

STIRLING PERMUTATION CODES

SHI-MEI MA, HAO QI, JEAN YEH, AND YEONG-NAN YEH

ABSTRACT. The development of the theories of the second-order Eulerian polynomials began with the works of Buckholtz and Carlitz in their studies of an asymptotic expansion. Gessel-Stanley introduced Stirling permutations and presented combinatorial interpretations of the second-order Eulerian polynomials. The Stirling permutations have been extensively studied by many researchers. The aim of this paper is to give substantial generalizations of the second-order Eulerian polynomials. We develop a general method for finding equidistributed statistics on Stirling permutations. Firstly, we show that the up-down-pair statistic is equidistributed with ascent-plateau statistic, and that the exterior up-down-pair statistic is equidistributed with left ascent-plateau statistic. Secondly, we introduce the Stirling permutation codes. Several equidistribution results follow from simple applications. In particular, we find that six bivariable set-valued statistics are equidistributed on the set of Stirling permutations. As applications, we extend a classical result independently established by Dumont and Bóna. Thirdly, we explore bijections among Stirling permutation codes, perfect matchings and trapezoidal words. We then show the e -positivity of the enumerators of Stirling permutations by left ascent-plateaux, exterior up-down-pairs and right plateau-descents. In the final part, the e -positivity of the multivariate k -th order Eulerian polynomials is established, which improves a result of Janson-Kuba-Panholzer and generalizes a recent result of Chen-Fu.

Keywords: Stirling permutations; Set-valued statistics; e -Positivity; Symmetric functions

CONTENTS

1. Introduction	2
1.1. Notation and preliminaries	2
1.2. Motivation and the organization of the paper	5
2. The ascent-plateau and up-down-pair statistics	6
3. Problem 3 and the Stirling permutation code (SP-code for short)	8
4. Bijections among SP-codes, trapezoidal words and perfect matchings	15
5. The e -positivity of the enumerators by (lap, eud, rpd)	17
5.1. Preliminary	17
5.2. Proof of Theorem 22	19
6. The e -positivity of multivariate k -th order Eulerian polynomials	21
References	25

1. INTRODUCTION

1.1. Notation and preliminaries.

The development of the theories of the second-order Eulerian polynomials began with the works of Buckholtz [3] and Carlitz [4] in their studies of an asymptotic expansion. Further developments continued with the contributions of Riordan [30], Gessel-Stanley [15], Dumont [11], Park [32], Bóna [1], Janson-Kuba-Panholzer [19], Haglund-Visontai [16] and Chen-Fu [7, 8]. The aim of this paper is to give substantial generalizations of these polynomials.

Put

$$e^{nx} = \sum_{r=0}^n \frac{(nx)^r}{r!} + \frac{(nx)^n}{n!} S_n(x). \quad (1)$$

where n is a positive integer and x an arbitrary complex number. The study of (1) was initiated by Ramanujan, see [29]. Buckholtz [3] found that

$$S_n(x) = \sum_{r=0}^{k-1} \frac{1}{n^r} U_r(x) + O(n^{-k}),$$

where

$$U_r(x) = (-1)^r \left(\frac{x}{1-x} \frac{d}{dx} \right)^r \frac{x}{1-x} = (-1)^r \frac{C_r(x)}{(1-x)^{2r+1}}.$$

Subsequently, Carlitz [4] discovered that

$$C_n(x) = (1-x)^{2n+1} \sum_{k=0}^{\infty} \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\} x^k,$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are the *Stirling numbers of the second kind*. The polynomials $C_n(x)$ are now known as the *second-order Eulerian polynomials* and they satisfy the recurrence relation

$$C_{n+1}(x) = (2n+1)x C_n(x) + x(1-x) \frac{d}{dx} C_n(x), \quad C_0(x) = 1.$$

The first few $C_n(x)$ are

$$C_1(x) = x, C_2(x) = x + 2x^2, C_3(x) = x + 8x^2 + 6x^3.$$

In [30], Riordan found that $C_n(x)$ are the enumerators of Riordan trapezoidal words of length n by number of distinct numbers. Subsequently, Gessel-Stanley [15] discovered that $C_n(x)$ are the descent polynomials of Stirling permutations in \mathcal{Q}_n . The Stirling permutations have been extensively studied by many researchers, see [8, 13, 22, 23, 27, 32] and references therein.

For $\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathbb{N}^n$, let $\mathbf{n} = \{1^{m_1}, 2^{m_2}, \dots, n^{m_n}\}$ be a multiset, where i appears m_i times. A multipermutation of \mathbf{n} is a sequence of its elements. Denote by $\mathfrak{S}_{\mathbf{n}}$ the set of multipermutations of \mathbf{n} . We say that the multipermutation σ of \mathbf{n} is a *Stirling permutation* if $\sigma_s \geq \sigma_i$ as soon as $\sigma_i = \sigma_j$ and $i < s < j$. Denote by $\mathcal{Q}_{\mathbf{n}}$ the set of Stirling permutations of \mathbf{n} . When $m_1 = \dots = m_n = 1$, the set $\mathcal{Q}_{\mathbf{n}}$ reduces to the symmetric group \mathfrak{S}_n , which is the set of permutations of $[n] = \{1, 2, \dots, n\}$. When $m_1 = \dots = m_n = 2$, the set $\mathcal{Q}_{\mathbf{n}}$ reduces to \mathcal{Q}_n , which is the set of ordinary Stirling permutations of $[n]_2 = \{1^2, 2^2, \dots, n^2\}$. Except where explicitly stated, we always assume that all Stirling permutations belong to \mathcal{Q}_n , and for $\sigma \in \mathcal{Q}_n$, we set $\sigma_0 = \sigma_{n+1} = 0$. For example,

$$\mathcal{Q}_1 = \{11\}, \quad \mathcal{Q}_2 = \{1122, 1221, 2211\}.$$

Definition 1. For $\sigma \in \mathfrak{S}_{\mathbf{n}}$, any entry σ_i is called

- (i) an ascent (resp. descent, plateau) if $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i > \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$), where $i \in \{0, 1, 2, \dots, m_1 + m_2 + \dots + m_n\}$ and we set $\sigma_0 = \sigma_{m_1+m_2+\dots+m_n+1} = 0$, see [1, 15];
- (ii) an ascent-plateau (resp. plateau-descent) if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$ (resp. $\sigma_{i-1} = \sigma_i > \sigma_{i+1}$), where $i \in \{2, 3, \dots, m_1 + m_2 + \dots + m_n - 1\}$, see [24, 26];
- (iii) a left ascent-plateau if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in [m_1 + m_2 + \dots + m_n - 1]$ and $\sigma_0 = 0$, see [24, 26];
- (iv) a right plateau-descent if $\sigma_{i-1} = \sigma_i > \sigma_{i+1}$, where $i \in \{2, 3, \dots, m_1 + m_2 + \dots + m_n\}$ and we set $\sigma_{m_1+m_2+\dots+m_n+1} = 0$, see [23, 27].

Let $\text{asc}(\sigma)$ (resp. $\text{des}(\sigma)$, $\text{plat}(\sigma)$, $\text{ap}(\sigma)$, $\text{pd}(\sigma)$, $\text{lap}(\sigma)$, $\text{rpd}(\sigma)$) denotes the number of ascents (resp. descents, plateaux, ascent-plateaux, plateau-descents, left ascent-plateaux, right plateau-descents) of σ . The reverse bijection $\sigma \rightarrow \sigma^r$ on \mathcal{Q}_n defined by $\sigma_i^r = \sigma_{2n+1-i}$ shows that

$$\begin{aligned} \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{pd}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{lap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{rpd}(\sigma)}. \end{aligned}$$

In [1], Bóna introduced the plateau statistic plat and discovered that

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)},$$

which leads to a remarkable equidistributed result on \mathcal{Q}_n :

$$\sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)}. \quad (2)$$

It should be noted that the plateau statistic has been considered by Dumont [11] in the name of the repetition statistic, and it went unnoticed until it was independently studied by Bóna. A trivariate version of the second-order Eulerian polynomial is defined by

$$C_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} z^{\text{plat}(\sigma)}. \quad (3)$$

Dumont [11, p. 317] found that

$$C_{n+1}(x, y, z) = xyz \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) C_n(x, y, z), \quad C_1(x, y, z) = xyz. \quad (4)$$

which implies that $C_n(x, y, z)$ is symmetric in the variables x, y and z , and so (2) holds. The symmetry of $C_n(x, y, z)$ was rediscovered by Janson [18, Theorem 2.1] by constructing an urn model. In [16], Haglund and Visontai introduced a refinement of the polynomial $C_n(x, y, z)$ by indexing each ascent, descent and plateau by the values where they appear. Recently, using the theory of context-free grammars, Chen and Fu [8] found that $C_n(x, y, z)$ is e -positive, i.e.,

$$C_n(x, y, z) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} (x+y+z)^i (xy+yz+zx)^j (xyz)^k, \quad (5)$$

where the coefficient $\gamma_{n,i,j,k}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices.

A *rooted tree* of order n with the vertices labelled $1, 2, \dots, n$, is an increasing tree if the node labelled 1 is distinguished as the root, and the labels along any path from the root are increasing. An *increasing plane tree*, usually called *plane recursive tree*, is an increasing tree with the children of each vertex are linearly ordered (from left to right, say). A *0-1-2- \dots - k increasing plane tree* on $[n]$ is an increasing plane tree with each vertex with at most k children. The *degree* of a vertex in a rooted tree is meant to be the number of its children (sometimes called outdegree). The *depth-first walk* of a rooted plane tree starts at the root, goes first to the leftmost child of the root, explores that branch (recursively, using the same rules), returns to the root, and continues with the next child of the root, until there are no more children left.

The following definition will be used repeatedly.

Definition 2 ([11]). *A ternary increasing tree of size n is an increasing plane tree with $3n + 1$ nodes in which each interior node has label and three children (a left child, a middle child and a right child), and exterior nodes have no children and no labels.*

Let \mathcal{T}_n denote the set of ternary increasing trees of size n , see Figure 1 for instance. For any $T \in \mathcal{T}_n$, it is clear that T has exactly $2n + 1$ exterior nodes. Let $\text{exl}(T)$ (resp. $\text{exm}(T)$, $\text{exr}(T)$) denotes the number of exterior left nodes (resp. exterior middle nodes, exterior right nodes) in T . Using a recurrence relation that is equivalent to (4), Dumont [11, Proposition 1] found that

$$C_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{plat}(\sigma)} z^{\text{des}(\sigma)} = \sum_{T \in \mathcal{T}_n} x^{\text{exl}(T)} y^{\text{exm}(T)} z^{\text{exr}(T)}. \quad (6)$$

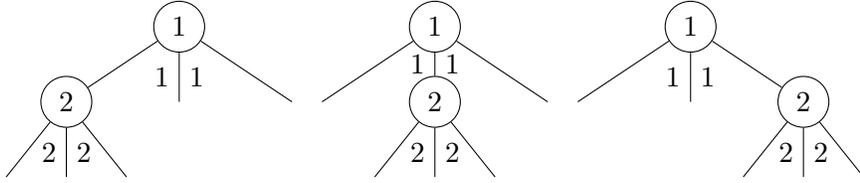


FIGURE 1. The ternary increasing trees of order 2 encoded by 2211, 1221, 1122, and their SP-codes are given by $((0, 0), (1, 1))$, $((0, 0)(1, 2))$ and $((0, 0)(1, 3))$, respectively .

