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Biased graphs with no two vertex-disjoint unbalanced cycles

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Abstract

Lovász has completely characterised the structure of graphs with no two vertex-disjoint cycles, while Slilaty has given a structural characterisation of graphs with no two vertex-disjoint odd cycles; his result is in fact more general, describing signed graphs with no two vertex-disjoint negative cycles. A *biased graph* is a graph with a distinguished set of cycles (called balanced) with the property that any theta subgraph does not contain exactly two balanced cycles. In this paper we characterise the structure of biased graphs with no two vertex-disjoint unbalanced cycles, answering a question by Zaslavsky and generalising the results of Lovász and Slilaty.

1 Introduction

By a *cycle* in a graph we mean a connected subgraph where every vertex has degree two. Throughout the paper we will say that two subgraphs are *disjoint* to mean that they are vertex-disjoint; this applies in particular to cycles and paths. A *biased graph* is a pair $(G, \mathcal{B})$, where G is a graph and $\mathcal{B}$ is a collection of cycles of G satisfying the *theta property*, which is as follows. For any two cycles C_1 and C_2 in $\mathcal{B}$ such that $C_1 \cap C_2$ is a path with at least one edge, the third cycle in $C_1 \cup C_2$ is also in $\mathcal{B}$. The cycles in $\mathcal{B}$ are called *balanced*, while those not in $\mathcal{B}$ are *unbalanced*. Biased graphs were introduced by Zaslavsky in [8]. Examples of biased graphs are graphs with all cycles balanced, graphs with all cycles unbalanced and biased graphs arising from group-labelled graphs. In a group-labelled graph each edge is oriented and assigned a value from a group; a cycle is balanced if multiplying the group values along the cycle (where we take the inverse values on edges traversed backwards) produces the group identity.

Biased graphs give rise to two main types of matroids, frame matroids and lift matroids (see [9]). We will not discuss these matroids here, but merely mention that these two matroids

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are the same, for a given biased graph Ω, if and only if Ω does not contain two vertex-disjoint unbalanced cycles. Hence the question arises of which biased graphs have this property. This question was first posed by Zaslavsky (Problem 3.5 in [9]) and is the subject of this paper.

There are two simple cases of biased graphs having no two vertex-disjoint unbalanced cycles. The first is biased graphs with no unbalanced cycles at all, i.e. biased graphs of the form $(G, \mathcal{B})$, where $\mathcal{B}$ is the set of all cycles of G. Biased graphs of this form are called *balanced*. The second simple example of biased graphs with no two disjoint unbalanced cycles is biased graphs where all unbalanced cycles use a specific vertex v, which is then called a *blocking vertex*. We will focus on biased graphs that have no two vertex-disjoint unbalanced cycles, but are not balanced and have no blocking vertex. Such biased graphs are called *tangled*.

A special type of biased graphs are those where all cycles are unbalanced. In this case, our question reduces to ask for the structure of graphs with no two vertex-disjoint cycles. This question was answered by Lovász in [3] (see [1] for a proof in English).

Theorem 1.1 (Lovász [3]). *Let G be a connected graph with no two disjoint cycles. Then either $G - v$ is a forest for some $v \in V(G)$, or G is a subgraph of a graph obtained from the ones in Figure 1 by possibly attaching trees on single vertices (where, in the figure, $k \geq 1$ and $\ell \geq 3$).*

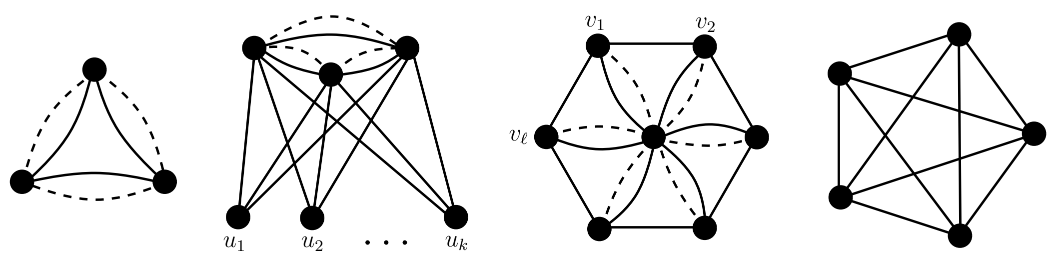

Figure 1: Graphs with no two disjoint cycles. A dotted edge indicates that any number of parallel edges may be added to that edge.

Another particular type of biased graphs are those arising from signed graphs. A signed graph is a pair (G, S), where $S \subseteq E(G)$. The associated biased graph is $(G, \mathcal{B}_S)$, where a cycle C is in $\mathcal{B}_S$ if and only if $|C \cap S|$ is even. If S is replaced by $S' = S \Delta D$, for an edge cut D of G, then $\mathcal{B}_S = \mathcal{B}_{S'}$. For simplicity we will sometimes identify the biased graph arising from a signed graph with the signed graph itself. A family of tangled signed graphs is that of *projective planar* signed graphs, which are signed graphs of the form (G, S), where G can be embedded in the projective plane so that S is a nonseparating cycle of the topological dual of G. In other words, G may be embedded in the projective plane so that the unbalanced cycles are exactly the nonseparating cycles. In [5] Slilaty characterized tangled signed graphs and showed that, saved for a specific case and simple decompositions, they are projective planar. We will make use of similar decompositions for biased graphs; such decomposition will be discussed in Section 5.

