Decompositions of complete multigraphs into cycles of varying lengths

Darryn Bryant^{*}, Daniel Horsley[†], Barbara Maenhaut^{*} and Benjamin R. Smith^{*}

Abstract

We establish necessary and sufficient conditions for the existence of a decomposition of a complete multigraph into edge-disjoint cycles of specified lengths, or into edge-disjoint cycles of specified lengths and a perfect matching.

1 Introduction

A decomposition of a graph K is a collection \mathcal{D} of subgraphs of K such that the edge sets of the graphs in \mathcal{D} partition the edge set of K. If the complete graph of order n, denoted K_n , has a decomposition into t cycles of specified lengths m_1, m_2, \ldots, m_t , then it is easy to see that $3 \leq m_i \leq n$ for $i = 1, 2, \ldots, t, n$ is odd, and $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$. Similarly, if K_n has a decomposition into t cycles of specified lengths m_1, m_2, \ldots, m_t and a perfect matching, then $3 \leq m_i \leq n$ for $i = 1, 2, \ldots, t, n$ is even, and $m_1 + m_2 + \cdots + m_t = \binom{n}{2} - \frac{n}{2}$. In [16] it was shown that these obvious necessary conditions are also sufficient for the existence of the desired decomposition, thereby solving a problem posed by Alspach in 1981 [1].

In this paper, the analogous problem for decompositions of complete multigraphs into cycles of specified lengths is completely solved, see Theorem 1.1. The complete multigraph of order n and multiplicity λ , which has λ distinct edges joining each pair of distinct vertices, is denoted λK_n .

Theorem 1.1 There is a decomposition $\{G_1, G_2, \ldots, G_t\}$ of λK_n in which G_i is an m_i -cycle for $i = 1, 2, \ldots, t$ if and only if

• $\lambda(n-1)$ is even;

^{*}School of Mathematics and Physics, The University of Queensland, QLD 4072, Australia. Email: db@maths.uq.edu.au, bmm@maths.uq.edu.au, bsmith.maths@gmail.com

[†]School of Mathematical Sciences, Monash University, VIC 3800, Australia. Email: daniel.horsley@monash.edu

- $2 \le m_1, m_2, \ldots, m_t \le n;$
- $m_1 + m_2 + \dots + m_t = \lambda \binom{n}{2};$
- $\max(m_1, m_2, \ldots, m_t) + t 2 \leq \frac{\lambda}{2} {n \choose 2}$ when λ is even; and
- $\sum_{m_i=2} m_i \leq (\lambda 1) \binom{n}{2}$ when λ is odd.

There is a decomposition $\{G_1, G_2, \ldots, G_t, I\}$ of λK_n in which G_i is an m_i -cycle for $i = 1, 2, \ldots, t$ and I is a perfect matching if and only if

- $\lambda(n-1)$ is odd;
- $2 \le m_1, m_2, \ldots, m_t \le n;$
- $m_1 + m_2 + \dots + m_t = \lambda \binom{n}{2} \frac{n}{2}$; and
- $\sum_{m_i=2} m_i \leq (\lambda 1) \binom{n}{2}$.

The necessity of the conditions of Theorem 1.1 are proved in Section 2 and sufficiency is proved in Section 6. Note that for $\lambda = 1$, the condition that $\sum_{m_i=2} m_i \leq (\lambda - 1) \binom{n}{2}$ implies that each $m_i \geq 3$, and so the necessary conditions of Theorem 1.1 reduce to the familiar necessary conditions for Alspach's problem, as described in the first paragraph.

There has been considerable work done on the existence of decompositions of complete multigraphs into cycles. For the case $\lambda = 1$, the eventual complete solution [16] was preceded by numerous partial results, dating back to 1847 [22, 24]. The special case where all of the cycles have uniform length was settled by Alspach, Gavlas and Šajna [2, 27]. Important preliminary results that contributed directly to the complete solution for $\lambda = 1$ are given in [11, 12, 13, 14], and other partial results can be found in the large number of cited papers in [16] and the surveys [9] and [19].

For $\lambda > 1$, there have been relatively few results for cases where there are cycles of varying lengths [15], but the case where all the cycles are of uniform length m has been studied extensively. Solutions for small values of m are given in [6, 7, 20, 21, 26], other partial results appear in [30, 31], and a complete solution for all m and all λ is given in [15].

Analogous problems concerning decompositions of λK_n into paths, matchings or stars of sizes m_1, m_2, \ldots, m_t have also been considered. The problem is completely solved for paths in [10]. Baranyai's Theorem [4] settles the problem for decompositions of λK_n into matchings when $\lambda = 1$, and an easy induction extends this result to a complete solution for all $\lambda \geq 1$. The problem of decomposing λK_n into isomorphic stars has been solved [32], as has the problem of decomposing K_n into stars of sizes m_1, m_2, \ldots, m_t [23].

We briefly mention some basic graph theory terminology that we will use. A graph G is a nonempty set V(G) of vertices and a set E(G) of edges, together with a

function which maps each edge in E(G) to a pair of distinct vertices in V(G) called its *endpoints*. The *size* of a graph G is |E(G)|, and the number of edges in G which have u and v as their endpoints is denoted by $\mu_G(uv)$ and called the *multiplicity* of edge uv. Note that the above definition of a graph distinguishes different edges with the same endpoints. Much of the time, however, distinguishing edges with the same endpoints is an unnecessary complication which we ignore. For example, if we wish to delete an edge with endpoints u and v from some graph G, then it will generally not matter which of the $\mu_G(uv)$ edges with endpoints u and v is deleted. We may also write, for example, that $uv \in E(G)$ when technically we should say there is an edge in E(G) with endpoints u and v. If $\mu_G(uv) \leq 1$ for each distinct u and v in V(G), we say that G is *simple*.

We denote the complete graph with vertex set V by K_V and the complete bipartite graph with parts U and V by $K_{U,V}$. If G is a graph and λ is a positive integer, then λG is the graph with vertex set V(G) and with $\mu_{\lambda G}(uv) = \lambda \mu_G(uv)$ for each pair of distinct u and v in V(G). If H is a subgraph of G, then G - H is the graph with vertex set V(G) and edge set $E(G) \setminus E(H)$. Similarly, if $E \subseteq E(G)$, then G - E is the graph obtained from G by deleting the edges in E. Note that $\mu_{G-H}(uv) = \mu_G(uv) - \mu_H(uv)$ for each pair of distinct u and v in V(H). Conversely, if H is a graph that is edge-disjoint from G, then $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Note that $\mu_{G \cup H}(uv) = \mu_G(uv) + \mu_H(uv)$ for each pair of distinct u and v in $V(G) \cap V(H)$. A cycle with m edges, $m \ge 2$, is called an *m*-cycle and is denoted (v_1, v_2, \ldots, v_m) , where v_1, v_2, \ldots, v_m are the vertices of the cycle and $v_1v_2, v_2v_3, \ldots, v_{m-1}v_m, v_mv_1$ are the edges (so $2K_2$ is a 2-cycle). A path with m edges is called an *m*-path and is denoted $[v_0, v_1, \ldots, v_m]$, where v_0, v_1, \ldots, v_m are the vertices of the path and $v_0v_1, v_1v_2, \ldots, v_{m-1}v_m$ are the edges. A graph is said to be *even* if every vertex of the graph has even degree and is said to be *odd* if every vertex of the graph has odd degree.

For brevity, we avoid having to make separate mention of the case where our decompositions are into cycles and a perfect matching (rather than just into cycles), as follows. Let K be a graph and let $M = (m_1, m_2, \ldots, m_t)$ be a list of integers with $m_i \ge 2$ for $i = 1, 2, \ldots, t$. If K is an even graph, then an (M)-decomposition of K is a decomposition $\{G_1, G_2, \ldots, G_t\}$ such that G_i is an m_i -cycle for $i = 1, 2, \ldots, t$. If K is an odd graph, then an (M)-decomposition of K is a decomposition $\{G_1, G_2, \ldots, G_t\}$ such that G_i is a decomposition $\{G_1, G_2, \ldots, G_t, I\}$ such that G_i is an m_i -cycle for $i = 1, 2, \ldots, t$. If K is a decomposition of K is a decomposition of K.

A packing of a graph K is a decomposition of some subgraph G of K, and the graph K - G is called the *leave* of the packing. An (M)-packing of λK_n is an (M)-decomposition of some subgraph G of λK_n such that G is an even graph if $\lambda(n-1)$ is even, and G is an odd graph otherwise. Thus the leave of an (M)-packing of λK_n is an even graph and, like an (M)-decomposition of λK_n , an (M)-packing of λK_n contains a perfect matching if and only if $\lambda(n-1)$ is odd.

Throughout the paper we denote by $\nu_m(M)$ the number of occurrences of m in the list M. We shall also sometimes use superscripts to specify the number of oc-

currences of a particular integer in a list. That is, we define $(m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t})$ to be the list comprised of α_i occurrences of m_i for $i = 1, 2, \ldots, t$. Let M and M' be lists of integers. It follows that for some distinct m_1, m_2, \ldots, m_t we may write $M = (m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t})$ and $M' = (m_1^{\beta_1}, m_2^{\beta_2}, \ldots, m_t^{\beta_t})$, where $\alpha_i, \beta_i \geq 0$ for $i = 1, 2, \ldots, t$. Then $\sum M = \alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_t m_t$, (M, M') is the list $(m_1^{\alpha_1+\beta_1}, m_2^{\alpha_2+\beta_2}, \ldots, m_t^{\alpha_t+\beta_2})$ and, if $0 \leq \beta_i \leq \alpha_i$ for $i = 1, 2, \ldots, t$, then M - M' is the list $(m_1^{\alpha_1-\beta_1}, m_2^{\alpha_2-\beta_2}, \ldots, m_t^{\alpha_t-\beta_t})$.

2 Necessity and admissible lists

For positive integers λ and n, we say that a list (m_1, m_2, \ldots, m_t) of integers is (λ, n) admissible if

(A1) $2 \le m_1, m_2, \dots, m_t \le n;$

(A2) $m_1 + m_2 + \dots + m_t = \lambda \binom{n}{2}$ when $\lambda(n-1)$ is even;

(A3) $m_1 + m_2 + \dots + m_t = \lambda \binom{n}{2} - \frac{n}{2}$ when $\lambda(n-1)$ is odd;

(A4) $\max(m_1, m_2, \ldots, m_t) + t - 2 \leq \frac{\lambda}{2} \binom{n}{2}$ when λ is even; and

(A5) $\sum_{m_i=2} m_i \leq (\lambda - 1) \binom{n}{2}$ when λ is odd.

It is clear that the conditions of Theorem 1.1 are satisfied if and only if the list (m_1, m_2, \ldots, m_t) is (λ, n) -admissible. Thus, with the above notation in hand, we can restate our main theorem (Theorem 1.1) as follows.

Theorem 2.1 For all positive integers λ and n, there is an (M)-decomposition of λK_n if and only if M is a (λ, n) -admissible list.

As noted above, Theorem 2.1 is known to hold when $\lambda = 1$ [16], and we include this result here for later reference.

Lemma 2.2 ([16]) Theorem 2.1 holds for $\lambda = 1$.

The necessity of conditions (A1)-(A3) is obvious, but proving that conditions (A4) and (A5) are necessary requires some work. The following lemma, which is a (slight) generalisation of a result in [25], is used to prove the necessity of condition (A4).

Lemma 2.3 Suppose G is a graph in which every edge has even multiplicity, \mathcal{D} is a cycle decomposition of G, and $C \in \mathcal{D}$. Then $|\mathcal{D}| \leq |E(G)|/2 - |E(C)| + 2$.

Proof Suppose for a contradiction that there is a graph G in which every edge has even multiplicity that admits a cycle decomposition \mathcal{D} such that $|\mathcal{D}| > |E(G)|/2 - |E(C)| + 2$ for some $C \in \mathcal{D}$. Suppose further that, of all such graphs, G has a minimal number of edges. Note that |E(G)| > 2|E(C)| because otherwise |E(G)| = 2|E(C)|and $|\mathcal{D}| = 2$, and thus G - C contains at least one edge of multiplicity at least 2. Let C' be a 2-cycle in G - C, let G' = G - C', let C_1 and C_2 be distinct cycles in \mathcal{D} such that $E(C') \subset E(C_1) \cup E(C_2)$, and let \mathcal{D}^* be a cycle decomposition of $(C_1 \cup C_2) - C'$. Observe that every edge in G' has even multiplicity, that $\mathcal{D}' = (\mathcal{D} \setminus \{C_1, C_2\}) \cup \mathcal{D}^*$ is a cycle decomposition of G', and that $C \in \mathcal{D}'$. Since |E(G')| = |E(G)| - 2, it follows that

$$|\mathcal{D}'| \ge (|\mathcal{D}| - 2) + 1 > |E(G)|/2 - |E(C)| + 1 = |E(G')|/2 - |E(C)| + 2.$$

This contradicts the minimality of G.

