# On the Orbital Diameter of Groups of Diagonal Type 

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#### Abstract

The orbital diameter of a primitive permutation group is the maximal diameter of its orbital graphs. There has been a lot of interest in bounds for the orbital diameter. In this paper we provide explicit bounds on the diameters of groups of simple diagonal type. As a consequence we obtain a classification of simple diagonal groups with orbital diameter less than or equal to 4 . As part of this, we classify all finite simple groups with covering number and conjugacy width at most 3 . We also prove some general bounds on the covering number and conjugacy width of groups of Lie type.


## 1 Introduction

Let $G$ be a group acting transitively on a finite set $\Omega$. Then $G$ acts on $\Omega \times \Omega$ componentwise. Define the orbitals to be the orbits of $G$ on $\Omega \times \Omega$. The diagonal orbital is the orbital of the form $\Delta=\{(\alpha, \alpha) \mid \alpha \in \Omega\}$. The number or orbitals is called the rank of $G$. Let us denote this by $\operatorname{rank}(G, \Omega)$. Let $\Gamma$ be a non-diagonal orbital. Define the corresponding orbital graph to be the undirected graph with vertex set $\Omega$ and edge set $\{\alpha, \beta\}$ for $(\alpha, \beta) \in \Gamma$. Note that $G$ acts transitively on the edges and the vertices of $\Gamma$ so it is an edge-transitive and vertex-transitive graph.

By [5. Thm 3.2A] the orbital graphs are all connected if and only if the action of $G$ is primitive. The orbital diameter of a primitive permutation group $G$ is the supremum of the diameters of its orbital graphs, see [21. Let us denote this by $\operatorname{orbdiam}(G)$.

The O'Nan-Scott theorem classifies the primitive permutation groups to be one of the following five types; affine, almost simple, simple diagonal actions, product actions and twisted wreath actions, see [5]. We call an infinite class $C$ of primitive permutation groups bounded if there exists $t \in \mathbb{N}$ such that $\operatorname{orbdiam}(G) \leq t$ for all $G \in C$. The paper [21] describes the O'NanScott classes which are bounded. The description is somewhat qualitative and does not contain explicit diameter bounds. Some explicit bounds were obtained in [26] for some almost simple groups. In this paper we study the orbital diameters of the class of simple diagonal type primitive permutation groups and provide bounds for these quantities. Further work along these lines by the author is under way for the other O'Nan-Scott classes.

Let us now describe primitive groups of simple diagonal type, following [22]. Let $T$ be a non-abelian simple group, $\Gamma=\{1, \ldots, k\}$ and $W=T w r_{\Gamma} S_{k}$ with base group $T^{k}$. Now let $D=\{(a, \ldots, a) \mid a \in T\}$ be a diagonal subgroup and let $\Omega$ be the set of right cosets of $D$ in $T^{k}$. Then $T^{k}$ acts on $\Omega$ by right multiplication, $S_{k}$ acts on $\Omega$ by permuting the components of the coset representatives, and $\alpha \in A u t(T)$ acts on $\Omega$ by $D\left(h_{1}, \ldots, h_{k}\right)^{\alpha}=D\left(h_{1}^{\alpha}, \ldots, h_{k}^{\alpha}\right)$ for $h_{i} \in T$. The groups $T^{k}, S_{k}$ and $\operatorname{Aut}(T)$ generate a group $N \cong T^{k} .\left(\operatorname{Out}(T) \times S_{k}\right)$ and this is the normalizer of $T^{k}$ in $\operatorname{Sym}(\Omega)$. We say $G \leq \operatorname{Sym}(\Omega)$ is a primitive permutation group of simple diagonal type if $T^{k} \leq G \leq N$ and $G$ acts primitively on $\Omega$, see [22]. Note $G=T^{k} \cdot X$ where $X \leq \operatorname{Out}(T) \times S_{k}$.

Write

$$
D(k, T)=N \leq \operatorname{Sym}(\Omega)
$$

The result in [21, Lemma 5.1] states that the class of simple diagonal groups $G=T^{k} \cdot X \leq$ $D(k, T)$ is bounded only if $k$ and the rank of $T$ are both bounded. However no explicit bounds are obtained.

Before we list our results, we need a few definitions. Notice that if $T$ is simple and $t \in T \backslash 1$, the conjugacy class $t^{T}$ generates $T$. For a subset $S$ of $T$ and $r \in \mathbb{N}$ we write $S^{r}=\left\{s_{1}, \ldots, s_{r} \mid s_{i} \in S\right\}$.

Definition 1.1. Let $T$ be a non-abelian simple group, and $S$ a generating set of $T$. Define the width of $T$ with respect to $S$, denoted $w_{S}(T)$, to be the minimal $k \in \mathbb{N}$ such that any element of $T$ can be expressed as a product of at most $k$ elements of $S$.

1. Let $t \in T \backslash 1$ and $C=t^{T}$ and put $c(T, t):=w_{C}(T)$. Define the conjugacy width of $T$ to be

$$
c(T)=\max _{t \in T \backslash 1} c(T, t) .
$$

2. For $t \in T \backslash 1$ let $C=t^{ \pm T}=t^{T} \cup\left(t^{-1}\right)^{T}$ and put $c_{i}(T, t):=w_{C}(T)$ Define the inverse conjugacy width of $T$ to be

$$
c_{i}(T)=\max _{t \in T \backslash 1} c_{i}(T, t)
$$

3. Let $X$ be such that $T \unlhd X \leq$ Aut $(T)$, let $C=t^{ \pm X}=\left\{t^{ \pm \alpha} \mid \alpha \in X\right\}$ and put $c_{X}(T, t):=w_{C}(T)$ Define the $X$-conjugacy width of $T$ to be

$$
c_{X}(T)=\max _{t \in T \backslash 1} c_{X}(T, t)
$$

When $X=\operatorname{Aut}(T)$ write $c_{A}(T)=c_{X}(T)$.
These are related to the concept of covering numbers introduced in [1]. The covering number is the lowest number $r \in \mathbb{N}$ such that $C^{r}=T$ for all conjugacy classes $C$. This is denoted $c n(T)$. Denote the lowest such number for a specific conjugacy class by $c n(T, C)$. Note that this is an upper bound for the conjugacy width. We also note that

$$
c_{A}(T) \leq c_{X}(T) \leq c_{i}(T) \leq c(T) \leq c n(T)
$$

Note also that the number $c_{i}(T)$ was introduced and studied in [19], where it was called the conjugacy diameter of $T$.

We now state our results. Let $G=T^{k} . X \leq D(k, T)$ where $X \leq \operatorname{Out}(T) \times S_{k}$. We know from [22] that

$$
D(k, T)=\left\{\left(\tau_{1}, \ldots, \tau_{k}\right) \cdot \pi \mid \tau_{i} \in \operatorname{Aut}(T), \pi \in S_{m} \text { and all } \tau_{i} \text { lie in the same } \operatorname{Inn}(T)-\operatorname{coset}\right\}
$$

Let $W=\left\{(\alpha, \ldots, \alpha) . \pi \mid \alpha \in A u t(T), \pi \in S_{k}\right\}$ and put $D_{A}=W \cap G$, so $D_{A}=D . X$. Since $G=D_{A} T^{k}$ the action of $G$ on $\Omega$ is equivalent to the action of $G$ on $\left(G: D_{A}\right)$.

For $a \in T$ write $\left(a^{k}\right)=(a, \ldots, a) \in T^{k}$. For $t \in T \backslash 1$ define the orbital graph

$$
\Gamma_{0}^{t}=\left\{D_{A}, D_{A}\left(1^{k-1}, t\right)\right\}^{G} .
$$

The following theorem gives lower and upper bounds on the diameter of $\Gamma_{0}^{t}$. In the statement we abuse notation and denote $c_{X_{0}}(T)$ by $c_{X}(T)$, where $X_{0}$ is defined as follows. Let $G=T^{k} . X$ with $X \leq \operatorname{Out}(T) \times S_{k}$. Let $\rho$ be the projection of $X$ onto $\operatorname{Out}(T)$ and let $\pi$ be the canonical map $\operatorname{Aut}(T) \rightarrow \operatorname{Out}(T)$. Define $X_{0}=\pi^{-1}(\rho(X)) \leq \operatorname{Aut}(T)$.

Theorem (3.1(3.2). Let $G=T^{k} . X$ as above. The diameter of $\Gamma_{0}^{t}$ satisfies the bounds

$$
\frac{1}{2}(k-1) c_{X}(T, t)+1 \leq \operatorname{diam}\left(\Gamma_{0}^{t}\right) \leq(k-1) c_{i}(T)
$$

Note that from this it follows that $\operatorname{orbdiam}(G) \geq \frac{1}{2}(k-1) c_{X}(T)+1$.
Using these bounds, we provide the following classification of simple diagonal groups of small orbital diameter.

Theorem (5.1). Let $G$ be a primitive group of simple diagonal type of the form $T^{k} \cdot X \leq D(k, T)$.

1. If $\operatorname{orbdiam}(G)=2$, then $k=2$ and $c_{A}(T)=2$.
2. If $\operatorname{orbdiam}(G)=3$, then $k=2$ and $c_{A}(T) \leq 3$.
3. If $\operatorname{orbdiam}(G)=4$, then one of the following holds:
(a) $k=2$ and $c_{A}(T) \leq 4$
(b) $k=3$ and $c_{A}(T)=2$.

The following result gives a partial converse to parts 1, 2 and $3(a)$.
Lemma (3.3). 1. If $G=T^{2}$ then $\operatorname{orbdiam}(G)=c_{i}(T)$.
2. If $G=D(2, T)$ then $\operatorname{orbdiam}(G)=c_{A}(T)$.

We have not determined whether there are examples of groups in case $3(b)$ of Theorem 5.1 with orbital diameter 4. In view of Lemma 3.3 and Theorem 5.1 to classify all diagonal type groups with orbital diameters 2 and 3 we need to classify all non-abelian simple groups with $X$-conjugacy width 2,3 for various $X$.

First we prove a general result on conjugacy widths and covering numbers of simple groups of Lie type. The upper bound in the following result was proved in [7]. By Lie rank we mean the rank of the corresponding simple algebraic group.

Theorem (4.1). There is a constant $d$ such that

$$
r-3 \leq c_{A}(T) \leq c n(T) \leq d r
$$

for all simple groups $T$ of Lie type of Lie rank $r$.
It was proved in [3] that the only finite simple group with covering number 2 is the sporadic group $J_{1}$. It turns out that $J_{1}$ is also the only finite simple group with (inverse) conjugacy width 2 (Proposition 4.3). However, there are infinitely many simple groups $T$ with $c_{A}(T)=2$.