Theorem 1.2 (Slilaty [5]). *Any connected tangled signed graph is either*

- *projective planar, or*
- *isomorphic to* $(G, E(G))$*, along with possibly some balanced loops, where* G *is obtained from* K_5 *by adding parallel edges, or*
- *a* 1-, 2- *or* 3-*sum of a tangled signed graph and a balanced signed graph having at least* $2, 3$ *or* 5 *vertices respectively.*

If a signed graph (G, S) is taken with $S = E(G)$, then the unbalanced cycles of (G, S) are exactly the odd cycles of G. Thus Theorem 1.2 also describes graphs having no two vertex-disjoint odd cycles. Such characterisation was also given for internally 4-connected graphs in [2].

Our main result is the proof of the following theorem, which generalises both Theorem 1.1 and Theorem 1.2. We make use of Theorem 1.1 in our proof (specifically, in the proof of Lemma 6.2), while Theorem 1.2 follows easily from our result.

We say that a biased graph is *simple* if it does not contain balanced loops and pairs of parallel edges e and f such that $\{e, f\}$ is a balanced cycle. The *simplification* of a biased graph Ω is a maximal subgraph of Ω which is simple. Thus the simplification of Ω is the biased graph obtained by deleting all balanced loops and all but one edge in any balanced parallel class. A biased graph is tangled if and only if its simplification is tangled. Thus we only consider simple biased graphs in our result. If $\Omega = (G, \mathcal{B})$ is a biased graph, we denote by $||\Omega||$ the graph G (called the *underlying graph* of Ω).

Theorem 1.3. *Let* Ω *be a simple connected tangled biased graph. Then either:*

(T1) Ω *is one of the following:*

- *(T1a) a projective planar signed graph,*
- *(T1b) a generalised wheel,*
- *(T1c) a criss-cross,*
- *(T1d) a fat triangle,*
- *(T1e) projective planar with a special vertex,*
- *(T1f) projective planar with a special pair,*
- *(T1g) projective planar with a special triple,*
- *(T1h) a tricoloured graph, or*

(T2) $||\Omega||$ *is obtained from* K_5 *by possibly adding edges in parallel to an edge of* K_5*, or*

(T3) Ω *is a* 1-, 2- *or* 3-*sum of a tangled biased graph and a balanced signed graph having at least* $2, 3$ *or* 4 *vertices respectively.*

The structures in (T1b)-(T1h) are described in Section 3; all these structures, except for the generalised wheel and fat triangle, occur when the underlying graph of Ω is projective planar. Somewhat surprisingly, all of (T1b)-(T1h) have the property that the removal of some set of at most three vertices leaves a balanced graph.

In the next section we provide basic definitions that will be used throughout the paper. The structures in Theorem 1.3 are defined in Section 3; figures for these structures may be found in the Appendix. For the proof of the main theorem we need to use and extend existing results on linkages; these are presented in Section 4. In Section 5 we consider the case when the tangled biased graph has small separations; we show that in this case Ω is either decomposable along a 1-, 2- or 3-sum or Ω is a generalised wheel. In Section 6 we show that Ω contains a 2-connected spanning balanced subgraph, unless Ω is a criss-cross or a projective planar biased graph with a special pair. In the same section we also show that such balanced subgraph may be chosen to be planar, unless Ω is a fat triangle. Finally, Section 7 contains the proof of Theorem 1.3.

2 Basic definitions

All graphs in this work are undirected and may have loops and parallel edges. Let G be a graph. For a set X of vertices we denote by $G[X]$ the subgraph of G induced by X and by $G-X$ the subgraph $G[V(G)-X]$; we use $G-v$ as shorthand notation for $G-\{v\}$. Moreover, we denote by $N_G(X)$ the set of vertices that are not in X but are adjacent to a vertex in X. If Y is a set of edges, we denote by $G[Y]$ the subgraph of G induced by Y and by $V_G(Y)$ the vertex set of $G[Y]$; we let $G-Y$ denote the graph obtained from G by deleting the edges in Y.

Given a graph G and $X \subseteq V(G)$, $\delta_G(X) := \{uv \in E(G) : u \in X, v \notin X\}$ and we write $\delta_G(v)$ for $\delta_G(\{v\})$. Throughout the paper we shall omit indices when there is no ambiguity. For instance we may write $\delta(v)$ for $\delta_G(v)$. Two edges in a graph are *independent* if they have no common endpoint. A set of edges U is *independent* if the edges in U are pairwise independent.

Let G be a graph and A and B be sets of vertices of G. An (A, B)*-path* is a path of G with one endpoint in A and one endpoint in B, and no other vertex in $A \cup B$. We use "(a, b)-path" as shorthand for "$(\{a\}, \{b\})$-path" and similarly, "(a, B)-path" as shorthand for "$(\{a\}, B)$-path". If $A = B$, then we refer to an (A, A)-path simply as an A-path. Sometimes we abuse notation and in the previous definition we replace one or both of A and B with subgraphs of G. So an (H_1, H_2)-path (for subgraphs H_1 and H_2 of G) is simply a $(V(H_1), V(H_2))$-path. A *theta graph* is a graph formed by three internally disjoint (a, b)-paths, for some distinct vertices a, b.