We are now ready to prove the necessity of (λ, n) -admissibility.

Lemma 2.4 If there is an (M)-decomposition of λK_n , then M is a (λ, n) -admissible list.

Proof Let M be the list (m_1, m_2, \ldots, m_t) and let \mathcal{D} be an (M)-decomposition of λK_n . To show that M is a (λ, n) -admissible list we need to show that conditions (A1)-(A5) hold. It is clear that conditions (A1)-(A3) hold. Without loss of generality we can assume $m_1 = \max(m_1, m_2, \ldots, m_t)$. If λ is even, then every edge in λK_n has even multiplicity and by Lemma 2.3, with $G = \lambda K_n$ and C a cycle of length m_1 in \mathcal{D} , we have $t \leq \frac{\lambda}{2} {n \choose 2} - m_1 + 2$. Thus (A4) holds. If λ is odd, it is clear that for any pair of distinct vertices $x, y \in V(\lambda K_n)$, at most $\lambda - 1$ of the edges joining x and y occur in 2-cycles in \mathcal{D} . Thus (A5) holds.

Later in the paper we will often need to establish the admissibility of certain lists, and the following lemma is a useful tool in this regard.

Lemma 2.5 Suppose $\lambda \geq 2$ and n are positive integers and $M = (m_1, m_2, \ldots, m_t)$ is a list of integers satisfying

(A1) $2 \le m_1, m_2, \ldots, m_t \le n;$

(A2) $m_1 + m_2 + \cdots + m_t = \lambda \binom{n}{2}$ when $\lambda(n-1)$ is even; and

(A3) $m_1 + m_2 + \dots + m_t = \lambda \binom{n}{2} - \frac{n}{2}$ when $\lambda(n-1)$ is odd.

If either $\nu_2(M) < n$, or λ is even and the two largest entries in M are equal, then M is (λ, n) -admissible.

Proof The result is trivially true for n = 1, and thus we may assume that $n \ge 2$. Let $M = (m_1, m_2, \ldots, m_t)$ be a list which satisfies the conditions of the lemma. Without loss of generality we may assume that M is non-increasing. By the definition of (λ, n) -admissibility, we need only show that $2\nu_2(M) \le (\lambda - 1)\binom{n}{2}$ if λ is odd and that $m_1 + t - 3 < \frac{\lambda}{2}\binom{n}{2}$ if λ is even (note that m_1 and t are integers).

Case 1. Suppose that λ is odd. Then $\nu_2(M) < n$. Since $\lambda \ge 3$ and $n \ge 2$ we have that $2\nu_2(M) < 2(n-1) \le (\lambda-1)\binom{n}{2}$ and the result follows.

Case 2a. Suppose that λ is even and $m_1 = m_2$. Then $m_1 + m_2 + \cdots + m_t = \lambda \binom{n}{2}$ and it follows that $m_3 + m_4 + \cdots + m_t = \lambda \binom{n}{2} - 2m_1$. Thus

$$t \leq \frac{1}{2} \left(\lambda \binom{n}{2} - 2m_1 \right) + 2 = \frac{\lambda}{2} \binom{n}{2} - m_1 + 2$$

and it follows that $m_1 + t - 3 < \frac{\lambda}{2} \binom{n}{2}$.

Case 2b. Suppose that λ is even and $\nu_2(M) < n$. Again $m_1 + m_2 + \cdots + m_t = \lambda \binom{n}{2}$. Let $t' = t - \nu_2(M)$. We have that $m_2 + m_3 + \cdots + m_{t'} = \lambda \binom{n}{2} - m_1 - 2\nu_2(M)$. Thus,

$$t \leq \frac{1}{3} \left(\lambda \binom{n}{2} - m_1 - 2\nu_2(M) \right) + \nu_2(M) + 1 = \frac{\lambda}{3} \binom{n}{2} - \frac{1}{3}m_1 + \frac{1}{3}\nu_2(M) + 1.$$

So $m_1 + t - 3 < \frac{\lambda}{2} {n \choose 2}$ will hold provided $\frac{2}{3}m_1 + \frac{1}{3}\nu_2(M) - 2 < \frac{\lambda}{6} {n \choose 2}$ holds. Because $m_1 \le n, \nu_2(M) \le n - 1$ and $\lambda \ge 2$, this latter does indeed hold. \Box

3 Preliminary results

In Section 4 we will prove that it is sufficient to establish our result for a certain subset of all (λ, n) -admissible lists. In this section we prove a number of preliminary results which we will need for this.

Lemma 3.1 Suppose λ and n are positive integers and $M = (m_1, m_2, \ldots, m_t)$ is a (λ, n) -admissible list. If either

- λ is even and $\max(m_1, m_2, \ldots, m_t) + t 2 = \frac{\lambda}{2} {n \choose 2}$; or
- λ is odd and $\sum_{m_i=2} m_i = (\lambda 1) {n \choose 2};$

then there is an (M)-decomposition of λK_n .

Proof Without loss of generality we may assume that $M = (m_1, m_2, \ldots, m_t)$ is non-increasing. Let $\nu = \nu_2(M)$, let $t' = t - \nu$ and let $M' = (m_1, m_2, \ldots, m_{t'})$.

Case 1. Suppose that λ is even and $m_1 + t - 2 = \frac{\lambda}{2} {n \choose 2}$. Since M is (λ, n) -admissible, $m_1 + m_2 + \cdots + m_t = \lambda {n \choose 2}$. It follows that

- $m_1 + m_2 + \dots + m_{t'} = m_1 + m_2 + \dots + m_t 2\nu = \lambda \binom{n}{2} 2\nu$ is even; and
- $m_1 + t' 2 = \frac{\lambda}{2} {n \choose 2} \nu = \frac{1}{2} (m_1 + m_2 + \dots + m_{t'}).$

It follows from Theorem 2.2 in [25] that there is an (M')-decomposition of 2G for some simple graph G satisfying

$$|V(G)| = \frac{1}{2}(m_1 + m_2 + \dots + m_{t'}) - t' + 2 = \frac{\lambda}{2} \binom{n}{2} - \nu - (t - \nu) + 2 = m_1.$$

Since $m_1 \leq n$, G is simple and $\lambda \geq 2$, we can relabel the vertices of G so that 2G is a subgraph of λK_n . It is clear that there is a (2^{ν}) -decomposition of $\lambda K_n - 2G$ and thus the result follows.

Case 2. Suppose that λ is odd and $2\nu = (\lambda - 1)\binom{n}{2}$. Since M is (λ, n) -admissible, $m_1 + m_2 + \cdots + m_t = \lambda\binom{n}{2}$ when n is odd and $m_1 + m_2 + \cdots + m_t = \lambda\binom{n}{2} - \frac{n}{2}$ when nis even. It follows that $m_1 + m_2 + \cdots + m_{t'} = \binom{n}{2}$ when n is odd, $m_1 + m_2 + \cdots + m_{t'} = \binom{n}{2} - \frac{n}{2}$ when n is even, and hence M' is a (1, n)-admissible list. Thus by Lemma 2.2 there is an (M')-decomposition of K_n . Furthermore, it is clear that there is a (2^{ν}) -decomposition of $(\lambda - 1)K_n$ and the result follows. \Box

The following two lemmas are taken directly from [15].

Lemma 3.2 ([15]) Let M be a list of integers and let λ , n, m_1 , m_2 , m'_1 and m'_2 be positive integers such that $m_1 \leq m'_1 \leq m'_2 \leq m_2$ and $m'_1 + m'_2 = m_1 + m_2$. If there is an (M, m_1, m_2) -decomposition of λK_n in which an m_1 -cycle and an m_2 -cycle share at least two vertices, then there is an (M, m'_1, m'_2) -decomposition of λK_n .

Lemma 3.3 ([15]) Let M be a list of integers and let λ , n, m, m' and h be positive integers such that $h \ge m + m'$ and $m + m' + h \le n + 1$. If there is an (M, h, m, m')-decomposition of λK_n , then there is an (M, h, m + m')-decomposition of λK_n .

In order to prove our next result we introduce the following definition. A graph G is an (a_1, a_2, \ldots, a_s) -flower if G is the union of $s \ge 1$ cycles A_1, A_2, \ldots, A_s such that

- A_i is a cycle of length $a_i \ge 2$ for $i = 1, 2, \ldots, s$; and
- if $s \ge 2$, then there is an $x \in V(G)$ such that $V(A_i) \cap V(A_j) = \{x\}$ for all $1 \le i < j \le s$.

Lemma 3.4 Let M and M' be lists of integers and let λ , n and $m \geq 3$ be positive integers. If there is an (M)-packing of λK_n whose leave has an (M', m, 2)-flower as its only nontrivial component, then there is an (M, 3)-packing of λK_n whose leave has an (M', m - 1)-flower as its only nontrivial component. **Proof** If m = 3 the result is obvious. Suppose then that $m \ge 4$. Let \mathcal{P} be an (M)-packing of λK_n which satisfies the conditions of the lemma and let L be its leave. Let A and B be cycles in L of lengths m and 2 respectively, and let [u, v, w, x, y] be a path in L with $u, v, w, x \in V(A)$ and $x, y \in V(B)$. By applying Lemma 2.1 from [15] to \mathcal{P} (performing the (v, y)-switch with origin w in the terminology of that paper) we can obtain an (M)-packing \mathcal{P}' of λK_n with a leave L' such that either $L' = (L - \{vw, vu\}) + \{yw, yu\}$ (if the switch has terminus u) or $L' = (L - \{vw, yx\}) + \{yw, vx\}$ (if the switch has terminus x). In either case, it is easy to check that $\mathcal{P}' \cup \{(w, x, y)\}$ is an (M, 3)-packing of λK_n whose leave has an (M', m - 1)-flower as its only nontrivial component.

The following lemma is a specific case of the more general Lemma 4.15 in [15].

Lemma 3.5 Let M be a list of integers and let λ and $n \geq 5$ be positive integers. If there is an (M, 2, 2, 2)-decomposition of λK_n in which two 2-cycles share at least one vertex, then there is an (M, 3, 3)-decomposition of λK_n .

We now use the above lemmas to prove some further results which will be used in Section 4.

Lemma 3.6 Let M be a list of integers and let λ , n and $m \geq 3$ be positive integers. If there is an (M, m, 2)-decomposition of λK_n in which an m-cycle and a 2-cycle share at least one vertex, then there is an (M, m - 1, 3)-decomposition of λK_n .

Proof If m = 3 the result is trivial. Suppose then that $m \ge 4$. If an *m*-cycle and a 2-cycle share at least two vertices, the result follows from Lemma 3.2. Otherwise, there is an (M)-packing of λK_n whose leave has an (m, 2)-floweras its only nontrivial component. Thus by Lemma 3.4 there is an (M, 3)-packing of λK_n whose leave has an (m-1)-cycle as its only nontrivial component. The result follows.

Lemma 3.7 Let M be a list of integers, let λ be odd, and let n, m, m'_1 and m'_2 be positive integers satisfying $2 \leq m'_1 \leq m'_2 \leq m, m'_1 + m'_2 = 2 + m$ and $2\nu_2(M) > (\lambda - 1)(\binom{n}{2} - \binom{m}{2})$. If there is an (M, m, 2)-decomposition of λK_n , then there is an (M, m_1, m_2) -decomposition of λK_n .

Proof Let \mathcal{D} be an (M, m, 2)-decomposition of λK_n an let C be an m-cycle in \mathcal{D} . There are $\binom{n}{2} - \binom{m}{2}$ pairs of distinct vertices of λK_n that are not subsets of V(C), and each such pair can be the vertex set of at most $(\lambda - 1)/2$ 2-cycles in \mathcal{D} . Since $2\nu_2(M) > (\lambda - 1)(\binom{n}{2} - \binom{m}{2})$, it follows that there is a 2-cycle in \mathcal{D} that shares two vertices with C. The result then follows by Lemma 3.2.

Lemma 3.8 Let M be a list of integers and let λ and $n \geq 5$ be positive integers satisfying $2\nu_2(M) \geq n-5$. If there is an (M, 2, 2, 2)-decomposition of λK_n , then there is an (M, 3, 3)-decomposition of λK_n .

Proof Let \mathcal{D} be an (M, 2, 2, 2)-decomposition of λK_n . Since $2\nu_2(M) \ge n-5$, the number of occurrences of vertices in 2-cycles in \mathcal{D} is at least n+1 and it follows that at least two 2-cycles share a vertex. The result then follows by Lemma 3.5. \Box

4 A reduction of the problem

In this section we show that to prove Theorem 2.1 it is sufficient to prove that the desired decompositions exist for what we call (λ, n) -ancestor lists, which we now define. For any positive integers λ and n, we shall call a list M a (λ, n) -ancestor list if it is (λ, n) -admissible and satisfies

(N1) if n = 4, then $\nu_3(M) = 0$;

(N2) if n = 5, then $\nu_3(M) + \nu_4(M) \in \{0, 1\}$

(N3) if $n \ge 6$, then $\nu_4(M) + \nu_5(M) + \dots + \nu_{n-1}(M) \in \{0, 1\};$

(N4) if $n \ge 6$ and $\nu_3(M) \ge 1$, then $\nu_{n-2}(M) + \nu_{n-1}(M) = 0$; and

(N5) if $n \ge 6$ and $\nu_3(M) \ge 2$, then $\nu_2(M) \le \lfloor \frac{n}{2} \rfloor - 3$.

