

The Brown–Colbourn Conjecture on Zeros of Reliability Polynomials is False

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Abstract

We give counterexamples to the Brown–Colbourn conjecture on reliability polynomials, in both its univariate and multivariate forms. The multivariate Brown–Colbourn conjecture is false already for the complete graph K_4 . The univariate Brown–Colbourn conjecture is false for certain simple planar graphs obtained from K_4 by parallel and series expansion of edges. We show, in fact, that a graph has the multivariate Brown–Colbourn property if and only if it is series-parallel.

Key Words: Reliability polynomial; all-terminal reliability; Brown–Colbourn conjecture; Tutte polynomial; Potts model.

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1 Introduction

Let us consider a connected (multi)graph¹ $G = (V, E)$ as a communications network with unreliable communication channels, in which edge e is operational with probability p_e and failed with probability $1 - p_e$, independently for each edge. Let $R_G(\mathbf{p})$ be the probability that every node is capable of communicating with every other node (this is the so-called *all-terminal reliability*). Clearly we have

$$R_G(\mathbf{p}) = \sum_{\substack{A \subseteq E \\ (V, A) \text{ connected}}} \prod_{e \in A} p_e \prod_{e \in E \setminus A} (1 - p_e), \quad (1.1)$$

where the sum runs over all connected spanning subgraphs of G , and we have written $\mathbf{p} = \{p_e\}_{e \in E}$. We call $R_G(\mathbf{p})$ the (multivariate) *reliability polynomial* [7] for the graph G ; it is a multi-affine polynomial, i.e. of degree at most 1 in each variable separately. If the edge probabilities p_e are all set to the same value p , we write the corresponding univariate polynomial as $R_G(p)$, and call it the univariate reliability polynomial. We are interested in studying the zeros of these polynomials when the variables p_e (or p) are taken to be *complex* numbers.

Brown and Colbourn [5] studied a number of examples and made the following conjecture:

Univariate Brown–Colbourn conjecture. For any graph G , the zeros of the univariate reliability polynomial $R_G(p)$ all lie in the closed disc $|p - 1| \leq 1$. In other words, if $|p - 1| > 1$, then $R_G(p) \neq 0$.

Subsequently, one of us [16] proposed a multivariate extension of the Brown–Colbourn conjecture:

Multivariate Brown–Colbourn conjecture. For any graph G , if $|p_e - 1| > 1$ for all edges e , then $R_G(\mathbf{p}) \neq 0$.

Not long ago, Wagner [18] proved, using an ingenious and complicated construction, that the univariate Brown–Colbourn conjecture holds for all series-parallel graphs.² Subsequently,

¹Henceforth we omit the prefix “multi”. In this paper a “graph” is allowed to have loops and/or multiple edges unless explicitly stated otherwise.

²Unfortunately, there seems to be no completely standard definition of “series-parallel graph”; a plethora of slightly different definitions can be found in the literature [9, 7, 13, 14, 4]. So let us be completely precise about our own usage: we shall call a loopless graph *series-parallel* if it can be obtained from a forest by a finite sequence of series and parallel extensions of edges (i.e. replacing an edge by two edges in series or two edges in parallel). We shall call a general graph (allowing loops) series-parallel if its underlying loopless graph is series-parallel. Some authors write “obtained from a tree”, “obtained from K_2 ” or “obtained from C_2 ” in place of “obtained from a forest”; in our terminology these definitions yield, respectively, all *connected* series-parallel graphs, all connected series-parallel graphs whose blocks form a path, or all *2-connected* series-parallel graphs. See [4, Section 11.2] for a more extensive bibliography.

one of us [16, Remark 3 in Section 4.1] showed, by a two-line induction, that the multivariate Brown–Colbourn conjecture holds for all series-parallel graphs.³ Both the univariate and multivariate conjectures remained open for general graphs, but most workers in the field suspected that they would be true. (At least the present authors did.)

In this short note we would like to report that both the univariate and multivariate Brown–Colbourn conjectures are false! The multivariate conjecture is false already for the simplest non-series-parallel graph, namely the complete graph K_4 . As a corollary we will deduce that the univariate conjecture is false for a 4-vertex, 16-edge planar graph that can be obtained from K_4 by adding parallel edges, and for a 1512-vertex, 3016-edge simple planar graph that can be obtained from K_4 by adding parallel edges and then subdividing edges. So the Brown–Colbourn conjecture is not true even for simple planar graphs.

Furthermore, for the multivariate property we are able to obtain a complete characterization: a graph has the multivariate Brown–Colbourn property *if and only if* it is series-parallel.

It is convenient to restate the Brown–Colbourn conjectures in terms of the generating polynomial for connected spanning subgraphs,

$$C_G(\mathbf{v}) = \sum_{\substack{A \subseteq E \\ (V, A) \text{ connected}}} \prod_{e \in A} v_e, \quad (1.2)$$

where we have written $\mathbf{v} = \{v_e\}_{e \in E}$. This is clearly related to the reliability polynomial by

$$R_G(\mathbf{p}) = \left[\prod_{e \in E} (1 - p_e) \right] C_G\left(\frac{\mathbf{p}}{\mathbf{1} - \mathbf{p}}\right) \quad (1.3)$$

$$C_G(\mathbf{v}) = \left[\prod_{e \in E} (1 + v_e) \right] R_G\left(\frac{\mathbf{v}}{\mathbf{1} + \mathbf{v}}\right) \quad (1.4)$$

where $\mathbf{1}$ denotes the vector with all entries 1, and division of vectors is understood componentwise. The multivariate Brown–Colbourn conjecture then states that if G is a *loopless* graph and $|1 + v_e| < 1$ for all edges e , then $C_G(\mathbf{v}) \neq 0$. Loops must be excluded because a loop e multiplies C_G by a factor $1 + v_e$ but leaves R_G unaffected. Some workers also prefer to use the failure probabilities $q_e = 1 - p_e$ as the variables.