Theorem (4.4). Let $T$ be a finite simple group. Then $c_{A}(T)=2$ if and only if $T \cong J_{1}$ or $T \cong P S L_{2}(q)$ with $q \equiv 1 \bmod 4$ or $q=2^{2 m}$.

We also classify simple groups with any of the numbers $c n(T), c(T), c_{i}(T)$ or $c_{A}(T)$ equal to 3 .

Theorem (4.5). Let $T$ be a finite simple group.

1. $c(T)=3$ if and only if $T$ is isomorphic to one of the following:

- $P S L_{2}(q)$ with $q>2$
- $P S L_{3}(q)$
- $P S U U_{3}(q)$ with $3 \mid q+1, q>2$
- ${ }^{2} B_{2}(q)$ with $q>2$
- ${ }^{2} G_{2}(q)$ with $q>3$
- $G_{2}\left(3^{n}\right)$ with $n \geq 2$
- $A_{5}, A_{6}, A_{7}$
- $M_{11}, M_{22}, M_{23}, M_{24}, J_{3}, J_{4}, M c l, R u, L y, O^{\prime} N, F i_{24^{\prime}}, T h, M$.

2. $\operatorname{cn}(T)=3$ if and only if $c(T)=3$.
3. $c_{i}(T)=3$ if and only if $c(T)=3$.
4. If $c_{A}(T)=3$ then one of the following holds:
(a) $c(T)=3$ and $T \neq P S L_{2}(q)$ with $q \equiv 1(\bmod 4)$ or $q=2^{2 m}$.
(b) $T \cong F_{4}\left(2^{n}\right)$.

Remark For part 3(b) we have not been able to determine whether $c_{A}\left(F_{4}\left(2^{n}\right)\right)=3$.
The result in [21, Lemma 5.1] on simple diagonal actions states that for a class of primitive groups $T^{k} . X$ with bounded orbital diameter the Lie rank of $T$ and $k$ are bounded. Conversely, if $T$ has bounded Lie rank, $k$ is bounded and a few more criteria are met, then one obtains a bounded class.The proof of this result is model theoretic and includes no explicit bounds on the orbital diameter. The following result provides an explicit upper bound, giving rise to many bounded families of primitive groups of simple diagonal type. For simplicity we restrict to the class of simple diagonal groups of the form $T^{k} \cdot S_{k} \leq D(k, T)$.

Theorem (6.1). Let $k \geq 3$ and let $T$ be a simple group. Then

$$
\operatorname{orbdiam}\left(T^{k} . S_{k}\right) \leq 24(k-1) c_{i}(T)^{2} .
$$

## 2 Preliminary Results

In this section we include some background material that we use in the proof of our theorems.
We begin with a well-known a character theoretic result which gives us a method to find conjugacy widths.

Lemma 2.1. [2, Lemma 10.1] Let $C_{1}, \ldots, C_{d}$ be conjugacy classes of a finite group $G$ with representatives $c_{1}, \ldots, c_{d}$. For $z \in G$, the number of solutions $\left(x_{1}, \ldots, x_{d}\right) \in C_{1} \times \cdots \times C_{d}$ to the equation $x_{1} \ldots x_{d}=z$ is

$$
\frac{\prod\left|C_{i}\right|}{|G|} \sum_{\chi \in \operatorname{Irr}(G)} \frac{\chi\left(c_{1}\right) \ldots \chi\left(c_{d}\right) \chi\left(z^{-1}\right)}{\chi(1)^{d-1}}
$$

Note the immediate corollary of this result.

Corollary 2.2. Let $C$ and $D$ be conjugacy classes of $G$ with representatives $c, d$. If

$$
\left|\sum_{\chi \in \operatorname{Irr}(G) \backslash 1_{G}} \frac{\chi(c)^{k} \chi\left(d^{-1}\right)}{\chi(1)^{k-1}}\right|<1
$$

then $D \subseteq C^{k}$.
We include another result that we use in our classification of groups with small conjugacy widths, namely the classification of strongly real groups. A group is strongly real if and only if any of its elements can be expressed as a product of at most two involutions.

Theorem 2.3 ( $8,10,12,13,17,23,25,27,28])$. Let $G$ be a non-abelian finite simple group. Then $G$ is strongly real if and only if it is isomorphic to one of

- $P S p_{2 n}(q)$ where $q \not \equiv 3(\bmod 4)$ and $n \geq 1$
- $P \Omega_{2 n+1}(q)$ where $q \equiv 1(\bmod 4)$ and $n \geq 3$
- $P \Omega_{9}(q)$ where $q \equiv 3(\bmod 4)$
- $P \Omega_{4 n}^{+}(q)$ where $q \not \equiv 3(\bmod 4)$ and $n \geq 3$
- $P \Omega_{4 n}^{-}(q)$ where $n \geq 2$
- $P \Omega_{8}^{+}(q)$ or ${ }^{3} D_{4}(q)$
- $A_{5}, A_{6}, A_{10}, A_{14}, J_{1}, J_{2}$.

The following result is on alternating groups from [4, Thm 2].
Theorem 2.4. [4] Let $n \geq 5, l$ odd and $l \leq n$. Then every permutation in the alternating group $A_{n}$ is a product of three $l-$ cycles if and only if either $\frac{n}{2} \leq l$ or $n=7$ and $l=3$.

We conclude by listing some existing results on the covering numbers of some finite simple groups.

Theorem 2.5 (3, 6, 20, 19). 1. If $n \geq 3$, then $\operatorname{cn}\left(P S L_{n}(q)\right)=n$ for $q \geq 4$. Also for $q>3$, $c n\left(P S L_{2}(q)\right)=3$.
2. $\operatorname{cn}\left(A_{n}\right)=\left\lfloor\frac{n}{2}\right\rfloor$ for $n \geq 6$, and $\operatorname{cn}\left(A_{5}\right)=3$.
3. $\operatorname{cn}\left({ }^{2} B_{2}(q)\right)=3$ for $q>2$.
4. $\operatorname{cn}\left({ }^{2} G_{2}(q)\right)=3$ for $q>3$.

Note that the covering numbers in Theorem 2.5 are upper bounds for the X -conjugacy widths, which we will be using later on.

## 3 The orbital diameter of a simple diagonal group

We begin with some notation. Let $T$ be simple, $k \geq 2$ and $G=T^{k} \cdot X \leq D(k, T)$ in a primitive simple diagonal action, where $X \leq O u t(T) \times S_{k}$ as in Section 1. Let $\Gamma$ be an orbital graph of $G$. Define $d_{\Gamma}(a, b)$ to be the distance between two vertices, $a$ and $b$ in $\Gamma$. Denote a path of length at most $m$ between $a$ and $b$ in $\Gamma$ by

$$
a \stackrel{m}{-} b .
$$

Denote the element $(t, \ldots, t) \in T^{k}$ as $\left(t^{k}\right)$.

Let $t \in T \backslash 1$ and $\Gamma_{0}^{t}$ be the orbital graph $\left\{D_{A}, D_{A}\left(1^{k-1}, t\right\}^{G}\right.$. Recall $X_{0}=\pi^{-1}((\rho(X))$ where $\pi$ is the canonical map $\operatorname{Aut}(T) \rightarrow \operatorname{Out}(T)$ and $\rho$ is the projection of $X$ to $\operatorname{Out}(T)$. For $g \in T$ we define the length of $g$ with respect to $t$ to be

$$
l_{t}^{X}(g)=\min \left\{a: g=t^{ \pm \alpha_{1}} \ldots t^{ \pm \alpha_{a}}, \text { for some } \alpha_{i} \in X_{0}\right\}
$$

Recall we write an element of $G$ as $\left(h_{1}, \ldots, h_{k}\right) \sigma_{h}$ where $h_{i} \in A u t(T), \sigma_{h} \in S_{k}$, and identify $T$ with $\operatorname{Inn}(T)$ when convenient.

The first result in this section is a lower bound for the diameter.
Theorem 3.1. Let $c=c_{X}(T, t)$. The diameter of $\Gamma_{0}^{t}$ satisfies diam $\left(\Gamma_{0}^{t}\right) \geq M$ where

$$
M= \begin{cases}\frac{1}{2}(k-1) c+1 & k \text { odd } \\ \frac{1}{2} k c & k \text { even }\end{cases}
$$

Proof. Note that $G=D_{A} T^{k}$ so every right coset of $D_{A}$ has a coset representative in $T^{k}$.
Claim 1 Every coset at distance $m$ away from $D_{A}$ is of the form

$$
D_{A}\left(t_{1}, \ldots, t_{k}\right)
$$

where $t_{i} \in T$ are such that $\sum_{i=1}^{k} l_{t}^{X}\left(t_{i}\right) \leq m$.
Proof of Claim 1 We prove Claim 1 by induction on $m$. We start with the base case $m=1$. Suppose $D_{A}\left(g_{1}, \ldots, g_{k}\right)$ is a neighbour of $D_{A}$ where $g_{i} \in T$. Then there exists $h=$ $\left(h_{1}, \ldots, h_{k}\right) \sigma_{h} \in G$ such that

$$
\left\{D_{A}, D_{A}\left(1^{k-1}, t\right)\right\} h=\left\{D_{A}, D_{A}\left(g_{1}, \ldots, g_{k}\right)\right\}
$$

Hence either $h \in D_{A}$ or $D_{A}\left(1^{k-1}, t\right) h=D_{A}$. If $h \in D_{A}$ then $h=\left(a^{k}\right) \sigma_{h}$, with $a \in \operatorname{Aut}(T)$. Now

$$
D_{A}\left(g_{1}, \ldots, g_{k}\right)=D_{A}\left(1^{k-1}, t\right) h=D_{A} h^{-1}\left(1^{k-1}, t\right) h=D_{A}\left(1^{k-1}, t^{a}\right)^{\sigma_{h}}
$$

as required. If $D_{A}\left(1^{k-1}, t\right) h=D_{A}$, then

$$
D_{A}\left(g_{1}, \ldots, g_{k}\right)=D_{A} h=D_{A} h^{-1}\left(1^{k-1}, t^{-1}\right) h=D_{A}\left(1^{k-1}, t^{-h_{k}}\right)^{\sigma_{h}}
$$