Theorem 4.1 For each pair of positive integers λ and n, if there exists an (M')decomposition of λK_n for each (λ, n) -ancestor list M', then there exists an (M)decomposition of λK_n for each (λ, n) -admissible list M.

Proof Let λ and n be positive integers. Throughout this proof we assume that any (λ, n) -admissible list is written in non-increasing order. For distinct (λ, n) -admissible lists (m_1, m_2, \ldots, m_t) and $(m'_1, m'_2, \ldots, m'_{t'})$, we say the list $(m'_1, m'_2, \ldots, m'_{t'})$ is larger than the list (m_1, m_2, \ldots, m_t) if t' > t or if t' = t and $m'_k > m_k$ where k is the smallest positive integer such that $m_k \neq m'_k$. Note that this defines a total order on the set of all non-increasing (λ, n) -admissible lists.

For a contradiction, suppose the theorem does not hold for λ and n. Then there exists a largest (λ, n) -admissible list $M = (m_1, m_2, \ldots, m_t)$ such that there is no (M)-decomposition of λK_n . By assumption, M is not a (λ, n) -ancestor list and so at least one of the following holds.

- (1) n = 4 and $\nu_3(M) \ge 1$.
- (2) n = 5 and $\nu_3(M) + \nu_4(M) \ge 2$.
- (3) $n \ge 6$ and $\nu_4(M) + \nu_5(M) + \dots + \nu_{n-1}(M) \ge 2$.
- (4) $n \ge 6$, $\nu_3(M) \ge 1$ and $\nu_{n-2}(M) + \nu_{n-1}(M) \ge 1$.
- (5) $n \ge 6, \nu_3(M) \ge 2$ and $\nu_2(M) \ge \lfloor \frac{n}{2} \rfloor 2$.

Furthermore, we may assume that $\lambda \geq 2$ (by Lemma 2.2), and that $m_1 + t - 2 < \frac{\lambda}{2} {n \choose 2}$ if λ is even, and $2\nu_2(M) < (\lambda - 1) {n \choose 2}$ if λ is odd (if we have equality in either of these there exists an (M)-decomposition of λK_n by Lemma 3.1). These strict inequalities allow us to modify the list M in a number of ways to obtain a larger list which still satisfies conditions (A4) and (A5) of (λ, n) -admissibility. We now show that there exists a (λ, n) -admissible list M' such that M' is larger than M and the existence of (M')-decomposition of λK_n implies the existence of an (M)-decomposition of λK_n . This will suffice to complete the proof, because an (M')-decomposition of λK_n must exist by the maximality of M and hence an (M)-decomposition of λK_n exists in contradiction to our assumption.

If (1) holds then $\nu_3(M) \ge 2$ (since $\sum M$ is even) and we define M' to be the list obtained from M by replacing two 3's with a 2 and a 4.

If (2) holds, then there exist integers x and y in M such that $3 \le x \le y \le 4$, and we define M' to be the list obtained from M by replacing an x and a y with an x - 1and a y + 1.

If (3) holds, then there exist integers x and y in M such that $4 \le x \le y \le n-1$, and we define M' as follows.

- (a) If $x + y \ge n + 2$, then M' is the list obtained from M by replacing an x and a y with an x 1 and a y + 1.
- (b) If $x + y \le n + 1$, then M' is the list obtained from M by replacing an x and a y with an x 2 and a y + 2 if x = 4, λ is odd and $2\nu_2(M) = (\lambda 1)\binom{n}{2} 2$, and replacing an x with a 2 and an x 2 otherwise.

If (4) holds, then there exists an $x \in \{n-2, n-1\}$ in M and we define M' to be the list obtained from M by replacing a 3 and an x with a 2 and an x + 1.

If (5) holds, and neither (3) nor (4) hold, we define M' as follows.

- (a) If either λ is even, or λ is odd and $2\nu_2(M) \leq (\lambda 1)\binom{n}{2} 6$, then M' is the list obtained from M by replacing two 3's with three 2's.
- (b) If λ is odd and $2\nu_2(M) \ge (\lambda 1)\binom{n}{2} 4$, then M' is the list obtained from M by replacing two 3's with a 4 and a 2.

It is easy to see that in each case M' is (λ, n) -admissible and M' is larger than M. We now show that we can construct an (M)-decomposition of λK_n from an (M')-decomposition \mathcal{D} of λK_n by applying one of Lemmas 3.2, 3.3, 3.6, 3.7 or 3.8.

If (1) holds then we can apply Lemma 3.2 (since n = 4 and thus any 4-cycle and 2-cycle in \mathcal{D} must share two vertices). If (2) holds and (x, y) = (3, 3), then we can apply Lemma 3.6 (since n = 5 and thus any 4-cycle and 2-cycle in \mathcal{D} must share at least one vertex). Similarly, if (2) holds and $(x, y) \neq (3, 3)$, then we can apply Lemma 3.2 (since $x + y \ge n + 2$). If (3) holds and $x + y \ge n + 2$, then we can apply Lemma 3.2. If (3) holds and $x + y \le n + 1$, λ is odd, x = 4 and $2\nu_2(M) = (\lambda - 1)\binom{n}{2} - 2$, then we can apply Lemma 3.7 with m = y + 2, $m'_1 = 4$ and $m'_2 = y$. Otherwise, if (3) holds and $x + y \le n + 1$, then we can apply Lemma 3.3 with m = 2, m' = x - 2and h = y. If (4) holds, then we can apply Lemma 3.6. If (5) holds and either λ is even, or λ is odd and $2\nu_2(M) \le (\lambda - 1)\binom{n}{2} - 6$, then we apply Lemma 3.8. Finally, if (5) holds, λ is odd and $2\nu_2(M) \ge (\lambda - 1)\binom{n}{2} - 4$, then we can apply Lemma 3.7 with m = 4 and $m'_1 = m'_2 = 3$.

5 The case $\lambda = 2$

In this section we give a proof of Theorem 2.1 in the case $\lambda = 2$. We first present two lemmas which are proved in Sections 7 and 8 respectively.

Lemma 5.1 If $n \ge 5$ and M is a (2, n)-ancestor list with $\nu_n(M) > (n - 3)/2$, then there is an (M)-decomposition of $2K_n$.

Lemma 5.2 If $n \ge 5$ and Theorem 2.1 holds for $2K_{n-1}$, then there is an (M)-decomposition of $2K_n$ for each (2, n)-ancestor list M satisfying $\nu_n(M) \le (n-3)/2$.

From these lemmas we can prove the following.

Lemma 5.3 Theorem 2.1 holds for $2K_n$.

Proof The proof is by induction on n. By Theorem 4.1 it suffices to prove the existence of an (M)-decomposition of $2K_n$ for each (2, n)-ancestor list M. The result is trivial for $n \in \{1, 2\}$. If n = 3 then $M \in \{(3, 3), (2, 2, 2)\}$ and in each case it is clear a suitable decomposition exists. If n = 4 then $M \in \{(4, 4, 4), (4, 4, 2, 2), (2, 2, 2, 2, 2, 2, 2)\}$ and in each case it is clear a suitable decomposition exists. Suppose then that $n \ge 5$ and assume Theorem 2.1 holds for all $2K_{n'}$ with n' < n. Lemma 5.1 covers each (2, n)-ancestor list M with $\nu_n(M) > (n - 3)/2$, and using the inductive hypothesis, Lemma 5.2 covers those with $\nu_n(M) \le (n - 3)/2$.

6 Proof of Theorem 2.1

Lemmas 2.2 and 5.3 allow us to prove our main result using induction on λ . The main ingredient in the inductive step is given in the following lemma.

Lemma 6.1 Let λ_1 , λ_2 and n be positive integers such that $\lambda_1 \in \{1, 2\}$ and λ_2 is even. If Theorem 2.1 holds for $\lambda_1 K_n$ and $\lambda_2 K_n$, then there is an (M)-decomposition of $(\lambda_1 + \lambda_2)K_n$ for each $(\lambda_1 + \lambda_2, n)$ -ancestor list M satisfying at least one of

(i) $\nu_n(M) \ge \lfloor \lambda_1(\frac{n-1}{2}) \rfloor;$

(ii) $2\nu_2(M) \ge \lambda_2\binom{n}{2}$; or

(iii) $\nu_3(M) \ge 2$.

Proof Let V be a vertex set of size n, let $\sigma_1 = n \lfloor \lambda_1(\frac{n-1}{2}) \rfloor$, let $\sigma_2 = \lambda_2 \binom{n}{2}$, and let M be a $(\lambda_1 + \lambda_2, n)$ -ancestor list satisfying at least one of (i), (ii) or (iii). We note that $\sum M = \sigma_1 + \sigma_2$, and that for each $i \in \{1, 2\}$, if M' is a (λ_i, n) -admissible list, then $\sum M' = \sigma_i$.

Case 1. Suppose that M satisfies (i). If $\nu_2(M) < n$, let $M_1 = (n^{\sigma_1/n})$ and let $M_2 = M - M_1$. It follows from Lemma 2.5 that M_i is (λ_i, n) -admissible for each $i \in \{1, 2\}$. Thus, by assumption there is an (M_i) -decomposition \mathcal{D}_i of $\lambda_i K_V$ for each $i \in \{1, 2\}$. Then $\mathcal{D}_1 \cup \mathcal{D}_2$ is an (M)-decomposition of $(\lambda_1 + \lambda_2)K_V$. If $\nu_2(M) \ge n$, let $M_1 = (n^{\sigma_1/n})$ and let $M_2 = (M, n, n) - (M_1, 2^n)$. It follows from Lemma 2.5 that M_i is (λ_i, n) -admissible for each $i \in \{1, 2\}$. Thus, by assumption there is an (M_i) -decomposition \mathcal{D}_i of $\lambda_i K_V$ for each $i \in \{1, 2\}$. Let H_1 be an *n*-cycle in \mathcal{D}_1 and let H_2 be an *n*-cycle in \mathcal{D}_2 . We may assume (by relabelling vertices in \mathcal{D}_2) that there is a (2^n) -decomposition \mathcal{D}' of $H_1 \cup H_2$. Then

$$(\mathcal{D}_1 \setminus \{H_1\}) \cup (\mathcal{D}_2 \setminus \{H_2\}) \cup \mathcal{D}'$$

is an (M)-decomposition of $(\lambda_1 + \lambda_2)K_V$.

Case 2. Suppose that M satisfies (ii). Let $M_2 = (2^{\sigma_2/2})$ and $M_1 = M - M_2$. If $\lambda_1 = 1$ then $\nu_2(M) \leq \lambda_2 {n \choose 2}$ because M is $(\lambda_1 + \lambda_2, n)$ -admissible, and thus $\nu_2(M) = \sigma_2/2$ and $\nu_2(M_1) = 0$. If $\lambda_1 = 2$, where M is the non-increasing list (m_1, m_2, \ldots, m_t) say, then $m_1 + t - 2 \leq (\frac{2+\lambda_2}{2}) {n \choose 2}$ and thus $m_1 + (t - \sigma_2/2) - 2 \leq (\frac{2+\lambda_2}{2}) {n \choose 2} - \sigma_2/2 = {n \choose 2}$. In either case it follows, by the definition of (λ, n) -admissible and Lemma 2.5, that M_i is (λ_i, n) -admissible for each $i \in \{1, 2\}$. Thus, by assumption there is an (M_i) -decomposition \mathcal{D}_i of $\lambda_i K_V$ for each $i \in \{1, 2\}$. Then $\mathcal{D}_1 \cup \mathcal{D}_2$ is an (M)-decomposition of $(\lambda_1 + \lambda_2)K_V$.

Case 3. Suppose that M satisfies (iii). We note that if n = 3 then M also satisfies (i) and the result follows from Case 1. Furthermore, by the properties of (λ, n) ancestor lists, it follows that $n \notin \{1, 2, 4, 5\}$ and thus we may assume that $n \ge 6$ and M satisfies $2\nu_2(M) \le n - 6$, $\nu_4(M) + \nu_5(M) + \cdots + \nu_{n-1}(M) \in \{0, 1\}$ and $\nu_{n-2}(M) + \nu_{n-1}(M) = 0$. It follows that $2\nu_2(M) + 3\nu_3(M) + n\nu_n(M) \ge \sigma_1 + \sigma_2 - (n-3)$, and thus $3\nu_3(M) + n\nu_n(M) \ge \sigma_1 + \sigma_2 - (n-3) - (n-6)$. Since $\sigma_2 > 2n$, we have $3\nu_3(M) + n\nu_n(M) > \sigma_1 + 9$. If $n\nu_n(M) \ge \sigma_1$ then M satisfies (i) and the result follows from Case 1. Suppose then that $n\nu_n(M) < \sigma_1$ and hence $3\nu_3(M) > 9$.