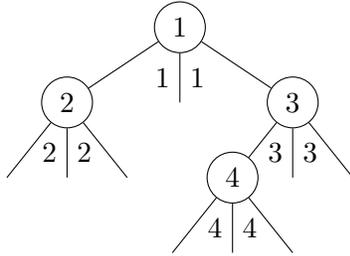


FIGURE 2. An order 3 ternary increasing tree encoded by 22114433, and its SP-code is $((0, 0), (1, 1), (1, 3), (3, 1))$.

1.2. Motivation and the organization of the paper.

In [18], Janson gave a bijection between plane recursive trees and Stirling permutations, which has previously been used by Koganov [20]. Subsequently, Janson-Kuba-Panholzer [19, Section 3] gave a detailed proof of the bijection between $(k + 1)$ -ary increasing trees and k -Stirling permutations, which was independently introduced by Gessel [32, p46]. Following the proof of [19, Theorem 1], let ϕ be the bijection between ternary increasing trees and Stirling permutations that is defined as follows:

- (i) Given $T \in \mathcal{T}_n$. Between the 3 edges of T going out from a node labelled v , we place 2 integers v . Now we perform the depth-first walk and code T by the sequence of the labels visited as we go around T . Let $\phi(T)$ be the code. In particular, the ternary increasing tree of order 1 is encoded by 11. A ternary increasing tree of order n is encoded by a string of $2n$ integers, where each of the labels $1, 2, \dots, n$ appears exactly 2 times. It is clear that for each $i \in [n]$, the elements occurring between the two occurrences of i are larger than i , since we can only visit nodes with higher labels. Hence the code $\phi(T)$ is a Stirling permutation, see Figures 1 and 2 for illustrations;
- (ii) The inverse of ϕ can be described as follows. Given $\sigma \in \mathcal{Q}_n$. We proceed recursively starting at step one by decomposing σ as $u_1 1 u_2 1 u_3$, where the u_i 's are again Stirling permutations. The smallest label in each u_i is attached to the root node labelled 1. One can recursively apply this procedure to each u_i to obtain the tree representation, and $\phi^{-1}(\sigma)$ is a ternary increasing tree.

Motivated by the work of Bóna [1], Dumont [10], Haglund-Visontai [16] and Janson-Kuba-Panholzer [19, Section 3], this paper is devoted to the following problem.

Problem 3. *Give some applications of the bijection ϕ and develop a general method for finding equidistributed statistics on Stirling permutations.*

In Section 2, we introduce the up-down-pair statistic ud and the exterior up-down-pair statistic eud on Stirling permutations, and we show that ud is equidistributed with ap and eud is equidistributed with lap . Therefore, we get

$$\begin{aligned} \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ud}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{pd}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{lap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{eud}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{rpd}(\sigma)}. \end{aligned} \quad (7)$$

In Section 3, by introducing the Stirling permutation code, we present various results concerning Problem 3. In particular, in Theorems 13 and 14, we present several bivariate generalizations of (2). The last two identities in Theorem 14 give the generalizations of (2) and (7) simultaneously. In Section 4, we establish bijections among SP-codes, trapezoidal words and perfect matchings. In Section 5, we show the e -positivity of the enumerators of Stirling permutations by (lap, eud, rpd) . In Section 6, we show the e -positivity of the multivariate k -th order Eulerian polynomials, which generalizes (5) and improves a classical result of Janson-Kuba-Panholzer [19].

2. THE ASCENT-PLATEAU AND UP-DOWN-PAIR STATISTICS

The number of elements in a set C is called the cardinality of C , written $\#C$. The *type A Eulerian polynomials* $A_n(x)$ [17], the *type B Eulerian polynomials* $B_n(x)$ [2], the *ascent-plateau polynomials* (they also known as 1/2-Eulerian polynomials) $M_n(x)$ [24, 31] and the *left ascent-plateau polynomials* $N_n(x)$ [24] can be respectively defined as follows:

$$\begin{aligned} A_n(x) &= \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)}, \quad B_n(x) = \sum_{\pi \in \mathcal{B}_n} x^{\text{des}_B(\pi)}, \\ M_n(x) &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)}, \quad N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{lap}(\sigma)}, \end{aligned}$$

where \mathcal{B}_n denotes the the hyperoctahedral group of rank n ,

$$\text{des}_B(\pi) = \#\{i \in \{0, 1, 2, \dots, n-1\} \mid \pi(i) > \pi(i+1)\}, \text{ and we set } \pi(0) = 0.$$

These polynomials share several similar properties, including recursions [15, 24], real-rootedness [1, 16], combinatorial expansions [8, 10, 22, 27, 34] and asymptotic distributions [17]. For convenience, we collect the recurrence relations of these polynomials:

$$\begin{aligned} A_{n+1}(x) &= (n+1)x A_n(x) + x(1-x) \frac{d}{dx} A_n(x), \\ B_{n+1}(x) &= (2nx+1+x) B_n(x) + 2x(1-x) \frac{d}{dx} B_n(x), \\ M_{n+1}(x) &= (2nx+1) M_n(x) + 2x(1-x) \frac{d}{dx} M_n(x), \\ N_{n+1}(x) &= (2n+1)x N_n(x) + 2x(1-x) \frac{d}{dx} N_n(x), \end{aligned}$$

with $A_0(x) = B_0(x) = C_0(x) = M_0(x) = N_0(x) = 1$. There are close connections among these polynomials (see [25] for details):

$$\begin{aligned} 2^n A_n(x) &= \sum_{i=0}^n \binom{n}{i} N_i(x) N_{n-i}(x), \\ B_n(x) &= \sum_{i=0}^n \binom{n}{i} M_i(x) N_{n-i}(x). \end{aligned} \tag{8}$$

Let $\mathcal{Q}_n^{(1)}$ be the set of Stirling permutations of the multiset $\{1, 2^2, 3^2, \dots, n^2, (n+1)^2\}$, i.e., this multiset has exactly one 1 and two copies of i for all $2 \leq i \leq n+1$. In particular,

$$\mathcal{Q}_2^{(1)} = \{12233, 12332, 13322, 33122, 22133, 22331, 23321, 33221\}.$$

By considering the position of the entry 1 in a Stirling permutation $\sigma \in \mathcal{Q}_n^{(1)}$, the following result immediately follows from (8).

Proposition 4. *For $n \geq 1$, we have*

$$\begin{aligned} 2^n A_n(x) &= \sum_{\sigma \in \mathcal{Q}_n^{(1)}} x^{\text{lap}(\sigma)}, \\ B_n(x) &= \sum_{\sigma \in \mathcal{Q}_n^{(1)}} x^{\text{ap}(\sigma)}. \end{aligned}$$

Definition 5. *Let $\sigma \in \mathcal{Q}_n$. An entry σ_i is called an up-down-pair entry if $\sigma_{i-1} < \sigma_i = \sigma_j > \sigma_{j+1}$, where $i < j$. The two equal entries σ_i and σ_j may appear arbitrarily far apart. The up-down-pair statistic ud and the exterior up-down-pair statistic eud are respectively defined as follows:*

$$\begin{aligned} \text{ud}(\sigma) &= \#\{i \in [2n-2] : \sigma_i \text{ is an up-down-pair entry, and we set } \sigma_0 = 0\}, \\ \text{eud}(\sigma) &= \#\{i \in [2n-1] : \sigma_i \text{ is an up-down-pair entry, and we set } \sigma_0 = \sigma_{2n+1} = 0\}. \end{aligned}$$

Example 6. *We have*

$$\begin{aligned} \text{ud}(123321) &= \text{ud}(0123321) = 2, \text{ud}(331221) = \text{ud}(0331221) = 2, \\ \text{eud}(123321) &= \text{eud}(01233210) = 3, \text{eud}(331221) = \text{eud}(03312210) = 2. \end{aligned}$$

The main result of this section is given as follows.

Theorem 7. *We have*

$$\begin{aligned} \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ud}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{lap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{eud}(\sigma)}. \end{aligned} \tag{9}$$

Proof. Let

$$\begin{aligned} M_n(x) &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)} = \sum_{i=0}^{n-1} M_{n,i} x^i, \\ N_n(x) &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{lap}(\sigma)} = \sum_{i=1}^n N_{n,i} x^i. \end{aligned}$$

Then the numbers $M_{n,i}$ and $N_{n,i}$ respectively satisfy the recurrence relations

$$\begin{aligned} M_{n+1,i} &= (2i+1)M_{n,i} + (2n-2i+2)M_{n,i-1}, \\ N_{n+1,i} &= 2iN_{n,i} + (2n-2i+3)N_{n,i-1}, \end{aligned} \tag{10}$$

with $M_{0,0} = N_{0,0} = 1$ and $M_{0,i} = N_{0,i} = 0$ if $i > 0$, see [24].

Let $m_{n,i} = \#\{\sigma \in \mathcal{Q}_n : \text{ud}(\sigma) = i\}$. It is clear that $m_{1,0} = M_{1,0} = 1$, since $\text{ud}(11) = \text{ud}(011) = 0$. There are two ways to obtain an element $\sigma' \in \mathcal{Q}_{n+1}$ with $\text{ud}(\sigma') = i$ from an element $\sigma \in \mathcal{Q}_n$ by inserting two copies of n into consecutive positions:

- (c_1) If $\text{ud}(\sigma) = i$, then we can insert the two copies of n before an up-down-pair entry or right after the second appearance of it. Moreover, we can insert the two copies of n at the end of σ . This accounts for $(2i+1)m_{n,i}$ possibilities;
- (c_2) If $\text{ud}(\sigma) = i-1$, then we insert the two consecutive copies of n into one of the remaining $2n+1 - (2(i-1)+1) = 2n-2i+2$ positions. This accounts for $(2n-2i+2)m_{n,i-1}$ possibilities.

Thus the numbers $m_{n,i}$ satisfy the same recursion and initial conditions as $M_{n,i}$, so they agree.

