $\{a_1,\ldots,a_i\}$. Notice that M_k is an infinite set, so we need to show that $m'_{k,i}$ is well defined.

Proposition 6.1. For $k \geq 0$ and $i \geq 1$, the number $m'_{k,i}$ is well defined, and it coincides with the stabilized entry $m_{k,i}$ in the table for A_n , for every $n \geq k+i-1$.

Proof. Let $M = A_{\infty}$ and let $b \in M_k^{(i)}$. The lex-representative of b starts with a letter from $\{a_1, \ldots, a_i\}$. We will show that a word representing b cannot contain the letter a_t for $t \ge k + i$.

Recall that all words representing b have length k, as the relations in A_{∞} are homogeneous. Also, all words representing b involve the same set of letters, as this set cannot be modified by applying a relation (no new letter can appear, and no letter can dissapear).

Suppose that some word w representing b contains the letter a_t , for some $t \geq k+i$. We know that some letter from $\{a_1, \ldots, a_i\}$ appears in w, as it appears in the lex-representative of b. But w has length k, and one of its letters already belongs to $\{a_1, \ldots, a_i\}$, so w cannot involve all letters from the set $\{a_{i+1}, \ldots, a_{i+k}\}$. Hence, there is some $a_j \in \{a_{i+1}, \ldots, a_{i+k}\}$ which does not appear in w.

Let $I = \{a_1, \ldots, a_{j-1}\}$ and $J = \{a_{j+1}, a_{j+1}, \ldots\}$. We know that the letters in w belong to $I \cup J$, and that w has letters from both sets. But every element in I commutes with every element in J. This means that we can obtain, from w, a word representing b having the form $w_J w_I$, where w_J only involves letters from J, and w_I only involves letters form I. By hypothesis, w_J is a nonempty word, so b admits some element from J as a prefix, and this contradicts that $b \in M_k^{(i)}$.

Therefore, every word representing $b \in M_k^{(i)}$ involves only letters from $\{a_1, \ldots, a_{k+i-1}\}$. Hence, the lex-representatives of b in A_{∞} and A_n coincide, for every $n \geq k+i-1$.

We remark that, although the growth rate in A_{∞} does not make sense, as it is not a finitely generated monoid (the number of elements of length one is already infinite), the numbers $m'_{k,i}$ in the *limit table* are all well defined. As $m'_{k,i} = m_{k,i}$, we will denote these numbers by $m_{k,i}$, from now on.

Now we will find a new way to describe each stabilized number $m_{k,i}$, depending only on the elements $m_{t,1}$ for $t = 0, \ldots, k-1$. For that purpose, we need some results from braid theory.

Proposition 6.2. Let $b \in A_n$ be a positive braid. For every $j \le 1$, there is a unique maximal braid γ_j such that $\gamma_j \le b$ and γ_j involves only the generators a_j, \ldots, a_n .

Proof. Let $\Delta_{[j,n]} = a_j \vee \cdots \vee a_n \in A_n$. It is well-known that a positive braid $\gamma \in A_n$ can be expressed as a word in the generators $\{a_j, \ldots, a_n\}$ if and only if it is a prefix of $(\Delta_{[j,n]})^m$ for some m > 0. Actually, if γ has length t, then γ can be expressed as a word in the generators $\{a_j, \ldots, a_n\}$ if and only if it is a prefix of $(\Delta_{[j,n]})^t$.

Let k be the length of the positive braid b. By the above arguments, a prefix $\gamma \leq b$ can be written as a word in $\{a_j, \ldots, a_n\}$ if and only if $\gamma \leq (\Delta_{[j,n]})^k$. Therefore, the set of prefixes of b which involve only the generators a_j, \ldots, a_n is the set of common prefixes of b and $(\Delta_{[j,n]})^k$. Since A_n is a lattice with respect to the prefix order, it follows that this set has a maximal element (with respect to \leq), namely $\gamma_j = b \wedge (\Delta_{[j,n]})^k$.

Notice that γ_j is also maximal in terms of length: it is the biggest prefix of b which involves only the generators a_j, \ldots, a_n .

Proposition 6.3. Let $b \in A_n$ be a positive braid. There is a unique decomposition $b = b_n b_{n-1} \cdots b_1$, so that $b_n b_{n-1} \cdots b_j$ is the biggest prefix of a which involves the generators $a_j, \ldots a_n$.

