PLANARITY CAN BE VERIFIED BY AN APPROXIMATE PROOF LABELING SCHEME IN CONSTANT-TIME

GÁBOR ELEK

Abstract. Approximate proof labeling schemes were introduced by Censor-Hillel, Paz and Perry [3]. Roughly speaking, a graph property $\mathcal{P}$ can be verified by an approximate proof labeling scheme in constant-time if the vertices of a graph having the property can be convinced, in a short period of time not depending on the size of the graph, that they are having the property $\mathcal{P}$ or at least they are not far from being having the property $\mathcal{P}$. The main result of this paper is that bounded-degree planar graphs (and also outer-planar graphs, bounded genus graphs, knotlessly embeddable graphs etc.) can be verified by an approximate proof labeling scheme in constant-time.

Keywords. approximate proof labeling schemes, planar graphs, Property A, hyperfiniteness

2010 *Mathematics Subject Classification.* 68M14, 05C85.

The author was partially supported by the ERC Consolidator Grant "Asymptotic invariants of discrete groups, sparse graphs and locally symmetric spaces" No. 648017.

Contents

1. Introduction

Our paper is about constant-time **distributed graph algorithms** introduced in the seminal work of Naor and Stockmeyer [16]. Such an algorithm runs on simple undirected graphs G of bounded degree. Each vertex of G collects information from vertices located within radius r from it and creates some output based on the collected information. In a **distributed decision algorithm** each vertex outputs a decision: accept or reject. Then, the graph G is accepted by the collective decision (verification) of the vertices if all the vertices accept and the graph is rejected if at least one of the vertices rejects. Unfortunately, there are not too much interesting graph families that can be verified in this way. However, one can consider a non-deterministic version of graph verification: **proof labeling schemes** (introduced by Korman, Kutten and Peleg [15]) or under the name of **locally checkable proofs** (due to Göös and Suomela [12]) with somewhat different conditions (see [7] for an extended survey). Here, a prover helps the vertices to make their decision.

The prover labels the vertices with an element of a finite set Q and the vertices can view their r-neighbourhoods as Q-labeled balls for some positive integer r. A graph family (note that in our paper we only consider graph families of bounded degrees) can be verified by a proof labeling scheme if there exists a labeling-verification protocol such that

• If a graph G is in the family then there exists a labeling that makes all vertices accept.

• If a graph G is not in the family then for all labelings there exists at least one vertices that rejects.

Proof labeling schemes help to verify more interesting graph classes. Nevertheless, one the most interesting classes, namely, the class of planar graphs still cannot be verified by a proof labeling scheme in constant-time [8]. Our graphs are of bounded-degree, the number of labelings are finite and the nodes explores a constant distance neighbourhood of themselves, that is why we use the *constant-time* terminology.

However, a relaxation of the above verification procedure, approximate proof labeling schemes, was introduced by Censor-Hillel, Paz and Perry [3](see also [5]). In the case of an approximate proof labeling scheme the vertices may accept a graph even if it is not in the graph family, provided that it is not far from the family in edit distance. One should note that in [3] the proof labeling schemes are used to certificate the approximation of certain parameters such as the maximum matching or a maximum independent set, nevertheless the idea is very similar.

Our result We will show (Theorem 1) that the class of **planar graphs** and in general all **monotone hyperfinite** (see Section 3) graph classes such as outer-planar graphs, bounded genus graphs or knotlessly embeddable graphs (or any other **minor-closed families**) can be verified with an approximate proof labeling scheme in constant-time.

Related work It is important to note that the class of planar graphs can be verified with a proof labeling scheme if we allow the vertices to be labeled by $O(\log(n))$-bits strings, where n is the size of the graph [8], even without the bounded degree assumption. This result can be extended to the class of bounded genus graphs as well ([9] and [6]).

Recently, Romero, Wrochna and Živný [19] constructed polynomial-time approximation schemes for certain maximum constraint satisfaction problems in the case of monotone hyperfinite graph classes. The main novelty of their approach was the application of strong hyperfiniteness, a strengthening of the hyperfiniteness property. Our proof is also based on strong hyperfiniteness, but in the form of **Property A.** This geometric property can be used for proof verification in a natural way and had already important applications in group theory, in algebraic topology and in the theory of operator algebras.

2. Approximate proof labeling schemes

In order to avoid any confusion, let us fix some terminologies. For an integer $d > 1$, let Gr_d be the set of all finite, simple graphs of maximum degree at most d. If x, y are adjacent vertices in a graph G, we use the notation $x \sim y$. For all graphs $G \in Gr_d$ we will consider the shortest path distance d_G on the vertex set $V(G)$ of G. In our paper we consider only properties $\mathcal{P}$ such that $\mathcal{P} \subset Gr_d$ for some d.

By a **ball** of radius r, we mean a finite connected graph B with a distinguished vertex v (the center) such that

$$\max_{y \in V(G)} d_G(v, y) = r\,.$$

For a fixed graph G and vertex $x \in V(G)$ the **neighbourhood of radius** s centered at x is the subgraph $B_s(x, G)$ induced on the vertices y such that

$$d_G(x, y) \leq s\,.$$

It is important to note that a neighbourhood $B_s(x, G)$ above is always a ball with center x, however, the radius of this ball can be equal to s only if the diameter of the graph G is at least s. If the diameter of G is less than s, then the radius of $B_s(x, G)$ as a ball is always less than s. We will denote by N_r^d the maximum size of a ball of radius r with maximum degree at most d.

For a graph $G \in Gr_d$ and a finite set Q, a Q**-proof** is a function $T : V(G) \to Q$.

A Q**-verifier** $\mathcal{V}$ of local horizon r is a subset of $B^Q_{r,d}$, where $B^Q_{r,d}$ is the set of all Q-vertex labeled balls of radius at most r and maximum degree at most d.

A Q-verifier $\mathcal{V}$ of local horizon r **accepts** a Q-proof T on the graph $G \in Gr_d$, if for all vertices $x \in V(G)$, $B_r(x, G, T) \in \mathcal{V}$, where $B_r(x, G, T)$ is the neighbourhood of radius r centered at x with vertex labelling induced by T.

A Q-verifier $\mathcal{V}$ of local horizon r **rejects** a Q-proof T on the graph $G \in Gr_d$, if for at least one vertex $x \in V(G)$, $B_r(x, G, T) \notin \mathcal{V}$.

