On the Orbital Diameter of Groups of Diagonal Type

Kamilla Rekvényi

Department of Mathematics, Imperial College London, London, SW7 2AZ k.rekvenyi19@imperial.ac.uk

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Abstract

The orbital diameter of a primitive permutation group is the maximal diameter of its orbital graphs. There has been a lot of interest in bounds for the orbital diameter. In this paper we provide explicit bounds on the diameters of groups of simple diagonal type. As a consequence we obtain a classification of simple diagonal groups with orbital diameter less than or equal to 4. As part of this, we classify all finite simple groups with covering number and conjugacy width at most 3. We also prove some general bounds on the covering number and conjugacy width of groups of Lie type.

1 Introduction

Let G be a group acting transitively on a finite set Ω . Then G acts on $\Omega \times \Omega$ componentwise. Define the *orbitals* to be the orbits of G on $\Omega \times \Omega$. The *diagonal orbital* is the orbital of the form $\Delta = \{(\alpha, \alpha) | \alpha \in \Omega\}$. The number or orbitals is called the rank of G. Let us denote this by $rank(G, \Omega)$. Let Γ be a non-diagonal orbital. Define the corresponding *orbital graph* to be the undirected graph with vertex set Ω and edge set $\{\alpha, \beta\}$ for $(\alpha, \beta) \in \Gamma$. Note that G acts transitively on the edges and the vertices of Γ so it is an *edge-transitive* and *vertex-transitive* graph.

By [5, Thm 3.2A] the orbital graphs are all connected if and only if the action of G is primitive. The *orbital diameter* of a primitive permutation group G is the supremum of the diameters of its orbital graphs, see [21]. Let us denote this by $\operatorname{orbdiam}(G)$.

The O'Nan-Scott theorem classifies the primitive permutation groups to be one of the following five types; affine, almost simple, simple diagonal actions, product actions and twisted wreath actions, see [5]. We call an infinite class C of primitive permutation groups bounded if there exists $t \in \mathbb{N}$ such that $\operatorname{orbdiam}(G) \leq t$ for all $G \in C$. The paper [21] describes the O'Nan-Scott classes which are bounded. The description is somewhat qualitative and does not contain explicit diameter bounds. Some explicit bounds were obtained in [26] for some almost simple groups. In this paper we study the orbital diameters of the class of simple diagonal type primitive permutation groups and provide bounds for these quantities. Further work along these lines by the author is under way for the other O'Nan-Scott classes.

Let us now describe primitive groups of simple diagonal type, following [22]. Let T be a non-abelian simple group, $\Gamma = \{1, \ldots, k\}$ and $W = Twr_{\Gamma}S_k$ with base group T^k . Now let $D = \{(a, \ldots, a) | a \in T\}$ be a diagonal subgroup and let Ω be the set of right cosets of D in T^k . Then T^k acts on Ω by right multiplication, S_k acts on Ω by permuting the components of the coset representatives, and $\alpha \in Aut(T)$ acts on Ω by $D(h_1, \ldots, h_k)^{\alpha} = D(h_1^{\alpha}, \ldots, h_k^{\alpha})$ for $h_i \in T$. The groups T^k , S_k and Aut(T) generate a group $N \cong T^k$. $(Out(T) \times S_k)$ and this is the normalizer of T^k in $Sym(\Omega)$. We say $G \leq Sym(\Omega)$ is a primitive permutation group of simple diagonal type if $T^k \leq G \leq N$ and G acts primitively on Ω , see [22]. Note $G = T^k.X$ where $X \leq Out(T) \times S_k$.

Write

$$D(k,T) = N < Sym(\Omega).$$

The result in [21, Lemma 5.1] states that the class of simple diagonal groups $G = T^k X \le D(k,T)$ is bounded only if k and the rank of T are both bounded. However no explicit bounds are obtained.

Before we list our results, we need a few definitions. Notice that if T is simple and $t \in T \setminus 1$, the conjugacy class t^T generates T. For a subset S of T and $r \in \mathbb{N}$ we write $S^r = \{s_1, \ldots, s_r | s_i \in S\}$.

Definition 1.1. Let T be a non-abelian simple group, and S a generating set of T. Define the width of T with respect to S, denoted $w_S(T)$, to be the minimal $k \in \mathbb{N}$ such that any element of T can be expressed as a product of at most k elements of S.

1. Let $t \in T \setminus 1$ and $C = t^T$ and put $c(T,t) := w_C(T)$. Define the conjugacy width of T to be

$$c(T) = \max_{t \in T \setminus 1} c(T, t).$$

2. For $t \in T \setminus 1$ let $C = t^{\pm T} = t^T \cup (t^{-1})^T$ and put $c_i(T,t) := w_C(T)$ Define the inverse conjugacy width of T to be

$$c_i(T) = \max_{t \in T \setminus 1} c_i(T, t).$$

3. Let X be such that $T \subseteq X \subseteq Aut(T)$, let $C = t^{\pm X} = \{t^{\pm \alpha} | \alpha \in X\}$ and put $c_X(T,t) := w_C(T)$ Define the X-conjugacy width of T to be

$$c_X(T) = \max_{t \in T \setminus 1} c_X(T, t).$$

When
$$X = Aut(T)$$
 write $c_A(T) = c_X(T)$.

These are related to the concept of covering numbers introduced in [1]. The covering number is the lowest number $r \in \mathbb{N}$ such that $C^r = T$ for all conjugacy classes C. This is denoted cn(T). Denote the lowest such number for a specific conjugacy class by cn(T,C). Note that this is an upper bound for the conjugacy width. We also note that

$$c_A(T) \le c_X(T) \le c_i(T) \le c(T) \le cn(T)$$
.

Note also that the number $c_i(T)$ was introduced and studied in [19], where it was called the conjugacy diameter of T.

We now state our results. Let $G=T^k.X\leq D(k,T)$ where $X\leq Out(T)\times S_k.$ We know from [22] that

$$D(k,T) = \{(\tau_1, \dots, \tau_k).\pi \mid \tau_i \in Aut(T), \pi \in S_m \text{ and all } \tau_i \text{ lie in the same } Inn(T) - \text{coset}\}.$$

Let $W = \{(\alpha, ..., \alpha).\pi | \alpha \in Aut(T), \pi \in S_k\}$ and put $D_A = W \cap G$, so $D_A = D.X$. Since $G = D_A T^k$ the action of G on Ω is equivalent to the action of G on G:

For
$$a \in T$$
 write $(a^k) = (a, \ldots, a) \in T^k$. For $t \in T \setminus 1$ define the orbital graph

$$\Gamma_0^t = \{D_A, D_A(1^{k-1}, t)\}^G.$$

The following theorem gives lower and upper bounds on the diameter of Γ_0^t . In the statement we abuse notation and denote $c_{X_0}(T)$ by $c_X(T)$, where X_0 is defined as follows. Let $G = T^k.X$ with $X \leq Out(T) \times S_k$. Let ρ be the projection of X onto Out(T) and let π be the canonical map $Aut(T) \to Out(T)$. Define $X_0 = \pi^{-1}(\rho(X)) \leq Aut(T)$.

Theorem (3.1,3.2). Let $G = T^k X$ as above. The diameter of Γ_0^t satisfies the bounds

$$\frac{1}{2}(k-1)c_X(T,t) + 1 \le diam(\Gamma_0^t) \le (k-1)c_i(T).$$

Note that from this it follows that $orbdiam(G) \ge \frac{1}{2}(k-1)c_X(T) + 1$.

Using these bounds, we provide the following classification of simple diagonal groups of small orbital diameter.

Theorem (5.1). Let G be a primitive group of simple diagonal type of the form $T^k.X \leq D(k,T)$.

- 1. If orbdiam(G) = 2, then k = 2 and $c_A(T) = 2$.
- 2. If orbdiam(G) = 3, then k = 2 and $c_A(T) \leq 3$.
- 3. If orbdiam(G) = 4, then one of the following holds:
 - (a) $k = 2 \text{ and } c_A(T) \le 4$
 - (b) k = 3 and $c_A(T) = 2$.

The following result gives a partial converse to parts 1, 2 and 3(a).