Hence Claim 1 holds for $m=1$.
Now let $m \geq 2$. Let $D_{A} h$ be a coset at distance $m$ from $D_{A}$. Then $D_{A} h$ is a neighbour of a coset at distance $m-1$ and by the induction hypothesis this coset has form

$$
D_{A}\left(x_{1}, \ldots, x_{k}\right)
$$

where $x_{i} \in T$ and $\sum_{i=1}^{k} l_{t}^{X}\left(x_{i}\right) \leq m-1$. There is an edge between $D_{A}\left(x_{1}, \ldots, x_{k}\right)$ and $D_{A} h$. Hence there is $f \in G$ such that

$$
\left\{D_{A}, D_{A}\left(1^{k-1}, t^{ \pm a}\right)^{\sigma}\right\} f=\left\{D_{A}\left(x_{1} \ldots, x_{k}\right), D_{A} h\right\}
$$

with $f=\left(f_{1}, \ldots, f_{k}\right) \pi$ where $f_{i} \in \operatorname{Aut}(T)$ and $\pi \in S_{k}$. Again either $D_{A} f=D_{A}\left(x_{1}, \ldots, x_{k}\right)$ or $D_{A}\left(1^{k-1}, t^{ \pm a}\right)^{\sigma} f=D_{A}\left(x_{1}, \ldots, x_{k}\right)$. If $D_{A} f=D_{A}\left(x_{1}, \ldots, x_{k}\right)$ then $f_{i} x_{i}^{-1}=f_{j} x_{j}^{-1}$ for all $i, j$, so

$$
D_{A} h=D_{A}\left(1^{k-1}, t^{ \pm a}\right)^{\sigma} f=D_{A}\left(x_{1}, \ldots, x_{k}\right) f^{-1}\left(1^{k-1}, t^{ \pm a}\right)^{\sigma} f=D_{A}\left(x_{1}, \ldots, x_{k}\right)\left(1^{k-1}, t^{ \pm a f_{k}}\right)^{\sigma \pi}
$$

and Claim 1 follows. When $D_{A}\left(1^{k-1}, t^{ \pm a}\right)^{\sigma} f=D_{A}\left(x_{1}, \ldots, x_{k}\right)$ we obtain the conclusion in a similar way.

Claim 2 There exist $h_{1}, \ldots, h_{k} \in T$ such that $d_{\Gamma_{0}^{t}}\left(D_{A}, D_{A}\left(h_{1}, \ldots, h_{k}\right)\right) \geq M$, where $M$ is as in the statement of the Theorem.

Proof of Claim 2 By Claim 1 it suffices to find $h_{1}, \ldots, h_{k} \in T$ such that $\min _{g \in T} \sum_{i=1}^{k} l_{t}^{X}\left(g h_{i}\right) \geq$ $M$. Let $h, h^{\prime} \in T$ with $h \neq h^{\prime}$ and $l_{t}^{X}(h)=l_{t}^{X}\left(h^{\prime}\right)=c$. Define

$$
\left(h_{1}, \ldots, h_{k}\right)= \begin{cases}(1, h, 1, h, \ldots, 1, h) & k \text { even } \\ \left(h, 1, h, 1, \ldots, h, 1, h^{\prime}\right) & k \text { odd }\end{cases}
$$

Then

$$
l_{t}^{X}\left(h_{1}^{-1} h_{2}\right)=l_{t}^{X}\left(h_{2}^{-1} h_{3}\right)=\cdots=l_{t}^{X}\left(h_{k-1}^{-1} h_{k}\right)=c \text { and } l_{t}^{X}\left(h_{1}^{-1} h_{k}\right) \geq 1
$$

Note that $l_{t}^{X}(x y) \leq l_{t}^{X}(x)+l_{t}^{X}(y)=l_{t}^{X}\left(x^{-1}\right)+l_{t}^{X}(y)$ for all $x, y \in T$, so it follows that for all $i \geq 1$ and any $g \in T$

$$
\left.c=l_{t}^{X}\left(h_{i}^{-1} h_{i+1}\right)\right) \leq l_{t}^{X}\left(g h_{i}\right)+l_{t}^{X}\left(g h_{i+1}\right)
$$

and

$$
1 \leq l_{t}^{X}\left(h_{1}^{-1} h_{k}\right) \leq l_{t}^{X}\left(g h_{1}\right)+l_{t}^{X}\left(g h_{k}\right)
$$

Summing these up gives

$$
2 \sum_{i=1}^{k} l_{t}^{X}\left(g h_{i}\right) \geq \sum_{i=1}^{k-1} l_{t}^{X}\left(h_{i}^{-1} h_{i+1}\right)+l\left(h_{1}^{-1} h_{k}\right) \geq(k-1) c+1
$$

The result now follows for $k$ odd, and for $k$ even we have $l_{t}^{X}\left(h_{1}^{-1} h_{k}\right)=c$, so we get $\frac{k}{2} c$ as a lower bound.

The following result is an upper bound.
Lemma 3.2. We have $\operatorname{diam}\left(\Gamma_{0}^{t}\right) \leq(k-1) c_{i}(T)$.

Proof. Claim 1 There exist $\alpha_{i} \in A u t(T)$ such that $D_{A}\left(\left(1^{i-1},\left(t^{\alpha_{i}}\right)^{ \pm a}, 1^{k-i}\right)\right.$ is adjacent to $D_{A}$ for all $a \in T$ and all $1 \leq i \leq k$.

Proof of Claim 1 This is clear for $k=2$ so we can assume $k \geq 3$. We have

$$
D_{A} \xrightarrow{1} D_{A}\left(1^{k-1}, t\right)
$$

by definition of $\Gamma_{0}$. Apply $\left(a^{k}\right) \in T^{k}$ to this to get

$$
D_{A} \xrightarrow{1} D_{A}\left(1^{k-1}, t^{a}\right)
$$

As $G$ is primitive, $X$ acts transitively on the symbols $1, \ldots, k$, so for $1 \leq i \leq k$ there is an element $\left(\alpha_{i}, \ldots, \alpha_{i}\right) . \sigma_{i} \in D_{A}$ such that

$$
D_{A}\left(1^{k-1}, t\right)\left(\alpha_{i}, \ldots, \alpha_{i}\right) \cdot \sigma_{i}=D_{A}\left(1^{i-1}, t^{\alpha_{i}}, 1^{k-i}\right)
$$

Applying ( $a^{k}$ ) gives

$$
D_{A} \xrightarrow{1} D_{A}\left(1^{i-1},\left(t^{\alpha_{i}}\right)^{a}, 1^{k-i}\right)
$$

Furthermore, applying $\left(1^{i-1},\left(t^{\alpha_{i}}\right)^{-a}, 1^{k-i}\right)$ to this gives

$$
D_{A}\left(1^{i-1},\left(t^{\alpha_{i}}\right)^{-a}, 1^{k-i}\right) \xrightarrow{1} D_{A}
$$

and as $a$ was arbitrary Claim 1 follows.
$\underline{\text { Claim } 2}$ Let $h_{i} \in T(1 \leq i \leq k)$ and let $a \in T$. Then $D_{A}\left(h_{1}, \ldots, h_{k}\right)$ is adjacent to $D_{A}\left(h_{1}, \ldots, h_{i-1},\left(t^{\alpha_{i}}\right)^{ \pm a} h_{i}, h_{i+1} \ldots, h_{k}\right)$ for $1 \leq i \leq k$.

Proof of Claim 2 Apply $\left(h_{1}, \ldots, h_{k}\right)$ to

$$
D_{A} \xrightarrow{1} D_{A}\left(1^{i-1},\left(t^{\alpha_{i}}\right)^{ \pm a}, 1^{k-i}\right)
$$

$\underline{\text { Claim } 3}$ Let $c=c_{i}(T)$. For any $h_{i}, \ldots, h_{k} \in T$ and $1 \leq i \leq k$,

$$
D_{A}\left(h_{1}, \ldots, h_{i-1}, 1, h_{i+1}, \ldots, h_{k}\right) \xrightarrow{c} D_{A}\left(h_{1}, \ldots, h_{k}\right)
$$

Proof of Claim 3 We know by definition of $c$ that $h_{i}$ can be expressed as a product of at most $c$ conjugates of $\left(t^{\alpha_{i}}\right)^{ \pm 1}$, so

$$
h_{i}=\left(t^{\alpha_{i}}\right)^{ \pm a_{1}} \ldots\left(t^{\alpha_{i}}\right)^{ \pm a_{c}}
$$

for some $a_{i} \in T$. Hence by repeatedly applying Claim 2

$$
D_{A}\left(h_{1}, \ldots, h_{i-1}, 1, h_{i+1}, \ldots, h_{k}\right) \stackrel{c}{-} D_{A}\left(h_{1}, \ldots, h_{i-1},\left(t^{\alpha_{i}}\right)^{ \pm a_{1}} \ldots\left(t^{\alpha_{i}}\right)^{ \pm a_{c}}, h_{i+1}, \ldots, h_{k}\right)
$$

so Claim 3 follows.
Using Claim 3 repeatedly we have the following path

$$
D_{A} \xrightarrow{c} D_{A}\left(1, h_{2}, 1^{k-2}\right) \xrightarrow{c} D_{A}\left(1, h_{2}, h_{3}, 1^{k-3}\right) \ldots \stackrel{c}{-} D_{A}\left(1, h_{2}, \ldots, h_{k}\right) .
$$

As $\left(1, h_{2}, \ldots, h_{k}\right)$ represents an arbitrary coset, the result follows.

We have an exact result for the orbital diameter for the case when $G=T^{2}$ or $G=D(2, T)$.
Lemma 3.3. 1. If $G=T^{2}$ then $\operatorname{orbdiam}(G)=c_{i}(T)$.
2. If $G=D(2, T)$ then $\operatorname{orbdiam}(G)=c_{A}(T)$.

Proof. 1. We first notice that in the case of $k=2$ all orbital graphs are of the form $\Gamma_{0}^{t}$. If $G=T^{2}$ then $X_{0}=T$, so $c_{X}(T)=c_{i}(T)$. Hence the bounds from Theorem 3.1 and Lemma 3.2 coincide, and the result follows.
2. Consider $G=D(2, T) \cong T^{2} \cdot\left(\operatorname{Out}(T) \times S_{2}\right)$. In this case $X_{0} \cong A u t(T)$ and also $D_{A} \cong \operatorname{Aut}(T) \times S_{2}$. Now Theorem 3.1 gives $\operatorname{orbdiam}(G) \geq c_{A}(T)$. We will show the other direction of this inequality. Let $t \in T \backslash 1$. Consider the orbital graph $\Gamma=\left\{D_{A}, D_{A}(1, t)\right\}^{G}$. Now

$$
D_{A}-D_{A}\left(1, t^{ \pm 1}\right)
$$

are edges in the graph. For all $a \in \operatorname{Aut}(T)$ apply $(a, a) \in G$ to these to get

$$
D_{A}-D_{A}\left(1, t^{ \pm a}\right)
$$

Now we can construct a path between $D_{A}$ and any arbitrary coset $D_{A}(1, h)$ where $h=t^{ \pm a_{1}} \ldots t^{ \pm a_{c}}$ with $c=c_{A}(T)$ such that $D_{A}{ }^{c} D_{A}(1, h)$. This shows that $\operatorname{orbdiam}(G) \leq c_{A}(T)$ and the result now follows.