We refer to subsets of Gr_d as "properties" and we say that a property $\mathcal{P} \subset Gr_d$ can be verified by a **proof labeling scheme** (PLS) in constant-time if there exists a finite set Q, a positive integer r and a verifier $\mathcal{V} \subset B^Q_{r,d}$ such that

- for any $G \in \mathcal{P}$ there exists a Q-proof $T : V(G) \to Q$ accepted by $\mathcal{V}$,
- for any $H \notin \mathcal{P}$ all the Q-proofs on H are rejected by $\mathcal{V}$.

Verifiability by a PLS in constant-time entails that the vertices of a graph $G \in \mathcal{P}$ can be convinced in a short period of time, that they are indeed vertices of a graph having the given property. Clearly, 3-colorability is such a property. Indeed, the proof $T : V(G) \to \{a, b, c\}$ will be the 3-coloring and the verifier will check the properness of the coloring on balls of radius 1. On the other hand, Feuilloley et al. showed ([8], Theorem 2.) that **planarity cannot be verified by a PLS in constant-time.**

In light of the result above we need a relaxation of the proof labeling scheme verification procedure. Such relaxation has been introduced by Censor-Hillel, Paz and Perry [3] (see also [5]) under the name approximate proof labeling scheme.

First we need some terminology. Recall that if $\mathcal{P} \subset Gr_d$ and $H \in Gr_d$, then the edit distance between the monotone property $\mathcal{P}$ and the graph H is defined by

$$e(H, \mathcal{P}) := \inf_{G \in \mathcal{P}, V(G)=V(H)} \frac{|E(G) \triangle E(H)|}{|V(H)|}.$$

Definition 2.1 (**Approximate Proof Labeling Scheme**). A property $\mathcal{P} \in Gr_d$ can be verified by an approximate proof labeling scheme in constant-time if for any $\varepsilon > 0$ there exists a set $Q_\varepsilon \geq 0$, some positive constant r_ε and a verifier $\mathcal{V}_\varepsilon \subset B^{Q_\varepsilon}_{r_\varepsilon,d}$ such that

- for any $G \in \mathcal{P}$ there exists a Q_ε-proof T on G accepted by $\mathcal{V}_\varepsilon$,
- for any $H, e(H, \mathcal{P}) > \varepsilon$, all Q_ε-proofs on H are rejected by $\mathcal{V}_\varepsilon$.

The main result of this paper is the following theorem.

Theorem 1. *Planarity and, in general, all monotone hyperfinite properties that are closed under taking disjoint unions can be verified by an approximate proof labeling scheme in constant-time for bounded-degree graphs.*

Note that a property is monotone if it is closed under taking subgraphs. Hyperfiniteness will be discussed in Section 3.

In [1] the authors showed that every minor-closed property is monotone hyperfinite (Theorem 1.1). In their paper they list several minor-closed properties e.g. planarity, outer-planarity, graphs with bounded genus or bounded tree-width. By definition, all of these properties are closed under taking disjoint unions. Hence, by our main theorem all of these properties can be verified with an approximate proof labeling scheme in constant-time.

Now, we introduce the notion of relative verifiability by PLS.

Definition 2.2. Let $\mathcal{P} \subset \mathcal{Q} \subset Gr_d$ be properties. We say that $\mathcal{P}$ can be verified by a PLS with respect to $\mathcal{Q}$ in constant-time if there exists a verifier $\mathcal{V} \subset B^{Q}_{r,d}$ such that

- for every $G \in \mathcal{P}$ there exists a proof $T : V(G) \to Q$ such that $\mathcal{V}$ accepts T,
- for every $H \notin \mathcal{Q}$ all proofs $T : V(H) \to Q$ are rejected by $\mathcal{V}$.

Now, let $G \in Gr_d, r \geq 1$, Q be a finite set and $\mathcal{V} \subset B^Q_{r,d}$. We say that $\mathcal{V}$ **verifies** G if there exists a proof $T : V(G) \to Q$ such that $\mathcal{V}$ accepts T.

Let $\mathcal{V} \subset B^Q_{r,d}$ be a verifier. We denote by $\mathcal{L}_\mathcal{V}$ the set of graphs in Gr_d verified by $\mathcal{V}$. So, a property $\mathcal{P}$ can be verified by a PLS in constant-time if there exists $\mathcal{V}$ such that $\mathcal{P} = \mathcal{L}_\mathcal{V}$.

Lemma 2.1. *If $\mathcal{V}_1 \subset B^{Q_1}_{r_1,d}$ and $\mathcal{V}_2 \subset B^{Q_2}_{r_2,d}$, then there exists a verifier $\mathcal{V}_3 \in B^{Q_1 \times Q_2}_{r_3,d}$ such that $\mathcal{L}_{\mathcal{V}_3} = \mathcal{L}_{\mathcal{V}_1} \cap \mathcal{L}_{\mathcal{V}_2}$ and $r_3 = \max(r_1, r_2)$.*

Proof. Let B be a $Q_1 \times Q_2$-labeled ball of radius at most r_3. Let B_1 be the r_1-neighbourhood of the center of B equipped with the Q_1-labeling inherited from the $Q_1 \times Q_2$-labeling of B using the first coordinate. Similarly, we can define B_2. Let $B \in \mathcal{V}_3$ if $B_1 \in \mathcal{V}_1$ and $B_2 \in \mathcal{V}_2$.

Assume that $G \in \mathcal{L}_{\mathcal{V}_3}$ and the proof $T = T_1 \times T_2 : V(G) \to Q_1 \times Q_2$ is accepted by $\mathcal{V}_3$. Then by definition, $T_1 : V(G) \to Q_1$ is accepted by $\mathcal{V}_1$ and $T_2 : V(G) \to Q_2$ is accepted by $\mathcal{V}_2$. Hence, $G \in \mathcal{L}_{\mathcal{V}_1} \cap \mathcal{L}_{\mathcal{V}_2}$.

Conversely, let $G \in \mathcal{L}_{\mathcal{V}_1} \cap \mathcal{L}_{\mathcal{V}_2}$ and let $S_1 : V(G) \to Q_1$ be accepted by $\mathcal{V}_1$ and $S_2 : V(G) \to Q_2$ be accepted by $\mathcal{V}_2$. Then by definition,

$$S = S_1 \times S_2 : V(G) \to Q_1 \times Q_2$$

is accepted by $\mathcal{V}_3$, thus $G \in \mathcal{L}_{\mathcal{V}_3}$. □

By definition, if $\mathcal{P} \subset \mathcal{Q}$, then there exists $\mathcal{V}$ such that

$$\mathcal{P} \subset \mathcal{L}_\mathcal{V} \subset \mathcal{Q}$$

if and only if $\mathcal{P}$ can be verified by a PLS in constant-time relative to $\mathcal{Q}$. If $\mathcal{P}$ is a property let $\mathcal{P}_\varepsilon$ be the set of graphs which are at most ε-far in edit-distance from having the property $\mathcal{P}$. Then $\mathcal{P}$ can be verified by an approximate proof labeling scheme in constant-time if for any $\varepsilon > 0$ there exists a verifier $\mathcal{V}_\varepsilon$ such that $\mathcal{P} \subset \mathcal{L}_{\mathcal{V}_\varepsilon} \subset \mathcal{P}_\varepsilon$.