Lemma (3.3). 1. If $G = T^2$ then $orbdiam(G) = c_i(T)$.

2. If
$$G = D(2,T)$$
 then $orbdiam(G) = c_A(T)$.

We have not determined whether there are examples of groups in case 3(b) of Theorem 5.1 with orbital diameter 4. In view of Lemma 3.3 and Theorem 5.1, to classify all diagonal type groups with orbital diameters 2 and 3 we need to classify all non-abelian simple groups with X-conjugacy width 2, 3 for various X.

First we prove a general result on conjugacy widths and covering numbers of simple groups of Lie type. The upper bound in the following result was proved in [7]. By Lie rank we mean the rank of the corresponding simple algebraic group.

Theorem (4.1). There is a constant d such that

$$r-3 \le c_A(T) \le cn(T) \le dr$$

for all simple groups T of Lie type of Lie rank r.

It was proved in [3] that the only finite simple group with covering number 2 is the sporadic group J_1 . It turns out that J_1 is also the only finite simple group with (inverse) conjugacy width 2 (Proposition 4.3). However, there are infinitely many simple groups T with $c_A(T) = 2$.

Theorem (4.4). Let T be a finite simple group. Then $c_A(T) = 2$ if and only if $T \cong J_1$ or $T \cong PSL_2(q)$ with $q \equiv 1 \mod 4$ or $q = 2^{2m}$.

We also classify simple groups with any of the numbers cn(T), c(T), $c_i(T)$ or $c_A(T)$ equal to 3.

Theorem (4.5). Let T be a finite simple group.

- 1. c(T) = 3 if and only if T is isomorphic to one of the following:
 - $PSL_2(q)$ with q > 2

- $PSL_3(q)$
- $PSU_3(q)$ with 3|q+1, q>2
- ${}^{2}B_{2}(q)$ with q > 2
- ${}^2G_2(q)$ with q>3
- $G_2(3^n)$ with $n \ge 2$
- A_5, A_6, A_7
- M_{11} , M_{22} , M_{23} , M_{24} , J_3 , J_4 , M_{cl} , Ru, Ly, O'N, $Fi_{24'}$, Th, M.
- 2. cn(T) = 3 if and only if c(T) = 3.
- 3. $c_i(T) = 3$ if and only if c(T) = 3.
- 4. If $c_A(T) = 3$ then one of the following holds:
 - (a) c(T) = 3 and $T \neq PSL_2(q)$ with $q \equiv 1 \pmod{4}$ or $q = 2^{2m}$.
 - (b) $T \cong F_4(2^n)$.

Remark For part 3(b) we have not been able to determine whether $c_A(F_4(2^n)) = 3$.

The result in [21, Lemma 5.1] on simple diagonal actions states that for a class of primitive groups $T^k.X$ with bounded orbital diameter the Lie rank of T and k are bounded. Conversely, if T has bounded Lie rank, k is bounded and a few more criteria are met, then one obtains a bounded class. The proof of this result is model theoretic and includes no explicit bounds on the orbital diameter. The following result provides an explicit upper bound, giving rise to many bounded families of primitive groups of simple diagonal type. For simplicity we restrict to the class of simple diagonal groups of the form $T^k.S_k \leq D(k,T)$.

Theorem (6.1). Let $k \geq 3$ and let T be a simple group. Then

$$orbdiam(T^k.S_k) \le 24(k-1)c_i(T)^2.$$

2 Preliminary Results

In this section we include some background material that we use in the proof of our theorems.

We begin with a well-known a character theoretic result which gives us a method to find conjugacy widths.

Lemma 2.1. [2, Lemma 10.1] Let C_1, \ldots, C_d be conjugacy classes of a finite group G with representatives c_1, \ldots, c_d . For $z \in G$, the number of solutions $(x_1, \ldots, x_d) \in C_1 \times \cdots \times C_d$ to the equation $x_1 \ldots x_d = z$ is

$$\frac{\prod |C_i|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(c_1) \dots \chi(c_d) \chi(z^{-1})}{\chi(1)^{d-1}}.$$

Note the immediate corollary of this result.

Corollary 2.2. Let C and D be conjugacy classes of G with representatives c, d. If

$$\left| \sum_{\chi \in Irr(G) \setminus 1_G} \frac{\chi(c)^k \chi(d^{-1})}{\chi(1)^{k-1}} \right| < 1,$$

then $D \subseteq C^k$.

We include another result that we use in our classification of groups with small conjugacy widths, namely the classification of strongly real groups. A group is *strongly real* if and only if any of its elements can be expressed as a product of at most two involutions.

Theorem 2.3 ([8, 10, 12, 13, 17, 23, 25, 27, 28]). Let G be a non-abelian finite simple group. Then G is strongly real if and only if it is isomorphic to one of

- $PSp_{2n}(q)$ where $q \not\equiv 3 \pmod{4}$ and $n \ge 1$
- $P\Omega_{2n+1}(q)$ where $q \equiv 1 \pmod{4}$ and $n \geq 3$
- $P\Omega_9(q)$ where $q \equiv 3 \pmod{4}$
- $P\Omega_{4n}^+(q)$ where $q \not\equiv 3 \pmod{4}$ and $n \geq 3$
- $P\Omega_{4n}^{-}(q)$ where $n \geq 2$
- $P\Omega_8^+(q)$ or $^3D_4(q)$
- \bullet A_5 , A_6 , A_{10} , A_{14} , J_1 , J_2 .

The following result is on alternating groups from [4, Thm 2].

Theorem 2.4. [4] Let $n \ge 5$, l odd and $l \le n$. Then every permutation in the alternating group A_n is a product of three l – cycles if and only if either $\frac{n}{2} \le l$ or n = 7 and l = 3.

We conclude by listing some existing results on the covering numbers of some finite simple groups.

Theorem 2.5 ([3, 6, 20, 19]). 1. If $n \ge 3$, then $cn(PSL_n(q)) = n$ for $q \ge 4$. Also for q > 3, $cn(PSL_2(q)) = 3$.

- 2. $cn(A_n) = \lfloor \frac{n}{2} \rfloor$ for $n \geq 6$, and $cn(A_5) = 3$.
- 3. $cn(^{2}B_{2}(q)) = 3 \text{ for } q > 2.$
- 4. $cn(^2G_2(q)) = 3$ for q > 3.

Note that the covering numbers in Theorem 2.5 are upper bounds for the X-conjugacy widths, which we will be using later on.

3 The orbital diameter of a simple diagonal group

We begin with some notation. Let T be simple, $k \geq 2$ and $G = T^k.X \leq D(k,T)$ in a primitive simple diagonal action, where $X \leq Out(T) \times S_k$ as in Section 1. Let Γ be an orbital graph of G. Define $d_{\Gamma}(a,b)$ to be the distance between two vertices, a and b in Γ . Denote a path of length at most m between a and b in Γ by

$$a \frac{m}{-} b$$
.

Denote the element $(t,...,t) \in T^k$ as (t^k) .

Let $t \in T \setminus 1$ and Γ_0^t be the orbital graph $\{D_A, D_A(1^{k-1}, t)\}^G$. Recall $X_0 = \pi^{-1}((\rho(X)))$ where π is the canonical map $Aut(T) \to Out(T)$ and ρ is the projection of X to Out(T). For $g \in T$ we define the length of g with respect to t to be

$$l_t^X(g) = \min\{a : g = t^{\pm \alpha_1} \dots t^{\pm \alpha_a}, \text{ for some } \alpha_i \in X_0\}.$$

Recall we write an element of G as $(h_1, \ldots, h_k)\sigma_h$ where $h_i \in Aut(T)$, $\sigma_h \in S_k$, and identify T with Inn(T) when convenient.

The first result in this section is a lower bound for the diameter.

Theorem 3.1. Let $c = c_X(T,t)$. The diameter of Γ_0^t satisfies $diam(\Gamma_0^t) \geq M$ where

$$M = \begin{cases} \frac{1}{2}(k-1)c + 1 & k \text{ odd} \\ \frac{1}{2}kc & k \text{ even} \end{cases}$$

Proof. Note that $G = D_A T^k$ so every right coset of D_A has a coset representative in T^k .