Let M' = M - (3, 3, 3). Since $n\nu_n(M) \leq \sigma_1 - n$, it follows by the definition of (λ, n) -ancestor lists that for some $\varepsilon \in \{3, 4, 5\}$, M' can be partitioned into lists M_1 and M_2 satisfying $\sum M_1 = \sigma_1 - \varepsilon$ and $\sum M_2 = \sigma_2 - 9 + \varepsilon$, with $\nu_2(M_1) = 0$ and $\nu_2(M_2) \leq \frac{n-6}{2} < n$. (In particular, if M' is the non-increasing list (m_1, m_2, \ldots, m_t))

say, then a suitable partition can be obtained by packing M_1 with lengths m_1 , m_2 , m_3 , and so on, until $\sum M_1 \in \{\sigma_1 - 5, \sigma_1 - 4, \sigma_1 - 3\}$.) If $\varepsilon = 3$, then by Lemma 2.5, $(M_1, 3)$ is (λ_1, n) -admissible and $(M_2, 3, 3)$ is (λ_2, n) -admissible, and so the result follows by taking the union of an $(M_1, 3)$ -decomposition of $\lambda_1 K_V$ and an $(M_2, 3, 3)$ decomposition of $\lambda_2 K_V$, which exist by assumption. Suppose then that $\varepsilon \in \{4, 5\}$. It follows by Lemma 2.5 that (M_1, ε) is (λ_1, n) -admissible and $(M_2, 9 - \varepsilon)$ is (λ_2, n) admissible. Thus, by assumption there is an (M_1, ε) -decomposition \mathcal{D}_1 of $\lambda_1 K_V$ and an $(M_2, 9 - \varepsilon)$ -decomposition \mathcal{D}_2 of $\lambda_2 K_V$. Let H_1 be an ε -cycle in \mathcal{D}_1 and let H_2 be a $(9 - \varepsilon)$ -cycle in \mathcal{D}_2 . By relabelling vertices we may assume that v, w, x, y and z are distinct vertices in V and that $\{H_1, H_2\} = \{(v, w, x, y, z), (v, w, z, x)\}$. Then

$$(\mathcal{D}_1 \setminus \{H_1\}) \cup (\mathcal{D}_2 \setminus \{H_2\}) \cup \{(v, w, x), (x, y, z), (v, w, z)\}$$

is an (M)-decomposition of $(\lambda_1 + \lambda_2)K_V$.

We now present the proof of our main Theorem.

Proof of Theorem 2.1

If there is an (M)-decomposition of λK_n , then by Lemma 2.4, M is a (λ, n) -admissible list. It remains to show that if M is a (λ, n) -admissible list, then there is an (M)decomposition of λK_n . By Theorem 4.1 we need only show there is an (M)-decomposition of λK_n for each (λ, n) -ancestor list M. The proof is by induction on λ . By Lemmas 2.2 and 5.3, Theorem 2.1 holds for K_n and $2K_n$. So let $\lambda \geq 3$ and assume Theorem 2.1 holds for $\lambda' K_n$ with $\lambda' < \lambda$.

Define λ_1 and λ_2 such that $\lambda_1 \in \{1, 2\}$ and $\lambda_2 = \lambda - \lambda_1$ is even. Let $\sigma_1 = n \lfloor \lambda_1(\frac{n-1}{2}) \rfloor$ and let $\sigma_2 = \lambda_2 \binom{n}{2}$. Thus σ_1 is a multiple of n, σ_2 is even, and $\sum M = \sigma_1 + \sigma_2$. By Lemma 6.1, we need only show that at least one of the following conditions holds.

- (1) $n\nu_n(M) \ge \sigma_1$.
- (2) $2\nu_2(M) \ge \sigma_2$.
- (3) $\nu_3(M) \ge 2$.

Note that (1) holds if n = 1 and (2) holds if n = 2. Similarly, if n = 3 then $2\nu_2(M) + n\nu_n(M) = \sum M = \sigma_1 + \sigma_2$ and thus (1) or (2) holds. Suppose then that $n \ge 4$. Let $k = \sum M - 2\nu_2(M) - 3\nu_3(M) - n\nu_n(M)$, and note by the definition of (λ, n) -ancestor lists, that $k \le n - 1$ and $k \le n - 3$ if $\nu_3(M) \ge 1$. Suppose, for a contradiction, that none of (1), (2) or (3) holds; that is, $n\nu_n(M) < \sigma_1, 2\nu_2(M) < \sigma_2$ and $\nu_3(M) < 2$. If $\nu_3(M) = 0$, then $2\nu_2(M) + 3\nu_3(M) + n\nu_n(M) \le (\sigma_2 - 2) + (\sigma_1 - n) = \sum M - (n + 2)$, and $k \ge n + 2$; a contradiction. Similarly if $\nu_3(M) = 1$, then $2\nu_2(M) + 3\nu_3(M) + n\nu_n(M) \le (\sigma_2 - 2) + (\sigma_1 - n) = \sum M - (n - 1)$, and $k \ge n - 1$; a contradiction. The result follows.

The remainder of the paper is devoted to filling in the details of the case $\lambda = 2$ by proving Lemmas 5.1 and 5.2.

7 The case of more than (n-3)/2 Hamilton cycles

The aim of this section is to prove Lemma 5.1 which states that, for $n \ge 5$, there is an (M)-decomposition of $2K_n$ for each (2, n)- ancestor list M satisfying $\nu_n(M) > (n-3)/2$. This will follow directly from Lemmas 7.6 and 7.13. Throughout this section we make frequent use of circulant graphs which we define as follows. For distinct $i, j \in \{0, 1, \ldots, n-1\}$, let $d_n(i, j)$ be the shortest distance from i to j in the n-cycle $(0, 1, \ldots, n-1)$. If $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, then $\langle S \rangle_n$ is the simple graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{\{i, j\} : d_n(i, j) \in S\}$.

7.1 Many 2-cycles

In this subsection we deal with the specific case of Lemma 5.1 in which the (2, n)ancestor list M satisfies $\nu_2(M) \ge n/2$.

For each positive integer n, we define a graph J_n by $V(J_n) = \{0, 1, \ldots, n+1\}$ and $E(J_n) = \{\{i, i+1\}, \{i, i+2\} : i = 0, 1, \ldots, n-1\}$. Let $M = (m_1, m_2, \ldots, m_t)$ be a list of integers with $m_i \ge 2$ for $i = 1, 2, \ldots, t$. A decomposition $\{A_1, A_2, \ldots, A_t, P_1, P_2\}$ of $2J_n$ such that

- A_i is a cycle of length m_i for $i = 1, 2, \ldots, t$, and
- P_i is a path from 0 to *n* such that $V(P_i) = \{0, 1, ..., n\}$ for i = 1, 2, ..., n

will be denoted $2J_n \to (M, n^*, n^*)$. We note the following basic properties of J_n .

- For any integers y and n such that $1 \leq y < n$, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the vertex map $x \mapsto x + (n-y)$. Thus, if there is a decomposition $2J_{n-y} \to (M, (n-y)^*, (n-y)^*)$ and a decomposition $2J_y \to (M', y^*, y^*)$, then there is a decomposition $2J_n \to (M, M', n^*, n^*)$. We will call this construction, and the similar constructions that follow, concatenations.
- For n ≥ 5, if for each i ∈ {0, 1} we identify vertex i of J_n with vertex i + n of J_n the resulting graph is ({1,2})_n. This means that for n ≥ 5, we can obtain an (M, n, n)-decomposition of 2({1,2})_n from a decomposition 2J_n → (M, n^{*}, n^{*}), provided that for each i ∈ {0,1} no cycle in the decomposition of 2J_n contains both vertex i and vertex i + n. Note in particular that this proviso holds if the decomposition 2J_n → (M, M', n^{*}, n^{*}) was formed as a non-trivial concatenation.

Lemma 7.1 The following decompositions exist.

- (i) $2J_k \to (k+1, 2^{(k-1)/2}, k^*, k^*)$, for any odd $k \ge 1$.
- (ii) $2J_k \rightarrow (2k_1 + 1, 2k_2 + 1, 2^{(k-2)/2}, k^*, k^*)$, for any positive k, k_1 and k_2 with $2k_1 + 2k_2 = k$.

Proof (i) It is easy to check that the decomposition $2J_1 \rightarrow (2, 1^*, 1^*)$ exists. Let $k \ge 3$ be odd and let A be the (k + 1)-cycle on vertices $\{0, 1, \ldots, k\}$ with

$$E(A) = \{\{0,1\}, \{k-1,k\}\} \cup \{\{i,i+2\} : i = 0, 1, \dots, k-2\}.$$

Let $\mathcal{P} = \{A, (2, 4)\}$ if k = 3, and $\mathcal{P} = \{A, (k - 1, k + 1)\} \cup \{(i, i + 1) : i = 2, 4, \dots, k - 3\}$ otherwise. Then \mathcal{P} is a $(k + 1, 2^{(k-1)/2})$ -packing of $2J_k$, and in each case it is straightforward to check that the leave of \mathcal{P} decomposes into two paths P_1 and P_2 , each from 0 to k, with $V(P_1) = V(P_2) = \{0, 1, \dots, k\}$ as required. (ii) Let k k_1 and k_2 be positive integers with $2k_1 + 2k_2 = k$. Let A_1 be the $(2k_1 + 1)$ -

(ii) Let k, k_1 and k_2 be positive integers with $2k_1 + 2k_2 = k$. Let A_1 be the $(2k_1 + 1)$ -cycle on vertices $\{0, 1, \ldots, 2k_1\}$ with

$$E(A_1) = \{\{0,1\}, \{2k_1 - 1, 2k_1\}\} \cup \{\{i, i + 2\} : i = 0, 1, \dots, 2k_1 - 2\},\$$

and let $\mathcal{P}_1 = \{A_1\}$ if $k_1 = 1$, and $\mathcal{P}_1 = \{A_1\} \cup \{(i, i+1) : i = 2, 4, \dots, 2k_1 - 2\}$ otherwise. Similarly, let A_2 be the $(2k_2 + 1)$ -cycle on vertices $\{2k_1, 2k_1 + 1, \dots, k\}$ with

$$E(A_2) = \{\{2k_1, 2k_1 + 1\}, \{k - 1, k\}\} \cup \{\{i, i + 2\} : i = 2k_1, 2k_1 + 2, \dots, k - 2\},\$$

and let $\mathcal{P}_2 = \{A_2\}$ if $k_2 = 1$, and $\mathcal{P}_2 = \{A_2\} \cup \{(i, i+1) : i = 2k_1+1, 2k_1+3, \dots, k-3\}$ otherwise. Then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \{(k-1, k+1)\}$ is a $(2k_1+1, 2k_2+1, 2^{(k-2)/2})$ -packing of $2J_k$, and in each case it is straightforward to check that the leave of \mathcal{P} decomposes into two paths P_1 and P_2 , each from 0 to k, with $V(P_1) = V(P_2) = \{0, 1, \dots, k\}$ as required. \Box

In the following result we use concatenations of decompositions from Lemma 7.1 to build decompositions of $2\langle \{1,2\}\rangle_n$.

Lemma 7.2 If $n \ge 5$ and $M = (m_1, m_2, \ldots, m_t)$ is any list satisfying $2 \le m_i \le n$ for $i = 1, 2, \ldots, t, \ 2\nu_2(M) \ge n$ and $\sum M = 2n$, then there is an (M, n, n)-decomposition of $2\langle \{1, 2\} \rangle_n$.

Proof We note that $n/2 < t \le n$ and thus $\nu_2(M) > n-t \ge 0$. Let $M' = M - (2^{n-t})$, and let r = t - (n-t) be the length of the list M', noting that $r \ge 2$ and that $\sum M' = 2t$. Take a partition of M' in which each part is either a single even integer or a pair of odd integers. This is possible since $\sum M'$ is even and, as M' contains at least one 2, this partition has at least two parts. For each part that is a single even integer, say $\{s\}$, use Lemma 7.1(i) to construct a decomposition $2J_{s-1} \to (s, 2^{(s-2)/2}, (s-1)^*, (s-1)^*)$, and for each part that is a pair of odd integers, say $\{s_1, s_2\}$, use Lemma 7.1(ii) to construct a decomposition $2J_{s_1+s_2-2} \to (s_1, s_2, 2^{(s_1+s_2-4)/2}, (s_1 + s_2 - 2)^*, (s_1 + s_2 - 2)^*)$. Concatenate all of these decompositions to obtain a decomposition $2J_{2t-r} \to (M', 2^{t-r}, (2t-r)^*, (2t-r)^*)$. Since 2t - r = n and t - r = n - t, we have $2J_n \to (M, n^*, n^*)$ and the result follows.