In the introduction we mentioned a second verification protocol introduced by Göös and Suomela [12] under the name of locally checkable proofs. Sometimes a unique identifier is provided for each node of the graph and the verifier can take the identifiers into consideration as well. Although there are certain properties for which one can benefit from the existence of the identifiers, it follows from ([10, Theorem 1.]) that if one cannot verify a monotone property without identifiers in constant-time, then one cannot verify that property in constant-time even if unique identifiers are provided.

3. Hyperfiniteness

First, we recall the notion of hyperfiniteness (see [1]) that plays an important role in our paper.

For $\varepsilon > 0$ and $K \geq 1$, a graph $G \in Gr_d$ is called (ε, K)-hyperfinite if there exists $W \subset E(G)$ such that

- $|W| \leq \varepsilon |E(G)|$,
- if we remove W from G, in the remaining graph all the components have size at most K.

A property $\mathcal{P} \subset Gr_d$ is (ε, K)-hyperfinite if all $G \in \mathcal{P}$ are (ε, K)-hyperfinite. The set of all (ε, K)-hyperfinite graphs of maximum degree at most d is denoted by $\mathcal{H}^d_{\varepsilon,K}$.

We call a property $\mathcal{P} \subset Gr_d$ **hyperfinite**, if for any $\varepsilon > 0$ there exists $K \geq 1$ such that $\mathcal{P} \subset \mathcal{H}^d_{\varepsilon,K}$.

The significance of hyperfiniteness in algorithm theory is highlighted by the following breakthrough result of Benjamini, Schramm and Shapira (Theorem 1.2 [1]): every monotone hyperfinite property is testable in constant-time (see [11] for property testing). Note that another proof of this statement is given in [13]. It is important to note that minor-closed families such as planar graphs, outer-planar graphs or bounded genus graphs are hyperfinite [1].

4. Property A

In order to avoid confusion, in this section we use the phrase "graph class" instead of "graph property", since we will talk about the notion of Property A.

First, let us formally define Property A. Let $G \in Gr_d$ be a graph. Then, $\text{Prob}(G)$ is the set of all probability measures on the vertices of G. If $f : V(G) \to \mathbb{R}$ and $g : V(G) \to \mathbb{R}$ are two real functions on the vertices then their l_1-distance is defined as $\|f - g\|_1 := \sum_{x \in V(G)} |f(x) - g(x)|$, also $\|f\| := \sum_{x \in V(G)} |f(x)|$.

Definition 4.1 (Property A)**.** For $\varepsilon > 0$ and $r \geq 1$, a graph $G \in Gr_d$ is called (ε, r)**-uniform** if there exists a probability measure valued function $\tilde{f} : V(G) \to \text{Prob}(G)$ such that

- for any adjacent pair of vertices $x, y \in V(G)$, $\|\tilde{f}(x) - \tilde{f}(y)\|_1 < \varepsilon$,
- for any $x \in V(G)$, we have that

$$\text{Supp}(\tilde{f}(x)) \subset B_r(x, G)\,,$$

where $\mathrm{Supp}(\tilde{f}(x))$ denotes the set of vertices z for which $\tilde{f}(x)(z) \neq 0$.

We call a class of graphs $\mathcal{P}$ (ε, r)-uniform if all $G \in \mathcal{P}$ are (ε, r)-uniform and we denote the class of all (ε, r)-uniform graphs by $\mathcal{A}^d_{\varepsilon,r}$. A graph class $\mathcal{P} \subset Gr_d$ is of **Property A** if for every $\varepsilon > 0$, there exists some $r \geq 1$ such that $\mathcal{P} \subset \mathcal{A}^d_{\varepsilon,r}$.

Interestingly, Property A can be defined for a single countable infinite graph or a finitely generated group as well. Actually, the notion of Property A has been introduced for finitely generated groups by Guoliang Yu [21] in the nineties with important applications in algebraic topology and operator algebras [18]. It is not hard to see that the set of paths $\{P_n\}_{n=1}^\infty$ forms a class of Property A, nevertheless later we will see that the class of planar graphs is of Property A as well.

As we will see in Section 8, for finite graph classes Property A is a strengthening of the notion of hyperfiniteness. The proof of our main theorem hinges on the fact that monotone hyperfinite graph classes are of Property A.

5. Property A and the Proof Labeling Schemes

The sole goal of this section is to prove the following proposition.

Proposition 5.1. *For any $0 < \varepsilon < \varepsilon' < 1$ and $r \geq 1$, $\mathcal{A}^d_{\varepsilon,r}$ can be verified by PLS in constant-time relative to $\mathcal{A}^d_{\varepsilon',r}$.*

Proof. A natural approach for such a PLS is to label every vertex x with its probability distribution, described as a list of $\tilde{f}(x)(z)$, and let the vertices check that the two conditions of Definition 4.1 hold. There are two obstacles to this approach: the precise value of $\tilde{f}(x)(z)$ might need a large number of bits to be encoded, and the vertices do not agree on which vertex is "z" (remember that the vertices do not have identifiers). The first lemma tackles the first obstacle via discretization. For the second obstacle, we will use a coloring.

Lemma 5.1. *Let $G \in Gr_d$, $r \geq 1$, $x \in V(G)$, $f : B_r(x,G) \to \mathbb{R}$ be a nonnegative function such that $\sum_{y \in B_r(x,G)} f(y) = 1$. Let $\alpha > \frac{3}{\varepsilon'-\varepsilon} N^d_r$ be a positive integer (see Section 2 for definition of N^d_r). Then, there exists a function $g : B_r(x,G) \to \mathbb{R}$ such that*

- $\sum_{y \in B_r(x,G)} g(y) = 1$,
- *for any $y \in B_r(x,G)$, $g(y) = \frac{i}{\alpha}$, where $0 \leq i \leq \alpha$ is an integer,*
- $\sum_{y \in B_r(x,G)} |f(y) - g(y)| \leq \frac{\varepsilon'-\varepsilon}{3}$.