<u>Claim 1</u> Every coset at distance m away from D_A is of the form

$$D_A(t_1,\ldots,t_k)$$

where $t_i \in T$ are such that $\sum_{i=1}^k l_t^X(t_i) \leq m$.

Proof of Claim 1 We prove Claim 1 by induction on m. We start with the base case m=1. Suppose $D_A(g_1,\ldots,g_k)$ is a neighbour of D_A where $g_i\in T$. Then there exists $h=(h_1,\ldots,h_k)\sigma_h\in G$ such that

$${D_A, D_A(1^{k-1}, t)}h = {D_A, D_A(g_1, \dots, g_k)}.$$

Hence either $h \in D_A$ or $D_A(1^{k-1},t)h = D_A$. If $h \in D_A$ then $h = (a^k)\sigma_h$, with $a \in Aut(T)$. Now

$$D_A(g_1, \dots, g_k) = D_A(1^{k-1}, t)h = D_Ah^{-1}(1^{k-1}, t)h = D_A(1^{k-1}, t^a)^{\sigma_h}$$

as required. If $D_A(1^{k-1},t)h=D_A$, then

$$D_A(g_1, \dots, g_k) = D_A h = D_A h^{-1}(1^{k-1}, t^{-1})h = D_A(1^{k-1}, t^{-h_k})^{\sigma_h}.$$

Hence Claim 1 holds for m = 1.

Now let $m \ge 2$. Let $D_A h$ be a coset at distance m from D_A . Then $D_A h$ is a neighbour of a coset at distance m-1 and by the induction hypothesis this coset has form

$$D_A(x_1,\ldots,x_k)$$

where $x_i \in T$ and $\sum_{i=1}^k l_t^X(x_i) \le m-1$. There is an edge between $D_A(x_1, \dots, x_k)$ and D_Ah . Hence there is $f \in G$ such that

$${D_A, D_A(1^{k-1}, t^{\pm a})^{\sigma}}f = {D_A(x_1, \dots, x_k), D_Ah}$$

with $f = (f_1, \ldots, f_k)\pi$ where $f_i \in Aut(T)$ and $\pi \in S_k$. Again either $D_A f = D_A(x_1, \ldots, x_k)$ or $D_A(1^{k-1}, t^{\pm a})^{\sigma} f = D_A(x_1, \ldots, x_k)$. If $D_A f = D_A(x_1, \ldots, x_k)$ then $f_i x_i^{-1} = f_j x_j^{-1}$ for all i, j, so

$$D_A h = D_A (1^{k-1}, t^{\pm a})^{\sigma} f = D_A (x_1, \dots, x_k) f^{-1} (1^{k-1}, t^{\pm a})^{\sigma} f = D_A (x_1, \dots, x_k) (1^{k-1}, t^{\pm af_k})^{\sigma \pi}$$

and Claim 1 follows. When $D_A(1^{k-1}, t^{\pm a})^{\sigma} f = D_A(x_1, \dots, x_k)$ we obtain the conclusion in a similar way.

<u>Claim 2</u> There exist $h_1, \ldots, h_k \in T$ such that $d_{\Gamma_0^t}(D_A, D_A(h_1, \ldots, h_k)) \geq M$, where M is as in the statement of the Theorem.

Proof of Claim 2 By Claim 1 it suffices to find $h_1, \ldots, h_k \in T$ such that $\min_{g \in T} \sum_{i=1}^k l_t^X(gh_i) \ge M$. Let $h, h' \in T$ with $h \ne h'$ and $l_t^X(h) = l_t^X(h') = c$. Define

$$(h_1, \dots, h_k) = \begin{cases} (1, h, 1, h, \dots, 1, h) & k \text{ even} \\ (h, 1, h, 1, \dots, h, 1, h') & k \text{ odd} \end{cases}.$$

Then

$$l_t^X(h_1^{-1}h_2) = l_t^X(h_2^{-1}h_3) = \dots = l_t^X(h_{k-1}^{-1}h_k) = c \text{ and } l_t^X(h_1^{-1}h_k) \ge 1.$$

Note that $l_t^X(xy) \leq l_t^X(x) + l_t^X(y) = l_t^X(x^{-1}) + l_t^X(y)$ for all $x, y \in T$, so it follows that for all $i \geq 1$ and any $g \in T$

$$c = l_t^X(h_i^{-1}h_{i+1})) \le l_t^X(gh_i) + l_t^X(gh_{i+1})$$

and

$$1 \le l_t^X(h_1^{-1}h_k) \le l_t^X(gh_1) + l_t^X(gh_k).$$

Summing these up gives

$$2\sum_{i=1}^{k} l_t^X(gh_i) \ge \sum_{i=1}^{k-1} l_t^X(h_i^{-1}h_{i+1}) + l(h_1^{-1}h_k) \ge (k-1)c + 1.$$

The result now follows for k odd, and for k even we have $l_t^X(h_1^{-1}h_k) = c$, so we get $\frac{k}{2}c$ as a lower bound.

The following result is an upper bound.

Lemma 3.2. We have $diam(\Gamma_0^t) \leq (k-1)c_i(T)$.

Proof. Claim 1 There exist $\alpha_i \in Aut(T)$ such that $D_A((1^{i-1}, (t^{\alpha_i})^{\pm a}, 1^{k-i}))$ is adjacent to D_A for all $a \in T$ and all $1 \le i \le k$.

<u>Proof of Claim 1</u> This is clear for k=2 so we can assume $k\geq 3$. We have

$$D_A \xrightarrow{1} D_A(1^{k-1}, t)$$

by definition of Γ_0 . Apply $(a^k) \in T^k$ to this to get

$$D_A \frac{1}{---} D_A(1^{k-1}, t^a).$$

As G is primitive, X acts transitively on the symbols $1, \ldots, k$, so for $1 \le i \le k$ there is an element $(\alpha_i, \ldots, \alpha_i).\sigma_i \in D_A$ such that

$$D_A(1^{k-1}, t)(\alpha_i, \dots, \alpha_i).\sigma_i = D_A(1^{i-1}, t^{\alpha_i}, 1^{k-i}).$$

Applying (a^k) gives

$$D_A = \frac{1}{1} D_A(1^{i-1}, (t^{\alpha_i})^a, 1^{k-i}).$$

Furthermore, applying $(1^{i-1}, (t^{\alpha_i})^{-a}, 1^{k-i})$ to this gives

$$D_A(1^{i-1}, (t^{\alpha_i})^{-a}, 1^{k-i}) - D_A$$

and as a was arbitrary Claim 1 follows.

<u>Claim 2</u> Let $h_i \in T$ $(1 \le i \le k)$ and let $a \in T$. Then $D_A(h_1, \ldots, h_k)$ is adjacent to $D_A(h_1, \ldots, h_{i-1}, (t^{\alpha_i})^{\pm a}h_i, h_{i+1}, \ldots, h_k)$ for $1 \le i \le k$.

<u>Proof of Claim 2</u> Apply (h_1, \ldots, h_k) to

$$D_A = \frac{1}{1} D_A(1^{i-1}, (t^{\alpha_i})^{\pm a}, 1^{k-i}).$$

Claim 3 Let $c = c_i(T)$. For any $h_i, \ldots, h_k \in T$ and $1 \le i \le k$,

$$D_A(h_1,\ldots,h_{i-1},1,h_{i+1},\ldots,h_k) - \frac{c}{} D_A(h_1,\ldots,h_k).$$

<u>Proof of Claim 3</u> We know by definition of c that h_i can be expressed as a product of at most c conjugates of $(t^{\alpha_i})^{\pm 1}$, so

$$h_i = (t^{\alpha_i})^{\pm a_1} \dots (t^{\alpha_i})^{\pm a_c}$$

for some $a_i \in T$. Hence by repeatedly applying Claim 2

$$D_A(h_1,\ldots,h_{i-1},1,h_{i+1},\ldots,h_k) = C D_A(h_1,\ldots,h_{i-1},(t^{\alpha_i})^{\pm a_1}\ldots(t^{\alpha_i})^{\pm a_c},h_{i+1},\ldots,h_k)$$

so Claim 3 follows.