Define $n_{n,i} = \#\{\sigma \in \mathcal{Q}_n : \text{eud}(\sigma) = i\}$. Clearly, $n_{1,1} = N_{1,1} = 1$, since $\text{eud}(0110) = 1$. Similarly, there are two ways to obtain an element $\sigma' \in \mathcal{Q}_{n+1}$ with $\text{eud}(\sigma') = i$ from an element $\sigma \in \mathcal{Q}_n$ by inserting two copies of n into consecutive positions:

- (c_1) If $\text{eud}(\sigma) = i$, then we can insert the two copies of n before an up-down-pair entry or right after the second appearance of it. This accounts for $2in_{n,i}$ possibilities;
- (c_2) If $\text{eud}(\sigma) = i-1$, then we insert the two consecutive copies of n into one of the remaining $2n+1 - 2(i-1) = 2n-2i+3$ positions. This accounts for $(2n-2i+3)n_{n,i-1}$ possibilities.

Thus the numbers $n_{n,i}$ satisfy the same recursion and initial conditions as $N_{n,i}$, so they agree. \square

3. PROBLEM 3 AND THE STIRLING PERMUTATION CODE (SP-CODE FOR SHORT)

Recall that a sequence (e_1, e_2, \dots, e_n) is an *inversion sequence* if $0 \leq e_i < i$ for all $i \in [n]$. It is well known that inversion sequences of length n are in bijection with permutations in \mathfrak{S}_n . As a dual of inversion sequence, by using the bijection ϕ , we shall introduce a common code for ternary increasing trees and Stirling permutations.

Recall that for any ternary increasing tree $T \in \mathcal{T}_n$, each interior node has label and three children (one at the left, a middle child and a right child), and exterior nodes have no children and no labels. For convenience, we introduce the following definition.

Definition 8. *A simplified ternary increasing tree is a ternary increasing tree with no exterior nodes. The degree of a vertex in a ternary increasing tree is meant to be the number of its children in the simplified ternary increasing tree.*

In fact, a simplified ternary increasing tree is the same as the ordinary ternary increasing tree, it is only a simplified version. A node in a simplified ternary increasing tree with no children is called a leaf, and any interior node has at most three children (left child, middle child or right child). For example, Figure 3 gives the set of simplified ternary increasing trees of order 2. In the following discussion, a ternary increasing tree is always meant to be a simplified ternary

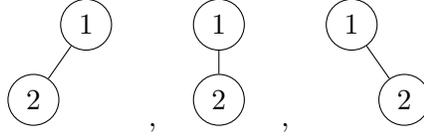


FIGURE 3. The simplified ternary increasing trees of order 2.

increasing tree. A ternary increasing tree of size n can be built up from the root 1 by successively adding nodes $2, 3, \dots, n$. Clearly, node 2 is a child of the root 1 and the root 1 has at most three children (a left, a middle or a right child), see Figure 3 for illustrations. For $2 \leq i \leq n$, when node i is inserted, we distinguish three cases:

- (c_1) if it is the left child of a node $v \in [i-1]$, then the node i is coded as $[v, 1]$;
- (c_2) if it is the middle child of a node $v \in [i-1]$, then the node i is coded as $[v, 2]$;
- (c_3) if it is the right child of a node $v \in [i-1]$, then the node i is coded as $[v, 3]$.

Thus the node i is coded as a 2-tuple (a_{i-1}, b_{i-1}) , where $1 \leq a_{i-1} \leq i-1$, $1 \leq b_{i-1} \leq 3$ and $(a_i, b_i) \neq (a_j, b_j)$ for all $1 \leq i < j \leq n-1$. By convention, the root 1 is coded as $(0, 0)$. Therefore, a ternary increasing tree of size n corresponds naturally to a *build-tree code* $((0, 0), (a_1, b_1), \dots, (a_{n-1}, b_{n-1}))$. Using the bijection ϕ between ternary increasing trees and Stirling permutations, one can see that the build-tree code is the same as the *Stirling permutation code*, which is defined as follows.

Definition 9. A 2-tuples sequence $C_n = ((0, 0), (a_1, b_1), (a_2, b_2), \dots, (a_{n-1}, b_{n-1}))$ of length n is a *Stirling permutation code* (SP-code for short) if $1 \leq a_i \leq i$, $1 \leq b_i \leq 3$ and $(a_i, b_i) \neq (a_j, b_j)$ for all $1 \leq i < j \leq n-1$.

Let CQ_n be the set of SP-codes of length n . In particular, $\text{CQ}_1 = \{(0, 0)\}$ and $\text{CQ}_2 = \{(0, 0)(1, 1), (0, 0)(1, 2), (0, 0)(1, 3)\}$, see Figure 1.

Theorem 10. The set CQ_n is in a natural bijection with the set \mathcal{Q}_n , i.e., $\text{CQ}_n \cong \mathcal{Q}_n$.

Proof. For $n \geq 2$, there are three cases to obtain an element of \mathcal{Q}_n from an element $\sigma \in \mathcal{Q}_{n-1}$ by putting the two copies of n between σ_i and σ_{i+1} : $\sigma_i < \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$, $\sigma_i > \sigma_{i+1}$. Set $\Gamma(11) = (0, 0)$. When $n \geq 2$, the bijection $\Gamma : \mathcal{Q}_n \rightarrow \text{CQ}_n$ can be defined as follows:

- (c_1) $\sigma_i < \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_{i+1}, 1)$;
- (c_2) $\sigma_i = \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_{i+1}, 2)$;
- (c_3) $\sigma_i > \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_i, 3)$.

See Figure 1 and Example 11 for illustrations. □

Example 11. Given $\sigma = 551443312662 \in \mathcal{Q}_5$. We give the procedure of creating its SP-code:

$$\begin{aligned} 11 &\Leftrightarrow (0, 0), \\ 1122 &\Leftrightarrow (0, 0)(1, 3), \\ 133122 &\Leftrightarrow (0, 0)(1, 3)(1, 2), \\ 14433122 &\Leftrightarrow (0, 0)(1, 3)(1, 2)(3, 1), \\ 5514433122 &\Leftrightarrow (0, 0)(1, 3)(1, 2)(3, 1)(1, 1), \\ 551443312662 &\Leftrightarrow (0, 0)(1, 3)(1, 2)(3, 1)(1, 1)(2, 2). \end{aligned}$$

Thus $\Gamma(\sigma) = (0, 0)(1, 3)(1, 2)(3, 1)(1, 1)(2, 2)$. Conversely, we get $\Gamma^{-1}(\Gamma(\sigma)) = \sigma$.

For $\sigma \in \mathcal{Q}_n$, let

$$\begin{aligned} \text{Asc}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i\}, \\ \text{Plat}(\sigma) &= \{\sigma_i \mid \sigma_i = \sigma_{i+1}\}, \\ \text{Des}(\sigma) &= \{\sigma_i \mid \sigma_i > \sigma_{i+1}\}, \\ \text{Lap}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_{i+1}\}, \\ \text{Rpd}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} = \sigma_i > \sigma_{i+1}\}, \\ \text{Eud}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_j > \sigma_{j+1}, i < j\}, \\ \text{Dasc}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i < \sigma_{i+1}\}, \\ \text{Dplat}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} > \sigma_i = \sigma_{i+1}\}, \\ \text{Ddes}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} > \sigma_i > \sigma_{i+1}\}, \\ \text{Pasc}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} = \sigma_i < \sigma_{i+1}\}, \\ \text{Apd}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_{i+1} > \sigma_{i+2}\}, \\ \text{Uu}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_j < \sigma_{j+1}, i < j\}, \\ \text{Dd}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} > \sigma_i = \sigma_j > \sigma_{j+1}, i < j\} \end{aligned}$$

denote the sets of ascents, plateaux, descents, left ascent-plateaux, right plateau-descents, exterior up-down-pairs, double ascents, descent-plateaux, double descents, plateau-ascents, ascent-plateau-descents, up-up-pairs and down-down-pairs of σ , respectively. We use $\text{dasc}(\sigma)$, $\text{dplat}(\sigma)$, $\text{ddes}(\sigma)$, $\text{pasc}(\sigma)$, $\text{apd}(\sigma)$, $\text{uu}(\sigma)$ and $\text{dd}(\sigma)$ to denote the number of double ascents, descent-plateaux, double descents, plateau-ascents, ascent-plateau-descents, up-up-pairs and down-down-pairs of σ , respectively, i.e., $\text{dasc}(\sigma) = \#\text{Dasc}(\sigma)$, $\text{dplat}(\sigma) = \#\text{Dplat}(\sigma)$, $\text{ddes}(\sigma) = \#\text{Ddes}(\sigma)$, $\text{pasc}(\sigma) = \#\text{Pasc}(\sigma)$, $\text{apd}(\sigma) = \#\text{Apd}(\sigma)$, $\text{uu}(\sigma) = \#\text{Uu}(\sigma)$ and $\text{dd}(\sigma) = \#\text{Dd}(\sigma)$.

Using the bijection ϕ , it is easily seen that the set-valued statistics on Stirling permutations listed in Table 1 correspond to the given set-valued statistics on SP-codes. We illustrate these correspondences in Example 12. By Table 1, a large number of properties of Stirling permutations can be easily deduced.

Statistics on Stirling permutation	Statistics on SP-code
Asc (ascent)	$[n] - \{a_i \mid (a_i, 1) \in C_n\}$
Plat (plateau)	$[n] - \{a_i \mid (a_i, 2) \in C_n\}$
Des (descent)	$[n] - \{a_i \mid (a_i, 3) \in C_n\}$
Lap (left ascent-plateau)	$[n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_n\}$
Rpd (right plateau-descent)	$[n] - \{a_i \mid (a_i, 2) \text{ or } (a_i, 3) \in C_n\}$
Eud (exterior up-down-pair)	$[n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 3) \in C_n\}$
Dasc (double ascent)	$\{a_i \mid (a_i, 1) \notin C_n \text{ \& } (a_i, 2) \in C_n\}$
Dplat (descent-plateau)	$\{a_i \mid (a_i, 1) \in C_n \text{ \& } (a_i, 2) \notin C_n\}$
Ddes (double descent)	$\{a_i \mid (a_i, 2) \in C_n \text{ \& } (a_i, 3) \notin C_n\}$
Pasc (plateau-ascent)	$\{a_i \mid (a_i, 2) \notin C_n \text{ \& } (a_i, 3) \in C_n\}$
Apd (ascent-plateau-descent)	$\{a_i \mid (a_i, 1) \notin C_n \text{ \& } (a_i, 2) \notin C_n \text{ \& } (a_i, 3) \notin C_n\}$
Uu (up-up-pair)	$\{a_i \mid (a_i, 1) \notin C_n \text{ \& } (a_i, 3) \in C_n\}$
Dd (down-down-pair)	$\{a_i \mid (a_i, 1) \in C_n \text{ \& } (a_i, 3) \notin C_n\}$