Proof. Let $g', g'' : B_r(x,G) \to \mathbb{R}$ be defined in the following way. $g'(y) = \frac{i}{\alpha}, g''(y) = \frac{i+1}{\alpha}$, where $\frac{i}{\alpha} \leq f(y) \leq \frac{i+1}{\alpha}$. Then, $\sum_{y \in B_r(x,G)} (f(y) - g'(y)) < |B_r(x,G)| \frac{1}{\alpha} \leq \frac{\varepsilon'-\varepsilon}{3}$. Similarly, $\sum_{y \in B_r(x,G)} (g"(y) - f(y)) < \frac{\varepsilon'-\varepsilon}{3}$.

Also,

$$\sum_{y\in B_r(x,G)} g'(y)=\frac{k}{\alpha}\leq 1 \text{ and } \sum_{y\in B_r(x,G)} g''(y)=\frac{l}{\alpha}\geq 1\,,$$

where k and l are integers such that $k\leq\alpha\leq l$. Note that for all $y\in B_r(x,G)$ we have $g''(y)-g'(y)=\frac{1}{\alpha}$, hence $l-k=|B_r(x,G)|$. Pick a subset $S\subseteq B_r(x,G)$ such that $|S|=\alpha-k$.

Let $g(y)=g'(y)$ if $y\notin S$ and $g(y)=g''(y)$ if $y\in S$. Then,

$$\sum_{y\in B_r(x,G)} g(y)=\frac{k}{\alpha}+\frac{\alpha-k}{\alpha}=1.$$

Also,

$$\sum_{y\in B_r(x,G)} |f(y)-g(y)|\leq |B_r(x,G)|\frac{1}{\alpha}\leq\frac{\varepsilon'-\varepsilon}{3}\,. \qquad \square$$

Corollary 5.1. *Let $0<\varepsilon<\varepsilon'<1$ and $G\in\mathcal{A}^d_{\varepsilon,r}$. Then, we have a probability measure valued function $\tilde{g}:V(G)\to\mathrm{Prob}(G)$ such that*

- *for any $x\in V(G)$, $\tilde{g}(x)(z)=\frac{i}{\alpha}$, where $0\leq i\leq\alpha$ is an integer and α is the integer defined by the previous lemma.*
- *for any adjacent pair of vertices $x,y\in V(G)$, $\|\tilde{g}(x)-\tilde{g}(y)\|_1<\epsilon'$,*
- *for any $x\in V(G)$, we have that $\mathrm{Supp}(\tilde{g}(x))\subset B_r(x,G)$.*

Proof. Let $\tilde{f}:V(G)\to\mathrm{Prob}(G)$ the probability measure valued function in Definition 4.1. For each $x\in V(G)$ we define $g_x:V(G)\to\mathbb{R}$ in such a way that g_x satisfies the conditions of the previous lemma with respect to the function $\tilde{f}(x)$. Now we define $\tilde{g}(x):=g_x$.

Let $x\sim y\in V(G)$. Then,

$$\|\tilde{g}(x)-\tilde{g}(y)\|_1\leq\|\tilde{g}(x)-\tilde{f}(x)\|_1+\|\tilde{f}(x)-\tilde{f}(y)\|_1+\|\tilde{f}(y)-\tilde{g}(y)\|_1\leq$$

$$\leq 2\frac{\varepsilon'-\varepsilon}{3}+\varepsilon<\varepsilon'. \qquad \square$$

Now we build the proof labeling scheme.

Lemma 5.2. *Let Q_1 be a finite set such that $|Q_1|=N^d_r+1$. Then, for any $G\in Gr_d$ there exists a coloring $S_1:V(G)\to Q_1$ such that if $d_G(x,y)\leq r$, then $S_1(x)\neq S_1(y)$.*

Proof. Let $\hat{G}$ be the graph with vertex set $V(G)$ such that $x,y\in V(G)$ are adjacent in $\hat{G}$ if and only if $d_G(x,y)\leq r$. Then, the degree of each vertex in $\hat{G}$ is at most N^d_r. Hence by the classical Brooks' Theorem, there is a proper coloring $S_1 : V(G) \to Q_1$ of the graph $\hat{G}$. By definition, S_1 defines a coloring of G satisfying the condition of our lemma. □

Let Q_2 denote the finite set $\{0, 1, 2, \dots, \alpha\}$, where α is the integer in Lemma 5.1. Also, let Q_3 be the set of all maps $\varphi : Q_1 \to Q_2$. Let $\pi_1 : Q_1 \times Q_3 \to Q_1$ be the first coordinate projection and $\pi_2 : Q_1 \times Q_3 \to Q_3$ be the second coordinate projection. For a $Q_1 \times Q_3$-proof $T : V(G) \to Q_1 \times Q_3$ let $T_1 : V(G) \to Q_1$ be defined as $\pi_1 \circ T$ and let $T_2 : V(G) \to Q_3$ be defined as $\pi_2 \circ T$.

We call $T : V(G) \to Q_1 \times Q_3$ proper if $T_1(x) \neq T_1(y)$ provided that $d_G(x, y) \leq r$. If T is proper then we can define a function of two variables $\tilde{T} : V(G) \times V(G) \to \mathbb{R}$ in the following way.

- If $d_G(x, y) \leq r$ then $\tilde{T}(x, y) = \frac{i}{\alpha}$, where $i = T_2(y)(T_1(x))$.
- If $d_G(x, y) > r$ then $\tilde{T}(x, y) = 0$.

Properness is used to break possible symmetries. The verifier $\mathcal{V} \subset B^d_{r+1, Q_1 \times Q_3}$ is defined in the following way. Let N be a ball of radius at most $r + 1$ and $C : V(N) \to Q_1 \times Q_3$ be a $Q_1 \times Q_3$-labeling on N. Again, let $C_1 = \pi_1 \circ C$ and $C_2 = \pi_2 \circ C$. Then the $Q_1 \times Q_3$-labeled ball (N, C) is in the verifier $\mathcal{V}$ if the following conditions are satisfied.