Using Claim 3 repeatedly we have the following path

$$D_A \xrightarrow{c} D_A(1, h_2, 1^{k-2}) \xrightarrow{c} D_A(1, h_2, h_3, 1^{k-3}) \dots \xrightarrow{c} D_A(1, h_2, \dots, h_k).$$

As $(1, h_2, \ldots, h_k)$ represents an arbitrary coset, the result follows.

We have an exact result for the orbital diameter for the case when $G = T^2$ or G = D(2, T).

Lemma 3.3. 1. If $G = T^2$ then $orbdiam(G) = c_i(T)$.

2. If
$$G = D(2,T)$$
 then $orbdiam(G) = c_A(T)$.

Proof. 1. We first notice that in the case of k=2 all orbital graphs are of the form Γ_0^t . If $G=T^2$ then $X_0=T$, so $c_X(T)=c_i(T)$. Hence the bounds from Theorem 3.1 and Lemma 3.2 coincide, and the result follows.

2. Consider $G = D(2,T) \cong T^2$.($Out(T) \times S_2$). In this case $X_0 \cong Aut(T)$ and also $D_A \cong Aut(T) \times S_2$. Now Theorem 3.1 gives $orbdiam(G) \geq c_A(T)$. We will show the other direction of this inequality. Let $t \in T \setminus 1$. Consider the orbital graph $\Gamma = \{D_A, D_A(1,t)\}^G$. Now

$$D_A - D_A(1, t^{\pm 1})$$

are edges in the graph. For all $a \in Aut(T)$ apply $(a, a) \in G$ to these to get

$$D_A - D_A(1, t^{\pm a}).$$

Now we can construct a path between D_A and any arbitrary coset $D_A(1,h)$ where $h = t^{\pm a_1} \dots t^{\pm a_c}$ with $c = c_A(T)$ such that $D_A \stackrel{c}{\longrightarrow} D_A(1,h)$. This shows that $\operatorname{orbdiam}(G) \leq c_A(T)$ and the result now follows.

4 Conjugacy Widths of Finite Simple Groups

4.1 Bounds on the Conjugacy Width and the Covering Number

In this section we prove bounds on the conjugacy widths c(T), $c_i(T)$ and $c_A(T)$ for simple groups as stated in the Introduction. We start with a result on conjugacy widths for simple groups of Lie type. In the following result, the upper bound is proved in [7].

Theorem 4.1. There is a constant d such that

$$r-3 \le c_A(T) \le cn(T) \le dr$$

for all simple groups T of Lie type of Lie rank r. More precisely, $c_A(T) \ge C_T$ where C_T is as in Table 1

T		C_T
$PSL_n(q)$	$(n,q) \neq (2,2) or (2,3)$	n
$PSU_n(q)$	$n \ge 3$	n
$PSp_n(q)$	$n \ge 4, (n,q) \ne (4,2)$	n
$PSp_4(2)'$		3
$P\Omega_n^{\epsilon}(q)$	$n \ge 7$	$\lfloor \frac{n}{2} \rfloor$
${}^{2}B_{2}(q)$	q > 2	$\bar{3}$
$^{2}G_{2}(q)$	q > 3	3
$G_2(3^n)$		3
$G_2(q)$	$3 \nmid q$	4
$^{3}D_{4}(q)$		4
$F_4(2^n)$		3
$F_4(q)$	$2 \nmid q$	4
${}^{2}F_{4}(q)$	q > 2	4
${}^{2}F_{4}(2)'$		4
$E_6^{\epsilon}(q)$		4
$E_7(q)$		4
$E_8(q)$		5

Table 1: Lower bounds

Note that the inequality $c_A(T) \le c_i(T) \le c(T) \le cn(T)$ is immediate from the definitions. Hence establishing a lower bound for $c_A(T)$ immediately gives a lower bound for $c_i(T)$, c(T) and cn(T).

4.1.1 The proof of the lower bound in Theorem 4.1

Let $V = V_n(q)$. For $x \in PGL(V)$ let $\tilde{x} \in GL(V)$ be a preimage of x and define

$$\nu(x) = n - \max_{\lambda \in \mathbb{F}_q^*} dim C_V(\lambda \widetilde{x}),$$

the minimal codimension of an \mathbb{F}_q -eigenspace of \widetilde{x} . For a subset $S \in PGL(V)$ define

$$\nu(S) = \max_{s \in S} \nu(s).$$

Proposition 4.2. Let $T \leq PGL(V) \cong PGL_n(q)$ be a simple group and let X be a group such that $InnT \leq X \leq AutT$. Let S_0 be a non-empty X-invariant subset of $T \setminus 1$ such that $\nu(s) = \nu(s')$ for all $s, s' \in S_0$, and let $S = S_0 \cup S_0^{-1}$. Then

$$c_X(T) \ge \frac{\nu(T)}{\nu(S)}.$$

Proof. Note that S is a union of X-conjugacy classes. Choose $s \in S$ and $t \in T$ such that $\nu(s) = \nu(S)$ and $\nu(t) = \nu(T)$. Put $C = s^X$. By hypothesis, $\nu(y) = \nu(S)$ for all $y \in C$. Let $k = n - \nu(S)$ and $r = n - \nu(T)$ so that $k = \max_{\lambda \in \mathbb{F}_q^*} dim C_V(\lambda \widetilde{s})$ and $r = \max_{\lambda \in \mathbb{F}_q^*} dim C_V(\lambda \widetilde{t})$. Let $s_1, \ldots, s_l \in C$. Using elementary linear algebra we see that

$$\max_{\lambda_i \in \mathbb{F}_q^{\star}} dim C_V(\lambda_1 \widetilde{s_1}, \dots, \lambda_l \widetilde{s_l}) \ge lk - (l-1)n.$$

Suppose that $w \in \mathbb{N}$ is minimal such that t can be expressed as the product of w elements of C, so $w \leq c_X(T)$. Hence

$$r \ge wk - (w-1)n$$
.

Rearranging gives

$$c_X(T) \ge w \ge \frac{n-r}{n-k} = \frac{\nu(T)}{\nu(S)}.$$

Now we prove the theorem.

Proof of Theorem 4.1. Case 1, Classical Groups

Let T be a classical simple group with natural module $V = V_n(q)$. Define S to be the set of long root elements in T and let X = Aut(T). Then S is X-invariant, provided $T \neq PSp_4(2^a)$.

Suppose first that T is $PSL_n(q)$, $PSp_n(q)$ or $PSU_n(q^{1/2})$ and $T \neq PSp_4(2^a)$. Then the long root elements of T are transvections, which have fixed space on V of dimension n-1, so $\nu(S)=1$. We claim that

$$\nu(T) = n.$$

This can be seen as follows. Provided $T \neq PSU_n(q^{1/2})$ with n even, by [14] T has a Singer element y (i.e. an element such that $\langle y \rangle$ is irreducible on V) and clearly $\nu(y) = n$. And if $T = PSU_n(q^{1/2})$ with $n = 2d \geq 4$, then $SU_n(q^{1/2})$ has a subgroup $SL_d(q)$, and a Singer element of this also satisfies $\nu(y) = n$. Hence by Proposition 4.2,

$$c_A(T) \ge \frac{\nu(T)}{\nu(S)} = n.$$

Next consider $T = PSp_4(2^a)$ with a > 1. Let \widetilde{S} be the set of all long or short root elements of T. Then \widetilde{S} in invariant under Aut(T). The generic character table of T is in the computer package Chevie [11]. This also contains a function, called ClassMult, which calculates the sum in Lemma 2.1. Using this it can be checked that $\widetilde{S}^3 \cup \widetilde{S}^2 \cup \widetilde{S} \cup 1 \neq T$. Hence $c_A(T) \geq 4$, as required.