The following lemma is from [8] (see Lemma 2.3 of [8] and its proof), also see [18].

Lemma 7.3 Let $n \ge 5$ be an integer and $M = (m_1, m_2, \ldots, m_t)$ be any list satisfying $3 \le m_i \le n$ for $i = 1, 2, \ldots, t$ and $\sum M = n$. Let $m_0 = 0$ and for each $j \in \{1, 2, \ldots, t\}$ let $s_j = \sum_{i=0}^{j-1} m_i$ and

$$C_{j} = \begin{cases} (s_{j}, s_{j} + 2, \dots, s_{j} + m_{j} - 1, s_{j} + m_{j} - 2, s_{j} + m_{j} - 4, \dots, s_{j} + 1), & \text{if } m_{j} \text{ is odd;} \\ (s_{j}, s_{j} + 2, \dots, s_{j} + m_{j} - 2, s_{j} + m_{j} - 1, s_{j} + m_{j} - 3, \dots, s_{j} + 1), & \text{if } m_{j} \text{ is even} \end{cases}$$

Then the leave H of the packing $\mathcal{P} = \{C_1, C_2, \dots, C_t\}$ of $\langle \{1, 2\} \rangle_n$ is an n-cycle and so $\mathcal{P} \cup \{H\}$ is an (M, n)-decomposition of $\langle \{1, 2\} \rangle_n$.

Lemma 7.4 If $n \ge 5$ and $M = (m_1, m_2, \ldots, m_t)$ is any list satisfying $2 \le m_i \le n$ for $i = 1, 2, \ldots, t$ and $\sum M = n$, then there is an (M, n, n, n)-decomposition of $2\langle \{1, 2\} \rangle_n$.

Proof If $\nu_2(M) = 0$ the result follows by combining an (M, n)-decomposition of $\langle \{1, 2\}\rangle_n$ and an (n, n)-decomposition of $\langle \{1, 2\}\rangle_n$, each of which exists by Lemma 7.3. If $\nu_2(M) = n/2$ the result follows from Lemma 7.2. Suppose then that $1 \leq \nu_2(M) < n/2$. Let $m \geq 2$ and $h \geq 0$ be integers such that $M' = M - (m, 2^{2h+1})$ satisfies $\nu_2(M') = 0$. (Note that m = 2 if $\nu_2(M)$ is even, and $3 \leq m \leq n-2$ if $\nu_2(M)$ is odd.) By Lemma 7.3 there is an $(M', m+2, 4^h, n)$ -decomposition, \mathcal{D}' say, of $\langle \{1, 2\}\rangle_n$. Furthermore, we may assume that if $h \geq 1$ then \mathcal{D}' contains the 4-cycles $C_i = (i, i+1, i+3, i+2)$ for $i = 0, 4, \ldots, 4(h-1)$, and that \mathcal{D}' contains an (m+2)-cycle C that has the path [4h+2, 4h, 4h+1, 4h+3] as a subgraph. Let $\{H_1, H_2\}$ be an (n, n)-decomposition of $\langle \{1, 2\}\rangle_n$, with $H_1 = (0, 1, \ldots, n-3, n-1, n-2)$ if n is even and $H_1 = (0, 1, \ldots, n-1)$ if n is odd. In either case, H_1 contains the path $[0, 1, \ldots, 4h+3]$ (note that, if n is even, $\sum M' + m \geq 4$ and hence $n \geq 4h+6$). Then

$$\mathcal{D} = (\mathcal{D}' \setminus \{C, C_0, C_4, \dots, C_{4(h-1)}\}) \cup \{C^*, H_1^*, H_2\} \cup \{(i, i+1) : i = 0, 2, \dots, 4h\}$$

is an (M, n, n, n)-decomposition of $2\langle\{1, 2\}\rangle_n$, where C^* is the *m*-cycle obtained from C by replacing the path [4h+2, 4h, 4h+1, 4h+3] with the path [4h+2, 4h+3], and H_1^* is the *n*-cycle obtained from H_1 by replacing the path [i, i+1, i+2, i+3] with the path [i, i+2, i+1, i+3], for each $i = 0, 4, \ldots, 4h$.

Lemma 7.5 If n, a and b are non-negative integers satisfying 2a + 2nb = n(n-5), then there is a $(2^a, n^{2b})$ -decomposition of $2\langle \{3, 4, \ldots, \lfloor n/2 \rfloor \} \rangle_n$.

Proof By Lemma 3.1 of [17], there is a decomposition \mathcal{D} of $\langle \{3, 4, \ldots, \lfloor n/2 \rfloor \} \rangle_n$ into $\lfloor \frac{n-5}{2} \rfloor$ *n*-cycles if *n* is odd, and into $\lfloor \frac{n-5}{2} \rfloor$ *n*-cycles and a 1-factor if *n* is even. In either case, let H_1, H_2, \ldots, H_b be distinct *n*-cycles in \mathcal{D} (noting that $b \leq \lfloor \frac{n-5}{2} \rfloor$) and let \mathcal{P} be the packing of $2\langle \{3, 4, \ldots, \lfloor n/2 \rfloor \} \rangle_n$ containing exactly two copies of each of H_1, H_2, \ldots, H_b . Clearly the leave of \mathcal{P} is a graph in which each edge has even multiplicity. Thus it can be decomposed into 2-cycles and the result follows. \Box

Finally, we have the following result.

Lemma 7.6 If $n \ge 5$ and M is a (2, n)-ancestor list satisfying $\nu_2(M) \ge n/2$ and $\nu_n(M) \ge 2$, then there is an (M)-decomposition of $2K_n$.

Proof Let M be a (2, n)-ancestor list with $\nu_2(M) \ge n/2$ and $\nu_n(M) \ge 2$. By the definition of (2, n)-ancestor lists it follows that

$$2\nu_2(M) + n\nu_n(M) \ge n(n-2).$$
 (A)

Let b be the largest integer such that $2b \leq \min(\nu_n(M) - 2, n - 5)$ (note that b is nonnegative) and let a = n(n-5-2b)/2. Because $2b \geq \nu_n(M) - 3$ or $2b \geq n-5$, it follows from (A) that $a \leq \nu_2(M)$, and thus $(2^a, n^{2b})$ is a sublist of M. Now 2a+2bn = n(n-5)and there exists a $(2^a, n^{2b})$ -decomposition of $2\langle\{3, 4, \ldots, \lfloor n/2 \rfloor\}\rangle_n$ by Lemma 7.5. Note that $M' = M - (2^a, n^{2b})$ satisfies $\sum M' = 4n$ and $\nu_n(M') \geq 2$. Further, since M is a (2, n)-ancestor list, M' contains either one 3 and at most one length in $\{4, 5, \ldots, n-3\}$, or no 3's and at most one length in $\{4, 5, \ldots, n-1\}$. It follows that either $\nu_n(M') = 2$, $2\nu_2(M') \geq n$ and there is an (M')-decomposition of $2\langle\{1,2\}\rangle_n$ by Lemma 7.2, or $\nu_n(M') \geq 3$ and there is an (M')-decomposition of $2\langle\{1,2\}\rangle_n$ by Lemma 7.4. The result follows.

7.2 Few 2-cycles

In this subsection we deal with the specific case of Lemma 5.1 in which the (2, n)ancestor list M satisfies $\nu_2(M) < n/2$.

Lemma 7.7 If there is an (M, 2, 2, 5)-decomposition of $2K_n$ in which vertices from two vertex disjoint 2-cycles are joined by an edge of a 5-cycle, then there is an (M, 3, 3, 3)-decomposition of $2K_n$.

Proof Our aim is to show there is an (M)-packing of $2K_n$ whose leave admits a (3, 4, 2)-decomposition in which the 4-cycle and 2-cycle share at least one vertex. Then there is an (M, 3, 4, 2)-decomposition of $2K_n$ in which a 4-cycle and a 2-cycle share at least one vertex and the result follows by Lemma 3.6.

By our hypothesis, there is an (M)-packing \mathcal{P} of $2K_n$ with leave L, such that Ladmits a decomposition $\{C_1, C_2, C\}$ in which C is a 5-cycle, say C = (u, v, x, y, z), and C_1 and C_2 are vertex disjoint 2-cycles with $u \in V(C_1)$ and $v \in V(C_2)$, say $C_1 = (u, u')$ and $C_2 = (v, v')$. If either $y \in \{u', v'\}$ or (u', v') = (x, z) then it is easy to check that the required (3, 4, 2)-decomposition of L exists. Suppose then that $y \notin \{u', v'\}$ and $(u', v') \neq (x, z)$. Without loss of generality (by suitably relabelling the vertices) we may assume that $u' \notin V(C)$. By applying Lemma 2.1 from [15] to \mathcal{P} (performing the (u', y)-switch with origin u in the terminology of that paper) we can obtain an (M)packing \mathcal{P}' of $2K_n$ with a leave L' such that one of $L' = (L - \{u'u, u'u\}) + \{yu, yu\}$ (if the switch has terminus u), $L' = (L - \{u'u, yx\}) + \{yu, u'x\}$ (if the switch has terminus x), or $L' = (L - \{u'u, yz\}) + \{yu, u'z\}$ (if the switch has terminus z). In each case, it is easy to check that the required decomposition of L' exists. \Box **Lemma 7.8** Suppose n and m are integers with $n \ge m \ge 3$, and $M = (m_1, m_2, ..., m_t)$ is a list of integers satisfying $\sum M = m$ and $m_i \ge 2$ for i = 1, 2, ..., t. Then there is a subgraph of $2K_n$ which admits both an (M, n)-decomposition and an (m, n)decomposition.

Proof If t = 1 then M = (m) and the result is obvious. Suppose then that $t \ge 2$. Let C_1, C_2, \ldots, C_t be pairwise vertex disjoint cycles, of lengths m_1, m_2, \ldots, m_t respectively, in $2K_n$. For each $i = 1, 2, \ldots, t$, partition the edges of C_i into three paths, say P_i, Q_i and R_i , of lengths 1, $m_i - 2$ and 1, respectively, and label the vertices of C_i so that $P_i = [u_i, v_i]$ and $R_i = [w_i, u_i]$ (with $w_i = v_i$ if $m_i = 2$). Let L be the leave of the packing $\{C_1, C_2, \ldots, C_t\}$ in $2K_n$. Define paths $X_i = [u_i, v_{i+1}]$ for $i = 1, 2, \ldots, t - 1$, $X_t = [u_t, v_1], Y_i = [w_i, u_{i+1}]$ for $i = 1, 2, \ldots, t - 2$ and $Y_{t-1} = [w_{t-1}, w_t]$ in L with length 1. Also define a path Y_t in L from u_t to u_1 with length n - m + 1 whose set of internal vertices is disjoint from the set $\{u_i, v_i, w_i : i = 1, 2, \ldots, t\}$. Observe that

- $C_i = P_i \cup Q_i \cup R_i$ is a cycle of length m_i for each i = 1, 2, ..., t;
- $H = \bigcup_{i=1}^{t} (Q_i \cup X_i \cup Y_i)$ is a cycle of length n;
- $C = \bigcup_{i=1}^{t} (Q_i \cup R_i \cup X_i)$ is a cycle of length m; and
- $H' = \bigcup_{i=1}^{t} (P_i \cup Q_i \cup Y_i)$ is a cycle of length n.

Then $G = \bigcup_{i=1}^{t} (P_i \cup 2Q_i \cup R_i \cup X_i \cup Y_i)$ is a subgraph of $2K_n$ which admits both an (M, n)-decomposition $\{C_1, C_2, \ldots, C_t, H\}$, and an (m, n)-decomposition $\{C, H'\}$. \Box

In order to prove the main result of this subsection we require some results on decompositions of circulant graphs of the form $\langle \{n/2-1, n/2\} \rangle_n$, where n is even. We obtain these results using graph concatenation methods similar to those in the previous subsection. Accordingly, we redefine J_n to suit our purposes in this subsection.

Let $V_i = \{i, i'\}$ for each nonnegative integer *i*. Then for each even integer $n \ge 2$, we define J_n by $V(J_n) = \bigcup_{i=0}^{n/2} V_i$ and $E(J_n) = \{\{i, i+1\}, \{i, i'\}, \{i', (i+1)'\} : i = 0, 1, \ldots, n/2 - 1\}$. Let $M = (m_1, m_2, \ldots, m_t)$ be a list of integers with $m_i \ge 2$ for $i = 1, 2, \ldots, t$. A decomposition $\{A_1, A_2, \ldots, A_t, P_1, P_2\}$ of $J_n \cup I$, where I is a 1regular graph with $V(I) = \bigcup_{i=0}^{n/2-1} V_i$, such that

- A_i is a cycle of length m_i with $V_{n/2} \cap V(A_i) = \emptyset$ for i = 1, 2, ..., t;
- P_1 and P_2 are vertex disjoint paths with end vertices in $V_0 \cup V_{n/2}$ such that $|E(P_1)| + |E(P_2)| = n;$

will be denoted $J_n \to (M, n^+)$ if each P_i has one end vertex in V_0 and one end vertex in $V_{n/2}$, and denoted $J_n \to (M, n^*)$ otherwise. We note the following basic properties of J_n .