If X is a set of edges of G, we define the *boundary* of X as $V_G(X) \cap V_G(\bar{X})$ (where $\bar{X}$ denotes the complement of X) and the *interior* of X as the vertices in $V_G(X)$ that are not on the boundary. We also define the boundary and interior of a subgraph H to be the boundary and interior of $E(H)$. A partition (A_1, A_2) of $E(G)$ is a *k-separation* if $G[A_1]$ and $G[A_2]$ are both connected with nonempty interior and the boundary of A_1 has size k. Sometimes we will abuse notation and say that $(G[A_1], G[A_2])$ is a k-separation. We may also omit one side of a k-separation and say, for example, that A is a k-separation if $(A, \bar{A})$ is a k-separation. A graph G is *k-connected* if it has no ℓ-separation for $\ell < k$.

A set of vertices X of G is a *vertex-cut* if $G - X$ is disconnected and X is minimal with this property. It is a *k-vertex-cut* if it is a vertex-cut of size k. If $X = \{v\}$ is a 1-vertex-cut, we call v a *cutvertex*. A *bridge* of a vertex set X is the subgraph of G formed by a component H of $G - X$ together with the edges between H and X and the endpoints of these edges. We also call a bridge of X an X-bridge. Note that our definition of a bridge differ slightly from the usual one, since we do not consider an edge connecting two vertices of X to be an X-bridge.

Given a graph G, the *blocks* of G are the maximal 2-connected subgraphs of G. If $H_1, \dots, H_k$ are all the blocks of G, then $E(H_1), \dots, E(H_k)$ is a partition of $E(G)$ and we may associate a tree $\mathcal{T}$ with this partition. Let $\{v_1, \dots, v_\ell\}$ be the set of cut-vertices of G and define $V(\mathcal{T}) = \{H_1, \dots, H_k\} \cup \{v_1, \dots, v_\ell\}$; block H_i and cut-vertex v_j are adjacent in $\mathcal{T}$ if $v_j \in V(H_i)$. A block is a *leaf block* if it corresponds to a leaf of $\mathcal{T}$.

Let $\Omega = (G, \mathcal{B})$ be a biased graph. The graph G is the *underlying* graph of Ω, denoted as $||\Omega||$. We will often refer to properties of $||\Omega||$ as being properties of Ω; for example, we may say that Ω is k-connected to mean that $||\Omega||$ is k-connected and we may write $\delta_\Omega(v)$ to mean $\delta_{||\Omega||}(v)$. We say that a biased graph $\Omega' = (G', \mathcal{B}')$ is a *subgraph* of Ω if G' is a subgraph of G and $\mathcal{B}' = \{B \in \mathcal{B} : B \subseteq G'\}$. Given a set X of edges of G, the biased graph *induced by X* is the subgraph of Ω with underlying graph $G[X]$. We will denote such subgraph as $\Omega[X]$. When referring to a subgraph of G, we will assume that such subgraph inherits the structure of balanced cycles of Ω. For example, when referring to a bridge of a set $X \subseteq V(G)$, we will often consider such bridge as a biased graph.

We say that two cycles in a biased graph *have the same bias* if they are both balanced or both unbalanced. Let C_1 and C_2 be cycles such that $C_1 \cup C_2$ is a theta subgraph. Let C_3 be the third cycle contained in $C_1 \cup C_2$. We say that C_3 is obtained from C_1 by *rerouting along* C_2. If $C_1, \dots, C_k$ is a sequence of cycles such that C_{i+1} is obtained from C_i be rerouting along some cycle C^i, then we say that C_k is obtained from C_1 by *rerouting along* $C^1, \dots, C^{k-1}$. By the theta property, if C_2 is obtained from C_1 by rerouting along a balanced cycle, then C_1 and C_2 have the same bias. Inductively, this is also the case if C_2 is obtained from C_1 by rerouting along a set of balanced cycles. We will make repeated use of this fact throughout the paper.

Let Ω be a biased graph and Ω' be a balanced subgraph of Ω. Given a set $A \subseteq E(\Omega) - E(\Omega')$, we say that a cycle C of Ω is an A-cycle for Ω' if $A \subseteq C \subseteq \Omega' \cup A$. We write e-cycle as a shorthand for $\{e\}$-cycle. We say that $F \subseteq E(\Omega) - E(\Omega')$ is *2-balanced* with respect to Ω' if, for all $A \subseteq F$ with $|A| = 2$, every A-cycle is balanced. The theta property implies that if $f \in E(\Omega) - E(\Omega')$ and some f-cycle for Ω' is unbalanced, then so are all the f-cycles for Ω' (see Proposition 3.1 in [8]). The same holds for A-cycles if A is a set of two edges sharing an endpoint. However, this is not true in general: as an example, choose $||\Omega|| = K_4$ and let Ω' be a 4-cycle of Ω. Let f_1 and f_2 be the diagonals of this 4-cycle. Then we may assign one of the 4-cycles using f_1 and f_2 to be balanced, and the other to be unbalanced (while all the triangles are unbalanced). We make use of 2-balanced sets in Lemma 7.2: suppose that Ω is a connected tangled biased graph and that Ω' is a maximal balanced subgraph of Ω. Then we show in the lemma that if $E(\Omega) - E(\Omega')$ is 2-balanced with respect to Ω' then

Ω is a signed graph (with signature $E(\Omega) - E(\Omega')$).