Proof. This is clear form the previous result. We just define $\gamma_{n+1}=1$ and $\gamma_j=b\wedge(\Delta_{[j,n]})^k$ for $j=1,\ldots,n$, where k is the length of b. Then

$$1 = \gamma_{n+1} \preccurlyeq \gamma_n \preccurlyeq \cdots \preccurlyeq \gamma_1 = b.$$

We must define each b_j so that $b_n \cdots b_j = \gamma_j$. Hence b_j is the only element such that $\gamma_{j+1}b_j = \gamma_j$.

Proposition 6.4. Let $b \in A_n$ be a positive braid, and let $b = b_n b_{n-1} \cdots b_1$ be the unique decomposition described in Proposition 6.3. For every $j = 1, \ldots, n$, the element b_j involves generators from $\{a_j, \ldots, a_n\}$ and, if $b_j \neq 1$, its lex-representative starts with a_j .

Proof. Let k be the length of b. By construction, b_j is the only element such that $\gamma_{j+1}b_j = \gamma_j$, where $\gamma_j = b \wedge \Delta_{[j,n]}^k$ for every $j = 1, \ldots, n$. Since b_j is a suffix of γ_j , it only involves generators from $\{a_j, \ldots, a_n\}$. Suppose that $b_j \neq 1$. From the above paragraph, the lex-representative of b_j can only start with a generator from $\{a_j, \ldots, a_n\}$. We must then show that b_j only admits the generator a_j as prefix. Suppose this is not the case. Then $a_i \leq b_j$ for some i > j. But then $\gamma_{j+1}a_i$ is a prefix of b which only involves generators from $\{a_{j+1}, \ldots, a_n\}$, contradicting the maximality of γ_{j+1} .

We can finally give a new interpretation of the numbers $m_{k,i}$ corresponding to the monoid $M = A_{\infty}$. Recall that $M_k^{(t)}$ is the set of elements of length k in A_{∞} whose lex-representative starts with a_j , for some $j \in \{1, \ldots, t\}$. On the other hand, let $P_k^{(t)}$ be the set of t-uples $(c_t, \ldots, c_1) \in (A_{\infty})^t$ such that $|c_1| + \cdots + |c_t| = k$ and, for $i = 1, \ldots, t$, either $c_i = 1$ or the lex-representative of c_i starts with a_1 .

Proposition 6.5. Given $k \geq 0$ and $t \geq 1$, the sets $M_k^{(t)}$ and $P_k^{(t)}$ have the same size.

Proof. If $b \in M_k^{(t)}$, then $b \in A_N$ for some big N (actually, we can take N = t + k - 1).

By Proposition 6.3, there is a unique decomposition $b = b_N b_{N-1} \cdots b_1$, so that $b_N \cdots b_j$ is the biggest prefix of b which involves the generators a_j, \ldots, a_N . Now, since the lex representative of b starts with a_i for some $i \leq t$, it follows that there cannot be a nontrivial prefix of b involving the generators a_{t+1}, \ldots, a_N . Therefore $b_j = 1$ for j > t, and we just have $b = b_t b_{t-1} \cdots b_1$.

Let $f: A_{\infty} \to A_{\infty}$ be the shifting homomorphism which sends a_i to a_{i+1} for every $i \geq 1$. Notice that f preserves the length of every element and, if the lex-representative of an element b starts with a_i , the lex-representative of $f^t(b)$ starts with a_{t+i} .

It is clear that $f^{-j}(c)$ is defined if $c \in A_{\infty}$ involves only generators a_i for i > j. By Proposition 6.4, $f^{1-j}(b_i)$ is well defined and, if it is nontrivial, its lex-representative starts with a_1 .

We can then define the following map:

$$\varphi: M_k^{(t)} \longrightarrow P_k^{(t)} \\ b \longmapsto (c_t, \dots, c_1),$$

where $c_j = f^{1-j}(b_j)$ for j = 1, ..., t. We see that φ is well defined from the above arguments, and also because $|c_t| + \cdots + |c_1| = |b_t| + \cdots + |b_1| = |b| = k$.

Now let us define the following map:

$$\begin{array}{cccc} \psi: & P_k^{(t)} & \longrightarrow & M_k^{(t)} \\ & (c_t, \dots, c_1) & \longmapsto & b = b_t \cdots b_1, \end{array}$$

where $b_j = f^{j-1}(c_j)$ for j = 1, ..., t. See Figure 5 for an example. We will show that ψ is well defined, and it is the inverse of φ , so both maps are bijections.