- For $n \geq 8$ and $n \equiv 0 \pmod{4}$, if we identify vertices 0 and 0' of J_n with vertices (n/2)' and (n/2) respectively of J_n , the resulting graph is isomorphic to $\langle \{n/2 1, n/2\} \rangle_n$. Similarly, for $n \geq 6$ and $n \equiv 2 \pmod{4}$, if we identify vertices 0 and 0' of J_n with vertices (n/2) and (n/2)' respectively of J_n , the resulting graph is isomorphic to $\langle \{n/2 1, n/2\} \rangle_n$. This means that for $n \geq 6$, we can obtain an (M, n)-decomposition of $\langle \{n/2 1, n/2\} \rangle_n \cup I_i$, for some perfect matching I_i in K_n , from a decomposition $J_n \to (M, n^*)$.
- For any even integers y and n such that 2 ≤ y < n, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the vertex map (x, x') → (x + (n y)/2, (x + (n y)/2)'). Similarly, if I₁ is a 1-regular graph with V(I₁) = ⋃_{i=0}^{(n-y)/2-1} V_i and I₂ is a 1-regular graph with V(I₂) = ⋃_{i=0}^{y/2-1} V_i, then the union of I₁ and the graph obtained from I₂ by applying the vertex map (x, x') → (x + (n y)/2, (x + (n y)/2)') is a 1-regular graph I with V(I) = ⋃_{i=0}^{n/2-1} V_i. Thus, if there is a decomposition J_{n-y} → (M, (n y)⁺) and a decomposition J_y → (M', y⁺), then there is a decomposition J_n → (M, M', n⁺). Similarly, if there is a decomposition J_{n-y} → (M, (n y)⁺) and a decomposition J_y → (M', y^{*}), then there is a decomposition J_n → (M, M', n^{*}). As before, we call this method of combining decompositions concatenation.

Lemma 7.9 The following decompositions exist.

- (i) $J_{2k} \to (2k, (2k)^*)$, for any $k \ge 2$.
- (ii) $J_{2k} \to (2k_1+1, 2k_2+1, (2k)^*)$, for any $k_1 \ge 2$ and $k_2 \ge 1$ with $k_1 + k_2 + 1 = k$.
- (iii) $J_4 \to (2, 2, 4^*), J_8 \to (2, 3, 3, 8^*) \text{ and } J_{12} \to (3, 3, 3, 3, 12^*).$
- (iv) $J_{2k} \to (2k, (2k)^+)$, for any $k \ge 1$.
- (v) $J_{2k} \to (2k_1 + 1, 2k_2 + 1, (2k)^+)$, for any $k_1, k_2 \ge 1$ with $k_1 + k_2 + 1 = k$.

Proof In each case we give only the decomposition \mathcal{D}_{2k} of $J_{2k} \cup I$, noting that it is then straightforward to check that (the implicitly defined) $I = (\bigcup_{G \in \mathcal{D}_{2k}} G) - J_{2k}$ is a 1-regular graph with $V(I) = \bigcup_{i=0}^{k-1} V_i$ as required.

- (i) Let $\mathcal{D}_{2k} = \{A, [0, 0'], [k, k-1, \dots, 1, 1', 2', \dots, k']\}$, where A = (0, 1, 0', 1') if k = 2, and $A = (1', 0', 0, 1, 2', 2, 3', 3, \dots, (k-1)', k-1)$ otherwise.
- (ii) Let $\mathcal{D}_{2k} = \{A_1, A_2, [0, 0'], P_2\}$, where
 - $A_1 = (1', 0', 0, 1, 2)$ if $k_1 = 2$, and $A_1 = (1', 0', 0, 1, 2', 2, 3', 3, \dots, (k_1 - 1)', (k_1 - 1), k_1)$ otherwise,
 - $A_2 = ((k-2)', (k-1)', k-1)$ if $k_2 = 1$, and $A_2 = (k'_1, (k_1+1)', (k_1+1), (k_1+2)', (k_1+2), \dots, (k-1)', k-1)$ otherwise,

• $P_2 = [k, k-1, \dots, k_1, k'_1, (k_1-1)', \dots, 1', 1, 2, \dots, k_1-1, (k_1+1)', (k_1+2)', \dots, k'].$

- (iii) Let $\mathcal{D}_4 = \{(0,1), (0',1'), [0,0'], [2,1,1',2']\},\$ let $\mathcal{D}_8 = \{(0,1), (0',1',2'), (2,3,3'), [0,0'], [4,3,1',1,2,2',3',4']\},\$ and let $\mathcal{D}_{12} = \{(0,1,2), (0',1',2'), (3,3',4'), (4,5,5'), [0,0'], [6,5,1',1,3',2',2,3,4,4',5',6']\}.$
- (iv) Let $\mathcal{D}_{2k} = \{A, [0, 1, \dots, k], [0', 1', \dots, k']\}$, where A = (0, 0') if k = 1, and $A = (0', 0, 1', 1, \dots, (k-1)', k-1)$ otherwise.
- (v) Let $\mathcal{D}_{2k} = \{A_1, A_2, P_1, P_2\}$, where
 - $A_1 = (0', 0, 1)$ if $k_1 = 1$ and $A_1 = (0', 0, 1', 1, \dots, (k_1 - 1)', k_1 - 1, k_1)$ otherwise,
 - $A_2 = (k'_1, (k-1)', k-1)$ if $k_2 = 1$ and $A_2 = (k'_1, (k_1+1)', k_1+1, (k_1+2)', k_1+2, \dots, (k-1)', k-1)$ otherwise,
 - $P_1 = [0, 2', 3', \dots, k']$ if $k_1 = 1$ and $P_1 = [0, 1, \dots, k_1 - 1, (k_1 + 1)', (k_1 + 2)', \dots, k']$ otherwise, and
 - $P_2 = [0', 1', \dots, k'_1, k_1, k_1 + 1, \dots, k].$

We now use the decompositions from Lemma 7.9 to build larger decompositions.

Lemma 7.10 Suppose $n \ge 6$ is even and $M = (m_1, m_2, ..., m_t)$ is a list of integers satisfying $\sum M = n$ and $m_i \ge 2$ for i = 1, 2, ..., t. Then there is a subgraph of $2K_n$ which admits both an (M, n)-decomposition, and a decomposition into $\langle \{n/2 - 1, n/2\} \rangle_n$ and a perfect matching.

Proof Suppose first that M = (3, 3). Thus n = 6 and we can easily choose I such that $\langle \{2,3\} \rangle_6 \cup I \cong K_6 - F$ where F is a perfect matching in K_6 . The required (3,3,6)-decomposition then exists by Lemma 2.2. Suppose then that $M \neq (3,3)$, and note that we need only show there is a decomposition $J_n \to (M, n^*)$. It is routine to check that, since M satisfies the hypotheses of the lemma, M can be written as M = (X, Y) where X is some (possibly empty) list and either Y = (2k) for some $k \ge 2$, $Y = (2k_1 + 1, 2k_2 + 1)$ for some $k_1 \ge 2$ and $k_2 \ge 1$, or $Y \in \{(2, 2), (2, 3, 3), (3, 3, 3, 3)\}$. Let $\sum Y = y$, then $J_y \to (Y, y^*)$ by Lemma 7.9 (i)–(iii). If X is empty then we are finished. If X is nonempty, take a partition of X in which each part is either a single even integer or a pair of odd integers. This is possible since $\sum X$ is even. For each part that is a single even integer, say $\{s\}$, use Lemma 7.9(iv) to construct a decomposition $J_s \to (s, s^+)$, and for each part that is a pair of odd integers, say $\{s_1, s_2\}$, use Lemma 7.9(v) to construct a decomposition $J_{s_1+s_2} \rightarrow (s_1, s_2, (s_1+s_2)^+)$. Concatenate all of these decompositions to obtain a decomposition $J_{n-y} \to (X, (n-y)^+)$. Then we can obtain the required decomposition by concatenating this decomposition with $J_y \to (Y, y^*).$

Lemma 7.11 If n is odd and (M_1, M_2, M_3) is a (2, n)-admissible list such that

- $\nu_n(M_1) \ge 1$,
- there is an (M_1) -decomposition of K_n , and
- there is an $(M_2, \sum M_3)$ -decomposition of K_n ,

then there is an (M_1, M_2, M_3) -decomposition of $2K_n$.

Proof Let V be a vertex set with |V| = n, and let $m = \sum M_3$. Since $n \ge m \ge 3$, it follows from Lemma 7.8 that there is a subgraph of $2K_V$ that admits both an (M_3, n) -decomposition, \mathcal{D}_3 say, and an (m, n)-decomposition, $\{C, H\}$ say, where C is an m-cycle and H is an n-cycle. Let D_1 be an (M_1) -decomposition of K_V which contains the n-cycle H, and let D_2 be an (M_2, m) -decomposition of K_V which contains the m-cycle C (such decompositions can be found by taking the decompositions given by our hypotheses and relabelling vertices). Then $(\mathcal{D}_1 \setminus \{H\}) \cup (\mathcal{D}_2 \setminus \{C\}) \cup \mathcal{D}_3$ is an (M_1, M_2, M_3) -decomposition of $2K_V$ as required. \Box

Lemma 7.12 If n is even and (M_1, M_2, M_3) is a (2, n)-admissible list such that

- there is an (M_1) -decomposition of $K_n \langle \{n/2 1, n/2\} \rangle_n$,
- there is an (M_2) -decomposition of K_n , and
- $\nu_n(M_3) \ge 1$,

then there is an (M_1, M_2, M_3) -decomposition of $2K_n$. Furthermore,

- (i) if $\nu_2(M_3) \ge 1$ and $\nu_4(M_2) \ge 1$, then there is an (M^{\dagger}) -decomposition of $2K_n$, where $M^{\dagger} = (M_1, M_2, M_3, 3, 3) - (2, 4)$; and
- (ii) if $\nu_2(M_3) \ge 2$ and $\nu_5(M_2) \ge 1$, then there is an (M^{\dagger}) -decomposition of $2K_n$, where $M^{\dagger} = (M_1, M_2, M_3, 3, 3, 3) - (2, 2, 5)$.

Proof Let V be a vertex set with |V| = n and let \mathcal{D}_1 be an (M_1) -decomposition of K_V-G , where $G \subseteq K_V$ is isomorphic to $\langle \{n/2-1, n/2\} \rangle_n$. Let $M_3 = (m_1, m_2, \ldots, m_t, n)$. Then $m_1 + m_2 + \ldots + m_t = n$ and by Lemma 7.10 there is an (M_3) -decomposition \mathcal{D}_3 of $G \cup I$, for some perfect matching I in K_V . Observe that the cycles of lengths m_1, m_2, \ldots, m_t in \mathcal{D}_3 are necessarily pairwise vertex disjoint. Finally, let \mathcal{D}_2 be an (M_2) -decomposition of $K_V - I$. Then $2K_V = (K_V - G) \cup (G \cup I) \cup (K_V - I)$ and $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ is an (M_1, M_2, M_3) -decomposition of $2K_V$.

Furthermore, if $\nu_4(M_2) \ge 1$ and $\nu_2(M_3) \ge 1$, then there is a 4-cycle C in \mathcal{D}_2 and a 2-cycle C' in \mathcal{D}_3 . We note that C' necessarily contains an edge of I, say xy. By relabelling vertices in \mathcal{D}_2 , we may assume that $x \in V(C)$. Case (i) then follows by Lemma 3.6. Similarly, if $\nu_5(M_2) \ge 1$ and $\nu_2(M_3) \ge 2$, then there is a 5-cycle C in \mathcal{D}_2 and two distinct 2-cycles C_1 and C_2 in \mathcal{D}_3 . We note that C_1 and C_2 contain distinct edges of I, say x_1y_1 and x_2y_2 . By relabelling vertices in \mathcal{D}_2 , we may assume that $x_1x_2 \in E(C)$. Case (ii) then follows by Lemma 7.7.

Finally, we have the following result.

Lemma 7.13 If $n \ge 5$ and M is a (2, n)-ancestor list satisfying $\nu_2(M) < n/2$ and $\nu_n(M) > (n-3)/2$, then there is an (M)-decomposition of $2K_n$.

Proof Let M be a (2, n)-ancestor list with $\nu_2(M) < n/2$ and $\nu_n(M) > (n-3)/2$. By the definition of (2, n)-ancestor lists it follows that

$$3\nu_3(M) + n\nu_n(M) > n(n-3).$$
 (B)

The problem now splits according to the parity of n.