## 4 Conjugacy Widths of Finite Simple Groups

### 4.1 Bounds on the Conjugacy Width and the Covering Number

In this section we prove bounds on the conjugacy widths $c(T), c_{i}(T)$ and $c_{A}(T)$ for simple groups as stated in the Introduction. We start with a result on conjugacy widths for simple groups of Lie type. In the following result, the upper bound is proved in [7].

Theorem 4.1. There is a constant d such that

$$
r-3 \leq c_{A}(T) \leq c n(T) \leq d r
$$

for all simple groups $T$ of Lie type of Lie rank $r$. More precisely, $c_{A}(T) \geq C_{T}$ where $C_{T}$ is as in Table 1.

| $T$ |  | $C_{T}$ |
| :---: | :---: | :---: |
| $P S L_{n}(q)$ | $(n, q) \neq(2,2)$ or $(2,3)$ | $n$ |
| $P S U_{n}(q)$ | $n \geq 3$ | $n$ |
| $P S p_{n}(q)$ | $n \geq 4,(n, q) \neq(4,2)$ | $n$ |
| $P S p_{4}(2)^{\prime}$ |  | 3 |
| $P \Omega_{n}^{\epsilon}(q)$ | $n \geq 7$ | $\left\lfloor\frac{n}{2}\right\rfloor$ |
| ${ }^{2} B_{2}(q)$ | $q>2$ | 3 |
| ${ }^{2} G_{2}(q)$ | $q>3$ | 3 |
| $G_{2}\left(3^{n}\right)$ |  | 3 |
| $G_{2}(q)$ | $3 \nmid q$ | 4 |
| ${ }^{3} D_{4}(q)$ |  | 4 |
| $F_{4}\left(2^{n}\right)$ |  | 3 |
| $F_{4}(q)$ | $2 \nmid q$ | 4 |
| ${ }^{2} F_{4}(q)$ | $q>2$ | 4 |
| ${ }^{2} F_{4}(2)^{\prime}$ |  | 4 |
| $E_{6}^{\epsilon}(q)$ |  | 4 |
| $E_{7}(q)$ |  | 4 |
| $E_{8}(q)$ |  | 5 |

Table 1: Lower bounds

Note that the inequality $c_{A}(T) \leq c_{i}(T) \leq c(T) \leq c n(T)$ is immediate from the definitions. Hence establishing a lower bound for $c_{A}(T)$ immediately gives a lower bound for $c_{i}(T), c(T)$ and $c n(T)$.

### 4.1.1 The proof of the lower bound in Theorem 4.1

Let $V=V_{n}(q)$. For $x \in P G L(V)$ let $\widetilde{x} \in G L(V)$ be a preimage of $x$ and define

$$
\nu(x)=n-\max _{\lambda \in \mathbb{F}_{q}^{\star}} \operatorname{dim} C_{V}(\lambda \widetilde{x})
$$

the minimal codimension of an $\mathbb{F}_{q}$-eigenspace of $\widetilde{x}$. For a subset $S \in P G L(V)$ define

$$
\nu(S)=\max _{s \in S} \nu(s)
$$

Proposition 4.2. Let $T \leq P G L(V) \cong P G L_{n}(q)$ be a simple group and let $X$ be a group such that Inn $T \leq X \leq A u t T$. Let $S_{0}$ be a non-empty $X$-invariant subset of $T \backslash 1$ such that $\nu(s)=\nu\left(s^{\prime}\right)$ for all $s, s^{\prime} \in S_{0}$, and let $S=S_{0} \cup S_{0}^{-1}$. Then

$$
c_{X}(T) \geq \frac{\nu(T)}{\nu(S)}
$$

Proof. Note that $S$ is a union of $X$-conjugacy classes. Choose $s \in S$ and $t \in T$ such that $\nu(s)=$ $\nu(S)$ and $\nu(t)=\nu(T)$. Put $C=s^{X}$. By hypothesis, $\nu(y)=\nu(S)$ for all $y \in C$. Let $k=n-\nu(S)$ and $r=n-\nu(T)$ so that $k=\max _{\lambda \in \mathbb{F}_{q}^{\star}} \operatorname{dim} C_{V}(\lambda \widetilde{s})$ and $r=\max _{\lambda \in \mathbb{F}_{q}^{\star}} \operatorname{dim} C_{V}(\lambda \widetilde{t})$. Let $s_{1}, \ldots, s_{l} \in C$. Using elementary linear algebra we see that

$$
\max _{\lambda_{i} \in \mathbb{F}_{q}^{\star}} \operatorname{dim} C_{V}\left(\lambda_{1} \widetilde{s_{1}}, \ldots, \lambda_{l} \widetilde{s_{l}}\right) \geq l k-(l-1) n .
$$

Suppose that $w \in \mathbb{N}$ is minimal such that $t$ can be expressed as the product of $w$ elements of $C$, so $w \leq c_{X}(T)$. Hence

$$
r \geq w k-(w-1) n
$$

Rearranging gives

$$
c_{X}(T) \geq w \geq \frac{n-r}{n-k}=\frac{\nu(T)}{\nu(S)}
$$

Now we prove the theorem.

## Proof of Theorem 4.1. Case 1, Classical Groups

Let $T$ be a classical simple group with natural module $V=V_{n}(q)$. Define $S$ to be the set of long root elements in $T$ and let $X=A u t(T)$. Then $S$ is $X$-invariant, provided $T \neq P S p_{4}\left(2^{a}\right)$.

Suppose first that $T$ is $P S L_{n}(q), P S p_{n}(q)$ or $P S U_{n}\left(q^{1 / 2}\right)$ and $T \neq P S p_{4}\left(2^{a}\right)$. Then the long root elements of $T$ are transvections, which have fixed space on $V$ of dimenstion $n-1$, so $\nu(S)=1$. We claim that

$$
\nu(T)=n
$$

This can be seen as follows. Provided $T \neq P S U_{n}\left(q^{1 / 2}\right)$ with $n$ even, by [14] $T$ has a Singer element $y$ (i.e. an element such that $\langle y\rangle$ is irreducible on $V$ ) and clearly $\nu(y)=n$. And if $T=P S U_{n}\left(q^{1 / 2}\right)$ with $n=2 d \geq 4$, then $S U_{n}\left(q^{1 / 2}\right)$ has a subgroup $S L_{d}(q)$, and a Singer element of this also satisfies $\nu(y)=n$. Hence by Proposition 4.2,

$$
c_{A}(T) \geq \frac{\nu(T)}{\nu(S)}=n
$$

Next consider $T=P S p_{4}\left(2^{a}\right)$ with $a>1$. Let $\widetilde{S}$ be the set of all long or short root elements of $T$. Then $\widetilde{S}$ in invariant under $\operatorname{Aut}(T)$. The generic character table of $T$ is in the computer package Chevie [11]. This also contains a function, called ClassMult, which calculates the sum in Lemma 2.1. Using this it can be checked that $\widetilde{S}^{3} \cup \widetilde{S}^{2} \cup \widetilde{S} \cup 1 \neq T$. Hence $c_{A}(T) \geq 4$, as required.

Finally suppose $T=P \Omega_{n}^{\epsilon}(q)$. For $n \leq 6, T$ is isomorphic to one of the groups we have already covered, so assume $n \geq 7$. The long root elements of $T$ have fixed point space of dimension $n-2$ on $V$, so $\nu(S)=2$. If $n$ is even, then $T$ has an element $y$ such that $\nu(y)=n$ : for $\epsilon=-$ take $y$ to be a Singer element of $\Omega_{n}^{-}(q)[14]$; and for $\epsilon=+$, take $y$ to be a Singer element of a subgroup $S L_{\frac{n}{2}}(q)$ of $\Omega_{n}^{+}(q)$. If $n$ is odd, then $T$ has an element $y$ such that $\nu(y)=n-1$ : for example choose a Singer element in a subgroup $\Omega_{n-1}^{-}(q)$. We conclude that

$$
\nu(T) \geq \begin{cases}n & n \text { even } \\ n-1 & n \text { odd }\end{cases}
$$

Now the conclusion follows from Proposition 4.2.

Case 2, Exceptional Groups There are only two families of exceptional groups whose covering number is known; the Suzuki groups, ${ }^{2} B_{2}(q)$ and the small Ree groups ${ }^{2} G_{2}(q)$ both have covering number 3 by Theorem 2.5. By Theorem 2.3 they are not strongly real, so in fact

$$
c_{A}\left({ }^{2} B_{2}(q)\right)=c_{A}\left({ }^{2} G_{2}(q)\right)=c_{i}\left({ }^{2} B_{2}(q)\right)=c_{i}\left({ }^{2} G_{2}(q)\right)=c\left({ }^{2} B_{2}(q)\right)=c\left({ }^{2} G_{2}(q)\right)=3
$$

Now consider $T=E_{8}(q), E_{7}(q)$ or $E_{6}^{\epsilon}(q)$. Let $V=V_{n}(q)$ be the adjoint module for $T$, of dimension 248, 133 or 78 , respectively, and let $S$ be the set of long root elements of $T$. Then $S$ is invariant under $\operatorname{Aut}(T)$. From Tables 9,8 and 6 of [18], we see that $\nu(S)$ is as in the table:

|  |  |  |  |
| :---: | :---: | :---: | :---: |
| $T$ | $E_{8}(q)$ | $E_{7}(q)$ | $E_{6}^{\epsilon}(q)$ |
| $\nu(S)$ | 58 | 34 | 22 |.

Also $T$ has regular unipotent elements $y$, and these have fixed point spaces of dimension 8,7 or 6 , respectively. Hence $\nu(T) \geq 240,126$ or 72 , and the bound $c_{A}(T) \geq \frac{\nu(T)}{\nu(S)}$ gives the conclusion of the theorem.