TABLE 1. The correspondences of statistics on Stirling permutations and SP-codes

Example 12. Let $\sigma = 77441223315665$. The corresponding SP-code is given by

$$C_7 = (0, 0)(1, 2)(2, 3)(1, 1)(1, 3)(5, 2)(4, 1).$$

Then we have

$$\begin{aligned} \text{Asc}(\sigma) &= [7] - \{a_i \mid (a_i, 1) \in C_7\} = \{2, 3, 5, 6, 7\}, \\ \text{Plat}(\sigma) &= [7] - \{a_i \mid (a_i, 2) \in C_7\} = \{2, 3, 4, 6, 7\}, \\ \text{Des}(\sigma) &= [7] - \{a_i \mid (a_i, 3) \in C_7\} = \{3, 4, 5, 6, 7\}, \\ \text{Lap}(\sigma) &= [7] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_7\} = \{2, 3, 6, 7\}, \\ \text{Rpd}(\sigma) &= [7] - \{a_i \mid (a_i, 2) \text{ or } (a_i, 3) \in C_7\} = \{3, 4, 6, 7\}, \\ \text{Eud}(\sigma) &= [7] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 3) \in C_7\} = \{3, 5, 6, 7\}, \\ \text{Dasc}(\sigma) &= \{a_i \mid (a_i, 1) \notin C_7 \text{ \& } (a_i, 2) \in C_7\} = \{5\}, \\ \text{Dplat}(\sigma) &= \{a_i \mid (a_i, 1) \in C_7 \text{ \& } (a_i, 2) \notin C_7\} = \{4\}, \\ \text{Ddes}(\sigma) &= \{a_i \mid (a_i, 2) \in C_7 \text{ \& } (a_i, 3) \notin C_7\} = \{5\}, \\ \text{Pasc}(\sigma) &= \{a_i \mid (a_i, 2) \notin C_7 \text{ \& } (a_i, 3) \in C_7\} = \{2\}, \\ \text{Apd}(\sigma) &= \{a_i \mid (a_i, 1) \notin C_7 \text{ \& } (a_i, 2) \notin C_7 \text{ \& } (a_i, 3) \notin C_7\} = \{3, 6\}, \\ \text{Uu}(\sigma) &= \{a_i \mid (a_i, 1) \notin C_7 \text{ \& } (a_i, 3) \in C_7\} = \{2\}, \\ \text{Dd}(\sigma) &= \{a_i \mid (a_i, 1) \in C_7 \text{ \& } (a_i, 3) \notin C_7\} = \{4\}. \end{aligned}$$

The following two results give several generalizations of (2), which can be proved by switching some 2-tuples in the corresponding SP-codes.

Theorem 13. *The six bivariable set-valued statistics are all equidistributed on \mathcal{Q}_n :*

$$\begin{aligned} & (\text{Asc}, \text{Dasc}), (\text{Plat}, \text{Dplat}), (\text{Des}, \text{Ddes}), \\ & (\text{Asc}, \text{Uu}), (\text{Plat}, \text{Pasc}), (\text{Des}, \text{Dd}). \end{aligned}$$

So we get the following four identities:

$$\begin{aligned} \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{dasc}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{dplat}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{ddes}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{dasc}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{pasc}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{ddes}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{uu}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{pasc}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{dd}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{uu}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{dplat}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{dd}(\sigma)}. \end{aligned}$$

Proof. Consider Table 1. For $C_n \in \text{CQ}_n$, if we switch the 2-tuples $(a_i, 1)$ and $(a_i, 2)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$\begin{aligned} & ([n] - \{a_i \mid (a_i, 1) \in C_n\}, \{a_i \mid (a_i, 1) \notin C_n \ \& \ (a_i, 2) \in C_n\}), \\ & ([n] - \{a_i \mid (a_i, 2) \in C_n\}, \{a_i \mid (a_i, 1) \in C_n \ \& \ (a_i, 2) \notin C_n\}). \end{aligned}$$

which yields $(\text{Asc}, \text{Dasc})$ and $(\text{Plat}, \text{Dplat})$ are equidistributed on \mathcal{Q}_n .

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 3)$ for all i (if any), then we find that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$\begin{aligned} & ([n] - \{a_i \mid (a_i, 1) \in C_n\}, \{a_i \mid (a_i, 1) \notin C_n \ \& \ (a_i, 2) \in C_n\}), \\ & ([n] - \{a_i \mid (a_i, 3) \in C_n\}, \{a_i \mid (a_i, 2) \in C_n \ \& \ (a_i, 3) \notin C_n\}), \end{aligned}$$

which yields $(\text{Asc}, \text{Dasc})$ and $(\text{Des}, \text{Ddes})$ are equidistributed on \mathcal{Q}_n .

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 2)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$\begin{aligned} & ([n] - \{a_i \mid (a_i, 1) \in C_n\}, \{a_i \mid (a_i, 1) \notin C_n \ \& \ (a_i, 3) \in C_n\}), \\ & ([n] - \{a_i \mid (a_i, 2) \in C_n\}, \{a_i \mid (a_i, 2) \notin C_n \ \& \ (a_i, 3) \in C_n\}), \end{aligned}$$

which yields (Asc, Uu) and $(\text{Plat}, \text{Pasc})$ are equidistributed on \mathcal{Q}_n .

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 3)$ for all i (if any), then we find that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$\begin{aligned} & ([n] - \{a_i \mid (a_i, 1) \in C_n\}, \{a_i \mid (a_i, 1) \notin C_n \ \& \ (a_i, 3) \in C_n\}), \\ & ([n] - \{a_i \mid (a_i, 3) \in C_n\}, \{a_i \mid (a_i, 1) \in C_n \ \& \ (a_i, 3) \notin C_n\}), \end{aligned}$$

which yields (Asc, Uu) and (Des, Dd) are equidistributed on \mathcal{Q}_n .

If we switch the 2-tuples $(a_i, 2)$ and $(a_i, 3)$ for all i (if any), then we find that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$\begin{aligned} & ([n] - \{a_i \mid (a_i, 2) \in C_n\}, \{a_i \mid (a_i, 1) \in C_n \ \& \ (a_i, 2) \notin C_n\}), \\ & ([n] - \{a_i \mid (a_i, 3) \in C_n\}, \{a_i \mid (a_i, 1) \in C_n \ \& \ (a_i, 3) \notin C_n\}), \end{aligned}$$

which yields (Plat, Dplat) and (Des, Dd) are equidistributed on \mathcal{Q}_n .

In conclusion, the proof is completed by using transitivity. \square

Theorem 14. *The six bivariable set-valued statistics are equidistributed on \mathcal{Q}_n :*

$$\begin{aligned} & (\text{Asc}, \text{Lap}), (\text{Plat}, \text{Lap}), (\text{Des}, \text{Rpd}), \\ & (\text{Asc}, \text{Eud}), (\text{Plat}, \text{Rpd}), (\text{Des}, \text{Eud}). \end{aligned}$$

So we get the following six identities:

$$\begin{aligned} \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{lap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{lap}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{rpd}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{lap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{rpd}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{rpd}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{eud}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{rpd}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{eud}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{eud}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{lap}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{eud}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{lap}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{rpd}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{eud}(\sigma)}, \\ \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{eud}(\sigma)} &= \sum_{\sigma \in \mathcal{Q}_n} x^{\text{plat}(\sigma)} y^{\text{lap}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des}(\sigma)} y^{\text{rpd}(\sigma)}, \end{aligned}$$

where the last two identities give the generalizations of (2) and (7) simultaneously.

Proof. Consider Table 1. For $C_n \in \text{CQ}_n$, if we switch the 2-tuples $(a_i, 1)$ and $(a_i, 2)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$\begin{aligned} & ([n] - \{a_i \mid (a_i, 1) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_n\}), \\ & ([n] - \{a_i \mid (a_i, 2) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_n\}), \end{aligned}$$

which yields (Asc, Lap) and (Plat, Lap) are equidistributed on \mathcal{Q}_n .

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 3)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$\begin{aligned} & ([n] - \{a_i \mid (a_i, 1) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_n\}), \\ & ([n] - \{a_i \mid (a_i, 3) \in C_n\}, [n] - \{a_i \mid (a_i, 2) \text{ or } (a_i, 3) \in C_n\}), \end{aligned}$$

which yields (Asc, Lap) and (Des, Rpd) are equidistributed on \mathcal{Q}_n .

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 2)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$\begin{aligned} & ([n] - \{a_i \mid (a_i, 1) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 3) \in C_n\}), \\ & ([n] - \{a_i \mid (a_i, 2) \in C_n\}, [n] - \{a_i \mid (a_i, 2) \text{ or } (a_i, 3) \in C_n\}), \end{aligned}$$

which yields (Asc, Eud) and (Plat, Rpd) are equidistributed on \mathcal{Q}_n .

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 3)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$([n] - \{a_i \mid (a_i, 1) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 3) \in C_n\}),$$

$$([n] - \{a_i \mid (a_i, 3) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 3) \in C_n\}),$$

which yields (Asc, Eud) and (Des, Eud) are equidistributed on \mathcal{Q}_n .

If we switch the 2-tuples $(a_i, 2)$ and $(a_i, 3)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n :

$$([n] - \{a_i \mid (a_i, 1) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_n\}),$$

$$([n] - \{a_i \mid (a_i, 1) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 3) \in C_n\}),$$

which yields (Asc, Lap) and (Asc, Eud) are equidistributed on \mathcal{Q}_n .

In conclusion, the proof is completed by using transitivity. \square

We say that a joint distribution of (set-valued) statistics or a multivariate polynomial is *symmetric* if it is invariant under any permutation of its indeterminates. We can now present the following two results, and the proofs follow in the same way as the proof of Theorem 14.

Theorem 15. *The six set-valued statistics are all equidistributed on \mathcal{Q}_n :*

$$\text{Dasc}, \text{Dplat}, \text{Ddes}, \text{Pasc}, \text{Uu}, \text{Dd}.$$

Moreover, if one select any two set-valued statistics from these six set-valued statistics, then the selected two set-valued statistics are symmetric on \mathcal{Q}_n .

Theorem 16. *The following triple set-valued statistics are all symmetric on \mathcal{Q}_n :*

$$(\text{Asc}(\sigma), \text{Plat}(\sigma), \text{Des}(\sigma)), (\text{Lap}(\sigma), \text{Rpd}(\sigma), \text{Eud}(\sigma)),$$

$$(\text{Dasc}(\sigma), \text{Pasc}(\sigma), \text{Dd}(\sigma)), (\text{Ddes}(\sigma), \text{Dplat}(\sigma), \text{Uu}(\sigma)).$$

We now give an example to illustrate the proof of the symmetric of (Ddes, Pasc).