- (Checking properness) If $y, z \in V(N)$ and $d_N(y, z) \leq r$ then $C_1(y) \neq C_2(z)$.
- (Checking probability) If x is the center of N then
$$\sum_{z \in B_r(x,N)} \frac{(C_2(z))(C_1(x))}{\alpha} = 1\,.$$
- (Checking l_1-distance) If x is the center of N and $x \sim y$, then
$$\sum_{z \in V(N)} \left| \frac{(C_2(z))(C_1(x))}{\alpha} - \frac{(C_2(z))(C_1(y))}{\alpha} \right| \leq \varepsilon'.$$

Therefore, if $\mathcal{V}$ accepts the proof T, then

- T is proper.
- $\sum_{z \in V(G)} \tilde{f}(x)(z) = 1$, where $\tilde{f}(x)(z) = \tilde{T}(x, z)$.
- For any adjacent pair $x \sim y \in V(G)$, $\|\tilde{f}(x) - \tilde{f}(y)\| \leq \varepsilon'$.

Hence, if $\mathcal{V}$ accepts T then $G \in \mathcal{A}^d_{\varepsilon', r}$.

In order to finish the proof of the proposition, we need to prove that if $G \in \mathcal{A}^d_{\varepsilon, r}$ then there exists a proof $T : V(G) \to Q_1 \times Q_3$ that is accepted by the verifier $\mathcal{V}$.

Let $\tilde{g}$ be the probability measure valued function defined in Corollary 5.1. Let $S_1 : V(G) \to Q_1$ be the function defined in Lemma 5.2. Finally, let $T : V(G) \to Q_1 \times Q_3$ be defined in the following way.

- $T_1 := S_1$.
- If $z \in V(G), q \in Q_1$ and there is no $x \in B_r(z, G)$ such that $S_1(x) = q$, then let $(T_2(z))(q) = 0$.
- If $z \in V(G), q \in Q_1$ and there exists $x \in B_r(z, G)$ such that $S_1(x) = q$, let $(T_2(z))(q) = i$, where $\tilde{g}(x)(z) = \frac{i}{\alpha}$.

Then, $\mathcal{V}$ accepts T. Therefore,

$$\mathcal{A}^d_{\varepsilon,r} \subset \mathcal{L}_{\mathcal{V}} \subset \mathcal{A}^d_{\varepsilon',r}\,,$$

hence our proposition follows. □

6. Strong hyperfiniteness

In this section we discuss some strengthenings of the notion of hyperfiniteness.

Definition 6.1. $G \in Gr_d$ is uniformly (ε, K)-hyperfinite if for all induced subgraph $F \subset G$, F is (ε, K)-hyperfinite as well.

We say that a graph property $\mathcal{P} \subset Gr_d$ is uniformly (ε, K)-hyperfinite if all $G \in \mathcal{P}$ are uniformly (ε, K)-hyperfinite. The set of all uniformly (ε, K)-hyperfinite graphs will be denoted by $\mathcal{UH}^d_{\varepsilon,K}$. We call a graph property $\mathcal{P}$ **uniformly hyperfinite** if for any $\varepsilon > 0$, there exists $K \geq 1$ such that $\mathcal{P} \subset \mathcal{UH}^d_{\varepsilon,K}$.

Monotone hyperfinite classes are, by definition, uniformly hyperfinite, since they are closed to taking subgraphs. In our paper, we will use another strengthening of hyperfiniteness introduced by Romero, Wrochna and Živný [19] under the name of *fractional-cc-fragility*. First, we need a definition. For a graph $G \in Gr_d$, we call $Y \subset V(G)$ a K-*separator* if by removing Y (and all the adjacent edges) the components of the remaining graph are of size at most K. We denote the set of all K-separators of G by $\mathrm{Sep}(G, K)$.

Definition 6.2. A graph $G \in Gr_d$ is strongly (ε, K)-hyperfinite if there exists a probability measure μ on $\mathrm{Sep}(G, K)$ such that for any $x \in V(G)$

$$\mu(Y \in \mathrm{Sep}(G, K) \mid x \in Y) < \varepsilon\,.$$

We say that a graph class $\mathcal{P} \subset Gr_d$ is strongly (ε, K)-hyperfinite if all $G \in \mathcal{P}$ are strongly (ε, K)-hyperfinite. The set of all strongly (ε, K)-hyperfinite graphs will be denoted by $\mathcal{SH}^d_{\varepsilon,K}$. We call a graph class $\mathcal{P}$ **strongly hyperfinite** if for any $\varepsilon > 0$, there exists $K \geq 1$ such that $\mathcal{P} \subset \mathcal{SH}^d_{\varepsilon,K}$.

It was first proved by Romero, Wrochna and Živný (Theorem 1.5 ,[19]) that monotone hyperfinite properties are strongly hyperfinite. The strong hyperfiniteness of monotone hyperfinite classes plays an important role in the proof of Theorem 1. Note that the author later proved in [4] that the notions of Property A, uniform hyperfiniteness and strong hyperfiniteness in fact coincide.

7. Property A implies hyperfiniteness

In this section we continue the study of Property A and prove the central technical proposition of our paper.

First, let us fix some notation, which will be used in the section. Let $G \in Gr_d$ and $A \subset V(G)$. Then, $\partial_G(A)$ is the set of vertices $x \in A$ such that there exists $y \notin A$, $x \sim y$. Also, $\partial^e_G(A)$ is the set of edges $e = (x, y)$, where $x \in \partial_G(A)$ and $y \notin A$. So, we have that

$$|\partial_G(A)| \leq |\partial^e_G(A)|. \tag{1}$$

Proposition 7.1. *For any $\varepsilon > 0$ and $r \geq 1$,*

$$\mathcal{A}^d_{\varepsilon,r} \subset \mathcal{H}^d_{\frac{d^2\varepsilon}{2}, N^d_{2r}},$$

where N^d_{2r} is defined in Section 2.

Proof. First, we need a technical lemma, which is very similar to Proposition 4.2 in [20]. Let $G \in Gr_d$ and $F \subset G$ be an induced subgraph. We say that F is (ε, r)-uniform relative to G if there exists a probability measure valued function $\tilde{f} : V(F) \to \mathrm{Prob}(F)$ such that for any pair of adjacent vertices $x, y \in V(F)$,

$$\|\tilde{f}(x) - \tilde{f}(y)\|_1 \leq \varepsilon \tag{2}$$

and for any $x \in V(F)$,

$$\mathrm{Supp}(\tilde{f}(x)) \subset B_r(x, G). \tag{3}$$

Lemma 7.1. *If G is (ε, r)-uniform and $F \subset G$ is an induced subgraph, then F is $(\varepsilon, 2r)$-uniform relative to G.*

Proof. For $x \in V(G)$, pick $\tau(x) \in V(F)$ in such a way that $d_G(x, \tau(x)) = d_G(x, F)$. Let $g : V(G) \to \mathrm{Prob}(G)$ be a probability measure valued function witnessing the fact that $G \in \mathcal{A}^d_{\varepsilon,r}$, that is,