Next consider $T=F_{4}(q), q$ odd. Again let $S$ be the set of long root elements, which is invariant under $A u t(T)$, and consider the action on the 26 -dimensional module $V=V_{26}(q)$, as given in [18, Table 3]. We see that $\nu(S)=6$ while regular unipotent elements show that $\nu(T) \geq 24$. Hence $c_{A}(T) \geq 4$.

Next we claim that $c_{A}(T) \geq 4$ for $T={ }^{3} D_{4}(q),{ }^{2} F_{4}(q)(q>2),{ }^{2} F_{4}(2)^{\prime}$ or $G_{2}(q)\left(q \neq 3^{a}\right)$. The character tables of these are available in Chevie and GAP. Using Lemma 2.1 we can compute that for a root element, $r, c_{A}(T, r) \geq 4$.

The last groups remaining to consider are $T=F_{4}\left(2^{a}\right)$ and $G_{2}\left(3^{a}\right)$. These groups are not strongly real by Theorem [2.3, so $c_{A}(T) \geq 3$ for these.

This completes the proof of Theorem 4.1.

### 4.2 Conjugacy Width 2

There has been some interest around classifying groups with a small covering numbers. In [3, Thm $2.1]$ it is proven that the only finite simple group with covering number 2 is $J_{1}$. It turns out that the same is true for the (inverse) conjugacy width.

Proposition 4.3. For a simple group $T$ the following are equivalent;
(i) $\mathrm{cn}(T)=2$
(ii) $c_{i}(T)=2$
(iii) $c(T)=2$
(iv) $T \cong J_{1}$.

Proof. Clearly $(i) \Rightarrow(i i) \Rightarrow(i i i)$ and $(i v) \Rightarrow(i)$ so it remains to prove (iii) $\Rightarrow$ (iv). So suppose $c(T)=2$. By Theorems 2.3 and 4.1 we get that $T \cong P S L_{2}(q), q \not \equiv 3 \bmod 4, A_{10}, A_{14}, J_{1}$ or $J_{2}$. The proof of [3, Thm 4.2 (a)] shows that there is a conjugacy class $C \subseteq P S L_{2}(q)$ such that $P S L_{2}(q) \neq C^{2} \cup C \cup 1$, so $c\left(P S L_{2}(q)\right) \geq 3$. Similarly, products of at most two 3 -cycles cannot express all elements in $A_{10}$ and $A_{14}$, so their conjugacy width is greater than 3 , and using GAP we can find an element $r$ in $J_{2}$ such that $c(T, r) \geq 3$. Hence $T=J_{1}$ and the result follows.

However, the next result shows that there is an infinite family of finite simple groups, such that the conjugacy width is not 2 but the automorphism conjugacy width is 2 .

Theorem 4.4. Let $T$ be a finite simple group. Then $c_{A}(T)=2$ if and only if either $T \cong J_{1}$ or $T \cong P S L_{2}(q)$ with $q \equiv 1 \bmod 4$ or $q=2^{2 m}$.

Proof. We begin by finding $c_{A}\left(P S L_{2}(q)\right)$ for all $q \geq 5$. We know by Proposition 2.5 that $c\left(P S L_{2}(q)\right)=$ 3. If $q \equiv 3 \bmod 4$, then Theorem 2.3 implies that the involution class has conjugacy width greater than 2 , hence $c_{A}\left(P S L_{2}(q)\right)=3$. If $q$ is a power of 2 , then by [3, Thm 4.2 (a)] the only conjugacy classes with conjugacy width 3 are those denoted by $R_{j}$ in [3] these have class representatives $\left(b^{j}\right)$ where $b$ is an element of order $q+1$ and $1 \leq j \leq \frac{q}{2}$. In fact, [3, Thm 4.2 (a)] gives that $R_{j}^{2}=G \backslash C_{2}$ where $C_{2}$ are the root elements. For $q=2^{2 m+1}$, the class $R_{\frac{q+1}{3}}$ is fixed by all outer automorphisms, so $c_{A}\left(P S L_{2}\left(2^{2 m+1}\right)=3\right.$. For $q=2^{2 m}$, there is no class of type $R_{j}$ which is fixed by all outer automorphisms. Using Lemma 2.1 and Chevie 11 we can show that $C_{2} \subseteq R_{j} R_{l}$, where $l=2 j$ if $2 j \leq \frac{q}{2}$ and $l=q+1-2 j$ if $2 j>\frac{q}{2}$, so $c_{A}\left(P S L_{2}\left(2^{2 m}\right)\right)=2$. For $P S L_{2}(q)$ with $q \equiv 1$ mod 4 we know from 3 that the only classes with conjugacy width equal to 3 are the two classes of root elements, and $P S L_{2}(q)$ has an outer automorphism that interchanges these two classes. Using Lemma 2.1 we can show that every element can be expressed as a product of two root elements, hence $c_{A}\left(P S L_{2}(q)\right)=2$. This proves the right to left implication of the theorem.

For the converse, suppose $c_{A}(T)=2$. Then any element of $T$ can be expressed as a product of at most two involutions, so $T$ is strongly real, hence is given by Theorem 2.3. If $T$ is a simple group of Lie type, then by Theorems 4.1 and 2.3 we have that $T \cong P S L_{2}(q)$ with $q \not \equiv 3$ $\bmod 4$. We proved above that $c_{A}\left(P S L_{2}(q)\right)=3$ for $q=2^{2 m+1}$, so $q \equiv 1 \bmod 4$ or $q=2^{2 m}$. If $T$ is not of Lie type then by Theorem [2.3, $T \cong A_{10}, A_{14}, J_{1}$ or $J_{2}$. All automorphisms of $A_{10}$ and $A_{14}$ fix the class of 3 -cycles, so their automorphism conjugacy width is not 2. Looking at the character table of $J_{2}$ and using Lemma 2.1 we conclude that $c\left(J_{2}\right)=c_{A}\left(J_{2}\right) \geq 3$. Hence $T=J_{1}$.

### 4.3 Conjugacy Width and Covering Number 3

The next result gives a similar classification of groups with conjugacy width 3 .
Theorem 4.5. Let $T$ be a finite simple group.

1. $c(T)=3$ if and only if $T$ is isomorphic to one of the following:

- $P S L_{2}(q)$ with $q>2$
- $P S L_{3}(q)$
- $P S U_{3}(q)$ with $3 \mid q+1, q>2$
- ${ }^{2} B_{2}(q)$ with $q>2$
- ${ }^{2} G_{2}(q)$ with $q>3$
- $G_{2}\left(3^{n}\right)$ with $n \geq 2$
- $A_{5}, A_{6}, A_{7}$
- $M_{11}, M_{22}, M_{23}, M_{24}, J_{3}, J_{4}, M c l, R u, L y, O^{\prime} N, F i_{24^{\prime}}, T h, M$.

2. $c n(T)=3$ if and only if $c(T)=3$.
3. $c_{i}(T)=3$ if and only if $c(T)=3$.
4. If $c_{A}(T)=3$ then one of the following holds:
(a) $c(T)=3$ and $T \neq P S L_{2}(q)$ with $q \equiv 1(\bmod 4)$ or $q=2^{2 m}$.
(b) $T \cong F_{4}\left(2^{n}\right)$.

## Proof. Case 1, Alternating Groups

By Theorem 2.5 the only alternating groups with covering number 3 are $A_{5}, A_{6}$ and $A_{7}$. Also by Theorem 2.4, $c_{A}\left(A_{n}\right) \geq 4$ for $n \geq 8$. Finally $c\left(A_{n}\right)=c_{i}\left(A_{n}\right)=3$ for $n=5,6$ and 7 while $c_{A}\left(A_{n}\right)=2,2,3$ respectively.

## Case 2, Groups of Lie type

If $T$ is a group of Lie type such that $c_{A}(T) \leq 3$ then by Theorem 4.1 $T$ is isomorphic to one of the following:

$$
P S L_{2}(q), P S L_{3}(q), P S U_{3}(q), P \Omega_{7}(q),{ }^{2} B_{2}(q),{ }^{2} G_{2}(q), G_{2}\left(3^{a}\right), F_{4}\left(2^{a}\right)
$$

By Theorem [2.5, $c n(T)=3$ for $T=P S L_{2}(q), P S L_{3}(q)$ with $q \geq 4,{ }^{2} B_{2}(q)$ and ${ }^{2} G_{2}(q)$; by Proposition 4.3 for these groups $c(T)=c_{i}(T)=3$; and by Theorem 4.4, $c_{A}(T)=3$ apart from $P S L_{2}(q)$ with $q \equiv 1(\bmod 4)$ or $q=2^{2 m}$. For $T=P S L_{3}(2)$ or $P S L_{3}(3)$ we can show using GAP that $c n(T)=c(T)=c_{i}(T)=c_{A}(T)=3$.

By [24, Cor 1.9$]$ for $T=P S U_{3}(q)(q>2)$

$$
c n(T)= \begin{cases}3 & 3 \mid q+1 \\ 4 & 3 \nmid q+1\end{cases}
$$

Hence if $3 \mid q+1$ then also $c(T)=c_{i}(T)=c_{A}(T)=3$. For $3 \nmid q+1$ it is shown in [24, Table 2] that for a transvection $t \in T, c(T, t)=4$ and hence $c_{A}(T)=4$ and $c_{i}(T)=4$.

Next consider $T=P \Omega_{7}(q), q$ odd. We claim that $c_{A}(T) \geq 4$. To prove this let $V$ be the 8 -dimensional spin representation of $\operatorname{Spin}_{7}(q)$. Let $S$ be the set of long root elements of $T$, which is invariant under $\operatorname{Aut}(T)$. Using triality we can show that the long root elements of $\operatorname{Spin}_{7}(q)$ act on $V$ as long root elements of $\Omega_{8}^{+}(q)$, hence they fix a 6 -dimensional subspace of $V$ pointwise, and so $\nu(S)=2$. We know that $\operatorname{Spin}_{6}^{+}(q) \cong S L_{4}(q)$ embeds into $\operatorname{Spin}_{7}(q)$. Put $V_{4}$ as the natural module of $S L_{4}(q)$. Then by [16, 2.2.8] $S L_{4}(q)$ acts on $V$ as on $V_{4} \oplus V_{4}^{*}$. Take $R$ to be a Singer cycle in $S L_{4}(q)$. Now $R$ on $V_{4}^{*}$ is also Singer cycle, and hence $\nu(R)=8$. Now by Proposition 4.2, $c_{A}(T) \geq 4$.