It is clear that b is a well-defined positive braid, of length $|b_t| + \cdots + |b_1| = |c_t| + \cdots + |c_1| = k$. Also, for every $j = 0, \ldots, t-1$, the element $b_t \cdots b_{j+1}$ involves only generators a_i for i > j.

Let us show that for $j=1,\ldots,t$, the braid $b_j\cdots b_1$ cannot start with a_i for i>j. If j=1 this is clear, as $b_1=c_1$ either is trivial or its lex-representative starts with a_1 . Suppose that j>1 and the claim is true for smaller values of j. If $a_i \preccurlyeq b_j \cdots b_1$ for some i>j, then $a_i \lor b_j \preccurlyeq b_j \cdots b_1$. By construction, b_j cannot start with a_i , hence $a_i \lor b_j = b_j d$ for some nontrivial positive braid d. Since a_i and b_j involve generators of index at least j, the braid d also involves generators of index at least j. But we have $b_j d \preccurlyeq b_j \cdots b_1$, hence $d \preccurlyeq b_{j-1} \cdots b_1$, which implies that $b_{j-1} \cdots b_1$ can start with a generator of index at least j, a fact that contradicts the induction hypothesis. The claim is then shown.

It follows that $b_t \cdots b_{j+1}$ is the biggest prefix of b involving generators a_i for i > j. This has two consequences. Firstly, the lex-representative of $b_t \cdots b_1$ starts with a_i for $i \le t$, hence $b \in M_k^{(t)}$, and ψ is

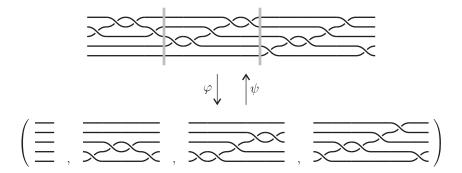


Figure 5: A braid $b = b_4b_3b_2b_1 = (1)(a_3a_4a_4a_3)(a_2a_2a_3a_4a_4)(a_1a_2a_2a_3a_4a_1) \in M_k^{(4)}$, and its corresponding 4-tuple $\varphi(b) = (1, \ a_1a_2a_2a_1, \ a_1a_1a_2a_3a_3, \ a_1a_2a_2a_3a_4a_1) \in P_k^{(4)}$.

well defined. Secondly, $b_t \cdots b_1$ is precisely the unique decomposition of b described in Proposition 6.3, which is used to define φ . So $\varphi(\psi((c_t,\ldots,c_1))) = \varphi(b) = (f^{1-t}(b_t),f^{2-t}(b_{t-1}),\ldots,f^0(b_1)) = (c_t,\ldots,c_1)$. As $\psi \circ \varphi$ is clearly equal to the identity map, it follows that ψ is the inverse of φ , as we wanted to show.

From Proposition 6.5, we can describe the number $m_{k,t}$ directly from the numbers $m_{l,1}$ for $l \leq k$. This will give us the desired connection of these numbers with the partial theta function f(x,y).

Recall that we denote $m_{k,t} = \left| M_k^{(t)} \right|$. Now let

$$\xi_0(y) = \sum_{k=0}^{\infty} m_{k,1} y^k = 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + \cdots$$

The coefficients of $\xi_0(y)$ are the numbers in the first column of the *limit table* containing the numbers $m_{k,i}$. Let us show that the series determined by the other columns of the table are, precisely, the powers of the series determined by the first column.

Proposition 6.6. For every
$$t > 0$$
, one has $(\xi_0(y))^t = \sum_{k=0}^{\infty} m_{k,t} y^k$.

Proof. The k-th coefficient of $(\xi_0(y))^t$ is equal to:

$$\sum_{\substack{(k_t, \dots, k_1) \\ k_t + \dots + k_1 = k}} m_{k_t, 1} \cdots m_{k_1, 1},$$

which is precisely the number of elements in $P_k^{(t)}$. By Proposition 6.5, this is also the number of elements in $M_k^{(t)}$, so the result follows.

We now denote $x_0(y) = -\xi_0(y)$, and we have the following:

Proposition 6.7. Let
$$f(x,y) = \sum_{n=0}^{\infty} y^{\binom{n}{2}} x^n$$
. Then $f(x_0(y), y) = 0$.

Proof. The expression $f(x_0(y), y)$ is a power series in the variable y. We need to show that all coefficients are zero.