- for any adjacent pair of vertices $x, y \in V(G)$

$$\|\tilde{g}(x) - \tilde{g}(y)\|_1 \leq \varepsilon\,. \tag{4}$$

- for any $x \in V(G)$

$$\mathrm{Supp}(\tilde{g}(x)) \subset B_r(x, G)\,. \tag{5}$$

We define the function $\tilde{f} : V(F) \to \mathrm{Prob}(F)$ by $\tilde{f}(x)(z) = \sum_{t \in \tau^{-1}(z)} \tilde{g}(x)(t)$. Note that $\tau^{-1}(z)$ denotes the set of vertices mapped to z by τ. Then by definition, $\mathrm{Supp}\, \tilde{f}(x) \subset V(F)$ and
for all $z \in V(F)$, $\tilde{f}(x)(z) \geq 0$. Also, since $\cup_{z \in V(F)} \tau^{-1}(z) = V(G)$ we have that

$$\sum_{z \in V(F)} \tilde{f}(x)(z) = \sum_{t \in V(G)} \tilde{g}(x)(t) = 1\,,$$

hence $\tilde{f} : V(F) \to \mathrm{Prob}(F)$. Also, if x, y are adjacent vertices, then

$$\|\tilde{f}(x) - \tilde{f}(y)\|_1 \leq \varepsilon.$$

Indeed,

$$\|\tilde{f}(x) - \tilde{f}(y)\|_1 = \sum_{z \in V(F)} |\tilde{f}(x)(z) - \tilde{f}(y)(z)| \leq$$

$$\leq \sum_{z \in V(F)} |\sum_{t \in \tau^{-1}(z)} \tilde{g}(x)(t) - \sum_{t \in \tau^{-1}(z)} \tilde{g}(y)(t)| \leq \sum_{z \in V(F)} \sum_{t \in \tau^{-1}(z)} |\tilde{g}(x)(t) - \tilde{g}(y)(t)| =$$

$$= \sum_{t \in V(G)} |\tilde{g}(x)(t) - \tilde{g}(y)(t)| = \|\tilde{g}(x) - \tilde{g}(y)\|_1 \leq \varepsilon.$$

Also,

$$\mathrm{Supp}(\tilde{f}(x)) \subset B_{2r}(x, G). \tag{6}$$

Indeed, if $\tilde{f}(x)(z) \neq 0$, then there exists $t \in \tau^{-1}(z)$ such that $\tilde{g}(x)(t) \neq 0$. Hence by (5), $d_G(t, x) \leq r$ and also, $d_G(t, z) \leq r$, since $d_G(t, z) \leq d_G(t, x)$ by the definition of τ. That is, $d_G(x, z) \leq 2r$, so our lemma follows. □

Lemma 7.2. *Let $G \in \mathcal{A}^d_{\varepsilon,r}$ and let $F \subset G$ be an induced subgraph. Then, there exists a non-empty subset $L \subset V(F)$ such that $|\partial_F(L)| \leq \frac{d\varepsilon}{2}|L|$ and $|L| \leq N^d_{2r}$.*

Proof. Let $\tilde{f} : V(F) \to \mathrm{Prob}(F)$ be a probability measure valued function satisfying (2) and (6). Such function exists by the previous lemma. Then,

$$\sum_{x \in V(F)} \sum_{x \sim y} \|\tilde{f}(x) - \tilde{f}(y)\|_1 \leq \sum_{x \in V(F)} d\varepsilon =$$

$$= \sum_{x \in V(F)} d\varepsilon \|\tilde{f}(x)\|_1\,.$$

Hence,

$$\sum_{z \in V(F)} \sum_{x \in V(F)} \sum_{x \sim y} |\tilde{f}(x)(z) - \tilde{f}(y)(z)| \leq d\varepsilon \sum_{z \in V(F)} \sum_{x \in V(F)} \tilde{f}(x)(z)\,.$$

Hence, there exists $z_0 \in V(F)$ such that

$$\sum_{x \in V(F)} \sum_{x \sim y} |\tilde{f}(x)(z_0) - \tilde{f}(y)(z_0)| \leq d\varepsilon \sum_{x \in V(F)} \tilde{f}(x)(z_0)\,.$$

We define the function $\zeta : V(F) \to [0,1]$ by $\zeta(x) = \tilde{f}(x)(z_0)$, and we have that

$$\sum_{x\in V(F)} \sum_{x\sim y} |\zeta(x) - \zeta(y)| \leq d\varepsilon \sum_{x\in V(F)} \zeta(x)\,. \tag{7}$$

So far, we followed the proof of Proposition 3.2 in [2], however, in order to avoid some heavy machinery, we now choose a different path. Let us recall the area and coarea formulas (Lemma 3.6 and Lemma 3.7) from [14]. If $F \in Gr_d$ and $\zeta : V(F) \to [0,1]$, then we have the following equations:

$$\sum_{x\in V(F)} \sum_{x\sim y} |\zeta(x) - \zeta(y)| = 2 \int_0^1 |\partial^e_F(\Omega_t(\zeta))|\, dt, \tag{8}$$

and

$$\sum_{x\in V(F)} \zeta(x) = \int_0^1 |\Omega_t(\zeta)|\, dt, \tag{9}$$

where

$$\Omega_t(\zeta) = \{x \in V(G) \mid \zeta(x) > t\}.$$

So, by (7)

$$2 \int_0^1 |\partial^e_F(\Omega_t(\zeta))|\, dt \leq d\varepsilon \int_0^1 |\Omega_t(\zeta)|\, dt.$$

Thus for some $t \geq 0$, we have

$$|\partial^e_F(\Omega_t(\zeta))| \leq \frac{d\varepsilon}{2} |\Omega_t(\zeta)|. \tag{10}$$

Now let $L = \Omega_t(\zeta)$. Then by (10) and (1) we have that $|\partial_F(L)| \leq \frac{d\varepsilon}{2}|L|$. Also by definition, if $x \in L$, then $\tilde{f}(x)(z_0) > 0$. Therefore, $x \in B_{2r}(z_0, G)$. So, $0 < |L| \leq N^d_{2r}$. Hence, our lemma follows. $\square$