Consider $T=G_{2}\left(3^{a}\right)$. In the case of $a=1$, the character table is in GAP, so using Lemma 2.1 we find that $c_{A}\left(G_{2}(3)\right)=c_{i}\left(G_{2}(3)\right)=c\left(G_{2}(3)\right)=4$. For $a \geq 2$, even though the generic character table is available in Chevie, solving this problem is not possible using only Chevie. This is due to the fact that when running the ClassMult function, Chevie outputs values for the character sum in Lemma 2.1 together with a set of many "possible exceptions" which give conditions under which this value might not hold. For some classes it is possible to deal with these exceptions, and for others it is not, so we used another method of solution. For the classes that are possible to handle with Chevie we used Chevie, for the rest we used Corollary 2.2. To describe this, we use the notation for conjugacy classes and characters of $T$ given in 9 . We partition the non-trivial irreducible characters into sets $\Delta_{k}$, where the degree of the characters in $\Delta_{k}$ is a polynomial in $q$ of degree $k$. We see from 9 that $\operatorname{Irr}(T) \backslash 1_{T}=\Delta_{4} \cup \Delta_{5} \cup \Delta_{6}$ and we get $\left|\Delta_{4}\right|=1$, $\left|\Delta_{5}\right|=2 q-7$ and $\left|\Delta_{5}\right|=q^{2}+13$. For $\chi \in \Delta_{4}$ we have $\chi(1) \geq q^{4}+q^{2}+1$, for $\chi \in \Delta_{5}$ we have $\chi(1) \geq \frac{1}{6} q(q-1)^{2}\left(q^{2}-q+1\right)$ and for $\chi \in \Delta_{6}$ we have $\chi(1) \geq q\left(q^{2}-q+1\right)\left(q^{3}-1\right)$. In Table 2 we give upper bounds for $|\chi(x)|$ for all non-identity conjugacy classes $x$. The first column lists the class representatives in the notation of 9 . The other three columns give upper bounds for $|\chi(x)|$

|  | $\Delta_{4}$ | $\Delta_{5}$ | $\Delta_{6}$ |
| :---: | :---: | :---: | :---: |
| $A_{2}$ | $q^{2}+1$ | $(q+1)\left(q^{2}+1\right)$ | $(q+1)\left(q^{2}+q+1\right)$ |
| $A_{31}$ | $q^{2}+1$ | $(q+1)\left(q^{2}+1\right)$ | $(q+1)\left(q^{2}+q+1\right)$ |
| $A_{41}$ | 1 | $\frac{1}{2} q(q+1)$ | $2 q+1$ |
| $A_{42}$ | 1 | $\frac{1}{2} q(q+1)$ | $2 q+1$ |
| $A_{51}$ | 1 | $\frac{4}{3} q$ | 1 |
| $A_{52}$ | 1 | $\frac{4}{3} q$ | 1 |
| $A_{53}$ | 1 | $\frac{4}{3} q$ | 1 |
| $B_{1}$ | $1+2 q$ | $q^{2}+3 q+2$ | $3(q+1)^{2}$ |
| $B_{2}$ | $1+q$ | $2 q+2$ | $3(q+1)$ |
| $B_{3}$ | $1+q$ | $2 q+2$ | $3(q+1)$ |
| $B_{4}$ | 1 | $2 q+2$ | 3 |
| $B_{5}$ | 1 | $2 q+2$ | 3 |
| $C_{11}$ | $2+q$ | $2 q+2$ | $3(q+1)$ |
| $C_{12}$ | 2 | 2 | 3 |
| $C_{21}$ | $2+q$ | $2 q+2$ | $3(q+1)$ |
| $C_{22}$ | 2 | 2 | 3 |
| $D_{11}$ | $2+q$ | $2 q+2$ | $3(q+1)$ |
| $D_{12}$ | 2 | 2 | 3 |
| $D_{21}$ | $2+q$ | $2 q+2$ | $3(q+1)$ |
| $D_{22}$ | 2 | 2 | 3 |
| $E_{1}(i, j)$ | 3 | 4 | 6 |
| $E_{2}(i)$ | 1 | 4 | 4 |
| $E_{3}(i)$ | 1 | 4 | 4 |
| $E_{4}(i, j)$ | 3 | 4 | 6 |
| $E_{5}(i)$ | 0 | 4 | 6 |
| $E_{6}(i)$ | 0 | 4 | 6 |
|  |  |  |  |

Table 2: Bounds on character values
for $\chi \in \Delta_{4}, \Delta_{5}$ and $\Delta_{6}$, respectively. We use these bounds to bound the sum in Corollary 2.2 with $k=3$. We find that this sum is less than 1 for all pairs of conjugacy classes $(C, D)$ with the following exceptions;


For these exceptions we can show $D \subseteq C^{3}$ using Chevie and Lemma 2.1 with the exception of showing $E_{2}(i) \subseteq B_{1}^{3}, E_{3}(i) \subseteq B_{1}^{3}, 1 \subseteq C_{11}(i)^{3}, 1 \subseteq C_{21}(i)^{3}, 1 \subseteq D_{11}(i)^{3}$, and $1 \subseteq D_{21}(i)^{3}$. For these exceptions we obtained more precise bounds for the character values than those in Table 2 and used Corollary 2.2 again. Hence we proved that $\operatorname{cn}(T)=3$. It follows by Theorem 4.1, $c(T)=c_{i}(T)=c_{A}(T)=3$ as well.

Finally we need to consider $F_{4}\left(2^{n}\right)$. We can use the argument given for $F_{4}(q), q$ odd in the proof of Theorem 4.1 to conclude that $c_{i}(T) \geq 4$. We have not been able to determine whether the automorphism conjugacy width of $F_{4}\left(2^{n}\right)$ is 3 .

## Case 3, Sporadic Groups

Zisser 29 and Karni 15 showed that the only sporadic simple groups with covering number 3 are the ones listed in Theorem 4.5. Using character tables in GAP and Lemma 2.1 we computed $c(T), c_{i}(T)$ and $c_{A}(T)$ for all sporadic groups and obtained the same list.

## 5 Simple diagonal groups with small orbital diameter

In this section we prove the following result, which classifies the primitive permutation groups of simple diagonal type with orbital diameter at most 4. Adopt the notation of the introduction: $T$ is simple, $k \geq 2, G=T^{k} \cdot X \leq D(k, T)$ with $X \leq O u t(T) \times S_{k}$ and $G$ acts primitively on $\Omega=\left(G: D_{A}\right)$.

Theorem 5.1. Let $G$ be a primitive group of simple diagonal type of the form $T^{k} \cdot X \leq D(k, T)$.

1. If $\operatorname{orbdiam}(G)=2$, then $k=2$ and $c_{A}(T)=2$.
2. If $\operatorname{orbdiam}(G)=3$, then $k=2$ and $c_{A}(T) \leq 3$.
3. If $\operatorname{orbdiam}(G)=4$, then one of the following holds:
(a) $k=2$ and $c_{A}(T) \leq 4$
(b) $k=3$ and $c_{A}(T)=2$.

Remark Note that Lemma 3.3 is a partial converse of this result, as it shows that there are some families of groups of simple diagonal type with $k=2$ such that $\operatorname{orbdiam}(G)=c_{A}(T)$, namely $G=D(2, T)$.

For the proof of this theorem need some preliminary lemmas.
Note that it follows from Theorem 3.1 that $k$ is a lower bound for the orbital diameter. In fact, for $c_{A}(T)=2$ it is a strict lower bound;

Lemma 5.2. Let $c_{A}(T)=2, k \geq 3$ and $G=T^{k} . X \leq D(k, T)$. Then $\operatorname{orbdiam}(G) \geq k+1$.

Proof. Recall our definition of the length of an element of $T$ with respect to $t \in T \backslash 1$ : for $g \in T$,

$$
l_{t}^{A}(g)=\min \left\{a: g=t^{ \pm \alpha_{1}} \ldots t^{ \pm \alpha_{a}}, \alpha_{i} \in \operatorname{Aut}(T)\right\}
$$

Since $c_{A}(T)=2$, we have $l_{t}^{A}(g) \leq 2$ and $l_{t}^{A}(g)=1$ if and only if $g \in t^{ \pm A u t(T)}$. Also by Proposition 4.4. $T=J_{1}$ or $P S L_{2}(q)$ with $q \equiv 1(\bmod 4)$ or $q=2^{2 m}$.

Let $t$ be an involution in $T$ and define $\Gamma_{0}=\left\{D_{A}, D_{A}\left(1^{k-1}, t\right)\right\}^{G}$.
Claim 1 We can choose $x, y, z \in T$ such that

$$
l_{t}^{A}(x)=l_{t}^{A}(y)=l_{t}^{A}\left(x^{-1} y\right)=l_{t}^{A}\left(t^{a} x\right)=2 \quad \forall a \in T
$$

and

$$
l_{t}^{A}(z)=l_{t}^{A}\left(x^{-1} z\right)=l_{t}^{A}\left(y^{-1} z\right)=2
$$

Proof of Claim 1 For $T=J_{1}$ and $P S L_{2}(4)$ this can be verified in GAP. Now let $T=$ $P S L_{2}(q)$, with $q \geq 8$ and $q \equiv 1(\bmod 4)$ or $q=2^{2 m}$. Let $Z=Z\left(S L_{2}(q)\right)$. Define $x, y, z \in T$ to be $\left(\begin{array}{cc}\omega & 0 \\ 0 & \omega^{-1}\end{array}\right) Z,\left(\begin{array}{cc}\omega^{2} & 0 \\ 0 & \omega^{-2}\end{array}\right) Z$ and $\left(\begin{array}{cc}0 & 1 \\ 1 & \omega\end{array}\right) Z$, respectively, where $\omega \in \mathbb{F}_{q}^{\star}$ has order $q-1$. To see that these elements satisfy Claim 1 consider the product of an involution in $P S L_{2}(q)$ with $x$ or $y$. An involution in $P S L_{2}(q)$ is of the form $\left(\begin{array}{cc}a & b \\ c & -a\end{array}\right) Z$, and

$$
\left(\begin{array}{cc}
a & b \\
c & -a
\end{array}\right)\left(\begin{array}{cc}
\omega^{i} & 0 \\
0 & \omega^{-i}
\end{array}\right) Z=\left(\begin{array}{cc}
a \omega^{i} & b \\
c & -a \omega^{-i}
\end{array}\right) Z
$$

which is not an involution. The other assertions in Claim 1 are easily verified.
$\underline{\text { Claim } 2}$ There is a coset $D_{A}\left(m_{1}, \ldots, m_{k}\right)$ such that $d_{\Gamma_{0}}\left(D_{A}, D_{A}\left(m_{1}, \ldots, m_{k}\right)\right) \geq k+1$.
Proof of Claim 2 Assume first that $k$ is odd and let and let $x, y, z \in T$ be as in Claim 1.
Set

$$
\left(m_{1}, \ldots, m_{k}\right)=(y, 1, x, 1, x, 1, \ldots, 1, x)
$$

Suppose that $d_{\Gamma_{0}}\left(D_{A}, D_{A}\left(m_{1}, \ldots, m_{k}\right)\right) \leq k$. Then by Claim 1 in the proof of Theorem 3.1, there exists $g \in T$ such that

$$
\begin{equation*}
\sum_{i=1}^{k} l_{t}^{A}\left(g m_{i}\right) \leq k \tag{1}
\end{equation*}
$$