Example 17. *Let σ and C_7 be the given in Example 12. Then we have $\text{Ddes}(\sigma) = \{5\}$ and $\text{Pasc}(\sigma) = \{2\}$. Let Φ be the bijection on CQ_7 that is defined by*

$$(a_i, 2) \leftrightarrow (a_i, 3), \text{ where } 1 \leq i \leq 6.$$

In other words, we just switch the 2-tuples $(a_i, 2)$ and $(a_i, 3)$ for all i (if any). Thus

$$\Phi((0, 0)(1, 2)(2, 3)(1, 1)(1, 3)(5, 2)(4, 1)) = (0, 0)(1, 3)(2, 2)(1, 1)(1, 2)(5, 3)(4, 1).$$

It is routine to verify that $\phi^{-1}(\Phi(C_7)) = 774415\mathbf{56612332}$. Therefore, we have

$$\text{Ddes}(\phi^{-1}(\Phi(C_7))) = \{2\}, \text{ Pasc}(\phi^{-1}(\Phi(C_7))) = \{5\}.$$

Corollary 18. *For $n \geq 1$, both of the following two polynomials are symmetric in their variables:*

$$C_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{asc}(\sigma)} y^{\text{plat}(\sigma)} z^{\text{des}(\sigma)},$$

$$N_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{lap}(\sigma)} y^{\text{rpd}(\sigma)} z^{\text{eud}(\sigma)}.$$

As discussed in introduction, the symmetry of $C_n(x, y, z)$ has been extensively studied, see [8, 16] and references therein. In Section 5, we shall show the e -positivity of $N_n(x, y, z)$.

4. BIJECTIONS AMONG SP-CODES, TRAPEZOIDAL WORDS AND PERFECT MATCHINGS

Following Riordan [30], we say that a word $t = t_1 t_2 \cdots t_n$ is a *Riordan trapezoidal word* if the element t_i takes the values $1, 2, \dots, 2i - 1$. Let RT_n be the set of Riordan trapezoidal words of length n . In particular,

$$\text{RT}_1 = \{1\}, \text{RT}_2 = \{11, 12, 13\}.$$

Besides (6), Dumont [11] gave another two interpretations of $C_n(x, y, z)$ in terms of Dumont trapezoidal words and perfect matchings. The Dumont trapezoidal word [11] is a variant Riordan trapezoidal word. We say that a word $w = w_1 w_2 \cdots w_n$ is a *Dumont trapezoidal word* of length n if $0 \leq |w_i| < i$, where w_i are all integers. Let DT_n denote the set of Dumont trapezoidal words of length n . For convenience, we set $\bar{i} = -i$. In particular,

$$\text{DT}_1 = \{0\}, \text{DT}_2 = \{00, 01, 0\bar{1}\}.$$

Given $w \in \text{DT}_n$. Let $\text{dist}(w)$ be the number of distinct elements in w , and we define

$$\text{nneg}(\sigma) = n - \{w_i \mid w_i < 0\}, \text{npos}(\sigma) = n - \{w_i \mid w_i > 0\}.$$

Dumont [11, Section 2.3] found that

$$C_n(x, y, z) = \sum_{w \in \text{DT}_n} x^{\text{dist}(w)} y^{\text{nneg}(\sigma)} z^{\text{npos}(\sigma)}.$$

A *perfect matching* of $[2n]$ is a set partition of $[2n]$ with blocks (disjoint nonempty subsets) of size exactly 2. Let \mathcal{M}_{2n} be the set of perfect matchings of $[2n]$, and let $M \in \mathcal{M}_{2n}$. The *standard form* of M is a list of blocks

$$\{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$$

such that $i_r < j_r$ for all $1 \leq r \leq n$ and $1 = i_1 < i_2 < \cdots < i_n$. As usual, we always write M in standard form. It is well known that M can be regarded as a fixed-point-free involution on $[2n]$. In particular,

$$\mathcal{M}_2 = \{(1, 2)\}, \mathcal{M}_4 = \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}.$$

Motivated by Theorem 10, it is natural to explore the bijections among SP-codes, trapezoidal words and perfect matchings.

Theorem 19. *For $n \geq 1$, we have*

$$\text{CQ}_n \cong \text{DT}_n \cong \text{RT}_n \cong \mathcal{M}_{2n}. \quad (11)$$

Proof. (i) Let $\varphi_1 : \text{DT}_n \rightarrow \text{RT}_n$ be the bijection defined by

$$\varphi_1(w_i) = \begin{cases} 1, & \text{if } w_i = 0; \\ 2k, & \text{if } w_i = k > 0; \\ 2k + 1, & \text{if } w_i = \bar{k} < 0, \end{cases} \quad (12)$$

which yields that $\text{DT}_n \cong \text{RT}_n$.

(ii) Now we start to construct a bijection, denoted by φ_2 , from RT_n to \mathcal{M}_{2n} . When $n = 1$, we have $\text{RT}_1 = \{1\}$. Set $\varphi_2(\mathbf{1}) = (\mathbf{1}, 2)$. When $n = 2$, we have $\text{RT}_2 = \{11, 12, 13\}$. We set

$$\varphi_2(\mathbf{11}) = (\mathbf{1}, 4)(2, 3), \varphi_2(\mathbf{12}) = (\mathbf{2}, 4)(1, 3), \varphi_2(\mathbf{13}) = (\mathbf{3}, 4)(1, 2).$$

We proceed by induction. Let $n = m$. Suppose that φ_2 is a bijection from RT_m to \mathcal{M}_{2m} . Given $M = (i_1, j_1)(i_2, j_2) \cdots (i_m, j_m) \in \mathcal{M}_{2m}$. Suppose that $\varphi_2(t) = M$, where $t = t_1 t_2 \cdots t_m \in \text{RT}_m$. For $1 \leq i \leq 2m + 1$, we let φ_2 be the following algorithm, which can be used to generate perfect matchings in \mathcal{M}_{2m+2} :

- $\varphi_2(t_1 t_2 \cdots t_m i) = (i, 2m + 2)(i'_1, j'_1)(i'_2, j'_2) \cdots (i'_m, j'_m)$, where $(i'_1, j'_1)(i'_2, j'_2) \cdots (i'_m, j'_m)$ is a perfect matching of the elements in $[2m + 2] - \{i, 2m + 2\}$ such that the elements in $(i'_1, j'_1) \cdots (i'_m, j'_m)$ keeps the same order relationships they have in $(i_1, j_1) \cdots (i_m, j_m)$.

Clearly, φ_2 is the desired bijection. See Example 20 for an illustration.

(iii) Now we start to construct a bijection, denoted by φ_3 , from DT_n to CQ_n . When $n = 1$, we set $\varphi_3(0) = (0, 0)$. When $n = m$, suppose φ_3 is a bijection from DT_m to CQ_m . Consider the case $n = m + 1$. Let $w = w_1 w_2 \cdots w_{m+1} \in \text{DT}_{m+1}$. Then $w' = w_1 w_2 \cdots w_m \in \text{DT}_m$ and $\varphi_3(w') = ((0, 0), (a_1, b_1), (a_2, b_2) \dots, (a_{m-1}, b_{m-1})) \in \text{CQ}_m$. We distinguish three cases:

- (c₁) $w_{m+1} = k$ and $k \in \{w_1, w_2, \dots, w_m\}$ if and only if

$$(a_m, b_m) = (j, 1), \text{ where } j = \max\{i \mid w_i = k, 1 \leq i \leq m\};$$

- (c₂) $w_{m+1} = -j$ and $j \notin \{w_1, w_2, \dots, w_m\}$ if and only if $(a_m, b_m) = (j, 2)$, where $1 \leq j \leq m$;

- (c₃) $w_{m+1} = j$ and $j \notin \{w_1, w_2, \dots, w_m\}$ if and only if $(a_m, b_m) = (j, 3)$, where $1 \leq j \leq m$.

It is routine to check that φ_3 is the desired bijection. In particular, when $n = 2, 3$, we have

$$\begin{aligned} \varphi_3(00) &= (0, 0)(1, 1), \quad \varphi_3(0\bar{1}) = (0, 0)(1, 2), \quad \varphi_3(01) = (0, 0)(1, 3); \\ \varphi_3(000) &= (0, 0)(1, 1)(2, 1), \quad \varphi_3(00\bar{1}) = (0, 0)(1, 1)(1, 2), \quad \varphi_3(001) = (0, 0)(1, 1)(1, 3), \\ \varphi_3(00\bar{2}) &= (0, 0)(1, 1)(2, 2), \quad \varphi_3(002) = (0, 0)(1, 1)(2, 3), \quad \varphi_3(0\bar{1}0) = (0, 0)(1, 2)(1, 1), \\ \varphi_3(0\bar{1}1) &= (0, 0)(1, 2)(1, 3), \quad \varphi_3(0\bar{1}\bar{1}) = (0, 0)(1, 2)(2, 1), \quad \varphi_3(0\bar{1}\bar{2}) = (0, 0)(1, 2)(2, 2), \\ \varphi_3(0\bar{1}2) &= (0, 0)(1, 2)(2, 3), \quad \varphi_3(010) = (0, 0)(1, 3)(1, 1), \quad \varphi_3(01\bar{1}) = (0, 0)(1, 3)(1, 2), \\ \varphi_3(011) &= (0, 0)(1, 3)(2, 1), \quad \varphi_3(01\bar{2}) = (0, 0)(1, 3)(2, 2), \quad \varphi_3(012) = (0, 0)(1, 3)(2, 3). \end{aligned}$$

This completes the proof. □

As illustrations of $\varphi_2(w)$ and $\varphi_3(w)$, we give another two examples.

Example 20. Given $t = 1-1-1-3-2-10$. We give the procedure of creating $\varphi_2(t)$.

$$\begin{aligned} \mathbf{1} &\Leftrightarrow (1, 2), \\ 1-\mathbf{1} &\Leftrightarrow (\mathbf{1}, 4)(2, 3), \\ 1-1-\mathbf{1} &\Leftrightarrow (\mathbf{1}, 6)(2, 5)(3, 4), \\ 1-1-1-\mathbf{3} &\Leftrightarrow (\mathbf{3}, 8)(1, 7)(2, 6)(4, 5), \\ 1-1-1-3-\mathbf{2} &\Leftrightarrow (\mathbf{2}, 10)(4, 9)(1, 8)(3, 7)(5, 6), \\ 1-1-1-3-2-\mathbf{10} &\Leftrightarrow (\mathbf{10}, 12)(2, 11)(4, 9)(1, 8)(3, 7)(5, 6). \end{aligned}$$

Thus $\varphi_2(t) = (10, 12)(2, 11)(4, 9)(1, 8)(3, 7)(5, 6)$. Conversely, we get $\varphi_2^{-1}(\varphi_2(t)) = t$.