The plan of this paper is as follows: In Section 2 we show that the multivariate Brown–Colbourn conjecture fails for the complete graph K_4 . In Section 3 we review the series and parallel reduction formulae for the reliability polynomial. In Section 4 we show that the univariate Brown–Colbourn conjecture fails for certain graphs that are obtained from K_4 by adding parallel edges and then optionally subdividing edges. In Section 5 we complete these results by showing that a graph has the multivariate Brown–Colbourn property if and only if it is series-parallel.

³This proof is reproduced here as Theorem 5.6(c) \implies (a).

2 The multivariate Brown–Colbourn conjecture is false for K_4

For the complete graph K_4 , the univariate polynomial $C_G(v)$ is

$$C_{K_4}(v) = 16v^3 + 15v^4 + 6v^5 + v^6. \quad (2.1)$$

The roots of this polynomial all lie outside the disc $|1 + v| < 1$, so the univariate Brown–Colbourn conjecture is true for K_4 .

Let us now consider the bivariate situation, in which the six edges receive two different weights a and b . There are five cases:

- (a) One edge receives weight a and the other five receive weight b :

$$C_{K_4}(a, b) = (8b^3 + 5b^4 + b^5) + (8b^2 + 10b^3 + 5b^4 + b^5)a \quad (2.2)$$

- (b) A pair of nonintersecting edges receive weight a and the other four edges receive weight b :

$$C_{K_4}(a, b) = (4b^3 + b^4) + (8b^2 + 8b^3 + 2b^4)a + (4b + 6b^2 + 4b^3 + b^4)a^2 \quad (2.3)$$

- (c) A pair of intersecting edges receive weight a and the other four edges receive weight b :

$$C_{K_4}(a, b) = (3b^3 + b^4) + (10b^2 + 8b^3 + 2b^4)a + (3b + 6b^2 + 4b^3 + b^4)a^2 \quad (2.4)$$

- (d) A 3-star receives weight a and the complementary triangle receives weight b :

$$C_{K_4}(a, b) = (9b^2 + 3b^3)a + (6b + 9b^2 + 3b^3)a^2 + (1 + 3b + 3b^2 + b^3)a^3 \quad (2.5)$$

- (e) A three-edge path receives weight a and the complementary three-edge path receives weight b :

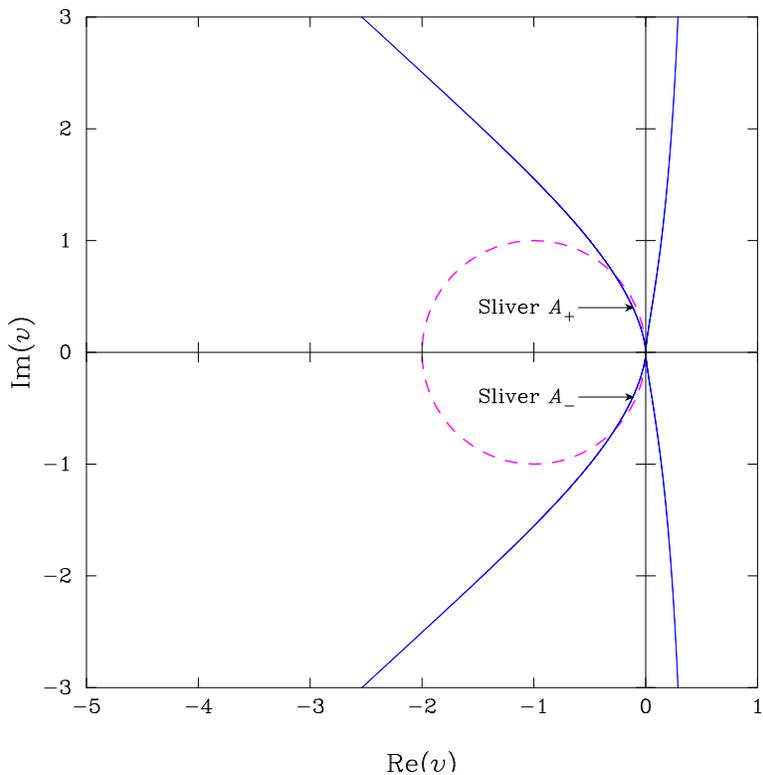
$$C_{K_4}(a, b) = b^3 + (7b^2 + 3b^3)a + (7b + 9b^2 + 3b^3)a^2 + (1 + 3b + 3b^2 + b^3)a^3 \quad (2.6)$$

We have plotted the roots a when b traces out the circle $|1 + b| = 1$, and vice versa. In cases (b) and (d) it turns out that the roots can enter the “forbidden discs” $|1 + a| < 1$ and $|1 + b| < 1$. This is shown in Figure 1 for case (b); blow-ups of the crucial regions are shown in Figure 2 both for case (b) and for case (d). As a result, counterexamples to the multivariate Brown–Colbourn conjecture can be obtained in these two cases: indeed, for any a lying in the region A_+ (resp. A_-), there exists $b \in B_-$ (resp. B_+) such that $C_{K_4}(a, b) = 0$, and conversely.

Let us note for future reference that the endpoint of the region A_{\pm} (resp. B_{\pm}) lies at $a = -1 + e^{\pm 2\pi i \alpha}$ (resp. $b = -1 + e^{\pm 2\pi i \beta}$), where $\alpha \approx 0.120692$ and $\beta \approx 0.164868$ in case (b), and $\alpha \approx 0.110198$ and $\beta \approx 0.030469$ in case (d).