Now we finish the proof of our proposition. Let $F_1 = G$. Using the previous lemma, we choose $L_1 \subset V(F_1)$ to be a set of size at most N^d_{2r} such that $|\partial_{F_1}(L_1)| \leq \frac{d\varepsilon}{2}|L_1|$. Then, we remove from G all the edges outgoing from L_1. The number of such edges is at most $d|\partial_{F_1}(L_1)| \leq \frac{d^2\varepsilon}{2}|L_1|$. Let F_2 be the subgraph of G induced on $V(G)\backslash L_1$. Let $L_2 \subset V(F_2)$ be a set of size at most N^d_{2r} such that $|\partial_{F_2}(L_2)| \leq \frac{d\varepsilon}{2}|L_2|$. Again, we remove from G all the edges outgoing from L_2. Inductively, we construct disjoint components $L_1, L_2, \dots, L_n$ of size at most N^d_{2r} such that $\cup^n_{i=1} L_i = V(G)$, by removing at most $\frac{d^2\varepsilon}{2}|V(G)|$ edges. Hence, our proposition follows. $\square$

8. Strong hyperfiniteness implies Property A

Proposition 8.1. *For any $\varepsilon > 0$ and $K \geq 1$,*

$$\mathcal{SH}^d_{\varepsilon,K} \subset \mathcal{A}^d_{4\varepsilon,K}\,.$$

Proof. Let $G \in \mathcal{SH}^d_{\varepsilon,K}$ and μ be a probability measure on the set $\mathrm{Sep}(G,K)$ of K-separators of G such that for any $x \in V(G)$ we have

$$\mu(Y \in \mathrm{Sep}(G,K) \mid x \in Y) \leq \varepsilon\,.$$

For a K-separator Y and $x \in V(G)$, let the non-negative function $f_{Y,x} : V(G) \to \mathbb{R}$ be defined in the following way.

- If $x \in Y$, let $f_{Y,x}(x) = \mu(Y)$ and if $z \neq x$ let $f_{Y,x}(z) = 0$.
- If $x \notin Y$, then let $C_{Y,x}$ be the component of $G\backslash Y$ containing the vertex x. If $z \in C_{Y,x}$ let $f_{Y,x}(z) = \frac{\mu(Y)}{|C_{Y,x}|}$. On the other hand, if $z \notin C_{Y,x}$, let $f_{Y,x}(z) = 0$.

Then, for any $x \in V(G)$ and $Y \in \mathrm{Sep}(G,K)$ we have that

$$\|f_{Y,x}\|_1 = \sum_{z \in V(G)} f_{Y,x}(z) = \mu(Y)\,. \tag{11}$$

Hence,

$$\tilde{f}(x) = \sum_{Y \in \mathrm{Sep}(G,K)} f_{Y,x}$$

defines a probability measure valued function $\tilde{f} : V(G) \to \mathrm{Prob}(G)$. Then, by the definition of K-separators, for any $x \in V(G)$,

$$\mathrm{Supp}(\tilde{f}(x)) \subset B_K(x,G)\,. \tag{12}$$

Also, for any pair of adjacent vertices $x \sim y$, we have that

$$\|\tilde{f}(x) - \tilde{f}(y)\|_1 \leq 4\varepsilon\,. \tag{13}$$

Indeed, if $x, y \notin Y$ and $x \sim y$, then by definition, $f_{Y,x} = f_{Y,y}$. If $x \in Y$ or $y \in Y$, then by (11),

$$\|f_{Y,x} - f_{Y,y}\|_1 \leq \|f_{Y,x}\|_1 + \|f_{Y,y}\|_1 \leq 2\mu(Y)\,.$$

Therefore,

$$\|\tilde{f}(x) - \tilde{f}(y)\|_1 \leq \sum_{Y \in \mathrm{Sep}(G,K)} \|f_{Y,x} - f_{Y,y}\|_1 \leq$$

$$\leq \sum_{Y \in \mathrm{Sep}(G,K), x \in Y} 2\mu(Y) + \sum_{Y \in \mathrm{Sep}(G,K), y \in Y} 2\mu(Y) + \sum_{Y \in \mathrm{Sep}(G,K), x,y \notin Y} \|f_{Y,x} - f_{Y,y}\|_1 \leq 4\varepsilon\,,$$

since if $x \sim y$ and $x, y \notin Y$, then $f_{Y,x} = f_{Y,y}$. Hence by (12) and (13), $G \in \mathcal{A}^d_{4\varepsilon,K}$. ☐

Lemma 8.1. *For any $\varepsilon > 0$, there exists $r > 0$ such that $\mathcal{P} \subset \mathcal{A}^d_{\varepsilon,r}$, where $\mathcal{P} \subset Gr_d$ is a monotone hyperfinite class of graphs.*

Proof. By Theorem 1.6 of [19], for any $\varepsilon > 0$, there exists $K > 0$ such that $\mathcal{P} \subset \mathcal{SH}^d_{\varepsilon,K}$. Hence our lemma follows, from Proposition 8.1. ☐

9. The Proof of the Main Theorem

In this section, we prove Theorem 1. Let $\mathcal{P} \subset Gr_d$ be a monotone hyperfinite property that is closed under taking disjoint unions and let $\varepsilon > 0$. We need to prove that there exists a verifier $\mathcal{V}_\varepsilon$ such that (using the notation of Section 2)

$$\mathcal{P} \subset \mathcal{L}_{\mathcal{V}_\varepsilon} \subset \mathcal{P}_\varepsilon \,. \tag{14}$$

For $r > 0$, let $L\mathcal{P}_r \subset Gr_d$ denote the class of *r-locally $\mathcal{P}$ graphs.* That is, $G \in L\mathcal{P}_r$ if for all $x \in V(G)$, $B_r(x, G) \in \mathcal{P}$. So, by definition, there exists a verifier $\mathcal{B}_K \subset B^d_{K,Q}$ such that $L\mathcal{P}_K = \mathcal{L}_{\mathcal{B}_K}$ and the finite set Q is empty.