A vertex v of a biased graph Ω that intersects all unbalanced cycles of Ω is called a *blocking vertex*. Two vertices v and w (neither of which is a blocking vertex) form a *blocking pair* if they intersect all unbalanced cycles. Suppose that v is a blocking vertex of Ω and $\Omega - v$ is connected. In this case we define a relation $\sim_v$ on the edges in $\delta_\Omega(v)$ by declaring $e \sim_v f$ if either $e = f$ or all cycles containing e and f are balanced. This is an equivalence relation, as we show next. Let e_1, e_2, e_3 be distinct edges in $\delta_\Omega(v)$ with $e_1 \sim_v e_2$ and $e_2 \sim_v e_3$. Let H be a theta subgraph of Ω containing all of e_1, e_2 and e_3; such theta subgraph exists because $\Omega - v$ is connected, thus it contains a spanning tree. The cycle in H containing both e_1 and e_2 is balanced, and so is the cycle containing both e_2 and e_3. Therefore the cycle C containing e_1 and e_3 is balanced. Any other cycle containing e_1 and e_3 may be obtained from C by rerouting along balanced cycles (contained in $\Omega - v$), hence all the cycles containing e_1 and e_3 are balanced and $e_1 \sim_v e_3$, showing that $\sim_v$ is an equivalence relation. The same argument shows that a cycle of Ω (that is not a loop) is unbalanced if and only if it contains two edges in $\delta_\Omega(v)$ which are not equivalent. We call the partition given by the equivalence classes of $\sim_v$ the *standard partition* of $\delta_\Omega(v)$.

3 Tangled structures

In this section we describe the possible structure of tangled biased graph. All the structures in this section are depicted in Appendix A.

We will make repeated use of the following definition (which will be repeated and extended in Section 4). Given two disjoint sets of vertices X and Y in a graph G, we say that $(G, (X, Y))$ is *planar* if G is a plane graph, $X \cup Y$ belongs to the same face F of G and there is some ordering $(x_1, \dots, x_k)$ of the vertices in X and some ordering $(y_1, \dots, y_\ell)$ of the vertices in Y such that $x_1, \dots, x_k, y_1, \dots, y_\ell$ appear on F in this circular order. If $X = \{x\}$ then we may abuse notation and write that $(G, (x, Y))$ is planar. This definition extends to the cases when $x_k = y_1$ and/or $y_\ell = x_1$. We also extend this notation in the obvious way to the case when we have more than two sets.

3.1 Generalized wheels

Let $\Omega = (G, \mathcal{B})$ be a biased graph. Suppose that G contains a special vertex w such that:

(a) $G - w$ is the union of 2-connected graphs $G_1, \dots, G_k$ (for $k \geq 2$) with $V(G_i) \cap V(G_{i+1}) = \{z_i\}$ for every $i \in [k]$ (where the indices are modulo k) and $z_1, \dots, z_k$ are all distinct;

(b) every G_i is balanced;

(c) every cycle of $G - w$ using all edges in $G_1, \dots, G_k$ is unbalanced.

Moreover, for every G_i that is not a single edge, the vertices in $(N_G(w) \cap V(G_i)) - \{z_{i-1}, z_i\}$ partition into two nonempty sets X_i and Y_i such that:

(d) $(G_i, (z_{i-1}, X_i, z_i, Y_i))$ is planar, and

(e) for every pair of edges wx and wy with $x, y \in V(G_i) - \{z_{i-1}, z_i\}$, and every (x, y)-path P in G_i, the cycle $P \cup \{wx, wy\}$ is unbalanced if and only if one of x and y is in X_i and the other is in Y_i.

The other cycles of Ω are not determined (as long as the theta property still holds). We say that such Ω is a *generalized wheel*. An example of a generalised wheel is given in Figure 3.

3.2 Criss-cross

Starting from a planar graph $(H, (u_1, u_2, u_3, u_4))$, where H is 2-connected, and a vertex w not in H, we construct a tangled biased graph Ω as follows. The graph $||\Omega||$ is obtained from H and w by adding four edges $e_i = wu_i$, for $i \in [4]$, and two more edges $f_1 = u_1u_3$ and $f_2 = u_2u_4$. Every cycle contained in H is balanced. Every f_1- and f_2-cycle for H is declared unbalanced, and so are cycles formed by a (u_i, u_j)-path in H together with e_i and e_j, where $i \neq j$. The two triangles $\{e_1, e_3, f_1\}$ and $\{e_2, e_4, f_2\}$ are balanced. The other cycles of Ω are not determined (as long as the theta property still holds). We call a biased graph constructed in this fashion a *criss-cross* (see Figure 4).