By Claim 1,

$$
2 k=\sum_{i=1}^{k-1} l_{t}^{A}\left(m_{i}^{-1} m_{i+1}\right)+l_{t}^{A}\left(m_{1}^{-1} m_{k}\right)
$$

and recall that

$$
l_{t}^{A}\left(m_{i}^{-1} m_{i+1}\right) \leq l_{t}^{A}\left(g m_{i+1}\right)+l_{t}^{A}\left(g m_{i}\right)
$$

Putting these together we get
$2 k=\sum_{i=1}^{k-1} l_{t}^{A}\left(m_{i}^{-1} m_{i+1}\right)+l_{t}^{A}\left(m_{1}^{-1} m_{k}\right) \leq 2 \sum_{i=1}^{k} l_{t}^{A}\left(g m_{i}\right)=2\left(l_{t}^{A}(g y)+\frac{k-1}{2} l_{t}^{A}(g x)+\frac{k-1}{2} l_{t}^{A}(g)\right) \leq 2 k$,
where the last inequality follows from (11). This tells us that $\sum_{i=1}^{k} l_{t}^{A}\left(g m_{i}\right)=k$. If $l_{t}^{A}\left(g m_{i}\right)=1$ for all $i$, then $l_{t}^{A}(g)=l_{t}^{A}(g x)=l_{t}^{A}(g y)=1$ which is a contradiction by Claim 1 . Hence there exist $j$ such that $l_{t}^{A}\left(g m_{j}\right)=0$ and so $g=1, x^{-1}$ or $y^{-1}$. If $g=x^{-1}$ then $\frac{k-1}{2} l_{t}^{A}(g)=k-1$ and $l_{t}^{A}\left(x^{-1} y\right)=2$, so $\sum_{i=1}^{k} l_{t}^{A}\left(g m_{i}\right) \geq k+1$. If $g=y^{-1}$ then $\frac{k-1}{2} l_{t}^{A}(g)=k-1$ and $\frac{k-1}{2} l_{t}^{A}\left(y^{-1} x\right)=k-1$, so $\sum_{i=1}^{k} l_{t}^{A}\left(g m_{i}\right) \geq 2 k-2 \geq k+1$. And if $g=1$ then $\frac{k-1}{2} l_{t}^{A}(x)=k-1$ and $l_{t}^{A}(y)=2$, so $\sum_{i=1}^{k} l_{t}^{A}\left(g m_{i}\right) \geq k+1$. These contradictions prove Claim 2 for the case where $k$ is odd.

Now assume $k$ is even. In this case let $\left(m_{1}, \ldots, m_{k}\right)=(y, z, x, 1, x, 1, \ldots, 1)$. Suppose that $d_{\Gamma_{0}}\left(D_{A}, D_{A}\left(m_{1}, \ldots, m_{k}\right)\right) \leq k$. As before, we deduce that there exists $g \in T$ such that $\sum_{i=1}^{k} l_{t}^{X}\left(g m_{i}\right)=k$ and we reach a contradiction in the same way. These contradictions establish Claim 2 and so the orbital diameter of $G$ is bounded below by $k+1$.

Now we include another result regarding simple groups with conjugacy width 3 .
Lemma 5.3. Let $T$ be a simple group and let $X$ be a group such that $\operatorname{Inn} T \leq X \leq \operatorname{Aut}(T)$ and $c_{X}(T)=3$. Then there exist $g, x, y \in T$ such that $c_{X}(T, g)=3$ and $l_{g}^{X}(x)=l_{g}^{X}(y)=\overline{l_{g}^{X}}\left(x^{-1} y\right)=3$.

Proof. By Theorem4.5 either $c n(T)=3$ or $T \cong F_{4}\left(2^{a}\right)$. Assume first that $c n(T)=3$ and choose any $g \in T$ such that $c_{X}(T, g)=3$. Let $C:=g^{ \pm X}$ and let $t \in T$ be such that $l_{g}^{X}(t)=3$ and let $D=t^{ \pm X}$.

First suppose that $c n(T)=3$. Assume $D^{2} \subseteq C^{2} \cup C \cup\{1\}=E$. Then

$$
T=D^{3} \subseteq E D \subseteq T
$$

and so

$$
E D=T=\left\{g^{ \pm a} g^{ \pm b} t^{ \pm c}, g^{ \pm b} t^{ \pm c}, t^{ \pm c} \mid a, b, c \in X\right\}
$$

However since $l_{g}^{X}(t)=3$ we have $1 \notin E D$, a contradiction. Hence $D^{2} \nsubseteq C^{2} \cup C \cup\{1\}$ and so there exist $x, y \in D$ satisfying the statement of the lemma.

Finally assume $T=F_{4}\left(2^{a}\right)$. Let $g \in T$ be an involution. As $F_{4}\left(2^{a}\right)$ is not strongly real, we have that $c_{X}(T, g) \geq 3$. Recall that an element is real if it is conjugate to its inverse. We shall show the existence of elements $x, y \in T$ such that none of $x, y, x^{-1} y$ is real. The proof of [27, Cor 4.5] shows the existence of a non-real order 12 element in $F_{4}\left(2^{a}\right)$. By construction, this has a conjugate in $F_{4}(2)$. From the character table in GAP, $F_{4}(2)$ has precisely two non-real classes, $C, C^{-1}$ of elements of order 12. A computation shows that we can find $x, y \in C$ such that $x^{-1} y \in C$. Hence by the previous observation, $x, y, x^{-1} y$ are also non-real in $T=F_{4}\left(2^{a}\right)$.

Lemma 5.4. Let $G=T^{3} . X \leq D(3, T)$ with $c_{X}(T)=3$. Choose $g, x, y \in T$ as in the statement of Lemma 5.3 and let $\Gamma_{0}=\left\{D_{A}, D_{A}(1,1, g)\right\}^{G}$. Then diam $\left(\Gamma_{0}\right)>4$.

Proof. Let $x, y \in T$ be as in Lemma 5.3. The for all $t \in T$ we have

$$
l_{g}^{X}(t x)+l_{g}^{X}(t)+l_{g}^{X}(t y) \geq \frac{l_{g}^{X}(x)+l_{g}^{X}(y)+l_{g}^{X}\left(x^{-1} y\right)}{2}=\frac{9}{2}>4
$$

Hence $d_{\Gamma_{0}}\left(D_{A}, D_{A}(x, 1, y)\right)>4$. The conclusion follows.

Proof of Theorem 5.1. Let $T$ be simple and $G=T^{k} \cdot X \leq D(k, T)$.

1. If $\operatorname{orbdiam}(G)=2$ then by Theorem 3.1 it follows that $k=2$ and $c_{A}(T)=2$.
2. Suppose $\operatorname{orbdiam}(G)=3$. Then $k \leq 3$ by Theorem 3.1. If $k=2$ then Theorem 3.1 shows $c_{A}(T) \leq 3$ as required. If $k=3$, then by Theorem 3.1, $c_{A}(T)=2$, but then Lemma 5.2 implies that $\operatorname{orbdiam}(G)>3$, a contradiction.
3. Assume that $\operatorname{orbdiam}(G)=4$. By Theorem 3.1 one of the following occurs:

$$
\begin{aligned}
& \text { I } k=2 \text { and } c_{A}(T) \leq 4 \\
& \text { II } k=3 \text { and } c_{A}(T)=2 \\
& \text { III } k=3 \text { and } c_{A}(T)=3 \\
& \text { IV } k=4 \text { and } c_{A}(T)=2 .
\end{aligned}
$$

In Cases I and II, conclusions $(a)$ and $(b)$ of Theorem 5.1 hold.
In Cases III and IV, Lemmas 5.2 and 5.4 imply that $\operatorname{orbdiam}(G) \geq 5$, which is a contradiction.

## 6 Upper bound on the orbital diameter

In this section we obtain a general upper bound for the orbital diameter of a primitive simple diagonal group. To keep things as simple as possible we restrict attention to groups of the form $G=T^{k} . S_{k} \leq D(k, T)$.

Theorem 6.1. Let $T$ be a simple group and let $G=T^{k} \cdot S_{k} \leq D(k, T)$ be a primitive simple diagonal group. Then

$$
\operatorname{orbdiam}(G) \leq 24(k-1) c_{i}(T)^{2}
$$

We first need a preliminary lemma.

Lemma 6.2. Let $T$ be an simple group and $u \in T$ an involution. Then there exists $x \in T$ such that $u u^{x}=[u, x]$ has order greater than 2.

Proof. Suppose that $u u^{x}$ has order less than or equal to 2 for all $x \in T$. Then $u$ commutes with all of its conjugates, hence it commutes with $\left\langle u^{T}\right\rangle=T$, as $T$ is simple, and so $u \in Z(T)$. This is a contradiction.

Proof of Theorem 6.1. For $k=2$ conclusion follows from Lemma 3.3 so assume $k \geq 3$. Let $\Gamma$ be an orbital graph of $G$.

As we already covered the case of $\Gamma_{0}^{t}$ in Lemma3.2. we may assume that $\Gamma=\left\{D_{A}, D_{A}\left(1^{i}, t_{i+1}, \ldots, t_{k}\right)\right\}^{G}$ where $i \leq k-2$ and $t_{j} \in T \backslash 1$ for $j \geq i+1$.