We can understand this behavior analytically as follows: For each of the five cases, let us solve the equation $C_{K_4}(a, b) = 0$ for a in terms of b , expanding in power series for b near 0. We obtain:

Locus of a as b moves around $|1+b|=1$



Locus of b as a moves around $|1+a|=1$

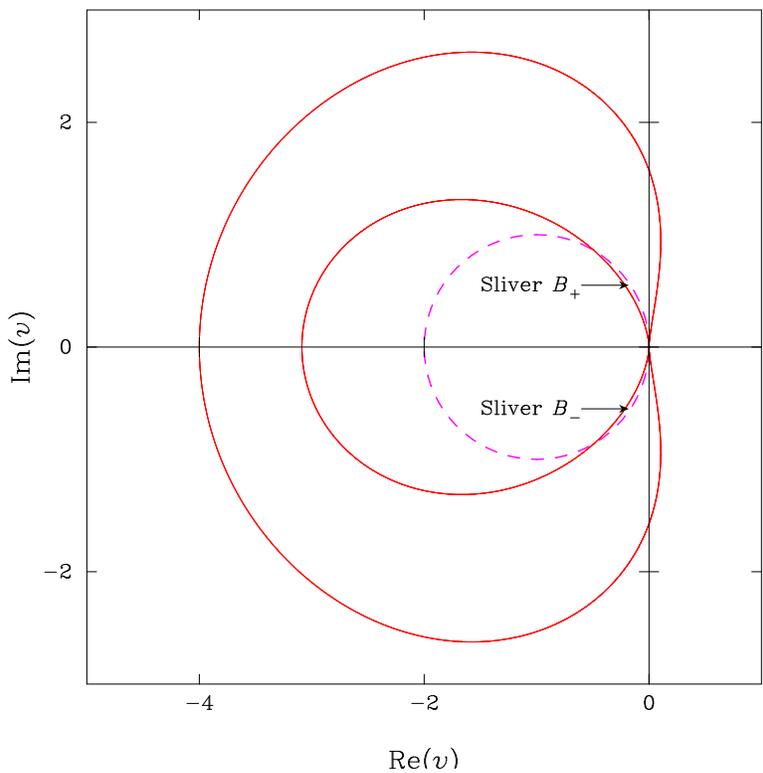


Figure 1: Curves for case (b). First plot shows the a -plane; second plot shows the b -plane. Dashed magenta curve is the circle $|1+v|=1$; solid blue curve is the locus of root a ; solid red curve is the locus of root b .

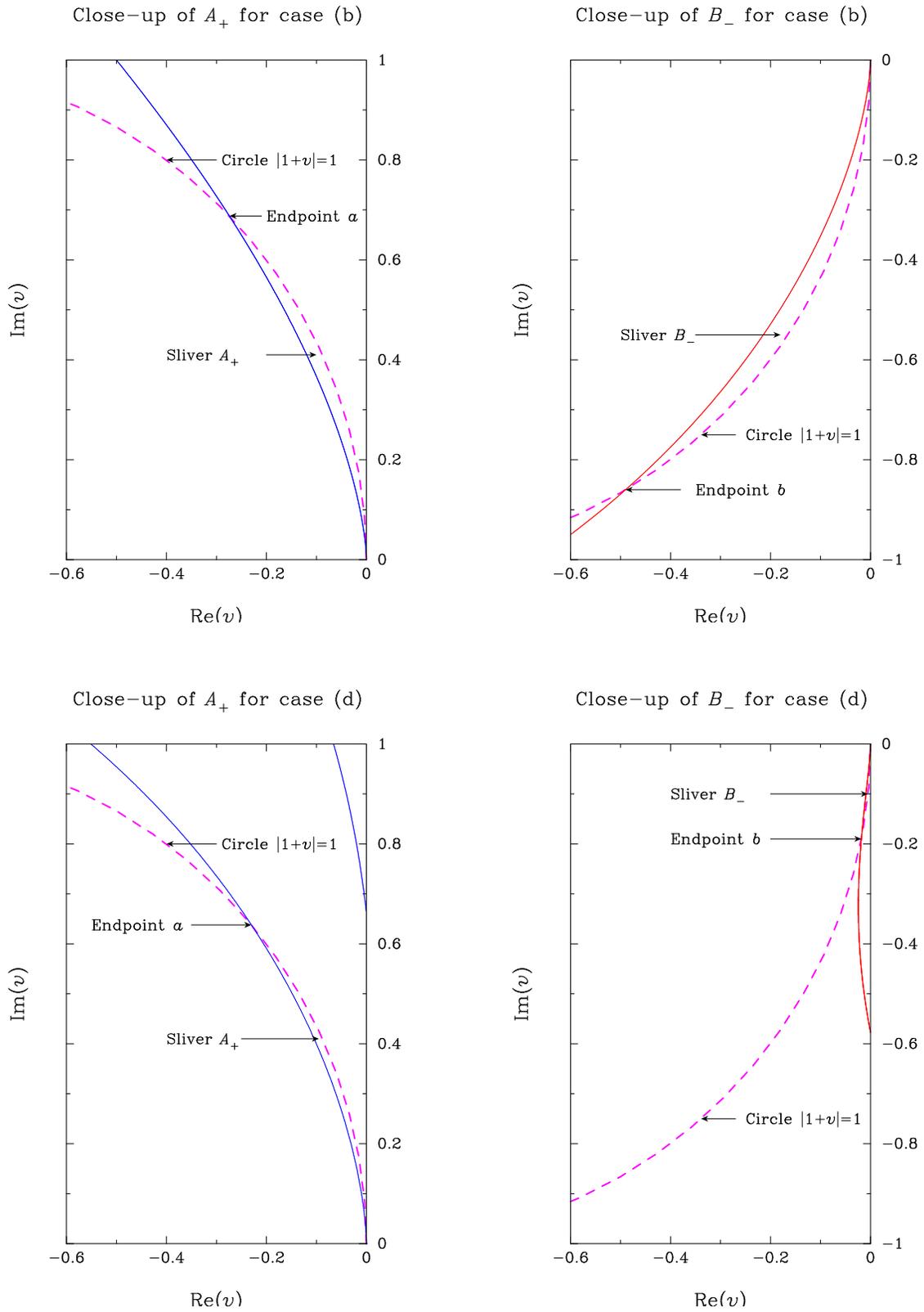


Figure 2: Blow-up of curves to show more clearly the “sliver” regions A_+ and B_- . Top row shows the a - and b -planes for case (b); bottom row shows the a - and b -planes for case (d).