Lemma 9.1. *For any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that if $K \geq M_\varepsilon$, then*

$$\mathcal{P} \subset \mathcal{H}^d_{\varepsilon,K} \cap L\mathcal{P}_K \subset \mathcal{P}_\varepsilon \,.$$

Proof. Pick $M_\varepsilon > 0$ such that $\mathcal{P} \subset \mathcal{H}^d_{\varepsilon,M_\varepsilon}$, let $K \geq M_\varepsilon$. Then, we have $\mathcal{P} \subset \mathcal{H}^d_{\varepsilon,K} \cap L\mathcal{P}_K$. Now, let $G \in \mathcal{H}^d_{\varepsilon,K} \cap L\mathcal{P}_K$. So, we can remove at most $\varepsilon|V(G)|$ edges from G to obtain a graph having components of size at most K and by monotonicity, all those components are in $\mathcal{P}$. Also, by our assumption about $\mathcal{P}$, the disjoint unions of the components are in $\mathcal{P}$ as well. Hence, $G \in \mathcal{P}_\varepsilon$. That is, $\mathcal{H}^d_{\varepsilon,K} \cap L\mathcal{P}_K \subset \mathcal{P}_\varepsilon$. $\square$

Lemma 9.2. *For any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ and a verifier $\mathcal{C}_\varepsilon$ such that if $K \geq N_\varepsilon$, then*

$$\mathcal{P} \subset \mathcal{L}_{\mathcal{C}_\varepsilon} \subset \mathcal{H}^d_{\varepsilon,K} \,.$$

Proof. Let $r \geq 1$ be an integer such that $\mathcal{P} \subset \mathcal{A}^d_{\frac{\varepsilon}{d^2},r}$. Such r exists by Lemma 8.1. Let $N_\varepsilon = N^d_{2r}$. By Proposition 5.1, there exists a verifier $\mathcal{C}_\varepsilon$ such that

$$\mathcal{A}^d_{\frac{\varepsilon}{d^2},r} \subset \mathcal{L}_{\mathcal{C}_\varepsilon} \subset \mathcal{A}^d_{\frac{2\varepsilon}{d^2},r} \,.$$

By Proposition 7.1,

$$\mathcal{A}^d_{\frac{2\varepsilon}{d^2},r} \subset \mathcal{H}^d_{\varepsilon,N^d_{2r}} = \mathcal{H}^d_{\varepsilon,N_\varepsilon} \,.$$

Therefore,

$$\mathcal{P} \subset \mathcal{L}_{\mathcal{C}_\varepsilon} \subset \mathcal{H}^d_{\varepsilon,K} \,. \quad \square$$

Now we finish the proof of Theorem 1. Let $K_\varepsilon = \max(N_\varepsilon, M_\varepsilon)$ and $\mathcal{D}_\varepsilon = \mathcal{B}_{K_\varepsilon}$, then by our previous lemmas,

$$\mathcal{P} \subset \mathcal{L}_{\mathcal{C}_\varepsilon} \cap \mathcal{L}_{\mathcal{D}_\varepsilon} \subset \mathcal{P}_\varepsilon \,.$$

Hence by Lemma 2.1, we have a verifier $\mathcal{V}_\varepsilon$ satisfying (14). $\square$

References

[1] *I. Benjamini, O. Schramm, and A. Shapira*, Every minor-closed property of sparse graphs is testable. *Adv. Math.* **223** (2010), no. 6, 2200–2218.

[2] *J. Brodzki, G. Niblo, J. Špakula, R. Willett and N. Wright*, Uniform local amenability. *J. Noncommut. Geom.* **7** (2013), no. 2, 583-603.

[3] *K. Censor-Hillel, A. Paz and M. Perry*, Approximate proof-labeling schemes. *Theoretical Computer Science*, **811**, (2020) 112-124.

[4] *G. Elek*, Uniform local amenability implies Property A. *Proc. Amer. Math. Soc.* **149** (2021), no. 6, 2573–2577.

[5] *Y. Emek and Y. Gil*, Twenty-two new approximate proof labeling schemes.*34th International Symposium on Distributed Computing, Art. No. 20, 14 pp., LIPIcs. Leibniz Int. Proc. Inform., 179, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, (2020).*

[6] *L. Esperet and B.Lévêque*, Local certification of graphs on surfaces. *preprint https://arxiv.org/pdf/2102.04133.pdf*

[7] *L. Feuilloley*, Introduction to local certification. *Discrete Math. Theor. Comput. Sci.* **23,** no. 3, (2021) Paper No. 9.

[8] *L. Feuilloley, P. Fraigniaud, P. Montealegre, I. Rapaport, É. Rémila and I. Todinca*, Compact distributed certification of planar graphs. *Algorithmica* **83** (2021), no. 7, 2215–2244.

[9] *L. Feuilloley, P. Fraigniaud, P. Montealegre, I. Rapaport, É. Rémila and I. Todinca,* Local certification of graphs with bounded genus. *preprint https://arxiv.org/pdf/2007.08084.pdf*

[10] *P. Fraigniaud, M. M. Halldórsson, and A. Korman*, On the impact of identifiers on local decision. *Principles of distributed systems, Lecture Notes in Comput. Sci.,* **7702**, Springer, Heidelberg, (2012), 224–238.

[11] *O. Goldreich and D. Ron*, Property testing in bounded degree graphs. *Algorithmica* **32** (2002), no. 2, 302-343.

[12] *M.Göös and J. Suomela*, Locally checkable proofs in distributed computing. *Theory Comput.* **12** Paper No. 19 (2016)

[13] *A. Hassidim, J. A. Kelner, H. N. Nguyen and K. Onak*, Local graph partitions for approximation and testing. *50th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2009 IEEE Computer Soc., Los Alamitos, CA*, (2009) 22-31.

[14] *M. Keller and D. Mugnolo*, General Cheeger inequalities for p-Laplacians on graphs. *Nonlinear Analysis* **147** (2016), 80-95.

[15] *A. Korman, S. Kutten and D. Peleg*, Proof labeling schemes. *Distributed Computing* **22**, (2010), 215-233.

[16] *M. Naor and L. Stockmeyer*, What can be computed locally? *SIAM J. Comput.* **24** (1995), no. 6, 1259-1277.

[17] *I. Newman and C. Sohler*, Every property of hyperfinite graphs is testable. *SIAM J. Comput.* **42** (2013), no. 3, 1095-1112.

[18] *P. Nowak and G. Yu,* What is ... property A? *Notices Amer. Math. Soc.* **55** (2008), no. 4, 474–475.

[19] *M. Romero, M. Wrochna and S. Živný*, Treewidth-Pliability and PTAS for Max-CSP's. *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA) [Society for Industrial and Applied Mathematics (SIAM)]* (2021), 473-483.

[20] *J-L. Tu*, Remarks on Yu's "property A" for discrete metric spaces and groups. *Bull. Soc. Math. France* **129** (2001), no. 1, 115–139.

[21] *G. Yu*, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.* **139** (2000), no. 1, 201-240.

Department of Mathematics And Statistics, Fylde College, Lancaster University, Lancaster, LA1 4YF, United Kingdom

and

Alfred Renyi Institute, Budapest, Hungary

Email address: g.elek@lancaster.ac.uk