3.3 Fat triangles

Consider any graph H with three distinct vertices v_1, v_2, v_3. Construct a biased graph Ω by adding nonempty sets of edges F_{12}, F_{23}, F_{31}, where every edge in F_{ij} is between v_i and v_j. Declare H to be balanced; every f-cycle is unbalanced, for all $f \in F_{12} \cup F_{23} \cup F_{13}$. The other cycles of Ω are not determined (as long as the theta property still holds). We call a biased graph constructed in this fashion a *fat triangle* (see Figure 5).

3.4 Projective planar with a special vertex

Consider two disjoint planar graphs $(H_1, (x_1, \dots, x_m, u_1, z_2))$ and $(H_2, (y_1, \dots, y_m, z_1, u_2))$, where consecutive vertices in $x_1, \dots, x_m$ and/or in $y_1, \dots, y_m$ may be repeated. Construct a graph H from H_1 and H_2 by adding edges z_1z_2 and u_1u_2 and adding a new vertex w and edges wz_1 and wz_2. We construct a biased graph Ω from H as follows. The underlying graph $||\Omega||$ is obtained from H by adding edges $g_1 = wu_1$ and $g_2 = wu_2$ and edges $f_i = x_iy_i$ for every $i \in [m]$. Denote by F the set of edges $\{f_1, \dots, f_m\}$. Every cycle contained in H is declared to be balanced. For every $f \in F \cup \{g_1, g_2\}$, every f-cycle for H is unbalanced. Every $\{g_1, g_2\}$-cycle for H is balanced and so is every $\{f_i, f_j\}$-cycle for $H - u_1u_2$, for all distinct $f_i, f_j \in F$. Finally every $\{g_i, f_j\}$-cycle for $H - u_1u_2$ is unbalanced. The bias of the other cycles in Ω may be chosen arbitrarily, as long as the theta property is preserved. We call a biased graph constructed in this fashion a *projective planar biased graph with a special vertex* (see the left of Figure 6).

3.5 Projective planar with a special pair

Consider a planar graph $(H, (x, y, X, Y))$, for some $X, Y \subseteq V(H)$ (sharing at most one vertex). We construct a biased graph Ω from H as follows. The underlying graph $||\Omega||$ is obtained from H by adding the following edges:

- edges xx' for every $x' \in X$; denote by F_x the set of edges added this way;
- edges yy' for every $y' \in Y$; denote by F_y the set of edges added this way;
- possibly adding edges $e_1, \ldots, e_\ell$ between x and y.

We declare H to be balanced, while F_x and F_y are 2-balanced for H. Every e_i-cycle for H is unbalanced. The bias of the other cycles in Ω may be chosen arbitrarily, as long as the theta property is preserved. We call a biased graph constructed in this fashion a *projective planar biased graph with a special pair* (see the middle of Figure 6).

3.6 Projective planar with a special triple

Consider a planar graph $(H, (y_1, x, y_2, X))$, for some $X \subseteq V(H)$. We construct a biased graph Ω from H as follows. The underlying graph $||\Omega||$ is obtained from H by adding the following edges:

- edges xx', for every $x' \in X$; denote by F the set of edges added this way;
- edges $e_1, \ldots, e_n$ between x and y_1;
- possibly edges $g_1, \ldots, g_m$ between x and y_2;
- an edge $f = y_1y_2$.

We declare H to be balanced, while F is 2-balanced for H. Every e_i-, g_i- and f-cycle for H is unbalanced. The bias of the other cycles in Ω may be chosen arbitrarily, as long as the theta property is preserved. We call a biased graph constructed in this fashion a *projective planar biased graph with a special triple* (see the right of Figure 6).

3.7 Tricoloured graphs

All the indices in this definition are modulo 6 and either $I = \{1, 2, 3\}$ or $I = \{1, 3, 5\}$. Let H be a 2-connected graph such that:

(a) H is the union of connected graphs $H_1, \ldots, H_6$ with $V(H_i) \cap V(H_{i+1}) = \{z_i\}$ for every $i \in [6]$ and $z_1, \ldots, z_6$ are all distinct.

(b) For every $i \in I$, let x_i be a vertex in H_i and Y_i be a set of vertices in H_{i+3} such that

  - If $I = \{1, 2, 3\}$, then $(H, (x_1, x_2, x_3, Y_1, Y_2, Y_3))$ is planar.
  - If $I = \{1, 3, 5\}$, then $(H, (x_1, Y_5, x_3, Y_1, x_5, Y_3))$ is planar.

  (Where we allow at most one of the Y_i's to be empty.)

We construct a biased graph Ω from H as follows. The underlying graph $||\Omega||$ is obtained from H by adding, for every $i \in I$, the set of edges $E_i = \{x_i y \mid y \in Y_i\}$. For every $i \in I$, we declare the graph with edge set $E_i \cup E(H_{i+3})$ to be balanced. For all distinct $i, j \in I$ and all $f_i \in E_i$ and $f_j \in E_j$, every $\{f_i, f_j\}$-cycle for $H_i \cup H_j \cup H_{i+3} \cup H_{j+3}$ is unbalanced. The bias of the other cycles in Ω may be chosen arbitrarily, as long as the theta property is preserved. In this definition we may replace some of the H_i's with a single vertex. If the vertices x_i for $i \in I$ are all distinct, we call a biased graph constructed in this fashion a *tricoloured biased graph* (see Figure 7).