- (a) $a = -b + \frac{5}{8}b^2 + O(b^3)$
- (b) $a = -b \pm \frac{1}{2}b^{3/2} + O(b^2)$
- (c) $a = -\frac{1}{3}b + \frac{1}{8}b^2 + O(b^3)$ and $a = -3b + \frac{31}{8}b^2 + O(b^3)$
- (d) $a = -3b \pm i\sqrt{3}b^{3/2} + O(b^2)$ and $a = 0$
- (e) $a = -b + \frac{3}{4}b^2 + O(b^3)$ and $a = (-3 \pm 2\sqrt{2})b + \frac{9}{16}(10 \mp 7\sqrt{2})b^2 + O(b^3)$

The behavior is thus different in cases (a,c,e) on the one hand and cases (b,d) on the other:

Cases (a,c,e): Here the solution is of the form

$$a = \gamma_1 b + \gamma_2 b^2 + O(b^3) \quad (2.7)$$

with γ_1, γ_2 real. Therefore, if we set $b = -1 + e^{i\theta}$ and expand in powers of θ , we obtain

$$|1 + a|^2 = 1 + (\gamma_1^2 - \gamma_1 - 2\gamma_2)\theta^2 + O(\theta^4). \quad (2.8)$$

Provided that $\gamma_1^2 - \gamma_1 - 2\gamma_2 > 0$ — as indeed holds for all the roots in cases (a,c,e) — we have $|1 + a| \geq 1$ for small θ , so no counterexample is found (at least for small θ).

Cases (b,d): Here, by contrast, the solution is of the form

$$a = \delta_1 b + \delta_2 b^{3/2} + O(b^2) \quad (2.9)$$

with $\delta_1 < 0$ and $\delta_2 \neq 0$. Therefore, if we set $b = -1 + e^{i\theta}$ and expand as before, we obtain

$$a = i\delta_1\theta + e^{\pm 3\pi i/4}\delta_2\theta^{3/2} + O(\theta^2). \quad (2.10)$$

Since $\operatorname{Re}(e^{\pm 3\pi i/4}\delta_2) < 0$ for at least one of the roots, we have $\operatorname{Re} a \propto -|\operatorname{Im} a|^{3/2}$ for small θ ; in particular, we have $|1 + a| < 1$ for small $\theta \neq 0$.

In fact, more can be said: suppose that we fix *any* $\lambda > 0$ and set $b = \lambda(-1 + e^{i\theta})$. Then we have

$$a = i\delta_1\lambda\theta + e^{\pm 3\pi i/4}\delta_2\lambda^{3/2}\theta^{3/2} + O(\theta^2), \quad (2.11)$$

so that once again $\operatorname{Re} a \propto -|\operatorname{Im} a|^{3/2}$ for small θ . In particular, we will have $|\lambda + a| < \lambda$ for small $\theta \neq 0$, irrespective of how small λ was chosen. This observation will play a crucial role in Section 5 (see Proposition 5.5).

3 Series and parallel reduction formulae

Suppose that G contains edges e_1, \dots, e_n (with corresponding weights v_1, \dots, v_n) in parallel between the same pair of vertices x, y . Then it is easy to see that the edges e_1, \dots, e_n can be replaced by a single edge of weight

$$v_1 \| v_2 \| \cdots \| v_n \equiv \prod_{i=1}^n (1 + v_i) - 1 \quad (3.1)$$

without changing the value of $C_G(\mathbf{v})$. [Reason: x is connected to y via this “super-edge” if and only if x is connected to y by at least one of the edges e_1, \dots, e_n .]

Suppose next that G contains edges e_1, \dots, e_n (with corresponding weights v_1, \dots, v_n) in series between the pair of vertices x, y : this means that the edges e_1, \dots, e_n form a path in which all the vertices except possibly the endvertices x and y have degree 2 in G . Let G' be the graph in which the edges e_1, \dots, e_n are replaced by a single edge e_* from x to y . Then it is not hard to see that

$$C_G(\mathbf{v}) = \left(\sum_{j=1}^n \prod_{i \neq j} v_i \right) C_{G'}(\mathbf{v}') \quad (3.2)$$

where the edge e_* is given weight

$$v'_* = v_1 \bowtie v_2 \bowtie \cdots \bowtie v_n \equiv \frac{1}{\sum_{i=1}^n 1/v_i} \quad (3.3)$$

and all edges other than e_1, \dots, e_n, e_* are given weight $v'_e = v_e$. [Reason: A connected spanning subgraph of G can omit at most one of the edges e_1, \dots, e_n , for otherwise at least one of the internal vertices of the path would be disconnected from both x and y . Moreover, x is connected to y via the “super-edge” e_* if and only if none of the edges e_1, \dots, e_n are omitted. The relative weight of the cases with and without x connected to y via e_* is thus $(\prod_{i=1}^n v_i) / (\sum_{j=1}^n \prod_{i \neq j} v_i) = v_*$; and there is an overall normalization factor $\sum_{j=1}^n \prod_{i \neq j} v_i$. See also [7, p. 35] for an equivalent formula.]

The formula for series reduction can be applied immediately to handle arbitrary subdivisions of a graph G . Given a finite graph $G = (V, E)$ and a family of integers $\mathbf{s} = \{s_e\}_{e \in E} \geq 1$, we define $G^{\bowtie \mathbf{s}}$ to be the graph in which each edge e of G is subdivided into s_e edges in series. If $s \geq 1$ is an integer, we define $G^{\bowtie s}$ to be the graph in which each edge of G is subdivided into s edges in series. All the edges in $G^{\bowtie \mathbf{s}}$ or $G^{\bowtie s}$ obtained by subdividing the edge $e \in E$ are assigned the same weight v_e as was assigned to e in the original graph G . It follows immediately from (3.2)/(3.3) that

$$C_{G^{\bowtie \mathbf{s}}}(\mathbf{v}) = \left(\prod_{e \in E} s_e v_e^{s_e - 1} \right) C_G(\mathbf{v}/\mathbf{s}) \quad (3.4)$$

where $(\mathbf{v}/\mathbf{s})_e \equiv v_e/s_e$.