Example 21. Given $w = 0-0-0-\bar{1}-1-5-1$. We give the procedure of creating $\varphi_3(w)$.

$$\begin{aligned} \mathbf{0} &\Leftrightarrow (0, 0), \\ 0-\mathbf{0} &\Leftrightarrow (0, 0)(\mathbf{1}, \mathbf{1}), \\ 0-0-\mathbf{0} &\Leftrightarrow (0, 0)(1, 1)(\mathbf{2}, \mathbf{1}), \\ 0-0-0-\bar{\mathbf{1}} &\Leftrightarrow (0, 0)(1, 1)(2, 1)(\mathbf{1}, \mathbf{2}), \\ 0-0-0-\bar{1}-\mathbf{1} &\Leftrightarrow (0, 0)(1, 1)(2, 1)(1, 2)(\mathbf{1}, \mathbf{3}), \\ 0-0-0-\bar{1}-1-5 &\Leftrightarrow (0, 0)(1, 1)(2, 1)(1, 2)(1, 3)(\mathbf{5}, \mathbf{3}), \\ 0-0-0-\bar{1}-1-5-1 &\Leftrightarrow (0, 0)(1, 1)(2, 1)(1, 2)(1, 3)(5, 3)(\mathbf{5}, \mathbf{1}). \end{aligned}$$

Thus $\varphi_3(w) = (0, 0)(1, 1)(2, 1)(1, 2)(1, 3)(5, 3)(5, 1)$. Conversely, we get $\varphi_3^{-1}(\varphi_3(w)) = w$.

5. THE e -POSITIVITY OF THE ENUMERATORS BY (lap, eud, rpd)

5.1. Preliminary.

Let $X_n = \{x_1, x_2, \dots, x_n\}$ be a set of commuting variables. Define

$$S_n(x) = \prod_{i=1}^n (x - x_i) = \sum_{k=0}^n (-1)^k e_k x^{n-k}.$$

Then the k -th elementary symmetric function associated with X_n is defined by

$$e_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

In particular,

$$e_0 = 1, \quad e_1 = \sum_{i=1}^n x_i, \quad e_n = x_1 x_2 \cdots x_n.$$

A function $f(x_1, x_2, \dots) \in \mathbb{R}[x_1, x_2, \dots]$ is said to be *symmetric* if it is invariant under any permutation of its indeterminates. We say that a symmetric function is *e -positive* if it can be written as a nonnegative linear combination of elementary symmetric functions.

For an alphabet A , let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A . Following Chen [6], a *context-free grammar* over A is a function $G : A \rightarrow \mathbb{Q}[[A]]$ that replaces each letter in A by a formal function over A . The formal derivative D_G with respect to G satisfies the derivation rules:

$$D_G(u + v) = D_G(u) + D_G(v), \quad D_G(uv) = D_G(u)v + uD_G(v).$$

So the Leibniz rule holds:

$$D_G^n(uv) = \sum_{k=0}^n \binom{n}{k} D_G^k(u) D_G^{n-k}(v).$$

See [12, 27] for some examples of context-free grammars.

Recently, two methods are developed in the theory of context-free grammars, i.e., grammatical labeling and the change of grammars. A *grammatical labeling* is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar (see [7] for details). A *change of grammars* is a substitution method in which the original grammars are replaced with functions of other grammars. In particular,

the following type of change of grammars can be used to study the γ -positivity and partial γ -positivity of several enumerative polynomials (see [8, 27, 28] for details):

$$\begin{cases} u = xy, \\ v = x + y. \end{cases}$$

Let G be the following grammar

$$G = \{x \rightarrow xyz, y \rightarrow xyz, z \rightarrow xyz\}. \quad (13)$$

It has been shown by Dumont [11], Haglund-Visontai [16] and Chen et al. [9] that

$$D_G^n(x) = C_n(x, y, z).$$

Very recently, Chen-Fu [8] introduced a new type of change of grammars:

$$\begin{cases} u = x + y + z, \\ v = xy + yz + zx, \\ w = xyz. \end{cases} \quad (14)$$

Combining (13) and (14), one can easily verify that $D_G(u) = 3w$, $D_G(v) = 2uw$, $D_G(w) = vw$, which yield a new grammar

$$H = \{u \rightarrow 3w, v \rightarrow 2uw, w \rightarrow vw\}. \quad (15)$$

For any $n \geq 1$, Chen-Fu [8,] discovered that

$$C_n(x, y, z) = D_G^n(x) = D_H^{n-1}(w) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} u^i v^j w^k, \quad (16)$$

which leads to the e -positive expansion (5).

We can now present the main result of this section.

Theorem 22. *Let*

$$N_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{lap}(\sigma)} y^{\text{eud}(\sigma)} z^{\text{rpd}(\sigma)}. \quad (17)$$

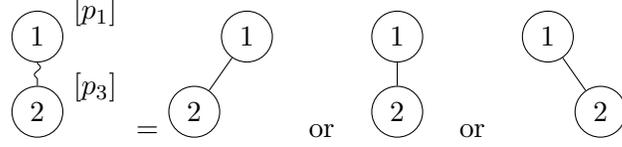
Then we have

$$N_n(x, y, z) = \sum_{i+2j+3k=2n+1} 3^i \gamma_{n,i,j,k} (x + y + z)^j (xyz)^k, \quad (18)$$

where $\gamma_{n,i,j,k}$ is the same as in (16), i.e., $\gamma_{n,i,j,k}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices.

Throughout this section, we always let

$$w_1 = x + y + z, \quad w_2 = xy + yz + zx, \quad w_3 = xyz.$$

FIGURE 4. $N_2(x, y, z) = (x + y + z)xyz$.

Below are the polynomials $N_n(x, y, z)$ for $n \leq 6$:

$$N_1(x, y, z) = w_3,$$

$$N_2(x, y, z) = w_1w_3,$$

$$N_3(x, y, z) = w_1^2w_3 + 6w_3^2,$$

$$N_4(x, y, z) = w_1^3w_3 + 24w_1w_3^2 + 6w_3^3,$$

$$N_5(x, y, z) = w_1^4w_3 + 66w_1^2w_3^2 + 42w_1w_3^3 + 144w_3^3,$$

$$N_6(x, y, z) = w_1^5w_3 + 156w_1^3w_3^2 + 192w_1^2w_3^3 + 1224w_1w_3^3 + 540w_3^4.$$

Example 23. For the elements in \mathcal{Q}_2 , we have

$$\text{lap}(1122) = \text{lap}(011220) = 2, \quad \text{eud}(1122) = \text{eud}(011220) = 1, \quad \text{rpd}(1122) = \text{rpd}(011220) = 1,$$

$$\text{lap}(1221) = \text{lap}(012210) = 1, \quad \text{eud}(1221) = \text{eud}(012210) = 2, \quad \text{rpd}(1221) = \text{rpd}(012210) = 1,$$

$$\text{lap}(2211) = \text{lap}(022110) = 1, \quad \text{eud}(2211) = \text{eud}(022110) = 1, \quad \text{rpd}(2211) = \text{rpd}(022110) = 2.$$

Thus $N_2(x, y, z) = xyz(x + y + z) = w_1w_3$. See Figure 4 for an illustration.

5.2. Proof of Theorem 22.

As discussed in Section 3, we shall use simplified ternary increasing trees for convenience. See Figure 4 for an illustration, where the left figure represents the three different figures in the right. The weights of $\sigma \in \mathcal{Q}_n$ and $C_n \in \text{CQ}_n$ are respectively defined as follows:

$$E_1(\sigma) = x^{\text{lap}(\sigma)}y^{\text{eud}(\sigma)}z^{\text{rpd}(\sigma)},$$

$$E_2(C_n) = x^{n - \#\{a_i | (a_i, 1) \text{ or } (a_i, 2) \in C_n\}}y^{n - \#\{a_i | (a_i, 1) \text{ or } (a_i, 3) \in C_n\}}z^{n - \#\{a_i | (a_i, 2) \text{ or } (a_i, 3) \in C_n\}}. \quad (19)$$

From Table 1, we see that $E_1(\sigma) = E_2(C_n)$, where C_n is the corresponding SP-code of σ . When $n = 1$, the SP-code $(0, 0)$ corresponds to the Stirling permutation 11. Clearly,

$$E_1(11) = E_2((0, 0)) = xyz = w_3.$$

When $n = 2$, the weights of elements in \mathcal{Q}_2 and CQ_n are given as follows:

$$\underbrace{2211 \leftrightarrow (0, 0)(1, 1)}_{xyz^2 = w_3z}, \quad \underbrace{1221 \leftrightarrow (0, 0)(1, 2)}_{xy^2z = w_3y}, \quad \underbrace{1122 \leftrightarrow (0, 0)(1, 3)}_{x^2yz = w_3x}$$

and the sum of weights is given by $w_3(x + y + z) = w_3w_1$.

Given $C_n = (0, 0)(a_1, b_1)(a_2, b_2) \cdots (a_{n-1}, b_{n-1}) \in \text{CQ}_n$. Consider the elements in CQ_{n+1} generated from C_n by appending the 2-tuples (a_n, b_n) , where $1 \leq a_n \leq n$ and $1 \leq b_n \leq 3$. Let T be the corresponding ternary increasing tree of C_n . We can add $n + 1$ to T as a child of a vertex, which is not of degree three. Let T' be the resulting ternary increasing tree. We first

give a labeling of T as follows. Label a leaf by p_3 , a degree one vertex by p_1 , a degree two vertex by p_2 and a degree three vertex by 1.

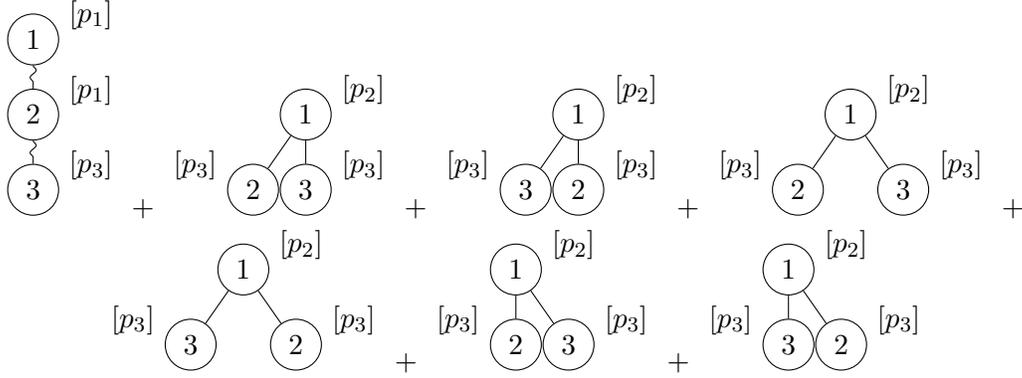


FIGURE 5. $N_3(x, y, z) = (x + y + z)^2xyz + 6(xyz)^2$

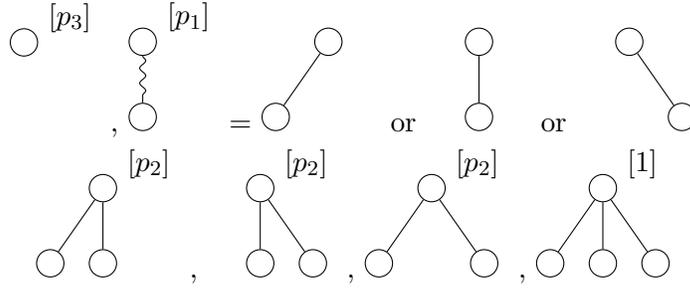


FIGURE 6. Labeling schemes of simplified ternary increasing trees.