4 Linkages and 3-planar graphs

Given four distinct vertices s_1, s_2, t_1, t_2 in a graph G, we say that two paths P_1 and P_2 form an $(s_1 - t_1, s_2 - t_2)$-*linkage* if P_1 is an (s_1, t_1)-path, P_2 is an (s_2, t_2)-path and P_1 and P_2 are disjoint. If S_1, S_2, T_1, T_2 are pairwise disjoint sets of vertices of G, then we say that G contains an $(S_1 - T_1, S_2 - T_2)$-linkage if G contains an $(s_1 - t_1, s_2 - t_2)$-linkage for some $s_1 \in S_1$, $s_2 \in S_2$, $t_1 \in T_1$ and $t_2 \in T_2$. Independently, Seymour [4] and Thomassen [6] characterised the graphs having no $(s_1 - t_1, s_2 - t_2)$-linkage. We state their result using the notation by Yu in [7]. We also use other results by Yu; in [7] linkages are allowed to be between pairs of vertices that are not necessarily disjoint. We will modify the results in [7] according to our setting.

We first need to define 3-planar graphs. Let G be a graph and let $\mathcal{A} = \{A_1, \dots, A_k\}$ be a (possibly empty) set of pairwise disjoint subsets of $V(G)$, such that $N_G(A_i) \cap A_j$ is empty for all $i, j \in [k]$. We define $\texttt{Proj}(G, \mathcal{A})$ to be the graph obtained from G by deleting all sets A_i and adding new edges joining each pair of vertices in $N_G(A_i)$.

We say that $(G, \mathcal{A})$ is a 3-*planar* graph if the following hold:

(a) $|N_G(A_i)| \leq 3$ for all $i \in [k]$;

(b) $\texttt{Proj}(G, \mathcal{A})$ is a planar graph and it can be embedded on the plane so that, for each A_i with $|N_G(A_i)| = 3$, $N_G(A_i)$ induces a facial triangle.

In addition, if $v_1, \dots, v_n$ are vertices in G such that $v_i \notin A_j$ for all $i \in [n]$ and $j \in [k]$, and $v_1, \dots, v_n$ occur in this circular order in a face boundary of $\texttt{Proj}(G, \mathcal{A})$ (for an embedding as in (b)), then we say that $(G, \mathcal{A}, (v_1, \dots, v_n))$ is 3-*planar*. Sometimes we will omit the set $\mathcal{A}$ and say that G or $(G, (v_1, \dots, v_n))$ is 3-planar. The vertices $v_1, \dots, v_n$ do not need to be all distinct in this definition. If $(G, \mathcal{A}, (v_1, \dots, v_n))$ is 3-planar for some empty set $\mathcal{A}$, then we say that $(G, (v_1, \dots, v_n))$ is *planar*.

Given two disjoint sets X and Y of vertices in a graph G, we say that $(G, (X, Y))$ is 3-*planar* if there is some ordering $(x_1, \dots, x_k)$ of the vertices in X and some ordering $(y_1, \dots, y_\ell)$ of the vertices in Y such that $(G, (x_1, \dots, x_k, y_1, \dots, y_\ell))$ is 3-planar. If $X = \{x\}$ then we may abuse notation and write that $(G, (x, Y))$ is 3-planar. This definition extends to the case when $x_k = y_1$ and/or $y_\ell = x_1$. We also extend this notation in the obvious way to the case when we have more than two sets.

Theorem 4.1 (Theorem 2.4 in [7])**.** *Let G be a graph and s_1, t_1, s_2, t_2 be distinct vertices in G. Then G contains no $(s_1 - t_1, s_2 - t_2)$-linkage if and only if $(G, (s_1, s_2, t_1, t_2))$ is 3-planar.*

Yu's paper [7] contains other useful results on linkages that we report next.

Let $(G, \mathcal{A})$ be 3-planar and let $A \in \mathcal{A}$. We say that A is *minimal* if there are no nonempty pairwise disjoint subsets $D_1, \ldots, D_k \subset A$ (where $k \geq 2$) such that $(G, (\mathcal{A} - A) \cup \{D_1, \ldots, D_k\})$ is 3-planar. If every $A \in \mathcal{A}$ is minimal, then we say that $\mathcal{A}$ is minimal.

Lemma 4.2 (Proposition 2.6 in [7]). *If $(G, (v_1, \ldots, v_n))$ is 3-planar, then there is a collection $\mathcal{A}$ of pairwise disjoint subsets of $V(G) - \{v_1, \ldots, v_n\}$ such that $(G, \mathcal{A}, (v_1, \ldots, v_n))$ is 3-planar and $\mathcal{A}$ is minimal.*

Lemma 4.3 (Proposition 3.1 in [7]). *Let $(G, \mathcal{A})$ be 3-planar, let $A \in \mathcal{A}$ with $N_G(A) = \{a_1, a_2, a_3\}$ and let $H = G[A \cup N_G(A)]$. Suppose that A is minimal. Then the following hold:*

(a) for any proper subset $X \subset N_G(A)$, $H - X$ is connected;

(b) for any $x \in A$, H contains an $(x - a_1, a_2 - a_3)$-linkage;

(c) for any $x \in A$, $N_G(A)$ is contained in a component of $H - x$.