Finally suppose $T=P\Omega_n^\epsilon(q)$. For $n\leq 6$, T is isomorphic to one of the groups we have already covered, so assume $n\geq 7$. The long root elements of T have fixed point space of dimension n-2 on V, so $\nu(S)=2$. If n is even, then T has an element y such that $\nu(y)=n$: for $\epsilon=-$ take y to be a Singer element of $\Omega_n^-(q)$ [14]; and for $\epsilon=+$, take y to be a Singer element of a subgroup $SL_{\frac{n}{2}}(q)$ of $\Omega_n^+(q)$. If n is odd, then T has an element y such that $\nu(y)=n-1$: for example choose a Singer element in a subgroup $\Omega_{n-1}^-(q)$. We conclude that

$$u(T) \ge \begin{cases} n & n \text{ even} \\ n-1 & n \text{ odd} \end{cases}$$

Now the conclusion follows from Proposition 4.2.

Case 2, Exceptional Groups There are only two families of exceptional groups whose covering number is known; the Suzuki groups, ${}^{2}B_{2}(q)$ and the small Ree groups ${}^{2}G_{2}(q)$ both have covering number 3 by Theorem 2.5. By Theorem 2.3 they are not strongly real, so in fact

$$c_A(^2B_2(q)) = c_A(^2G_2(q)) = c_i(^2B_2(q)) = c_i(^2G_2(q)) = c(^2B_2(q)) = c(^2G_2(q)) = 3.$$

Now consider $T = E_8(q)$, $E_7(q)$ or $E_6^{\epsilon}(q)$. Let $V = V_n(q)$ be the adjoint module for T, of dimension 248, 133 or 78, respectively, and let S be the set of long root elements of T. Then S is invariant under Aut(T). From Tables 9, 8 and 6 of [18], we see that $\nu(S)$ is as in the table:

$$\begin{array}{c|cccc} T & E_8(q) & E_7(q) & E_6^{\epsilon}(q) \\ \hline \nu(S) & 58 & 34 & 22 \\ \end{array}.$$

Also T has regular unipotent elements y, and these have fixed point spaces of dimension 8, 7 or 6, respectively. Hence $\nu(T) \geq 240$, 126 or 72, and the bound $c_A(T) \geq \frac{\nu(T)}{\nu(S)}$ gives the conclusion of the theorem.

Next consider $T=F_4(q), q$ odd. Again let S be the set of long root elements, which is invariant under Aut(T), and consider the action on the 26-dimensional module $V=V_{26}(q)$, as given in [18, Table 3]. We see that $\nu(S)=6$ while regular unipotent elements show that $\nu(T)\geq 24$. Hence $c_A(T)\geq 4$.

Next we claim that $c_A(T) \ge 4$ for $T = {}^3D_4(q)$, ${}^2F_4(q)$ (q > 2), ${}^2F_4(2)'$ or $G_2(q)$ $(q \ne 3^a)$. The character tables of these are available in Chevie and GAP. Using Lemma 2.1 we can compute that for a root element, r, $c_A(T,r) \ge 4$.

The last groups remaining to consider are $T = F_4(2^a)$ and $G_2(3^a)$. These groups are not strongly real by Theorem 2.3, so $c_A(T) \geq 3$ for these.

This completes the proof of Theorem 4.1.

4.2 Conjugacy Width 2

There has been some interest around classifying groups with a small covering numbers. In [3, Thm 2.1] it is proven that the only finite simple group with covering number 2 is J_1 . It turns out that the same is true for the (inverse) conjugacy width.

Proposition 4.3. For a simple group T the following are equivalent;

- (i) cn(T) = 2
- (ii) $c_i(T) = 2$
- (iii) c(T) = 2
- (iv) $T \cong J_1$.

Proof. Clearly $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$ so it remains to prove $(iii) \Rightarrow (iv)$. So suppose c(T) = 2. By Theorems 2.3 and 4.1 we get that $T \cong PSL_2(q)$, $q \not\equiv 3 \mod 4$, A_{10} , A_{14} , J_1 or J_2 . The proof of [3, Thm 4.2 (a)] shows that there is a conjugacy class $C \subseteq PSL_2(q)$ such that $PSL_2(q) \not\equiv C^2 \cup C \cup 1$, so $c(PSL_2(q)) \geq 3$. Similarly, products of at most two 3-cycles cannot express all elements in A_{10} and A_{14} , so their conjugacy width is greater than 3, and using GAP we can find an element r in J_2 such that $c(T, r) \geq 3$. Hence $T = J_1$ and the result follows. \square

However, the next result shows that there is an infinite family of finite simple groups, such that the conjugacy width is not 2 but the automorphism conjugacy width is 2.

Theorem 4.4. Let T be a finite simple group. Then $c_A(T) = 2$ if and only if either $T \cong J_1$ or $T \cong PSL_2(q)$ with $q \equiv 1 \mod 4$ or $q = 2^{2m}$.

Proof. We begin by finding $c_A(PSL_2(q))$ for all $q \geq 5$. We know by Proposition 2.5 that $c(PSL_2(q)) = 3$. If $q \equiv 3 \mod 4$, then Theorem 2.3 implies that the involution class has conjugacy width greater than 2, hence $c_A(PSL_2(q)) = 3$. If q is a power of 2, then by [3, Thm 4.2 (a)] the only conjugacy classes with conjugacy width 3 are those denoted by R_j in [3]; these have class representatives (b^j) where b is an element of order q+1 and $1 \leq j \leq \frac{q}{2}$. In fact, [3, Thm 4.2 (a)] gives that $R_j^2 = G \setminus C_2$ where C_2 are the root elements. For $q = 2^{2m+1}$, the class $R_{\frac{q+1}{3}}$ is fixed by all outer automorphisms, so $c_A(PSL_2(2^{2m+1}) = 3$. For $q = 2^{2m}$, there is no class of type R_j which is fixed by all outer automorphisms. Using Lemma 2.1 and Chevie[11] we can show that $C_2 \subseteq R_j R_l$, where l = 2j if $2j \leq \frac{q}{2}$ and l = q + 1 - 2j if $2j > \frac{q}{2}$, so $c_A(PSL_2(2^{2m})) = 2$. For $PSL_2(q)$ with $q \equiv 1 \mod 4$ we know from [3] that the only classes with conjugacy width equal to 3 are the two classes of root elements, and $PSL_2(q)$ has an outer automorphism that interchanges these two classes. Using Lemma 2.1 we can show that every element can be expressed as a product of two root elements, hence $c_A(PSL_2(q)) = 2$. This proves the right to left implication of the theorem.

For the converse, suppose $c_A(T)=2$. Then any element of T can be expressed as a product of at most two involutions, so T is strongly real, hence is given by Theorem 2.3. If T is a simple group of Lie type, then by Theorems 4.1 and 2.3 we have that $T\cong PSL_2(q)$ with $q\not\equiv 3$ mod 4. We proved above that $c_A(PSL_2(q))=3$ for $q=2^{2m+1}$, so $q\equiv 1\mod 4$ or $q=2^{2m}$. If T is not of Lie type then by Theorem 2.3, $T\cong A_{10}$, A_{14} , A_{14} or A_{14} . All automorphisms of A_{10} and A_{14} fix the class of 3-cycles, so their automorphism conjugacy width is not 2. Looking at the character table of A_{10} and using Lemma 2.1 we conclude that $C(A_{10})=C_{11}$ and $C(A_{11})=C_{11}$.

4.3 Conjugacy Width and Covering Number 3

The next result gives a similar classification of groups with conjugacy width 3.

Theorem 4.5. Let T be a finite simple group.

- 1. c(T) = 3 if and only if T is isomorphic to one of the following:
 - $PSL_2(q)$ with q > 2
 - $PSL_3(q)$
 - $PSU_3(q)$ with 3|q+1, q>2
 - ${}^{2}B_{2}(q)$ with q>2
 - ${}^2G_2(q)$ with q>3
 - $G_2(3^n)$ with $n \ge 2$
 - A_5, A_6, A_7
 - M_{11} , M_{22} , M_{23} , M_{24} , J_3 , J_4 , M_{cl} , Ru, Ly, O'N, $Fi_{24'}$, Th, M.
- 2. cn(T) = 3 if and only if c(T) = 3.
- 3. $c_i(T) = 3$ if and only if c(T) = 3.