The 2-tuples (a_n, b_n) can be divided into three classes:

- if $a_n \neq a_i$ for all $1 \leq i \leq n - 1$, then we must add $n + 1$ to a leaf of T . This operation corresponds to the change of weights

$$E_2(C_n) \rightarrow E_2(C_{n+1}) = E_2(C_n)(x + y + z), \tag{20}$$

which yields the substitution $p_3 \rightarrow p_1p_3$, see Figure 4 and the first case in Figure 5 for illustrations. Thus the contribution of any leaf to the weight is xyz and that of a degree one vertex is $x + y + z$ (represents this vertex may has a left, middle or right child). When we compute the corresponding enumerative polynomials of Stirling permutations, it follows from (20) that we need to set

$$p_1 = x + y + z, \quad p_3 = xyz; \tag{21}$$

- if there is exactly one 2-tuple (a_i, b_i) in C_n such that $a_n = a_i$, then we must add $n + 1$ to T as a child of the node a_i . Note that the node a_i already has one child $i + 1$, and $n + 1$ becomes the second child of a_i . There are six cases to add $n + 1$. As illustrations, the last six cases in Figure 5 are the total possibilities when we add 3 to the simplified ternary increasing trees in Figure 4 as the second child of the node 1. This operation

corresponds to the substitution $p_1 \rightarrow 6p_2p_3$. From (19), we see that each degree two vertex makes no contribution to the weight. Thus we need to set $p_2 = 1$ when we compute the corresponding enumerative polynomial of the joint distribution of (lap, eud, rpd);

- if there are exactly two 2-tuples (a_i, b_i) and (a_j, b_j) in C_n such that $a_n = a_i = a_j$ and $i < j$, then we must add $n + 1$ to T as the third child of a_i , and $n + 1$ becomes a leaf with label p_3 . This operation corresponds to the substitution $p_2 \rightarrow p_3$. From (19), we see that each degree three vertex makes no contribution to the weight, and so we label all degree three vertices by 1.

The aforementioned three cases exhaust all the possibilities to construct a SP-code of length $n + 1$ from a SP-code of length n by appending 2-tuples (a_n, b_n) . In conclusion, each case corresponds to an application of a substitution rule in the following grammar:

$$I = \{p_3 \rightarrow p_1p_3, p_1 \rightarrow 6p_2p_3, p_2 \rightarrow p_3\}, \quad (22)$$

and the corresponding labeling schemes are illustrated in Figure 6.

We can now conclude the following lemma.

Lemma 24. *Let I be the context-free grammar given by (22). For any $n \geq 1$, we have*

$$D_I^{n-1}(p_3) |_{p_1=x+y+z, p_2=1, p_3=xyz} = N_n(x, y, z).$$

In particular, $D_I(p_3) = p_1p_3$, $D_I^2(p_3) = p_1^2p_3 + 6p_2p_3^2$ and $D_I^3(p_3) = p_1^3p_3 + 24p_1p_2p_3^2 + 6p_3^3$.

A proof Theorem 22:

Proof. Consider a change of the grammar H , which is given by (15). Setting $w = p_3, v = p_1$ and $u = 3p_2$, we get

$$D_H(p_3) = p_1p_3, D_H(p_1) = 6p_2p_3, D_H(p_2) = p_3,$$

which yield the grammar I . It follows from (16) that

$$D_I^{n-1}(p_3) = D_H^{n-1}(w) |_{w=p_3, v=p_1, u=3p_2} = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} 3^i p_2^j p_1^k.$$

By (21) and Lemma 24, we obtain

$$N_n(x, y, z) = \sum_{i+2j+3k=2n+1} 3^i \gamma_{n,i,j,k} (x+y+z)^j (xyz)^k.$$

This completes the proof of Theorem 22. \square

6. THE e -POSITIVITY OF MULTIVARIATE k -TH ORDER EULERIAN POLYNOMIALS

A bivariate version of the Eulerian polynomial over the symmetric group is given as follows:

$$A_n(x, y) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{asc}(\pi)} y^{\text{des}(\pi)}.$$

Clearly, $A_n(x, 1) = A_n(1, x) = A_n(x)$. Carlitz and Scoville [5] showed that

$$A_{n+1}(x, y) = xy \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) A_n(x, y), \quad A_1(x, y) = xy.$$

Foata and Schützenberger [14] found that $A_n(x, y)$ has the gamma-expansion

$$A_n(x, y) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \gamma(n, k)(xy)^k(x+y)^{n+1-2k},$$

where $\gamma(n, k)$ counts permutations in \mathfrak{S}_n with k descents, but with no double descents.

In this section, we always let k be a given positive integer. A k -Stirling permutation of order n is a multiset permutation of $\{1^k, 2^k, \dots, n^k\}$ with the property that all elements between two occurrences of i are at least i , where $i \in [n]$, see [22, 23, 33] for the recent study on k -Stirling permutations and their variants. Let $\mathcal{Q}_n(k)$ be the set of k -Stirling permutations of order n . It is clear that $\mathcal{Q}_n(1) = \mathfrak{S}_n$, $\mathcal{Q}_n(2) = \mathcal{Q}_n$.

Let $\sigma \in \mathcal{Q}_n(k)$. The ascents, descents and plateaux of σ of are defined as before, where we always set $\sigma_0 = \sigma_{kn+1} = 0$. More precisely, an index i is called an ascent (resp. descent, plateau) of σ if $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i > \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$). It is clear that $\text{asc}(\sigma) + \text{des}(\sigma) + \text{plat}(\sigma) = kn + 1$. As a natural refinement of ascents, descents and plateaux, Janson-Kuba-Panholzer [19] introduced the following definition, and related the distribution of j -ascents, j -descents and j -plateaux in k -Stirling permutations with certain parameters in $(k+1)$ -ary increasing trees.

Definition 25 ([19]). *An index i is called a j -plateau (resp. j -descent, j -ascent) if i is a plateau (resp. descent, ascent) and there are exactly $j-1$ indices $\ell < i$ such that $a_\ell = a_i$.*

Let $\text{plat}_j(\sigma)$ be the number of j -plateaux of σ . For $\sigma \in \mathcal{Q}_n(k)$, it is clear that $\text{plat}_j(\sigma) \leq k-1$.

Example 26. *Consider the 4-Stirling permutation $\sigma = 111223333221$. The set of 1-plateaux is given by $\{1, 4, 6\}$, the set of 2-plateaux is given by $\{2, 7\}$, and the set of 3-plateaux is given by $\{8, 10\}$. Thus $\text{plat}_1(\sigma) = 3$ and $\text{plat}_2(\sigma) = \text{plat}_3(\sigma) = 2$.*

The multivariate k -th order Eulerian polynomials $C_n(x_1, \dots, x_{k+1})$ are defined by

$$C_n(x_1, x_2, \dots, x_{k+1}) = \sum_{\sigma \in \mathcal{Q}_n(k)} x_1^{\text{plat}_1(\sigma)} x_2^{\text{plat}_2(\sigma)} \dots x_{k-1}^{\text{plat}_{k-1}(\sigma)} x_k^{\text{des}(\sigma)} x_{k+1}^{\text{asc}(\sigma)}.$$

In particular, when $x_1 = z$, $x_2 = \dots = x_{k-1} = 0$, $x_k = y$ and $x_{k+1} = x$, the polynomials $C_n(x_1, x_2, \dots, x_{k+1})$ reduce to $C_n(x, y, z)$; when $x_1 = x_2 = \dots = x_{k-1} = 0$, $x_k = 1$ and $x_{k+1} = x$, the polynomials $C_n(x_1, x_2, \dots, x_{k+1})$ reduce to the Eulerian polynomials $A_n(x)$.

In the following, we always set $X_{k+1} = \{x_1, x_2, \dots, x_{k+1}\}$. Let e_i be the i -th elementary symmetric function associated with X_{k+1} . In particular,

$$e_0 = 1, \quad e_1 = x_1 + x_2 + \dots + x_{k+1}, \quad e_k = \sum_{i=1}^k \frac{e_{k+1}}{x_i}, \quad e_{k+1} = x_1 x_2 \dots x_{k+1}.$$

Lemma 27. *Let $G_1 = \{x_1 \rightarrow e_{k+1}, x_2 \rightarrow e_{k+1}, \dots, x_{k+1} \rightarrow e_{k+1}\}$ be a grammar, where $e_{k+1} = x_1 x_2 \dots x_{k+1}$. For $n \geq 1$, one has $D_{G_1}^n(x_1) = C_n(x_1, x_2, \dots, x_{k+1})$.*

Proof. We shall show that the grammar G_1 can be used to generate k -Stirling permutations. We first introduce a grammatical labeling of $\sigma \in \mathcal{Q}_n(k)$ as follows:

- (L₁) If i is an ascent, then put a superscript label x_{k+1} right after σ_i ;
- (L₂) If i is a descent, then put a superscript label x_k right after σ_i ;

(L_3) If i is a j -plateau, then put a superscript label x_j right after σ_i .