Lemma 4.4 (Proposition 3.2 in [7]). *Let $(G, \mathcal{A})$ be 3-planar, where $\mathcal{A}$ is minimal. Let $v_1, v_2 \in V(G)$ be distinct. Let $u_1 \in V(G)$ and define $u_1^* = u_1$ when $u_1 \in V(\mathtt{Proj}(G, \mathcal{A}))$ and u_1^* to be an arbitrary vertex in $N_G(A_1)$ if $u_1 \in A_1$ for some $A_1 \in \mathcal{A}$. Let $u_2 \in V(G)$ and define $u_2^* = u_2$ when $u_2 \in V(\mathtt{Proj}(G, \mathcal{A}))$ and u_2^* to be an arbitrary vertex in $N_G(A_2)$ if $u_2 \in A_2$ for some $A_2 \in \mathcal{A}$. Suppose that v_1, v_2, u_1^*, u_2^* are all distinct and*

(i) $A_1 \neq A_2$ if A_1 and A_2 are both defined, and

(ii) $\mathtt{Proj}(G, \mathcal{A})$ contains a $(v_1 - u_1^*, v_2 - u_2^*)$-linkage L^*.

Then G contains a $(v_1 - u_1, v_2 - u_2)$-linkage L such that, for any vertex z of $\mathtt{Proj}(G, \mathcal{A})$, z is contained in a path in L if and only if z is contained in a path in L^*.

We conclude this section with some results on linkages and 3-planar graphs. The proofs are similar to the proofs of results in [7].

Lemma 4.5. *Let $(G, \mathcal{A}, (v_1, \ldots, v_n))$ be 3-planar, where G is 2-connected and $\mathcal{A}$ is minimal. Then G contains a cycle C such that $v_1, \ldots, v_n$ appear in C in this circular order.*

Proof. Let F be a face boundary of $\mathtt{Proj}(G, \mathcal{A})$ containing $v_1, \ldots, v_n$ (in this circular order). Since G is 2-connected, so is $\mathtt{Proj}(G, \mathcal{A})$. Therefore F is a cycle of $\mathtt{Proj}(G, \mathcal{A})$. Suppose that $e = a_1 a_2 \in E(F)$ is not an edge of G. Then $a_1, a_2 \in N_G(A)$ for some $A \in \mathcal{A}$. If $|N_G(A)| = 2$, let P be an (a_1, a_2)-path in $G[A]$. Otherwise let $N_G(A) = \{a_1, a_2, a_3\}$ and let P be an (a_1, a_2)-path in $G[A - a_3]$. Substituting e with P in F produces a cycle. Repeating this process for every edge of F that is not in G we obtained the desired cycle. □

Lemma 4.6. *Let G be a 2-connected graph. Let $k \geq 3$ be an integer and $x, y, v_1, v_2, \ldots, v_k$ be distinct vertices in G such that, for all distinct $i, j \in [k]$, there is no $(x - y, v_i - v_j)$-linkage in G. Then $\{x, y\}$ is a 2-vertex-cut of G and each $\{x, y\}$-bridge contains at most one of $v_1, v_2, \cdots, v_k$.*

Proof. Since there is no $(x-y, v_1-v_2)$-linkage in G, Theorem 4.1 implies that $(G, (x, v_1, y, v_2))$ is 3-planar. Lemma 4.5 implies that G contains a cycle C such that x, v_1, y, v_2 appear in C in this circular order. If there is a $(v_3, C - \{x, y\})$-path in G, then G contains either an $(x - y, v_1 - v_3)$-linkage or an $(x - y, v_2 - v_3)$-linkage. It follows that $\{x, y\}$ is a 2-vertex-cut of G separating v_3 from v_1 and v_2. Let $B_1, \ldots, B_n$ be the $\{x, y\}$-bridges of G. Since $B_i - \{x, y\}$ is connected for every $i \in [n]$, if B_i contains vertices v_j and v_ℓ (for distinct $j, \ell \in [k]$) then G contains an $(x - y, v_j - v_\ell)$-linkage. It follows that every $\{x, y\}$-bridge contains at most one of $v_1, \ldots, v_k$, and the result holds. □

The following lemma will be used in the proof of Lemma 6.2.

Lemma 4.7. *Let G be a 2-connected graph. Let X, Y be subsets of $V(G)$ with at least three vertices and with $|X \cup Y| \geq 4$. Then for some $x, x' \in X$ and $y, y' \in Y$ with $\{x, x'\} \cap \{y, y'\} = \emptyset$ there is an $(x - x', y - y')$-linkage.*

Proof. Let x_1, x_2, x_3 be vertices in X and y_1, y_2, y_3 vertices in Y with $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$. Assume that G has no (x_1-x_2, y_1-y_2)-linkage. Then Theorem 4.1 and Lemma 4.5 imply that $(G, (x_1, y_1, x_2, y_2))$ is 3-planar and x_1, y_1, x_2, y_2 appear in a cycle of G in this order. When $X \cup Y = \{x_1, y_1, x_2, y_2\}$, the lemma is obviously true. So we may assume that $y_3 \notin \{x_1, x_2\}$ and G has no $(x_1 - x_2, y_i - y_j)$-linkage for all distinct $i, j \in [3]$. By Lemma 4.6, $\{x_1, x_2\}$ is a 2-vertex-cut of G and each $\{x_1, x_2\}$-bridge contains at most one of y_1, y_2, y_3. For $i = 1, 2, 3$, let B_i be the $\{x_1, x_2\}$-bridge containing y_i. Without loss of generality we may assume that $x_3 \notin V(B_1) \cup V(B_2)$. Then G contains an $(x_1 - x_3, y_1 - y_2)$-linkage. □