Remarks. 1. Series and parallel reduction formulae can be derived in the more general context of the q -state Potts model (also known as the multivariate Tutte polynomial): see e.g. [17, Section 2]. Parallel reduction is always given by (3.1), independently of the value of the parameter q . Series reduction is given by

$$v_1 \bowtie v_2 \bowtie \cdots \bowtie v_n = \frac{q}{\prod_{i=1}^n (1 + q/v_i) - 1} . \quad (3.5)$$

Please note that (3.5) reduces to (3.3) when $q \rightarrow 0$, which is precisely the limit in which the multivariate Tutte polynomial $Z_G(q, \mathbf{v})$ tends (after division by q) to $C_G(\mathbf{v})$.

2. If one takes in $C_G(\mathbf{v})$ the further limit of \mathbf{v} infinitesimal, one obtains the generating polynomial of *minimal* connected spanning subgraphs, i.e. spanning trees. Now, spanning trees are intimately related to linear electrical circuits, as was noticed by Kirchoff in 1847 [10, 12]. For v infinitesimal, the parallel reduction formula (3.1) becomes

$$v_1 || v_2 || \cdots || v_n \equiv v_1 + v_2 + \cdots + v_n , \quad (3.6)$$

which is precisely the law for putting electrical conductances in parallel. And the series reduction formula (3.3) is precisely the law for putting electrical conductances in series!

4 The univariate Brown–Colbourn conjecture is false as well

Let $K_4^{(a,p_1,p_2)}$ be the graph obtained from K_4 by replacing one edge by p_1 parallel edges and replacing each of the other five edges by p_2 parallel edges. Let $K_4^{(b,p_1,p_2)}$ be the graph obtained from K_4 by replacing each of two nonintersecting edges by p_1 parallel edges and replacing each of the remaining four edges by p_2 parallel edges. Define in a similar manner $K_4^{(c,p_1,p_2)}$, $K_4^{(d,p_1,p_2)}$ and $K_4^{(e,p_1,p_2)}$ for the cases (c), (d) and (e) discussed in Section 2.

We saw in Section 2 that in cases (b) and (d) one can obtain a counterexample to the multivariate Brown–Colbourn conjecture by choosing the weight a to lie anywhere in the region A_+ ; this leads to a root b lying in the region B_- (see Figures 1 and 2). Note now that the p th power of the region $1 + A_+$ will overlap the region $1 + B_-$ whenever $p > (1 - \beta)/\alpha$ [just choose any point $b \in B_-$ close enough to the endpoint $-1 + e^{-2\pi i\beta} = -1 + e^{2\pi i(1-\beta)}$; then one of the p th roots of $1 + b$ will lie in the region $1 + A_+$]. And (3.1) tells us that p edges in parallel, each with weight v , are equivalent to a single edge with weight v_{eff} satisfying $1 + v_{\text{eff}} = (1 + v)^p$. This reasoning suggests that counterexamples to the univariate Brown–Colbourn conjecture might be found for the graphs $K_4^{(b,1,p)}$ and $K_4^{(d,1,p)}$: for all $p > (1 - \beta)/\alpha$ they should have a root $v \in A_+$.⁴ Likewise, the graphs $K_4^{(b,p,1)}$ and $K_4^{(d,p,1)}$ are expected to

⁴We do not claim that this is a proof, though we suspect that a suitable topological argument might be able to turn it into a proof.

Graph	Value of p									
	6	7	8	9	10	11	12	13	14	15
$K_4^{(b,1,p)}$	1	0.999765	0.997818	0.996996	0.996734	0.996749	0.996897	0.997102	0.997326	0.997547
$K_4^{(b,p,1)}$	0.998274	0.997234	0.997001	0.997083	0.997284	0.997519	0.997753	0.997971	0.998169	0.998345
$K_4^{(d,1,p)}$	1	1	1	0.999956	0.999813	0.999746	0.999718	0.999713	0.999718	0.999730
$K_4^{(d,p,1)}$	1	1	1	1	1	1	1	1	1	1

Table 1: Minimum value of $|1 + v|$ for a zero of $C_G(v)$ for selected graphs $G = K_4^{(b/d,p_1,p_2)}$. For $1 \leq p \leq 5$ the value equals 1. A value strictly less than 1 indicates a counterexample to the univariate Brown–Colbourn conjecture. For $K_4^{(d,p,1)}$ a counterexample can be found for $p \geq 30$.

have, for all $p > (1 - \alpha)/\beta$, a root $v \in B_+$. These guesses are in fact correct, and we find the following counterexamples to the univariate Brown–Colbourn conjecture:

- $G = K_4^{(b,1,7)}$, 30 edges: $v \approx -0.269253 \pm 0.682304i$, $|1 + v| \approx 0.999765$
- $G = K_4^{(b,6,1)}$, 16 edges: $v \approx -0.405015 \pm 0.801589i$, $|1 + v| \approx 0.998274$
- $G = K_4^{(d,1,9)}$, 30 edges: $v \approx -0.220759 \pm 0.626655i$, $|1 + v| \approx 0.999956$
- $G = K_4^{(d,30,1)}$, 93 edges: $v \approx -0.017476 \pm 0.185846i$, $|1 + v| \approx 0.999946$

Counterexamples are also obtained for each larger p ; some typical numbers are shown in Table 1. Please note that all these counterexample graphs are planar.

The graphs $G = K_4^{(b/d,p_1,p_2)}$ are, of course, non-simple (except when $p_1 = p_2 = 1$); so one might cling to the hope that the univariate Brown–Colbourn conjecture is true at least for *simple* graphs (or, weaker yet, for *simple planar* graphs). But these hopes too are false. To see why this is the case, consider the following procedure:

- 1) Choose p_1, p_2 so that the graph $K_4^{(b,p_1,p_2)}$ has a root v_1 satisfying $|1 + v_1| < 1$.
- 2) Choose any integer $s \geq 2$.
- 3) Find an integer k large enough so that $v_k \equiv -1 + (1 + v_1)^{1/k}$ — defined using the root with $|\arg[(1 + v_1)^{1/k}]| \leq \pi/k$ — lies in the disc $|1/s + v_k| < 1/s$. [It is always possible to find such a k , because the points v_k lie on a logarithmic spiral that approaches the point $v = 0$ making a nonzero angle with the imaginary axis, while all the circles $|1/s + v| = 1/s$ pass through $v = 0$ tangent to the imaginary axis.]