The weight of σ is defined as the product of the labels, that is

$$w(\sigma) = x_1^{\text{plat}_1(\sigma)} x_2^{\text{plat}_2(\sigma)} \dots x_{k-1}^{\text{plat}_{k-1}(\sigma)} x_k^{\text{des}(\sigma)} x_{k+1}^{\text{asc}(\sigma)}.$$

Recall that we always set $\sigma_0 = \sigma_{kn+1} = 0$. Thus the index 0 is always an ascent and the index kn is always a descent. Thus $\mathcal{Q}_1(k) = \{x_{k+1} 1^{x_1} 1^{x_2} 1^{x_3} \dots 1^{x_k}\}$. There are $k+1$ elements in $\mathcal{Q}_2(k)$ and they can be labeled as follows, respectively:

$$\begin{aligned} & x_{k+1} 1^{x_1} 1^{x_2} \dots 1^{x_{k-1}} 1^{x_{k+1}} 2^{x_1} 2^{x_2} \dots 2^{x_{k-1}} 2^{x_k}, \\ & x_{k+1} 1^{x_1} 1^{x_2} \dots 1^{x_{k-2}} 1^{x_{k+1}} 2^{x_1} 2^{x_2} \dots 2^{x_{k-1}} 2^{x_k} 1^{x_k}, \dots \\ & x_{k+1} 2^{x_1} 2^{x_2} \dots 2^{x_{k-1}} 2^{x_k} 1^{x_1} 1^{x_2} \dots 1^{x_{k-1}} 1^{x_k}. \end{aligned}$$

Note that $D_{G_1}(x_1) = e_{k+1}$ and $D_{G_1}^2(x_1) = e_k e_{k+1}$. Then the weight of the element in $\mathcal{Q}_1(k)$ is given by $D_{G_1}(x_1)$, and the sum of weights of the elements in $\mathcal{Q}_2(k)$ is given by $D_{G_1}^2(x_1)$. Hence the result holds for $n = 1, 2$. We proceed by induction on n . Suppose we get all labeled permutations in $\mathcal{Q}_{n-1}(k)$, where $n \geq 3$. Let σ' be obtained from $\sigma \in \mathcal{Q}_{n-1}(k)$ by inserting the string $nn \dots n$ with length k . Then the changes of labeling are illustrated as follows:

$$\begin{aligned} & \dots \sigma_i^{x_j} \sigma_{i+1} \dots \mapsto \dots \sigma_i^{x_{k+1}} n^{x_1} n^{x_2} \dots n^{x_k} \sigma_{i+1} \dots; \\ & \sigma^{x_k} \mapsto \sigma^{x_{k+1}} n^{x_1} n^{x_2} \dots n^{x_k}; \quad x_{k+1} \sigma \mapsto x_{k+1} n^{x_1} n^{x_2} \dots n^{x_k} \sigma. \end{aligned}$$

In each case, the insertion of the string $nn \dots n$ corresponds to one substitution rule in G_1 . Then the action of D_{G_1} on the set of weights of all elements in $\mathcal{Q}_{n-1}(k)$ gives the set of weights of all elements in $\mathcal{Q}_n(k)$. Therefore, we get the desired description of $C_n(x_1, x_2, \dots, x_{k+1})$. \square

It should be noted that in [19], there is no explicit connection to the k -th order Eulerian polynomials is brought up. By combining an urn model for the exterior leaves of $(k+1)$ -ary increasing trees and a bijection between $(k+1)$ -ary increasing trees and k -Stirling permutations, Janson-Kuba-Panholzer [19, Theorem 2, Theorem 8] found that the variables in $C_n(x_1, x_2, \dots, x_{k+1})$ are exchangeable. We can now present the main result of this section.

Theorem 28. *For $n \geq 2$ and $k \geq n - 2$, we have*

$$C_n(x_1, x_2, \dots, x_{k+1}) = \sum \gamma(n; i_1, i_2, \dots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \dots e_k^{i_2} e_{k+1}^{i_1}, \quad (23)$$

where the summation is over all sequences (i_1, i_2, \dots, i_n) of nonnegative integers such that $i_1 + i_2 + \dots + i_n = n$, $1 \leq i_1 \leq n - 1$, $i_n = 0$ or $i_n = 1$. When $i_n = 1$, one has $i_1 = n - 1$. The coefficients $\gamma(n; i_1, i_2, \dots, i_n)$ equal the numbers of 0 - 1 - 2 - \dots - k - $(k+1)$ increasing plane trees on $[n]$ with i_j degree $j - 1$ vertices for all $1 \leq j \leq n$.

Proof. Let G_1 be the grammar given in Lemma 27. Consider a change of G_1 . Note that $D_{G_1}(x_1) = e_{k+1}$, $D_{G_1}(e_i) = (k - i + 2)e_{i-1}e_{k+1}$ for $1 \leq i \leq k + 1$. Thus we get a new grammar

$$G_2 = \{x_1 \rightarrow e_{k+1}, e_i \rightarrow (k - i + 2)e_{i-1}e_{k+1} \text{ for } 1 \leq i \leq k + 1\}, \quad (24)$$

Note that $G_2(x_1) = e_{k+1}$, $G_2^2(x_1) = e_k e_{k+1}$, $G_2^3(x_1) = e_k^2 e_{k+1} + 2e_{k-1} e_{k+1}^2$,

$$D_{G_2}^4(x_1) = e_k^3 e_{k+1} + 8e_{k-1} e_k e_{k+1}^2 + 6e_{k-2} e_{k+1}^3,$$

$$D_{G_2}^5(x_1) = e_k^4 e_{k+1} + 22e_k^2 e_{k-1} e_{k+1}^2 + 16e_{k-1}^2 e_{k+1}^3 + 42e_{k-2} e_k e_{k+1}^3 + 24e_{k-3} e_{k+1}^4.$$

By induction, we assume that

$$G_2^m(x_1) = \sum \gamma(n; i_1, i_2, \dots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1}. \quad (25)$$

Note that

$$\begin{aligned} G_2^{m+1}(x_1) &= G_2 \left(\sum \gamma(n; i_1, i_2, \dots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1} \right) \\ &= \sum n i_n \gamma(n; i_1, i_2, \dots, i_n) e_{k-n+1} e_{k-n+2}^{i_n-1} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1+1} + \\ &\quad \sum (n-1) i_{n-1} \gamma(n; i_1, i_2, \dots, i_n) e_{k-n+2}^{i_n+1} e_{k-n+3}^{i_{n-1}-1} \cdots e_k^{i_2} e_{k+1}^{i_1+1} + \cdots + \\ &\quad \sum 2 i_2 \gamma(n; i_1, i_2, \dots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_{k-1}^{i_3+1} e_k^{i_2-1} e_{k+1}^{i_1+1} + \\ &\quad \sum i_1 \gamma(n; i_1, i_2, \dots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2+1} e_{k+1}^{i_1}, \end{aligned}$$

which yields that the expansion (25) holds for $n+1$. Combining Lemma 27 and (25), we get (23). By induction, one can verify that $i_1 + i_2 + \cdots + i_n = n$, $1 \leq i_1 \leq n-1$, $i_n = 1$ or $i_n = 0$.

Using (24), the combinatorial interpretation of $\gamma(n; i_1, i_2, \dots, i_n)$ can be proved along the same lines as the proof of [8, Theorem 4.1]. However, we give a direct proof of it for our purpose. Let T be a 0-1-2- \cdots - k -($k+1$) increasing plane tree on $[n]$. The labeling of T is given by labeling a degree i vertex by e_{k-i+1} for all $0 \leq i \leq k+1$. In particular, label a leaf by e_{k+1} and label a degree $k+1$ vertex by 1. Let T' be a 0-1-2- \cdots - k -($k+1$) increasing plane tree on $[n+1]$ by adding $n+1$ to T as a leaf. We can add $n+1$ to T only as a child of a vertex v that is not of degree $k+1$. For $1 \leq i \leq k+1$, if the vertex v is a degree $k-i+1$ vertex with label e_i , there are $k-i+2$ cases to attach $n+1$ (from left to right, say). In either case, in T' , the vertex v becomes a degree $k-i+2$ with label e_{i-1} and $n+1$ becomes a leaf with label e_{k+1} . Hence the insertion of $n+1$ corresponds to the substitution rule $e_i \rightarrow (k-i+2)e_{i-1}e_{k+1}$. Therefore, $G_2(x_1)$ equals the sum of the weights of 0-1-2- \cdots -($k+1$) increasing plane trees on $[n]$, and the combinatorial interpretation of $\gamma(n; i_1, i_2, \dots, i_n)$ follows. This completes the proof. \square

By using $G_2^{m+1}(x_1) = G_2(G_2^m(x_1))$, it is routine to verify that

$$\begin{aligned} \gamma(n+1; 1, n, 0, \dots, 0) &= \gamma(n; 1, n-1, 0, \dots, 0) = 1, \\ \gamma(n+1; n, 0, \dots, 0, 1) &= n\gamma(n; n-1, 0, \dots, 0, 1) = n!, \\ \gamma(n+1; i_1, i_2, \dots, i_n, 0) &= i_1 \gamma(n; i_1, i_2-1, i_3, \dots, i_n) + \\ &\quad \sum_{j=2}^{n-1} j(i_j+1) \gamma(n; i_1-1, i_2, \dots, i_{j-1}, i_j+1, i_{j+1}-1, i_{j+2}, \dots, i_n). \end{aligned}$$

Note that $\gamma(3; 2, 0, 1, 0, \dots, 0) = 2$, $\gamma(4; 2, 1, 1, 0, \dots, 0) = 8$ and

$$\gamma(n+1; 2, n-2, 1, 0, \dots, 0) = 2\gamma(n; 2, n-3, 1, 0, \dots, 0) + 2(n-1)\gamma(n; 1, n-1, 0, \dots, 0).$$

By induction, it is easy to verify that

$$\gamma(n; 2, n-3, 1, 0, \dots, 0) = 2^n - 2n \text{ for } n \geq 3. \quad (26)$$

Recall that the second-order Eulerian numbers $C_{n,j}$ satisfy the recurrence relation

$$C_{n+1,j} = jC_{n,j} + (2n + 2 - j)C_{n,j-1},$$

with the initial conditions $C_{1,1} = 1$ and $C_{1,j} = 0$ if $j \neq 1$ (see [1, 15]). In particular,

$$C_{n,2} = 2^{n+1} - 2(n + 1).$$

Comparing this with (26), we see that $\gamma(n; 2, n - 3, 1, 0, \dots, 0) = C_{n-1,2}$ for $n \geq 3$. Following Janson [18], the number $C_{n,j}$ equals the number of increasing plane trees on $[n + 1]$ with j leaves. So we immediately get the following result.

Proposition 29. *For $n \geq 2$ and $1 \leq j \leq n - 1$, we have*

$$C_{n-1,j} = \sum_{i_2+i_3+\dots+i_n=n-j} \gamma(n; j, i_2, \dots, i_{n-1}, i_n).$$

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SCHOOL OF MATHEMATICS AND STATISTICS, NORTHEASTERN UNIVERSITY AT QINHUANGDAO, HEBEI 066000, P.R. CHINA

Email address: shimeimapapers@163.com (S.-M. Ma)

COLLEGE OF MATHEMATICS AND PHYSICS, WENZHOU UNIVERSITY, WENZHOU 325035, P.R. CHINA

Email address: qihao@wzu.edu.cn (H. Qi)

DEPARTMENT OF MATHEMATICS, NATIONAL KAOHSIUNG NORMAL UNIVERSITY, KAOHSIUNG 82444, TAIWAN

Email address: chunchenyeh@nknu.edu.tw

COLLEGE OF MATHEMATICS AND PHYSICS, WENZHOU UNIVERSITY, WENZHOU 325035, P.R. CHINA

Email address: mayeh@math.sinica.edu.tw (Y.-N. Yeh)