Case 1. Suppose that *n* is odd. Let $M_1 = (n^{(n-1)/2})$. Thus M_1 is a sublist of M and there is an (M_1) -decomposition of K_n by Lemma 2.2. Furthermore, the list $M' = M - M_1$ satisfies $\sum M' = \binom{n}{2}$ and $2\nu_2(M') = 2\nu_2(M) \leq (n-1)/2$. It is straightforward to show there is a sublist M_3 of M' satisfying $\nu_2(M_3) = \nu_2(M')$ and $3 \leq \sum M_3 \leq n$. Let $M_2 = M' - M_3$, and thus $(M_2, \sum M_3)$ is (1, n)-admissible and there exists an $(M_2, \sum M_3)$ -decomposition of K_n by Lemma 2.2. The result then follows by Lemma 7.11, noting that $M = (M_1, M_2, M_3)$ and $\nu_n(M_1) = (n-1)/2 > 1$.

Case 2. Suppose that *n* is even. Let $M_1 = (n^{n/2-2})$. Thus M_1 is a sublist of *M* and there is an (M_1) -decomposition of $K_n - \langle \{n/2 - 1, n/2\} \rangle_n$ by Lemma 3.1 of [17]. Consider the list $M' = M - M_1$. Since $2\nu_2(M') = 2\nu_2(M) < n$, *n* divides $\sum M'$ and M' contains at most one length in $\{4, 5, \ldots, n-1\}$, it follows that for some $\varepsilon \in \{0, 1, 2\}$ there is a partition of M' into M'_2 and M'_3 satisfying $\sum M'_2 = n(n-2)/2 - \varepsilon$, $\sum M'_3 = 2n + \varepsilon$, $\nu_2(M'_3) = \nu_2(M')$, $\nu_3(M'_3) \ge \varepsilon$ and $\nu_n(M'_3) \ge 1$. Let $M_2 = (M'_2, 3 + \varepsilon) - (3)$ and let $M_3 = (M_3, 2^{\varepsilon}) - (3^{\varepsilon})$. Thus M_2 is (1, n)-admissible and there is an (M_2) -decomposition of K_n by Lemma 2.2. The result then follows by Lemma 7.12, noting that $M = (M_1, M_2, M_3)$ when $\varepsilon = 0$, that $M = (M_1, M_2, M_3, 3, 3) - (2, 2, 5)$ when $\varepsilon = 2$.

8 The case of at most (n-3)/2 Hamilton cycles

The aim of this section is to prove Lemma 5.2 which states that if $n \ge 5$ and Theorem 2.1 holds for $2K_{n-1}$, then there is an (M)-decomposition of $2K_n$ for each (2, n)-ancestor list M satisfying $\nu_n(M) \le (n-3)/2$. We begin with the following useful lemmas.

Lemma 8.1 If there is an (M)-decomposition of $2K_{n-1}$, then there is an $(M, 2^{n-1})$ decomposition of $2K_n$.

Proof Let U be a vertex set with |U| = n - 1, let ∞ be a vertex not in U, let $V = U \cup \{\infty\}$, and let \mathcal{D} be an (M)-decomposition of $2K_U$. Then $\mathcal{D} \cup \mathcal{D}_1$ is an $(M, 2^{n-1})$ -decomposition of $2K_V$, where \mathcal{D}_1 is a (2^{n-1}) -decomposition of $2K_{\{\infty\},U}$. \Box

Lemma 8.2 If there is an (M, h)-decomposition of $2K_{n-1}$, then there is an $(M, 2^{n-1-h}, 3^h)$ -decomposition of $2K_n$.

Proof Let U be a vertex set with |U| = n - 1, let ∞ be a vertex not in U, let $V = U \cup \{\infty\}$ and let \mathcal{D} be an (M, h)-decomposition of $2K_U$. Let C be an h-cycle in \mathcal{D} . Then

$$(\mathcal{D} \setminus \{C\}) \cup \mathcal{D}_1 \cup \mathcal{D}_2$$

is an $(M, 2^{n-1-h}, 3^h)$ -decomposition of $2K_V$, where

- \mathcal{D}_1 is a (2^{n-1-h}) -decomposition of $2K_{\{\infty\},U\setminus V(C)}$; and
- \mathcal{D}_2 is a (3^h) -decomposition of $2K_{\{\infty\},V(C)} \cup C$.

These decompositions are straightforward to construct.

Lemma 8.3 Let $n \ge 5$ and $h \ge 2$ be integers. If there is an (M)-packing of $2K_n$ with leave L of size 3h such that either

- (i) L has a vertex of degree 2h and admits a decomposition into a (2^{h-1}) -flower and an (h+2)-cycle, or
- (ii) L admits a decomposition into a (2^h) -flower and an h-cycle,

then there is an $(M, 3^h)$ -decomposition of $2K_n$.

Proof If h = 2 and (i) holds then there is an (M, 4, 2)-decomposition of $2K_n$ in which a 4-cycle and a 2-cycle share at least one vertex and the result follows from Lemma 3.4. If h = 2 and (ii) holds then there is an (M, 2, 2, 2)-decomposition of $2K_n$ in which two 2-cycles share a vertex and the result follows from Lemma 3.5. Suppose then that $h \ge 3$ and that the result holds for any h' < h. Our aim is to show that there is an (M, 3)-packing of $2K_n$ with leave of size 3(h - 1) such that either

- (a) L has a vertex of degree 2(h-1) and admits a decomposition into a (2^{h-2}) -flower and an (h+1)-cycle, or
- (b) L admits a decomposition into a (2^{h-1}) -flower and an (h-1)-cycle.

The result will then follow by our inductive hypothesis. Let V be a vertex set with |V| = n and let \mathcal{P} be an (M)-packing of $2K_V$ with leave L of size 3h which satisfies (i) or (ii).

Case 1. Suppose that L satisfies (i). Let $\{F, C\}$ be a decomposition of L in which F is a (2^{h-1}) -flower and C is an (h+2)-cycle, let v be the vertex of degree 2h in L, and let $X = V(F) \cap V(C)$. Let [u, v, w, x, y] be a path in C and observe that $u, w \notin X$. If $x \in X$ then L contains the 3-cycle (v, w, x), and hence $\mathcal{P}' = \mathcal{P} \cup \{(v, w, x)\}$ is an (M, 3)-packing of $2K_V$ with leave L' of size 3(h-1). Furthermore, L' decomposes into the (2^{h-1}) -flower F - (v, x) and the (h-1)-cycle $(C - [v, w, x]) \cup [v, x]$ and thus satisfies (b). Suppose then that $x \notin X$. It follows that |X| < h and hence there is a vertex $z \in V(F) \setminus X$. By applying Lemma 2.1 from [15] to \mathcal{P} (performing the (z, x)-switch with origin w in the terminology of that paper) we can obtain an (M)-packing \mathcal{P}' of $2K_V$ with leave L' such that either $L' = L_1 = (L - \{wx, yx\}) + \{wz, yz\}$ (if the switch has terminus y), or $L' = L_2 = (L - \{wx, vz\}) + \{wz, vx\}$ (if the switch has terminus v). In either case, $\mathcal{P}' \cup \{(z, v, w)\}$ is an (M, 3)-packing of $2K_V$ with leave L' - (z, v, w) decomposes into the (2^{h-2}) -flower F - (z, v) and either the (h+1)-cycle $(C - [v, w, x, y]) \cup [v, z, y]$ (if $L' = L_1$) or the (h+1)-cycle $(C - [v, w, x]) \cup [v, x]$ (if $L' = L_2$), and thus satisfies (a).

Case 2. Suppose that *L* satisfies (*ii*). Let $\{F, C\}$ be a decomposition of *L* in which *F* is a (2^h) -flower and *C* is an *h*-cycle, let *v* be the vertex of degree 2h in *F*, and let $X = V(F) \cap V(C)$. If h = 3 then $\mathcal{P} \cup \{C\}$ is an (M, 3)-packing of $2K_V$ whose leave satisfies (*b*). Suppose then that $h \geq 4$.

Subcase 2a. Suppose that $X \neq \emptyset$ and $v \notin X$. Thus there are distinct vertices $w, x, y \in V$ such that $w \in X$ and [w, x, y] is a path in C. If $x \in X$ then L contains the 3-cycle (v, w, x), and hence $\mathcal{P}' = \mathcal{P} \cup \{(v, w, x)\}$ is an (M, 3)-packing of $2K_V$ with leave L' of size 3(h - 1). Furthermore, L' decomposes into the (2^{h-2}) -flower $F - ((v, w) \cup (v, x))$ and the (h + 1)-cycle $(C - [w, x]) \cup [w, v, x]$ and thus satisfies (a). Suppose then that $x \notin X$. It follows that there is a vertex $u \in V(F) \setminus (X \cup \{v\})$. By applying Lemma 2.1 from [15] to \mathcal{P} (performing the (u, x)-switch with origin w in the terminology of that paper) we can obtain an (M)-packing \mathcal{P}' of $2K_V$ with leave L' such that either $L' = L_1 = (L - \{wx, yx\}) + \{wu, yu\}$ (if the switch has terminus y), or $L' = L_2 = (L - \{wx, vu\}) + \{wu, vx\}$ (if the switch has terminus v). In either case, $\mathcal{P}' \cup \{(u, v, w)\}$ is an (M, 3)-packing of $2K_V$ with leave L' - (u, v, w) of size 3(h - 1). Furthermore, L' - (u, v, w) decomposes into the (2^{h-2}) -flower $F - ((v, u) \cup (v, w))$ and either the (h + 1)-cycle $(C - [w, x, y]) \cup [w, v, u, y]$ (if $L' = L_1$) or the (h + 1)-cycle $(C - [w, x, y]) \cup [w, v, u, y]$ (if $L' = L_1$) or the (h + 1)-cycle $(C - [w, x]) \cup [w, v, x]$ (if $L' = L_2$), and thus satisfies (a).

Subcase 2b. Suppose that $v \in X$. Thus there are distinct vertices $u, w, x, y \in V$ such that $u \in V(F) \setminus X$ and [v, w, x, y] is a path in C. Note that $w \notin V(F)$. If $x \in X$ then

L contains the 3-cycle (v, w, x), and hence $\mathcal{P}' = \mathcal{P} \cup \{(v, w, x)\}$ is an (M, 3)-packing of $2K_V$ with leave L' of size 3(h-1). Furthermore, L' decomposes into the (2^{h-1}) -flower F - (v, x) and the (h-1)-cycle $(C - [v, w, x]) \cup [v, x]$ and thus satisfies (b). Suppose then that $x \notin X$. By applying Lemma 2.1 from [15] to \mathcal{P} (performing the (u, x)-switch with origin w in the terminology of that paper) we can obtain an (M)-packing \mathcal{P}' of $2K_V$ with leave L' such that either $L' = L_1 = (L - \{wx, yx\}) + \{wu, yu\}$ (if the switch has terminus y), or $L' = L_2 = (L - \{wx, vu\}) + \{wu, vx\}$ (if the switch has terminus v). In either case, $\mathcal{P}' \cup \{(u, v, w)\}$ is an (M, 3)-packing of $2K_V$ with leave L' - (u, v, w) of size 3(h-1). Furthermore, L' - (u, v, w) decomposes into the (2^{h-1}) -flower F - (u, v) and either the (h-1)-cycle $(C - [v, w, x, y]) \cup [v, u, y]$ (if $L' = L_1$) or the (h-1)-cycle $(C - [v, w, x, y]) \cup [v, u, y]$ (if $L' = L_1$) or the (h-1)-cycle $(C - [v, w, x]) \cup [v, x]$ (if $L' = L_2$), and thus satisfies (b).

Subcase 2c. Suppose that $X = \emptyset$. Thus there are distinct vertices $u, w, x, y \in V$ such that (u, v) is a 2-cycle in F and [w, x, y] is a path in C. By applying Lemma 2.1 from [15] to \mathcal{P} (performing the (u, x)-switch with origin w in the terminology of that paper) we can obtain an (M)-packing \mathcal{P}' of $2K_V$ with leave L' of size 3h such that either $L' = L_1 = (L - \{wx, yx\}) + \{wu, yu\}$ (if the switch has terminus y), or $L' = L_2 = (L - \{wx, vu\}) + \{wu, vx\}$ (if the switch has terminus v). If $L' = L_1$, then L' decomposes into the (2^h) -flower F and the h-cycle $C_1 = (C - [w, x, y]) \cup [w, u, y]$ and, since $V(F) \cap V(C_1) = \{u\}$, we can continue as in Subcase 2a. If $L' = L_2$, then $\deg_{L'}(v) = 2h$ and L' decomposes into the (2^{h-1}) -flower F - (u, v) and the (h+2)-cycle $C_2 = (C - [w, x]) \cup [w, u, v, x]$ and we can continue as in Case 1.

Lemma 8.4 If $n \ge 2s+3 \ge 5$ and there is an $(M, (n-1)^s)$ -decomposition of $2K_{n-1}$, then there is an $(M, 3^s, n^s)$ -packing of $2K_n$ whose leave has a (2^{n-2s-1}) -flower as its only nontrivial connected component.