Then v_k is a root for the graph $K_4^{(b, kp_1, kp_2)}$, by the rules for parallel reduction; and sv_k is a root for the graph $(K_4^{(b, kp_1, kp_2)})^{\bowtie s}$, by the rules for series reduction. And by construction we have $|1 + sv_k| < 1$. Therefore, the graph $(K_4^{(b, kp_1, kp_2)})^{\bowtie s}$, which is simple and planar, is the desired counterexample.

For example, if we take $(p_1, p_2) = (11, 1)$ and $s = 2$, counterexamples can be obtained for $k \geq 58$:

- $v_1 \approx -0.140\,970\,808\,664 + 0.507\,062\,767\,880i$, $|1 + v_1| \approx 0.997\,518\,822\,949$
- $v_{58} \approx -0.000\,085\,091\,565 + 0.009\,193\,226\,407i$, $|1 + 2v_{58}| \approx 0.999\,998\,862\,173$

This shows that the graph $(K_4^{(b, 638, 58)})^{\bowtie 2}$, which has 1512 vertices and 3016 edges, is a counterexample to the univariate Brown–Colbourn conjecture. Similarly, if we take $(p_1, p_2) = (1, 12)$ and $s = 2$, counterexamples can be obtained for $k \geq 36$:

- $v_1 \approx -0.112\,358\,418\,620 + 0.453\,757\,934\,703i$, $|1 + v_1| \approx 0.996\,897\,106\,175$
- $v_{36} \approx -0.000\,172\,469\,038 + 0.013\,125\,252\,246i$, $|1 + 2v_{36}| \approx 0.999\,999\,665\,908$

Therefore, the graph $(K_4^{(b, 36, 432)})^{\bowtie 2}$, which has 1804 vertices and 3600 edges, is a counterexample to the univariate Brown–Colbourn conjecture.

Smaller counterexamples of the forms $(K_4^{(b/d, p, 1)})^{\bowtie (s, 1)}$ or $(K_4^{(b/d, 1, p)})^{\bowtie (1, s)}$ can probably be found by direct search. But the foregoing construction has the advantage that there is no need to compute the roots of extremely-high-degree polynomials; it suffices to compute the roots for the base case $K_4^{(b/d, p_1, p_2)}$ (for which the polynomials are large but not huge) and then make simple manipulations on them.

Methodological remark. In this work we needed to compute accurately the roots of polynomials of fairly high degree (up to 93) with very large integer coefficients (up to about 10^{27}). To do this we used the package MPSolve 2.0 developed by Dario Bini and Giuseppe Fiorentino [2, 3]. MPSolve is much faster than MATHEMATICA’s `NSolve` for high-degree polynomials (this is reported in [3], and we confirm it); it gives guaranteed error bounds for the roots, based on rigorous theorems [3]; its algorithms are publicly documented [3]; and its source code is freely available [2].

Let us mention, finally, that counterexamples with smaller values of $|1 + v|$ can be found. Consider, for example, the complete graph K_6 in which a pair of vertex-disjoint triangles receives weight a and the remaining nine edges receive weight b . We have

$$\begin{aligned}
C_{K_6}(a, b) = & \\
& (81b^5 + 78b^6 + 36b^7 + 9b^8 + b^9) + \\
& (324b^4 + 594b^5 + 480b^6 + 216b^7 + 54b^8 + 6b^9)a + \\
& (486b^3 + 1314b^4 + 1665b^5 + 1224b^6 + 540b^7 + 135b^8 + 15b^9)a^2 +
\end{aligned}$$

$$\begin{aligned}
& (324b^2 + 1188b^3 + 2160b^4 + 2376b^5 + 1656b^6 + 720b^7 + 180b^8 + 20b^9)a^3 + \\
& (81b + 432b^2 + 1134b^3 + 1800b^4 + 1854b^5 + 1254b^6 + 540b^7 + 135b^8 + 15b^9)a^4 + \\
& (54b + 216b^2 + 504b^3 + 756b^4 + 756b^5 + 504b^6 + 216b^7 + 54b^8 + 6b^9)a^5 + \\
& (9b + 36b^2 + 84b^3 + 126b^4 + 126b^5 + 84b^6 + 36b^7 + 9b^8 + b^9)a^6
\end{aligned} \tag{4.1}$$

If we then substitute $a = (1+v)^{p_1} - 1$ and $b = (1+v)^{p_2} - 1$, counterexamples to the univariate Brown–Colbourn conjecture can be found for many pairs (p_1, p_2) . For example, for $(p_1, p_2) = (1, 6)$ we obtain a 60-edge non-planar graph whose roots include $v \approx -0.357514 \pm 0.713815i$, yielding $|1+v| \approx 0.960375$.

It would be interesting to know whether examples can be found in which $|1+v|$ is arbitrarily small. More generally, one can ask:

Question 4.1 *What is the closure of the set of all roots of the polynomials $C_G(v)$ as G ranges over all graphs? Over all planar graphs? Over all simple planar graphs?*

Brown and Colbourn [5] pointed out that the graphs $G = C_n^{(p)}$ (the n -cycle with each edge replaced by p parallel edges) have roots that, taken together, are dense in the region $|1+v| \geq 1$. We have shown here that roots can also enter the region $|1+v| < 1$. But how far into this latter region can they penetrate? Might the roots actually be dense in the whole complex plane? If this is indeed the case, it would mean that the univariate Brown–Colbourn conjecture is as false as it can possibly be.