4. If $c_A(T) = 3$ then one of the following holds:

- (a) c(T) = 3 and $T \neq PSL_2(q)$ with $q \equiv 1 \pmod{4}$ or $q = 2^{2m}$.
- (b) $T \cong F_4(2^n)$.

Proof. Case 1, Alternating Groups

By Theorem 2.5 the only alternating groups with covering number 3 are A_5 , A_6 and A_7 . Also by Theorem 2.4, $c_A(A_n) \ge 4$ for $n \ge 8$. Finally $c(A_n) = c_i(A_n) = 3$ for n = 5, 6 and 7 while $c_A(A_n) = 2,2,3$ respectively.

Case 2, Groups of Lie type

If T is a group of Lie type such that $c_A(T) \leq 3$ then by Theorem 4.1, T is isomorphic to one of the following:

$$PSL_2(q), \ PSL_3(q), \ PSU_3(q), \ P\Omega_7(q), \ ^2B_2(q), \ ^2G_2(q), \ G_2(3^a), \ F_4(2^a).$$

By Theorem 2.5, cn(T)=3 for $T=PSL_2(q)$, $PSL_3(q)$ with $q\geq 4$, $^2B_2(q)$ and $^2G_2(q)$; by Proposition 4.3 for these groups $c(T)=c_i(T)=3$; and by Theorem 4.4, $c_A(T)=3$ apart from $PSL_2(q)$ with $q\equiv 1\ (mod\ 4)$ or $q=2^{2m}$. For $T=PSL_3(2)$ or $PSL_3(3)$ we can show using GAP that $cn(T)=c(T)=c_i(T)=c_A(T)=3$.

By [24, Cor 1.9] for $T = PSU_3(q)$ (q > 2)

$$cn(T) = \begin{cases} 3 & 3|q+1\\ 4 & 3 \nmid q+1 \end{cases}$$

Hence if 3|q+1 then also $c(T)=c_i(T)=c_A(T)=3$. For $3\nmid q+1$ it is shown in [24, Table 2] that for a transvection $t\in T$, c(T,t)=4 and hence $c_A(T)=4$ and $c_i(T)=4$.

Next consider $T = P\Omega_7(q)$, q odd. We claim that $c_A(T) \geq 4$. To prove this let V be the 8-dimensional spin representation of $Spin_7(q)$. Let S be the set of long root elements of T, which is invariant under Aut(T). Using triality we can show that the long root elements of $Spin_7(q)$ act on V as long root elements of $\Omega_8^+(q)$, hence they fix a 6-dimensional subspace of V pointwise, and so $\nu(S) = 2$. We know that $Spin_6^+(q) \cong SL_4(q)$ embeds into $Spin_7(q)$. Put V_4 as the natural module of $SL_4(q)$. Then by [16, 2.2.8] $SL_4(q)$ acts on V as on $V_4 \oplus V_4^*$. Take R to be a Singer cycle in $SL_4(q)$. Now R on V_4^* is also Singer cycle, and hence $\nu(R) = 8$. Now by Proposition 4.2, $c_A(T) \geq 4$.

Consider $T=G_2(3^a)$. In the case of a=1, the character table is in GAP, so using Lemma 2.1 we find that $c_A(G_2(3))=c_i(G_2(3))=c(G_2(3))=4$. For $a\geq 2$, even though the generic character table is available in Chevie, solving this problem is not possible using only Chevie. This is due to the fact that when running the ClassMult function, Chevie outputs values for the character sum in Lemma 2.1 together with a set of many "possible exceptions" which give conditions under which this value might not hold. For some classes it is possible to deal with these exceptions, and for others it is not, so we used another method of solution. For the classes that are possible to handle with Chevie we used Chevie, for the rest we used Corollary 2.2. To describe this, we use the notation for conjugacy classes and characters of T given in [9]. We partition the non-trivial irreducible characters into sets Δ_k , where the degree of the characters in Δ_k is a polynomial in q of degree k. We see from [9] that $Irr(T) \setminus 1_T = \Delta_4 \cup \Delta_5 \cup \Delta_6$ and we get $|\Delta_4| = 1$, $|\Delta_5| = 2q - 7$ and $|\Delta_5| = q^2 + 13$. For $\chi \in \Delta_4$ we have $\chi(1) \geq q^4 + q^2 + 1$, for $\chi \in \Delta_5$ we have $\chi(1) \geq \frac{1}{6}q(q-1)^2(q^2-q+1)$ and for $\chi \in \Delta_6$ we have $\chi(1) \geq q(q^2-q+1)(q^3-1)$. In Table 2 we give upper bounds for $|\chi(x)|$ for all non-identity conjugacy classes x. The first column lists the class representatives in the notation of [9]. The other three columns give upper bounds for $|\chi(x)|$

	Δ_4	Δ_5	Δ_6
A_2	$q^2 + 1$	$(q+1)(q^2+1)$	$(q+1)(q^2+q+1)$
A_{31}	$q^2 + 1$	$(q+1)(q^2+1)$	$(q+1)(q^2+q+1)$
A_{41}	1	$\frac{1}{3}q(q+1)$	2q + 1
A_{42}	1	$ \frac{\frac{1}{2}q(q+1)}{\frac{4}{3}q} $ $ \frac{\frac{4}{3}q}{\frac{4}{3}q} $	2q + 1
A_{51}	1	$\frac{4}{3}q$	1
A_{52}	1	$\frac{4}{3}q$	1
A_{53}	1	$\frac{4}{3}q$	1
B_1	1+2q	$q^2 + 3q + 2$	$3(q+1)^2$
B_2	1+q	2q+2	3(q+1)
B_3	1+q	2q + 2	3(q+1)
B_4	1	2q + 2	3
B_5	1	2q + 2	3
C_{11}	2+q	2q + 2	3(q+1)
C_{12}	2	2	3
C_{21}	2+q	2q + 2	3(q+1)
C_{22}	2	2	3
D_{11}	2+q	2q + 2	3(q+1)
D_{12}	2	2	3
D_{21}	2+q	2q + 2	3(q+1)
D_{22}	2	2	3
$E_1(i,j)$	3	4	6
$E_2(i)$	1	4	4
$E_3(i)$	1	4	4
$E_4(i,j)$	3	4	6
$E_5(i)$	0	4	6
$E_6(i)$	0	4	6

Table 2: Bounds on character values

for $\chi \in \Delta_4$, Δ_5 and Δ_6 , respectively. We use these bounds to bound the sum in Corollary 2.2 with k=3. We find that this sum is less than 1 for all pairs of conjugacy classes (C,D) with the following exceptions;

For these exceptions we can show $D \subseteq C^3$ using Chevie and Lemma 2.1, with the exception of showing $E_2(i) \subseteq B_1^3$, $E_3(i) \subseteq B_1^3$, $E_3(i) \subseteq E_1^3$, and $E_3(i) \subseteq E_1^3$. It follows by Theorem 4.1, $E_3(i) \subseteq E_1^3$, $E_3($

Finally we need to consider $F_4(2^n)$. We can use the argument given for $F_4(q)$, q odd in the proof of Theorem 4.1 to conclude that $c_i(T) \geq 4$. We have not been able to determine whether the automorphism conjugacy width of $F_4(2^n)$ is 3.

Case 3, Sporadic Groups

Zisser[29] and Karni[15] showed that the only sporadic simple groups with covering number 3 are the ones listed in Theorem 4.5. Using character tables in GAP and Lemma 2.1 we computed c(T), $c_i(T)$ and $c_A(T)$ for all sporadic groups and obtained the same list.

5 Simple diagonal groups with small orbital diameter

In this section we prove the following result, which classifies the primitive permutation groups of simple diagonal type with orbital diameter at most 4. Adopt the notation of the introduction: T is simple, $k \geq 2$, $G = T^k.X \leq D(k,T)$ with $X \leq Out(T) \times S_k$ and G acts primitively on $\Omega = (G:D_A)$.

Theorem 5.1. Let G be a primitive group of simple diagonal type of the form $T^k.X \leq D(k,T)$.

- 1. If orbdiam(G) = 2, then k = 2 and $c_A(T) = 2$.
- 2. If orbdiam(G) = 3, then k = 2 and $c_A(T) \leq 3$.
- 3. If orbdiam(G) = 4, then one of the following holds:
 - (a) $k = 2 \text{ and } c_A(T) \le 4$
 - (b) k = 3 and $c_A(T) = 2$.