The coefficient of y^0 in $f(x_0(y), y)$ comes from the values n = 0 and n = 1, and it is 1 - 1 = 0, as desired. Let k > 0. By Proposition 6.6, the coefficient of y^k in $f(x_0(y), y)$ is

$$-m_{k,1}+m_{k-1,2}-m_{k-\binom{3}{2},3}+m_{k-\binom{4}{2},4}-\cdots$$

where there are summands as long as the first subindex of $m_{k-\binom{r}{2},r}$ is non-negative. This sum equals zero by (3) for i=1, and the result follows.

The following result follows immediately:

Theorem 6.8. Let $x_0(y)$ be the only solution to the classical partial theta function $\sum_{k=0}^{\infty} y^{\binom{k}{2}} x^k$, and let $\xi_0(y) = -x_0(y) = 1 + y + 2y^2 + 4y^3 + 9y^4 + \cdots$ For every $k \geq 0$, the coefficient of y^k in the series $\xi_0(y)$ is equal to the number of braids of length k, in the monoid A_{∞} , whose maximal lexicographic representative starts with a_1 .

6.3 Limit of growth rates of braid monoids

We are considering the partial theta function $f(x,y) = \sum_{k=0}^{\infty} y^{\binom{k}{2}} x^k$, whose only nontrivial root is the series $x_0(y)$. The coefficients of the series $\xi_0(y) = -x_0(y) = 1 + y + 2y^2 + 4y^3 + 9y^4 + \cdots$ form the sequence $(L_k)_{k>0} = (1,1,2,4,9,21,52,\ldots)$.

The growth rate of this sequence is known to be the KLV-constant q_{∞} [33], which can be computed with arbitrary precision:

Theorem 6.9. [33] Let $(L_k)_{k\geq 0}$ be the sequence of coefficients of $\xi_0(y)$. Then its growth rate is:

$$\lim_{k \to \infty} \sqrt[k]{L_k} = \lim_{k \to \infty} \frac{L_{k+1}}{L_k} = q_{\infty} = 3.233636\dots$$

Notice that the above result states that $(L_k)_{k\geq 0}$ grows like $(q_\infty)^k$. In other words:

$$0 < \lim_{k \to \infty} \frac{L_k}{(q_{\infty})^k} < \infty \tag{4}$$

We want to relate the constant q_{∞} to the growth rate of the monoids A_n , for $n \geq 1$. Recall that we are counting the elements in A_n by considering their lex-representatives (their maximal lexicographic representatives with $a_1 < a_2 < \cdots < a_n$). The following is an important property of this set of words:

Theorem 6.10. [17] For every $n \ge 1$, the set of lex-representatives of the braid monoid A_n is a regular language.

In [17], an automaton accepting this regular language is defined, having the minimal possible number of states. Moreover, in [14], the automaton is described in detail for every $n \ge 1$, and the following result is shown:

Theorem 6.11. [14, Corollary 5.5] For every $n \ge 1$, the proportion of lex-representatives of length k, in the braid monoid A_n , finishing at the same state as a_1 , tends to a limit $p > \frac{1}{32}$ when k tends to infinity.

The above result can be described as follows. Let M_n be the incidence matrix of the automaton accepting the regular language of lex-representatives of A_n . Each row (resp. column) of M_n corresponds to a state of the automaton. We can assume that the first row (and also the first column) corresponds to the state determined by the word a_1 .

By Perron-Frobenius theory, there is a unique left eigenvector $\mathbf{v_n}$ of M_n , all of whose coordinates are non-negative and such that the sum of these coordinates is equal to 1. The *i*th coordinate of $\mathbf{v_n}$ is precisely the limit, as k tends to infinity, of the proportion of lex-representatives of length k finishing at the *i*th state (see [14]). Theorem 6.11 states that the first coordinate of $\mathbf{v_n}$ is greater than $\frac{1}{32}$, for all n > 1.

Now let us fix some $n \ge 1$. We will denote by $m_{k,i}(A_n)$ the numbers appearing in Corollary 4.2, which are also the numbers appearing in Figure 4 for n = 5 and n = 6. We know that the growth rate of the column n + 1 is precisely ρ_n , the growth rate of the monoid A_n . Let us show that all the columns of the table corresponding to A_n have the same growth rate.