Proof Let U be a vertex set with |U| = n - 1, let ∞ be a vertex not in U, let $V = U \cup \{\infty\}$, let \mathcal{D} be an $(M, (n-1)^s)$ -decomposition of $2K_U$, and let H_1, H_2, \ldots, H_s be distinct (n-1)-cycles in \mathcal{D} . We begin by showing there is a subgraph G of $2K_U$ such that E(G) contains precisely one edge from each of the cycles H_1, H_2, \ldots, H_s , and such that each nontrivial connected component of G is a path. (As an aside, a similar result concerning the existence of such a graph G, in the case where $\{H_1, H_2, \ldots, H_s\}$ is a 2-factorisation of a graph, is given in Theorem 4.5 of [3].) Construct a sequence $(G_0, U_0), (G_1, U_1), \ldots, (G_s, U_s)$, where

- each G_i is a subgraph of $2K_U$ of size *i* having the property that each of its nontrivial connected components is a path, and
- each U_i is a subset of U of size i,

as follows. Define $V(G_0) = U$, $E(G_0) = \emptyset$ and $U_0 = \emptyset$. Then for each $i \in \{1, 2, ..., s\}$ let G_i be the graph obtained from G_{i-1} by adding an edge, $x_i y_i$ say, from $E(H_i)$,

such that $x_i, y_i \in U \setminus U_{i-1}$, and let U_i be a subset of U containing every vertex of degree 2 in G_i and exactly one vertex of degree 1 from each nontrivial connected component of G_i . Observe that $E(H_i)$ always contains such an edge since $V(H_i) = U$ and $|U| = n - 1 > 2s > 2|U_{i-1}|$, and that adding such an edge to G_{i-1} ensures that each nontrivial connected component of G_i is a path. Then $G = G_s$ is a graph with the required properties.

Let t be the number of nontrivial connected components of G, let p_1, p_2, \ldots, p_t be their respective sizes, and let $U' = U \setminus \{x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_s\}$ (the set of vertices of degree 0 in G). Observe that $t \leq s$, that $p_1 + p_2 + \cdots + p_t = s$, and that $|U'| = n - 1 - s - t \geq n - 2s - 1$. Then

$$\mathcal{P} = (\mathcal{D} \setminus \{H_1, H_2, \dots, H_s\}) \cup \{H'_1, H'_2, \dots, H'_s\}$$

is an (M, n^s) -packing of $2K_V$, where $H'_i = (H_i - [x_i, y_i]) \cup [x_i, \infty, y_i]$, for $i \in \{1, 2, \ldots, s\}$. Furthermore, the only nontrivial connected component of the leave of \mathcal{P} is a $(p_1 + 2, p_2 + 2, \ldots, p_t + 2, 2^{n-1-s-t})$ -flower. Let $\mathcal{P}_0 = \mathcal{P}$ and for each $i = 1, 2, \ldots, s$, let \mathcal{P}_i be the $(M, 3^i, n^s)$ -packing obtained by applying Lemma 3.4 to \mathcal{P}_{i-1} , choosing $m \geq 3$. Then \mathcal{P}_s is the required packing.

Lemma 8.5 If $n \ge 2s + 3 \ge 5$ and there is an $(M, h, (n-1)^s)$ -decomposition of $2K_{n-1}$ with $h \le n-2s-1$, then there is an $(M, 2^{n-2s-1-h}, 3^{s+h}, n^s)$ -decomposition of $2K_n$.

Proof Since $h \leq n - 2s - 1$, it follows from Lemma 8.4 that there is an $(M, h, 2^{n-2s-1-h}, 3^s, n^s)$ -packing of $2K_n$ whose leave has a (2^h) -flower as its only non-trivial connected component. The result then follows from Lemma 8.3 (*ii*).

Proof of Lemma 5.2 Let M be a (2, n)-ancestor list with $\nu_n(M) \leq (n-3)/2$. Since M contains at most one cycle of length in $\{4, 5, \ldots, n-1\}$, we have

$$2\nu_2(M) + 3\nu_3(M) + n\nu_n(M) \ge (n-1)^2.$$
 (C)

Case 1. Suppose that $\nu_n(M) = 0$. It follows from (C) that $2\nu_2(M) + 3\nu_3(M) \ge (n-1)^2$, and since $n \ge 5$, that $\nu_2(M) + \nu_3(M) \ge n$. Let h = 0 if $\nu_2(M) \ge n-1$, let h = 2 if $\nu_2(M) = n-2$, let $h = n-1-\nu_2(M)$ if $\nu_2(M) \le n-3$, and let $M' = M - (3^h, 2^{n-1-h})$. If h = 0, then the fact that M is (2, n)-admissible implies that M' is (2, n-1)-admissible. Thus, by assumption there is an (M')-decomposition of $2K_{n-1}$ and the result follows by Lemma 8.1. Otherwise, $2 \le h \le n-1$ and $\nu_2(M') \le 1$. Then (M', h) is (2, n-1)-admissible (by Lemma 2.5) and by assumption there is an (M', h)-decomposition of $2K_{n-1}$. The result then follows by Lemma 8.2.

Case 2. Suppose that $\nu_n(M) = 1$. It follows from (C) that $2\nu_2(M) + 3\nu_3(M) \ge (n-1)^2 - n$, and since $n \ge 5$, that $\nu_2(M) + \nu_3(M) \ge n-1$. Furthermore, by the properties of (2, n)-admissible lists, it is clear that $\nu_3(M) \ge 1$. Let h = 0 if $\nu_2(M) \ge n-3$,

let h = 2 if $\nu_2(M) = n - 4$, let $h = n - 3 - \nu_2(M)$ if $\nu_2(M) \leq n - 5$, and let $M' = M - (n, 3^{h+1}, 2^{n-3-h})$. If h = 0, then the fact that M is (2, n)-admissible implies that (M', n - 1) is (2, n - 1)-admissible. Thus, by assumption there is an (M', n - 1)-decomposition of $2K_{n-1}$ and the result follows by Lemma 8.4 (with s = 1). Otherwise, $2 \leq h \leq n-1$ and $\nu_2(M') \leq 1$. Then (M', h, n-1) is (2, n-1)-admissible (by Lemma 2.5) and by assumption there is an (M', h, n - 1)-decomposition of $2K_{n-1}$. The result then follows by Lemma 8.5 (with s = 1).

Case 3. Suppose that $\nu_n(M) = s \ge 2$. If $\nu_2(M) \ge n/2$ then the required decomposition exists by Lemma 7.6. Suppose then that $\nu_2(M) < n/2$. Because $s \le (n-3)/2$, it follows from (C) that $\nu_3(M) \ge s$. It also follows from (C) that $2\nu_2(M) + 3\nu_3(M) \ge (n-1)^2 - sn$, and since $n \ge 2s + 3$, that $\nu_2(M) + \nu_3(M) \ge n - s$. Let h = 0 if $\nu_2(M) \ge n - 2s - 1$, let h = 2 if $\nu_2(M) = n - 2s - 2$, let $h = n - 2s - 1 - \nu_2(M)$ if $\nu_2(M) \le n - 2s - 3$, and let $M' = M - (n^s, 3^{s+h}, 2^{n-2s-1-h})$. If h = 0, then $(M', (n-1)^s)$ is (2, n-1)-admissible (by Lemma 2.5). Thus, by assumption there is an $(M', (n-1)^s)$ -decomposition of $2K_{n-1}$ and the result follows by Lemma 8.4. Otherwise, $2 \le h \le n - 1$ and $\nu_2(M') \le 1$. Then $(M', h, (n-1)^s)$ is (2, n-1)-admissible (by Lemma 8.5.

Acknowledgements

The authors acknowledge the support of the Australian Research Council via grants DP150100530, DP150100506, DP120100790, DP120103067, DE120100040 and DP130102987.

References

- [1] B. Alspach, Research Problem 3, Discrete Math. 36 (1981), 333.
- [2] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n I$, J. Combin. Theory Ser. B 81 (2001), 77–99.
- [3] B. Alspach, K. Heinrich and G.Z. Liu, Orthogonal factorizations of graphs, in Contemporary Design Theory: A Collection of Surveys, (Eds. J. Dinitz, D. Stinson), Wiley, New York (1992), pp. 1340.
- [4] Zs. Baranyai, On the factorization of the complete uniform hypergraph, Colloq. Math. Soc. Janos Bolyai 10 (1975), 91–108.
- [5] J.-C. Bermond, O. Favaron and M. Maheo, Hamiltonian decomposition of Cayley graphs of degree 4, J. Combin. Theory Ser. B. 46 (1989), 142–153.

- [6] J.C. Bermond and D. Sotteau, Cycle and circuit designs odd case, *Contributions to graph theory and its applications* (Internat. Colloq., Oberhof, 1977) (German), pp. 11–32, Tech. Hochschule Ilmenau, Ilmenau, 1977.
- [7] J.C. Bermond, C. Huang and D. Sotteau, Balanced cycle and circuit designs: even cases, Ars Combin. 5 (1978), 293–318.
- [8] D. Bryant, Hamilton cycle rich two-factorisations of complete graphs, J. Combin. Des. 12 (2004), 147–155.
- [9] D. Bryant, Cycle decompositions of complete graphs, in Surveys in Combinatorics 2007, A. Hilton and J. Talbot (Editors), London Mathematical Society Lecture Note Series 346, Proceedings of the 21st British Combinatorial Conference, Cambridge University Press, 2007, pp 67–97.
- [10] D. Bryant, Packing paths in complete graphs, J. Combin. Theory Ser. B 100 (2010), 206–215.
- [11] D. Bryant and D. Horsley, Packing cycles in complete graphs, J. Combin. Theory Ser. B 98 (2008), 1014–1037.
- [12] D. Bryant and D. Horsley, Decompositions of complete graphs into long cycles, Bull. London Math. Soc. 41 (2009), 927–934.
- [13] D. Bryant and D. Horsley, An asymptotic solution to the cycle decomposition problem for complete graphs, J. Combin. Theory Ser. A 117 (2010), 1258–1284.
- [14] D. Bryant, D. Horsley and B. Maenhaut, Decompositions into 2-regular subgraphs and equitable partial cycle decompositions, J. Combin. Theory Ser. B 93 (2005), 67–72.
- [15] D. Bryant, D. Horsley, B. Maenhaut and B.R. Smith, Cycle decompositions of complete multigraphs, J. Combin. Des. 19 (2011), 42–69.
- [16] D. Bryant, D. Horsley and W. Pettersson, Cycle decompositions V: Complete graphs into cycles of arbitrary lengths, *Proc. London Math. Soc.* (2013), doi 10.1112/plms/pdt051.
- [17] D. Bryant and B. Maenhaut, Decompositions of complete graphs into triangles and Hamilton cycles, J. Combin. Des. 12 (2004), 221-232.
- [18] D. Bryant and G. Martin, Some results on decompositions of low degree circulant graphs, Austral. J. Combin. 45 (2009), 251–261.
- [19] D. Bryant and C. A. Rodger, Cycle decompositions, in *The CRC Handbook of Combinatorial Designs*, 2nd edition (Eds. C. J. Colbourn, J. H. Dinitz), CRC Press, Boca Raton (2007), pp 373–382.

- [20] H. Hanani, The existence and construction of balanced incomplete block designs, Ann. Math. Statist. 32 (1961), 361–386.
- [21] C. Huang and A. Rosa, On the existence of balanced bipartite designs, Utilitas Math. 4 (1973), 55–75.
- [22] T. P. Kirkman, On a problem in combinations, Cambridge and Dublin Math. J. 2 (1847), 191–204.
- [23] C. Lin and T-W Shyu, A necessary and sufficient condition for the star decomposition of complete graphs, J. Graph Theory 23 (1996), 361–364.
- [24] E. Lucas, "Récreations Mathématiqués," Vol II, Gauthier-Villars, Paris, 1892.
- [25] B. Maenhaut and B.R. Smith, Face 2-colourable embeddings with faces of specified lengths, J. Graph Theory (to appear).
- [26] A. Rosa and C. Huang, Another class of balanced graph designs: balanced circuit designs, *Discrete Math.* 12 (1975), 269–293.
- [27] M. Sajna, Cycle decompositions III: complete graphs and fixed length cycles, J. Combin. Des. 10 (2002), 27–78.
- [28] N. Shalaby, Skolem and Langford Sequences, in *The CRC Handbook of Combinatorial Designs*, 2nd edition (Eds. C. J. Colbourn, J. H. Dinitz), CRC Press, Boca Raton (2007), pp 612–616.
- [29] J.E. Simpson, Langford sequences: Perfect and hooked, Discrete Math 44 (1983), 97–104.
- [30] B.R. Smith, Cycle decompositions of complete multigraphs, J. Combin. Des. 18 (2010), 85–93.
- [31] B.R. Smith, Decompositions of generalised complete graphs, *PhD Thesis*, The University of Queensland, (2009).
- [32] M. Tarsi, Decomposition of complete multigraphs into stars, Discrete Math. 26 (1979), 273–278.