Note Added (April 2004). Building on the examples constructed in this section, Chang and Shrock [6, Sections 5.17 and 5.18] have recently devised families of strip graphs in which the limiting curve of zeros of $C_G(v)$, as the strip length tends to infinity, penetrates into the “forbidden region” $|1+v| < 1$. Some of these families consist of planar graphs.

5 Series-parallel is necessary and sufficient

In this section we shall prove that a graph has the multivariate Brown–Colbourn property *if and only if* it is series-parallel.

Let us begin by defining a weakened version of the Brown–Colbourn property:

Definition 5.1 *Let G be a graph, and let $\lambda > 0$. We say that G*

- *has the univariate property BC_λ if $C_G(v) \neq 0$ whenever $|\lambda + v| < \lambda$;*
- *has the multivariate property BC_λ if $C_G(\mathbf{v}) \neq 0$ whenever $|\lambda + v_e| < \lambda$ for all edges e .*

Properties BC_1 are, of course, the original univariate and multivariate Brown–Colbourn properties; the properties BC_λ become increasingly weaker as λ is decreased.

The properties BC_λ are intimately related to subdivisions:

Lemma 5.2 *Let $\lambda > 0$ and let s be a positive integer. Then the following are equivalent for a graph G :*

- (a) G has the univariate property BC_λ .
- (b) $G^{\boxtimes s}$ has the univariate property $BC_{s\lambda}$.

Lemma 5.3 *Let $\lambda > 0$ and let s be a positive integer. Then the following are equivalent for a graph G :*

- (a) G has the multivariate property BC_λ .
- (b) $G^{\boxtimes s}$ has the multivariate property $BC_{s\lambda}$.
- (c) $G^{\boxtimes s}$ has the multivariate property $BC_{s\lambda}$ for all vectors \mathbf{s} satisfying $s_e \geq s$ for all edges e .

Indeed, Lemmas 5.2 and 5.3 are an immediate consequence of the formula (3.4) for subdivisions — which states that subdivision by \mathbf{s} moves the nonzero roots from \mathbf{v} to \mathbf{sv} — together with the fact that $|s\lambda + sv| < s\lambda$ is equivalent to $|\lambda + v| < \lambda$.

In the preceding section we have shown that not all graphs have the univariate property BC_1 . It is nevertheless true — and virtually trivial — that every connected graph has the univariate property BC_λ for *some* $\lambda > 0$. (Since a non-identically-vanishing univariate polynomial has finitely many roots, it suffices to choose λ small enough so that none of the roots of $C_G(v)$ lie in the disc $|\lambda + v| < \lambda$.) By Lemma 5.2, an equivalent assertion is that $G^{\boxtimes s}$ has the univariate property BC_1 for all sufficiently large integers s .⁵

The situation is very different, however, when we consider the multivariate property BC_λ . We begin with a simple but important lemma:

Lemma 5.4 *Let $\lambda > 0$, and suppose that the connected graph G has the multivariate property BC_λ . Then every connected subgraph $H \subseteq G$ also has the multivariate property BC_λ .*

PROOF. Consider first the case in which H is a connected *spanning* subgraph (i.e. its vertex set is the same as that of G). Let us write $\mathbf{v} = (\mathbf{v}', \mathbf{v}'')$ where $\mathbf{v}' = \{v_e\}_{e \in E(H)}$ and $\mathbf{v}'' = \{v_e\}_{e \in E(H) \setminus E(G)}$. Then

$$C_H(\mathbf{v}') = C_G(\mathbf{v}', \mathbf{0}) = \lim_{\mathbf{v}'' \rightarrow \mathbf{0}} C_G(\mathbf{v}', \mathbf{v}''). \quad (5.1)$$

⁵Brown and Colbourn [5, Proposition 4.4 and Theorem 4.5] have proven a result also for *non-uniform* subdivisions $G^{\boxtimes \mathbf{s}}$: namely, for each graph G there exists an integer s such that $G^{\boxtimes \mathbf{s}}$ has the univariate property BC_1 whenever $s_e \geq s$ for all e . This is significantly stronger than the just-mentioned trivial result, and it would be worth trying to understand it better. Brown and Colbourn’s method looks very different from ours, at least at first glance; it would be interesting to try to translate it into our language. In particular, there may be a “partially multivariate” result hiding underneath their apparently univariate proof.

By hypothesis, $C_G(\mathbf{v}', \mathbf{v}'') \neq 0$ whenever $|\lambda + v_e| < \lambda$ for all $e \in E(G)$. Now take $\mathbf{v}'' \rightarrow \mathbf{0}$ from within this product of discs ($\mathbf{0}$ lies on its boundary). By Hurwitz's theorem⁶, either $C_H(\mathbf{v}')$ is nonvanishing whenever $|\lambda + v_e| < \lambda$ for all $e \in E(H)$, or else C_H is identically zero. But the latter is impossible since H is connected.

Now let H be an arbitrary connected subgraph of G (spanning or not). Construct a connected spanning subgraph \widehat{H} of G by hanging trees off some or all of the vertices of H without creating any new circuits.⁷ Let us write $\mathbf{v} = \{v_e\}_{e \in E(\widehat{H})} = (\mathbf{v}', \mathbf{v}'')$ where $\mathbf{v}' = \{v_e\}_{e \in E(H)}$ and $\mathbf{v}'' = \{v_e\}_{e \in E(\widehat{H}) \setminus E(H)}$. Then

$$C_{\widehat{H}}(\mathbf{v}) = C_H(\mathbf{v}') \prod_{e \in E(\widehat{H}) \setminus E(H)} v_e. \quad (5.2)$$

Since \widehat{H} has multivariate property BC_λ , so does H . \square

The following is the fundamental fact from which all else flows:

Proposition 5.5 *The complete graph K_4 does not have the multivariate property BC_λ for any $\lambda > 0$.*

PROOF. This is an almost immediate consequence of the observations made at the end of Section 2. In cases (b) and (d), for any $\lambda > 0$ there exists b with $|\lambda + b| = \lambda$ for which at least one of the solutions to $C_{K_4}(a, b) = 0$ satisfies $|\lambda + a| < \lambda$. By slightly perturbing this pair, we can find a pair (a, b) with $C_{K_4}(a, b) = 0$ satisfying $|\lambda + a| < \lambda$ and $|\lambda + b| < \lambda$. So K_4 does not even have the bivariate property BC_λ . \square