Proposition 6.12. Let $n \ge 1$. For every i = 1, ..., n + 1, we have:

$$\lim_{k \to \infty} \frac{m_{k+1,i}(A_n)}{m_{k,i}(A_n)} = \rho_n.$$

Proof. The result is trivial if i = n + 1 or if i = n, as these columns contain precisely the number of braids of given length.

By definition, the number $m_{k,i}(A_n)$ is the number of lex-representatives in A_n starting with a generator from $\{a_1,\ldots,a_i\}$. If we consider the incidence matrix M_n , and assume that the rows $1,2,\ldots,n$ correspond to the states of the lex-representatives a_1,a_2,\ldots,a_n respectively, then $m_{k,i}(A_n)$ is the sum of the entries of the rows $1,2,\ldots,i$ in the matrix $(M_n)^{k-1}$.

By Perron-Frobenius theory, all rows of $(M_n)^k$ have the same growth rate as k tends to infinity, which is precisely the Perron-Frobenius eigenvalue. In this case, this eigenvalue is precisely ρ_n (the growth rate of the sum of the first n rows). Therefore, all sequences $(m_{k,i}(A_n))_{k\geq 0}$ grow like $(\rho_n)^k$, as we wanted to show.

Now we will relate the numbers ρ_n to the constant q_{∞} . Recall from [19] that ρ_n is an increasing sequence of real numbers, whose limit we denote ρ :

$$\lim_{n\to\infty} \rho_n = \rho.$$

We need the following:

Definition 6.13. Given a real number $\mu \ge \rho$, let $c(n,\mu) = \sum_{k=0}^{\infty} \left(\frac{\rho_n}{\mu}\right)^k = \frac{1}{1 - \frac{\rho_n}{\mu}} = \frac{\mu}{\mu - \rho_n}$.

Lemma 6.14. Given $\mu \geq \rho$, one has $\lim_{n \to \infty} c(n, \mu) = \infty$ if and only if $\mu = \rho$.

Proof. It follows trivially from Definition 6.13.

Now let us denote by ${\bf e}$ the (column) vector (of any desired length) which consists only of 1's. Hence, from the definition of the Perron-Frobenius eigenvector ${\bf v_n}$, we have ${\bf v_n} \cdot {\bf e} = 1$.

Lemma 6.15. For every $\mu \ge \rho$ and every $n \ge 2$, we have:

$$c(n,\mu) = \mathbf{v_n}(I - \mu^{-1}M_n)^{-1}\mathbf{e}$$

Proof. We know [14, Lemma 5.2] that

$$(I - \mu^{-1}M_n)^{-1} = I + (\mu^{-1}M_n) + (\mu^{-1}M_n)^2 + (\mu^{-1}M_n)^3 + \cdots$$

If we multiply any summand by $\mathbf{v_n}$ from the left, and by \mathbf{e} from the right, we obtain

$$\mathbf{v_n}(\mu^{-1}M_n)^k \mathbf{e} = \frac{1}{\mu^k} \left(\mathbf{v_n} M_n^k \right) \mathbf{e} = \frac{1}{\mu^k} \left(\rho_n^k \mathbf{v_n} \right) \mathbf{e} = \left(\frac{\rho_n}{\mu} \right)^k.$$

Therefore

$$\mathbf{v_n}(I-\mu^{-1}M_n)^{-1}\mathbf{e} = 1 + \left(\frac{\rho_n}{\mu}\right) + \left(\frac{\rho_n}{\mu}\right)^2 + \left(\frac{\rho_n}{\mu}\right)^3 + \dots = c(n,\mu).$$

Now recall that the number $m_{k,1}(A_n)$ is equal to the sum of the entries of the first row of $(M_n)^{k-1}$. That is, $m_{k,1}(A_n)$ is the first entry of the column vector $(M_n)^{k-1}\mathbf{e}$. Recall also that the sequence $\{m_{n,1}(A_n)\}_{n\geq 0}$ is equal to $\{m_{n,1}(A_\infty)\}_{n\geq 0} = \{L_n\}_{n\geq 0} = \{1,1,2,4,9,\ldots\}$. By Theorem 6.9, the growth rate of $\{m_{n,1}(A_n)\}_{n>0}$ is equal to the constant q_∞ .