Suppose there is a path of length at most $m$ from $D_{A}$ to $D_{A}\left(m_{1}, \ldots, m_{k}\right)$ where $m_{i} \in T$. Recall our notation for this:

$$
\begin{equation*}
D_{A} \xrightarrow{m} D_{A}\left(m_{1}, \ldots, m_{k}\right) \tag{2}
\end{equation*}
$$

Applying $\left(m_{1}^{-1}, \ldots, m_{k}^{-1}\right)$ we get

$$
\begin{equation*}
D_{A} \frac{m}{} D_{A}\left(m_{1}^{-1}, \ldots, m_{k}^{-1}\right) \tag{3}
\end{equation*}
$$

If we apply $\left(a^{k}\right)$ to (2) or (3) for any $a \in T$, we get

$$
\begin{equation*}
D_{A} \xrightarrow{m} D_{A}\left(m_{1}^{ \pm a}, \ldots, m_{k}^{ \pm a}\right) \tag{4}
\end{equation*}
$$

$\underline{\text { Claim } 1}$ There exists $t \in T \backslash 1$ and a path in $\Gamma$ of the following form;

$$
D_{A} \frac{2}{} D_{A}\left(1^{k-2}, t^{-1}, t\right)
$$

## Proof of Claim 1

Suppose first that there exist $l, j \geq i+1$ such that $t_{l} \neq t_{j}$. Apply the permutation ( $l j$ ) and $\left(1^{i}, t_{i+1}, \ldots, t_{k}\right)$ to the edge $D_{A}-D_{A}\left(1^{i}, t_{i+1}^{-1}, \ldots, t_{k}^{-1}\right)$ to get a path

$$
D_{A} \longrightarrow D_{A}\left(1^{i}, t_{i+1}, \ldots, t_{k}\right) \longrightarrow D_{A}\left(1^{l-1}, t_{j}^{-1} t_{l}, 1^{j-l-1}, t_{l}^{-1} t_{j}, 1^{k-j}\right)
$$

Putting $u=t_{j}^{-1} t_{l}$ and applying a suitable permutation yields Claim 1 in this case. Now assume $t_{i+1}=\cdots=t_{k}=t$. Apply $(1 k)$ and $\left(1^{i}, t^{k-i}\right)$ to the edge $D_{A} \longrightarrow D_{A}\left(1^{i},\left(t^{-1}\right)^{k-i}\right)$ to get

$$
D_{A}=D_{A}\left(1^{i}, t^{k-i}\right) \longrightarrow D_{A}\left(t^{-1}, 1^{k-2}, t\right)
$$

Claim 1 again follows.
Claim 2 There exists $t^{\prime} \in T \backslash 1$ and a path in $\Gamma$

$$
D_{A} \frac{24 c}{} D_{A}\left(1^{k-1}, t^{\prime}\right)
$$

where $c=c_{i}(T)$.
Proof of Claim 2 We know from Claim 1 that for some $t \in T \backslash 1$ there is a path

$$
D_{A} \frac{2}{} D_{A}\left(1^{k-2}, t^{-1}, t\right)
$$

Let $c_{u} \leq c$ such that $u=t^{ \pm a_{1}} \ldots t^{ \pm a_{c_{u}}}$ is an involution in $T$ where $a_{i} \in T$. We can construct a path in a similar way as in the proof of Claim 3 of Lemma 3.2 to get that

$$
D_{A} \frac{2 c}{} D_{A}\left(1^{k-2}, t^{ \pm a_{1}} \ldots t^{ \pm a_{c_{u}}}, t^{\mp a_{1}} \ldots t^{\mp a_{c_{u}}}\right)=D_{A}\left(1^{k-2}, u, u^{\prime}\right)
$$

If there exist $b_{1}, b_{2} \in T$ such that $u^{b_{1}} u^{b_{2}}=1$ and $u^{\prime n_{1}} u^{b_{2}} \neq 1$ then

$$
D_{A} \frac{4 c}{} D_{A}\left(1^{k-1}, u^{\prime b_{1}} u^{b_{2}}\right) \neq D_{A}
$$

If no such $b_{1}, b_{2}$ exist then for $b \in T, u u^{b}=1$ implies $u^{\prime} u^{\prime b}=1$. Hence $u^{\prime}$ is also an involution and $C_{G}(u)=C_{G}\left(u^{\prime}\right)$. So assume now that this is the case. Applying $\left(1^{k-2}, u^{\prime}, u\right)$ to $D_{A} \xrightarrow[2 c]{ } D_{A}\left(1^{k-2}, u, u^{\prime}\right)$, we see that here is a path

$$
D_{A} \xrightarrow{4 c} D_{A}\left(1^{k-2}, w, w\right),
$$

where $w \neq 1$ and either $w=u=u^{\prime}$ or $w=u u^{\prime}$. Note that if $k=3$, then the claim follows, so assume $k \geq 4$.

We know that $w$ is an involution, as $u$ and $u^{\prime}$ commute. By Lemma 6.2 there is $x \in T$ such that $w w^{x}$ has order strictly greater than 2 . Now

$$
D_{A} \frac{4 c}{} D_{A}\left(1^{k-2}, w^{x}, w^{x}\right) .
$$

Apply $\left(1^{k-2}, w, w\right)$ to get

$$
D_{A} \frac{4 c}{} D_{A}\left(1^{k-2}, w, w\right) \frac{4 c}{} D_{A}\left(1^{k-2}, w^{x} w, w^{x} w\right)
$$

Similarly,

$$
D_{A} \frac{8 c}{} D_{A}\left(1^{k-2}, w w^{x}, w w^{x}\right)
$$

Let $h=w w^{x}$. Then $h^{-1}=w^{x} w$, so we have

$$
D_{A} \frac{8 c}{} D_{A}\left(1^{k-2}, h^{-1}, h^{-1}\right)
$$

Apply $\left(1^{k-3}, h, h, 1\right)$ to get

$$
D_{A}\left(1^{k-3}, h, h, 1\right)-\frac{8 c}{} D_{A}\left(1^{k-3}, h, 1, h^{-1}\right)
$$

so

$$
D_{A} \frac{16 c}{} D_{A}\left(1^{k-3}, h, 1, h^{-1}\right)
$$

Apply $\left(1^{k-3}, h^{-1}, 1, h^{-1}\right)$ to get

$$
D_{A}\left(1^{k-3}, h^{-1}, 1, h^{-1}\right)-16 c \quad D_{A}\left(1^{k-3}, 1,1, h^{-2}\right)
$$

so

$$
D_{A} \frac{24 c}{} D_{A}\left(1^{k-3}, 1,1, h^{-2}\right),
$$

where $h^{-2} \neq 1$. Hence Claim 2 follows.
At this point we can apply Lemma 3.2 to deduce that the diameter of $\Gamma$ is bounded above by $24(k-1) c^{2}$, completing the proof.

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## References

[1] Z. Arad and M. Herzog, editors. Products of conjugacy classes in groups, volume 1112 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1985.
[2] Z. Arad, J. Stavi, and M. Herzog. Powers and products of conjugacy classes in groups. In [1], pages 6-51. Springer, Berlin, 1985.
[3] Zvi Arad, David Chillag, and Gadi Moran. Groups with a small covering number. In [1], pages 222-244. Springer, Berlin, 1985.
[4] Edward Bertram and Marcel Herzog. Powers of cycle-classes in symmetric groups. J. Comb. Theory, Series A, 94:87-99, 2001.
[5] John D. Dixon and Brian Mortimer. Permutation groups, volume 163 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1996.
[6] Yoav Dvir. Covering properties of permutation groups. In [1], pages 197-221. Springer, Berlin, 1985.
[7] Erich W. Ellers, Nikolai Gordeev, and Marcel Herzog. Covering numbers for Chevalley groups. Israel J. Math., 111:339-372, 1999.
[8] Erich W. Ellers and Wolfgang Nolte. Bireflectionality of orthogonal and symplectic groups. Arch. Math. (Basel), 39:113-118, 1982.
[9] Hikoe Enomoto. The characters of the finite Chevalley group $G_{2}(q), q=3{ }^{f}$. Japan. J. Math., pages 191-248, 1976.
[10] A. A. Galt. Strongly real elements in finite simple orthogonal groups. Sibirsk. Mat. Zh., 51:241-248, 2010.
[11] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer. CHEVIE - A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. Appl. Algebra Engrg. Comm. Comput., 7:175-210, 1996.
[12] R. Gow. Products of two involutions in classical groups of characteristic 2. J. Algebra, 71:583591, 1981.
[13] R. Gow. Commutators in the symplectic group. Arch. Math. (Basel), 50:204-209, 1988.
[14] Bertram Huppert. Singer-zyklen in klassischen gruppen. Math. Z. 117, 141-150), 1970.
[15] Shimy Karni. Covering numbers of groups of small order and sporadic groups. In [1], pages 52-196. Springer, Berlin, 1985.
[16] Peter Kleidman. The maximal subgroups of the finite 8-dimensional orthogonal groups $\mathrm{P} \Omega_{8}^{+}(q)$ and of their automorphism groups. J. Algebra, 110:173-242, 1987.
[17] S. G. Kolesnikov and Ja. N. Nuzhin. On strong reality of finite simple groups. Acta Appl. Math., 85:195-203, 2005.
[18] R. Lawther. Jordan block sizes of unipotent elements in exceptional algebraic groups. Comm. in Alg., 23:4125-4156, 1995.
[19] R. Lawther and Martin W. Liebeck. On the diameter of a Cayley graph of a simple group of Lie type based on a conjugacy class. J. Combin. Theory Ser. A, 83:118-137, 1998.
[20] Arieh Lev. The covering number of the group $\mathrm{PSL}_{n}(F)$. J. Algebra, 182:60-84, 1996.
[21] Martin W. Liebeck, Dugald Macpherson, and Katrin Tent. Primitive permutation groups of bounded orbital diameter. Proc. Lond. Math. Soc., 100:216-248, 2010.
[22] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl. On the O'Nan-Scott theorem for finite primitive permutation groups. J. Austral. Math. Soc. Ser. A, 44:389-396, 1988.
[23] Alexander J. Malcolm. The involution width of finite simple groups. J. Algebra, 493:297340, 2018.
[24] S. Yu. Orevkov. Products of conjugacy classes in finite unitary groups $G U\left(3, q^{2}\right)$ and $S U\left(3, q^{2}\right)$. Ann. Fac. Sci. Toulouse Math., 22:219-251, 2013.
[25] Johanna Rämö. Strongly real elements of orthogonal groups in even characteristic. J. Group Theory, 14:9-30, 2011.
[26] Atiqa Sheikh. Orbital diameters of the symmetric and alternating groups. J. Algebraic Combin., 45:1-32, 2017.
[27] Pham Tiep and Alexandre Zalesski. Real conjugacy classes in algebraic groups and finite groups of lie type. J. Group Theory, pages 291-315, 2005.
[28] E. P. Vdovin and A. A. Galt. Strong reality of finite simple groups. Sibirsk. Mat. Zh., 51:769-777, 2010.
[29] Ilan Zisser. The covering numbers of the sporadic simple groups. Israel J. Math., 67:217-224, 1989.