We can deduce from Lemma 5.4 and Proposition 5.5 a necessary and sufficient condition for G to have various forms of the multivariate Brown–Colbourn property:

Theorem 5.6 *Let G be a loopless connected graph. Then the following are equivalent:*

(a) *G has the multivariate property BC_1 .*

⁶Hurwitz's theorem states that if D is a domain in \mathbb{C}^n and (f_k) are nonvanishing analytic functions on D that converge to f uniformly on compact subsets of D , then f is either nonvanishing or else identically zero. Hurwitz's theorem for $n = 1$ is proved in most standard texts on the theory of analytic functions of a single complex variable (see e.g. [1, p. 176]). Surprisingly, we have been unable to find Hurwitz's theorem proven for general n in any standard text on several complex variables (but see [11, p. 306] and [15, p. 337]). So here, for completeness, is the sketch of a proof: Suppose that $f(c) = 0$ for some $c = (c_1, \dots, c_n) \in D$, and let $D' \subset D$ be a small polydisc centered at c . Applying the single-variable Hurwitz theorem, we conclude that $f(z_1, c_2, \dots, c_n) = 0$ for all z_1 such that $(z_1, c_2, \dots, c_n) \in D'$. Applying the same argument repeatedly in the variables z_2, \dots, z_n , we conclude that f is identically vanishing on D' and hence, by analytic continuation, also on D .

⁷This can be done, for instance, by running breadth-first search with the vertices of H initially on the queue.

(b) G has the multivariate property BC_λ for some $\lambda > 0$.

(c) G is series-parallel.

PROOF. (a) \implies (b) is trivial.

(b) \implies (c): Let G be a loopless connected graph that is not series-parallel. Then G contains a subgraph H that is a subdivision of K_4 .⁸ Suppose that $H = (K_4)^{\boxtimes \mathbf{s}}$ with $\mathbf{s} = (s_1, \dots, s_6)$, and define $s = \max(s_1, \dots, s_6)$. Now fix any $\lambda > 0$; then, by Proposition 5.5 we can find a vector $\mathbf{v} = (v_1, \dots, v_6)$ that is a zero of $C_{K_4}(\mathbf{v})$ and satisfies $|\lambda/s + v_i| < \lambda/s$ for $i = 1, \dots, 6$. It then follows that the vector $\mathbf{v}' = (v'_1, \dots, v'_6)$ defined by $v'_i = s_i v_i$ satisfies $C_H(\mathbf{v}') = 0$ and $|\lambda + v'_i| < \lambda$ for $i = 1, \dots, 6$. Therefore H does not have the multivariate property BC_λ . By Lemma 5.4, G cannot have this property either.

(c) \implies (a): This is proven in [16, Remark 3 in Section 4.1], but for the convenience of the reader we repeat the proof here. Suppose that G is a loopless connected series-parallel graph; this means that G can be obtained from a tree by a finite sequence of series and parallel extensions of edges (i.e. replacing an edge by two edges in series or two edges in parallel). We will prove that G has the multivariate property BC_1 , by induction on the length of this sequence of series and parallel extensions. The base case is when G is a tree: then $C_G(\mathbf{v}) = \prod_{e \in E(G)} v_e$ and G manifestly has the multivariate property BC_1 . Now suppose that G is obtained from a smaller graph G' by replacing an edge e_* of G' by two parallel edges e_1, e_2 . Use the parallel reduction formula (3.1): since $|1 + v_1| < 1$ and $|1 + v_2| < 1$ imply $|1 + v_*| < 1$, we deduce that G has the multivariate property BC_1 if G' does. Suppose, finally, that G is obtained from a smaller graph G' by replacing an edge e_* of G' by two edges e_1, e_2 in series. Use the series reduction formula (3.2)/(3.3) and the fact that $|1 + v| < 1$ is equivalent to $\operatorname{Re}(1/v) < -1/2$: then $\operatorname{Re}(1/v_i) < -1/2$ for $i = 1, 2$ implies that $\operatorname{Re}(1/v_*) < -1 < -1/2$, and moreover the prefactor $v_1 + v_2$ is nonzero; so we deduce that G has the multivariate property BC_1 if G' does. \square

For each graph G , let us define $\lambda_*(G)$ to be the maximum λ for which G has the multivariate property BC_λ . Then Theorem 5.6 states a surprising (at first sight) dichotomy: either $\lambda_*(G) = 0$ [when G is not series-parallel] or else $\lambda_*(G) \geq 1$ [when G is series-parallel].

Some series-parallel graphs have $\lambda_*(G) = 1$ exactly: for example, the graphs $K_2^{(n)}$ (a pair of vertices connected by n parallel edges) have $C_{K_2^{(n)}}(v) = (1 + v)^n - 1$ and hence even have univariate roots on the circle $|1 + v| = 1$. On the other hand, some series-parallel graphs have $\lambda_*(G) > 1$: for example, the cycles C_n have $\lambda_*(G) = n/2$. [PROOF: We have

$$C_{C_n}(\mathbf{v}) = \left(\prod_{i=1}^n v_i \right) \left(1 + \sum_{i=1}^n \frac{1}{v_i} \right), \quad (5.3)$$

⁸The relevant fact is the following [8, Exercise 8.16 and Proposition 1.7.2]: G is series-parallel $\iff G$ has no K_4 minor $\iff G$ has no K_4 topological minor. And the latter statement says precisely that G contains no subgraph H that is a subdivision of K_4 . See also [9, 13].

which is nonvanishing if $\operatorname{Re}(1/v_i) < -1/n$ for all i . But this is equivalent to $|n/2 + v_i| < n/2$. It is an interesting open problem to characterize the graphs that have $\lambda_*(G) = 1$ or, more ambitiously, to find a simple graph-theoretic formula for $\lambda_*(G)$.

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