Notice that, for every $n \geq 1$ and every $k \geq 0$, we have $m_{k,1}(A_n) \leq m_{k,1}(A_\infty)$. Hence, the sequence $\{m_{k,1}(A_n)\}_{k\geq 0}$ is dominated by the sequence $\{m_{k,1}(A_\infty)\}_{k\geq 0}$. It follows that the growth rate of the former sequence cannot be bigger than the growth rate of the latter. In other words: $\rho_n \leq q_\infty$ for every $n \geq 1$. Therefore,

$$\rho = \lim_{n \to \infty} \rho_n \le q_{\infty},$$

and we can consider the numbers $c(n, q_{\infty})$.

Proposition 6.16. We have:

$$\lim_{n \to \infty} c(n, q_{\infty}) = \infty.$$

Proof. We know from Lemma 6.15 that

$$c(n,q_{\infty}) = \mathbf{v_n} I \mathbf{e} + \mathbf{v_n} (q_{\infty}^{-1} M_n) \mathbf{e} + \mathbf{v_n} (q_{\infty}^{-1} M_n)^2 \mathbf{e} + \mathbf{v_n} (q_{\infty}^{-1} M_n)^3 \mathbf{e} + \cdots$$

That is,

$$c(n, q_{\infty}) = 1 + \frac{\mathbf{v_n} M_n \mathbf{e}}{q_{\infty}} + \frac{\mathbf{v_n} M_n^2 \mathbf{e}}{q_{\infty}^2} + \frac{\mathbf{v_n} M_n^3 \mathbf{e}}{q_{\infty}^3} + \cdots$$

We now recall from Theorem 6.11 that the first coordinate of $\mathbf{v_n}$ is greater than $\frac{1}{32}$. Therefore, if we denote \mathbf{e}_1 the first row of the identity matrix, we have, for every k > 0:

$$\mathbf{v_n} M_n^k \mathbf{e} > \frac{1}{32} \left(\mathbf{e}_1 M_n^k \mathbf{e} \right) = \frac{m_{k+1,1} (A_n)}{32}.$$

The above inequality holds since all the coordinates of vectors and matrices involved are nonnegative. Recall also that $m_{1,1}(A_n) = 1$, so $1 > \frac{1}{32} = \frac{m_{1,1}(A_n)}{32}$.

Finally, we obtain:

$$c(n, q_{\infty}) > \frac{m_{1,1}(A_n)}{32} + \frac{m_{2,1}(A_n)}{32 q_{\infty}} + \frac{m_{3,1}(A_n)}{32 q_{\infty}^2} + \frac{m_{4,1}(A_n)}{32 q_{\infty}^3} + \cdots$$

Now recall that for every $k \leq n$, we have $m_{k,1}(A_n) = m_{k,1}(A_k)$. Also, $q_{\infty} > 3$, so we can divide the above expression by q_{∞} and truncate at the *n*th term, to obtain:

$$c(n,q_{\infty}) > \frac{m_{1,1}(A_1)}{32 \, q_{\infty}} + \frac{m_{2,1}(A_2)}{32 \, q_{\infty}^2} + \frac{m_{3,1}(A_3)}{32 \, q_{\infty}^3} + \dots + \frac{m_{n,1}(A_n)}{32 \, q_{\infty}^n}.$$

The right hand side is the truncation at n of the infinite sum:

$$\frac{1}{32} \sum_{k=1}^{\infty} \frac{m_{k,1}(A_k)}{q_{\infty}^k}$$

Since the numerators correspond to the sequence $\{L_k\}_{k\geq 0}$ whose growth rate is q_{∞} , it follows that the fractions tends to a positive real number, so the above infinite sum does not converge. In other words, its truncations tend to infinity, and this implies that $c(n, q_{\infty})$ also tends to infinity, as n grows.

We can finally show the main result of this section.

Theorem 6.17. Let $\rho = \lim_{n \to \infty} \rho_{A_n}$. Then $\rho = 3.23363...$ is the growth rate of the coefficients of $\xi_0(y)$. That is, ρ is equal to the KLV-constant q_{∞} .

Proof. By Proposition 6.16, we have that $\lim_{n\to\infty} c(n,q_\infty) = \infty$. By Lemma 6.14, this can only happen if $q_\infty = \rho$.

We finish this paper with a question concerning the remaining Artin–Tits monoids:

Question: Is it true that $\rho(A_n) = \rho(B_n) = \rho(D_n) = q_\infty$?

Using injective maps $A_n \to B_{n+1}$ and $A_n \to D_{n+1}$, which send a_i to a_i , it is easy to prove that $\rho(B_n) \leq q_\infty$; however, our methods are unable to state the opposite inequalities.

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