Remark Note that Lemma 3.3 is a partial converse of this result, as it shows that there are some families of groups of simple diagonal type with k = 2 such that $\operatorname{orbdiam}(G) = c_A(T)$, namely G = D(2,T).

For the proof of this theorem need some preliminary lemmas.

Note that it follows from Theorem 3.1 that k is a lower bound for the orbital diameter. In fact, for $c_A(T) = 2$ it is a strict lower bound;

Lemma 5.2. Let $c_A(T) = 2$, $k \ge 3$ and $G = T^k X \le D(k,T)$. Then $orbdiam(G) \ge k+1$.

Proof. Recall our definition of the length of an element of T with respect to $t \in T \setminus 1$: for $g \in T$,

$$l_t^A(g) = \min\{a : g = t^{\pm \alpha_1} \dots t^{\pm \alpha_a}, \alpha_i \in Aut(T)\}.$$

Since $c_A(T)=2$, we have $l_t^A(g)\leq 2$ and $l_t^A(g)=1$ if and only if $g\in t^{\pm Aut(T)}$. Also by Proposition 4.4, $T=J_1$ or $PSL_2(q)$ with $q\equiv 1\ (mod\ 4)$ or $q=2^{2m}$.

Let t be an involution in T and define $\Gamma_0 = \{D_A, D_A(1^{k-1}, t)\}^G$.

Claim 1 We can choose $x, y, z \in T$ such that

$$l_t^A(x) = l_t^A(y) = l_t^A(x^{-1}y) = l_t^A(t^ax) = 2 \ \forall a \in T$$

and

$$l_t^A(z) = l_t^A(x^{-1}z) = l_t^A(y^{-1}z) = 2.$$

Proof of Claim 1 For $T=J_1$ and $PSL_2(4)$ this can be verified in GAP. Now let $T=PSL_2(q)$, with $q\geq 8$ and $q\equiv 1\ (mod\ 4)$ or $q=2^{2m}$. Let $Z=Z(SL_2(q))$. Define $x,y,z\in T$ to be $\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} Z$, $\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix} Z$ and $\begin{pmatrix} 0 & 1 \\ 1 & \omega \end{pmatrix} Z$, respectively, where $\omega\in \mathbb{F}_q^\star$ has order q-1. To see that these elements satisfy Claim 1 consider the product of an involution in $PSL_2(q)$ with x or y. An involution in $PSL_2(q)$ is of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} Z$, and

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix} Z = \begin{pmatrix} a\omega^i & b \\ c & -a\omega^{-i} \end{pmatrix} Z,$$

which is not an involution. The other assertions in Claim 1 are easily verified.

Claim 2 There is a coset $D_A(m_1, \ldots, m_k)$ such that $d_{\Gamma_0}(D_A, D_A(m_1, \ldots, m_k)) \ge k+1$.

<u>Proof of Claim 2</u> Assume first that k is odd and let and let $x, y, z \in T$ be as in Claim 1.

Set

$$(m_1,\ldots,m_k)=(y,1,x,1,x,1,\ldots,1,x).$$

Suppose that $d_{\Gamma_0}(D_A, D_A(m_1, \dots, m_k)) \leq k$. Then by Claim 1 in the proof of Theorem 3.1, there exists $g \in T$ such that

$$\sum_{i=1}^{k} l_t^A(gm_i) \le k. \tag{1}$$

By Claim 1,

$$2k = \sum_{i=1}^{k-1} l_t^A(m_i^{-1}m_{i+1}) + l_t^A(m_1^{-1}m_k)$$

and recall that

$$l_t^A(m_i^{-1}m_{i+1}) \le l_t^A(gm_{i+1}) + l_t^A(gm_i).$$

Putting these together we get

$$2k = \sum_{i=1}^{k-1} l_t^A(m_i^{-1}m_{i+1}) + l_t^A(m_1^{-1}m_k) \le 2\sum_{i=1}^k l_t^A(gm_i) = 2\left(l_t^A(gy) + \frac{k-1}{2}l_t^A(gx) + \frac{k-1}{2}l_t^A(g)\right) \le 2k,$$

where the last inequality follows from (1). This tells us that $\sum_{i=1}^k l_t^A(gm_i) = k$. If $l_t^A(gm_i) = 1$ for all i, then $l_t^A(g) = l_t^A(gx) = l_t^A(gy) = 1$ which is a contradiction by Claim 1. Hence there exist j such that $l_t^A(gm_j) = 0$ and so $g = 1, x^{-1}$ or y^{-1} . If $g = x^{-1}$ then $\frac{k-1}{2}l_t^A(g) = k-1$ and $l_t^A(x^{-1}y) = 2$, so $\sum_{i=1}^k l_t^A(gm_i) \ge k+1$. If $g = y^{-1}$ then $\frac{k-1}{2}l_t^A(g) = k-1$ and $\frac{k-1}{2}l_t^A(y^{-1}x) = k-1$, so $\sum_{i=1}^k l_t^A(gm_i) \ge 2k-2 \ge k+1$. And if g = 1 then $\frac{k-1}{2}l_t^A(x) = k-1$ and $l_t^A(y) = 2$, so $\sum_{i=1}^k l_t^A(gm_i) \ge k+1$. These contradictions prove Claim 2 for the case where k is odd.

Now assume k is even. In this case let $(m_1, \ldots, m_k) = (y, z, x, 1, x, 1, \ldots, 1)$. Suppose that $d_{\Gamma_0}(D_A, D_A(m_1, \ldots, m_k)) \leq k$. As before, we deduce that there exists $g \in T$ such that $\sum_{i=1}^k l_t^X(gm_i) = k$ and we reach a contradiction in the same way. These contradictions establish Claim 2 and so the orbital diameter of G is bounded below by k+1.

Now we include another result regarding simple groups with conjugacy width 3.

Lemma 5.3. Let T be a simple group and let X be a group such that $InnT \le X \le Aut(T)$ and $c_X(T) = 3$. Then there exist $g, x, y \in T$ such that $c_X(T, g) = 3$ and $l_q^X(x) = l_q^X(y) = l_q^X(x^{-1}y) = 3$.

Proof. By Theorem 4.5 either cn(T)=3 or $T\cong F_4(2^a)$. Assume first that cn(T)=3 and choose any $g\in T$ such that $c_X(T,g)=3$. Let $C:=g^{\pm X}$ and let $t\in T$ be such that $l_g^X(t)=3$ and let $D=t^{\pm X}$.

First suppose that cn(T)=3. Assume $D^2\subseteq C^2\cup C\cup\{1\}=E$. Then

$$T = D^3 \subseteq ED \subseteq T$$

and so

$$ED = T = \{ g^{\pm a} g^{\pm b} t^{\pm c}, g^{\pm b} t^{\pm c}, t^{\pm c} \mid a, b, c \in X \}.$$

However since $l_g^X(t) = 3$ we have $1 \notin ED$, a contradiction. Hence $D^2 \nsubseteq C^2 \cup C \cup \{1\}$ and so there exist $x, y \in D$ satisfying the statement of the lemma.

Finally assume $T = F_4(2^a)$. Let $g \in T$ be an involution. As $F_4(2^a)$ is not strongly real, we have that $c_X(T,g) \geq 3$. Recall that an element is real if it is conjugate to its inverse. We shall show the existence of elements $x, y \in T$ such that none of $x, y, x^{-1}y$ is real. The proof of [27, Cor 4.5] shows the existence of a non-real order 12 element in $F_4(2^a)$. By construction, this has a conjugate in $F_4(2)$. From the character table in GAP, $F_4(2)$ has precisely two non-real classes, C, C^{-1} of elements of order 12. A computation shows that we can find $x, y \in C$ such that $x^{-1}y \in C$. Hence by the previous observation, $x, y, x^{-1}y$ are also non-real in $T = F_4(2^a)$.

Lemma 5.4. Let $G = T^3.X \le D(3,T)$ with $c_X(T) = 3$. Choose $g, x, y \in T$ as in the statement of Lemma 5.3 and let $\Gamma_0 = \{D_A, D_A(1,1,g)\}^G$. Then $diam(\Gamma_0) > 4$.

Proof. Let $x, y \in T$ be as in Lemma 5.3. The for all $t \in T$ we have

$$l_g^X(tx) + l_g^X(t) + l_g^X(ty) \geq \frac{l_g^X(x) + l_g^X(y) + l_g^X(x^{-1}y)}{2} = \frac{9}{2} > 4.$$

Hence $d_{\Gamma_0}(D_A, D_A(x, 1, y)) > 4$. The conclusion follows.

Proof of Theorem 5.1. Let T be simple and $G = T^k X \leq D(k,T)$.

- 1. If orbdiam(G) = 2 then by Theorem 3.1 it follows that k = 2 and $c_A(T) = 2$.
- 2. Suppose orbdiam(G) = 3. Then $k \leq 3$ by Theorem 3.1. If k = 2 then Theorem 3.1 shows $c_A(T) \leq 3$ as required. If k = 3, then by Theorem 3.1, $c_A(T) = 2$, but then Lemma 5.2 implies that orbdiam(G) > 3, a contradiction.
- 3. Assume that orbdiam(G) = 4. By Theorem 3.1 one of the following occurs:

I
$$k=2$$
 and $c_A(T) < 4$

II
$$k=3$$
 and $c_A(T)=2$

III
$$k = 3$$
 and $c_A(T) = 3$

IV
$$k=4$$
 and $c_A(T)=2$.

In Cases I and II, conclusions (a) and (b) of Theorem 5.1 hold.

In Cases III and IV, Lemmas 5.2 and 5.4 imply that $orbdiam(G) \geq 5$, which is a contradiction.

6 Upper bound on the orbital diameter

In this section we obtain a general upper bound for the orbital diameter of a primitive simple diagonal group. To keep things as simple as possible we restrict attention to groups of the form $G = T^k.S_k \leq D(k,T)$.

Theorem 6.1. Let T be a simple group and let $G = T^k.S_k \leq D(k,T)$ be a primitive simple diagonal group. Then

$$orbdiam(G) \le 24(k-1)c_i(T)^2$$
.

We first need a preliminary lemma.

Lemma 6.2. Let T be an simple group and $u \in T$ an involution. Then there exists $x \in T$ such that $uu^x = [u, x]$ has order greater than 2.

Proof. Suppose that uu^x has order less than or equal to 2 for all $x \in T$. Then u commutes with all of its conjugates, hence it commutes with $\langle u^T \rangle = T$, as T is simple, and so $u \in Z(T)$. This is a contradiction.

Proof of Theorem 6.1. For k=2 conclusion follows from Lemma 3.3 so assume $k\geq 3$. Let Γ be an orbital graph of G.

As we already covered the case of Γ_0^t in Lemma 3.2, we may assume that $\Gamma = \{D_A, D_A(1^i, t_{i+1}, \dots, t_k)\}^G$ where $i \leq k-2$ and $t_j \in T \setminus 1$ for $j \geq i+1$.

Suppose there is a path of length at most m from D_A to $D_A(m_1, \ldots, m_k)$ where $m_i \in T$. Recall our notation for this:

$$D_A \xrightarrow{m} D_A(m_1, \dots, m_k) \tag{2}$$

Applying $(m_1^{-1}, \ldots, m_k^{-1})$ we get

$$D_A = -m D_A(m_1^{-1}, \dots, m_k^{-1}).$$
 (3)

If we apply (a^k) to (2) or (3) for any $a \in T$, we get

$$D_A \xrightarrow{m} D_A(m_1^{\pm a}, \dots, m_k^{\pm a}). \tag{4}$$

<u>Claim 1</u> There exists $t \in T \setminus 1$ and a path in Γ of the following form;

$$D_A = D_A(1^{k-2}, t^{-1}, t).$$

Proof of Claim 1

Suppose first that there exist $l, j \geq i+1$ such that $t_l \neq t_j$. Apply the permutation (l j) and $(1^i, t_{i+1}, \dots, t_k)$ to the edge D_A $D_A(1^i, t_{i+1}^{-1}, \dots, t_k^{-1})$ to get a path

$$D_A - D_A(1^i, t_{i+1}, \dots, t_k) - D_A(1^{l-1}, t_i^{-1}t_l, 1^{j-l-1}, t_l^{-1}t_j, 1^{k-j}).$$

Putting $u = t_j^{-1} t_l$ and applying a suitable permutation yields Claim 1 in this case. Now assume $t_{i+1} = \cdots = t_k = t$. Apply (1 k) and $(1^i, t^{k-i})$ to the edge D_A ————— $D_A(1^i, (t^{-1})^{k-i})$ to get

$$D_A$$
 $D_A(1^i, t^{k-i})$ $D_A(t^{-1}, 1^{k-2}, t)$.

Claim 1 again follows.

Claim 2 There exists $t' \in T \setminus 1$ and a path in Γ

$$D_A - D_A(1^{k-1}, t')$$

where $c = c_i(T)$.

<u>Proof of Claim 2</u> We know from Claim 1 that for some $t \in T \setminus 1$ there is a path

$$D_A - D_A(1^{k-2}, t^{-1}, t).$$

Let $c_u \leq c$ such that $u = t^{\pm a_1} \dots t^{\pm a_{c_u}}$ is an involution in T where $a_i \in T$. We can construct a path in a similar way as in the proof of Claim 3 of Lemma 3.2 to get that

$$D_A - \frac{2c}{D_A(1^{k-2}, t^{\pm a_1} \dots t^{\pm a_{c_u}}, t^{\mp a_1} \dots t^{\mp a_{c_u}})} = D_A(1^{k-2}, u, u').$$

If there exist $b_1, b_2 \in T$ such that $u^{b_1}u^{b_2} = 1$ and $u'^{n_1}u'^{b_2} \neq 1$ then

$$D_A - \frac{4c}{D_A(1^{k-1}, u'^{b_1}u'^{b_2})} \neq D_A.$$

If no such b_1, b_2 exist then for $b \in T$, $uu^b = 1$ implies $u'u'^b = 1$. Hence u' is also an involution and $C_G(u) = C_G(u')$. So assume now that this is the case. Applying $(1^{k-2}, u', u)$ to $D_A \xrightarrow{2c} D_A(1^{k-2}, u, u')$, we see that here is a path

$$D_A$$
 $D_A(1^{k-2}, w, w),$

where $w \neq 1$ and either w = u = u' or w = uu'. Note that if k = 3, then the claim follows, so assume $k \geq 4$.

We know that w is an involution, as u and u' commute. By Lemma 6.2 there is $x \in T$ such that ww^x has order strictly greater than 2. Now

$$D_A = D_A(1^{k-2}, w^x, w^x).$$

Apply $(1^{k-2}, w, w)$ to get

$$D_A = D_A(1^{k-2}, w, w) = D_A(1^{k-2}, w^x w, w^x w).$$

Similarly,

$$D_A = \frac{8c}{D_A(1^{k-2}, ww^x, ww^x)}.$$

Let $h = ww^x$. Then $h^{-1} = w^x w$, so we have

$$D_A = \frac{8c}{D_A(1^{k-2}, h^{-1}, h^{-1})}.$$

Apply $(1^{k-3}, h, h, 1)$ to get

$$D_A(1^{k-3}, h, h, 1) - \frac{8c}{D_A(1^{k-3}, h, 1, h^{-1})}$$

so

$$D_A = D_A(1^{k-3}, h, 1, h^{-1}).$$

Apply $(1^{k-3}, h^{-1}, 1, h^{-1})$ to get

$$D_A(1^{k-3}, h^{-1}, 1, h^{-1}) = 16c D_A(1^{k-3}, 1, 1, h^{-2}),$$

so

$$D_A = D_A(1^{k-3}, 1, 1, h^{-2}),$$

where $h^{-2} \neq 1$. Hence Claim 2 follows.

At this point we can apply Lemma 3.2 to deduce that the diameter of Γ is bounded above by $24(k-1)c^2$, completing the proof.

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