Lemma 4.8. *Let G be a 2-connected graph, let X and Y be disjoint nonempty sets of vertices of G and let $v_1, v_2 \in V(G) - (X \cup Y)$. Suppose that for every $x \in X$ and $y \in Y$, G has no $(v_1 - v_2, x - y)$-linkage. Then one of the following occurs.*

(a) G contains a 2-separation (A_1, A_2) with $v_1, v_2 \in V(A_1)$ and either $X \subset V(A_1)$ and $y \in V(A_2)$ for some $y \in Y$, or $Y \subset V(A_1)$ and $x \in V(A_2)$ for some $x \in X$.

(b) $(G, (v_1, X, v_2, Y))$ is 3-planar.

Proof. We prove the result by induction on $|X| + |Y|$. If $|X| = |Y| = 1$ the result holds by Theorem 4.1. Now suppose that $|X| \geq 2$ and pick some $x \in X$. By induction one of the following occurs.

(1) G contains a 2-separation (A_1, A_2) with $v_1, v_2 \in V(A_1)$ and either

(1.1) $X - \{x\} \subset V(A_1)$ and $y \in V(A_2)$ for some $y \in Y$, or

(1.2) $Y \subset V(A_1)$ and $x' \in V(A_2)$ for some $x' \in X - \{x\}$.

(2) $(G, \mathcal{A}, (v_1, X - \{x\}, v_2, Y))$ is 3-planar for some set $\mathcal{A}$.

If (1.2) occurs, then the same separation (A_1, A_2) satisfies (a) for X and Y. Now suppose that (1.1) occurs. If $x \in V(A_1)$, then again (A_1, A_2) satisfies (a) for X and Y. So now assume that $x \in V(A_2) - V(A_1)$. If no vertex of Y is in $V(A_2) - V(A_1)$, then (A_1, A_2) satisfies (a) for X and Y. Now suppose some $y \in Y$ is in $V(A_2) - V(A_1)$. Then either G contains a $(v_1 - v_2, x - y)$-linkage (which is not possible), or $V(A_1) \cap V(A_2) = \{v_1, v_2\}$ and x and y are in different $\{v_1, v_2\}$-bridges (since G is 2-connected). Let B be the $\{v_1, v_2\}$-bridge containing x and define $A'_1 = A_1 \cup B$ and $A'_2 = A_2 - (B - \{v_1, v_2\})$. Since $y \in V(A'_2) - V(A'_1)$, (A'_1, A'_2) is a 2-separation of G satisfying (a).

Now suppose that (2) occurs. We may choose $\mathcal{A}$ to be minimal. Set $G' = \texttt{Proj}(G, \mathcal{A})$ and let F be the face boundary of G' containing $v_1, v_2, X - \{x\}$ and Y. Let P_X be the (v_1, v_2)-path in F containing $X - \{x\}$ and P_Y be the (v_1, v_2)-path in F containing Y.

If $x \in P_X$, then (b) holds and we are done. Suppose this is not the case. Define $Z = \{x\}$ if $x \in V(G')$ and $Z = N_{G'}(A_x)$ if $x \in A_x$ for some $A_x \in \mathcal{A}$. Suppose that there exists a path Q in G' (possibly with no edges) joining some $x^* \in Z$ to P_Y and such that Q and P_X are disjoint. Then $F \cup Q$ contains a $(v_1 - v_2, y - x^*)$-linkage for any $y \in Y$. By Lemma 4.4, G contains a $(v_1 - v_2, y - x)$-linkage, a contradiction. Thus every path joining some $x^* \in Z$ to P_Y intersects P_X. By the planarity of G', this implies that G' contains a 2-separation (H_1, H_2) such that $\{v_1, v_2\} \subseteq V(H_1)$ and such that P_Y is contained in H_1 and $Z \subseteq V(H_2)$. Then (H_1, H_2) naturally extends to a 2-separation in G satisfying (a). □

For the proof of the next lemma we require some new terminology. This terminology will be used throughout the paper. Let C be a cycle of a 2-connected graph G and let x be a vertex of C. A vertex $y \in V(G)$ *attaches to C at x* if there exists an (x, y)-path P of G such that $|V(C) \cap V(P)| = 1$. In this case x is an *attachment* of y on C. Given a path P in C, we say that y *only attaches to P* if all the attachments of y on C are in P. Since G is 2-connected, y has only one attachment if and only if $y \in V(C)$.

Let C be a cycle and suppose that $x_1, x_2, \ldots, x_n \in V(C)$ occur on C in this cyclic order. For any two distinct x_i and x_j, C contains two (x_i, x_j)-paths. Let $C[x_i, x_{i+1}, \ldots, x_j]$ denote the (x_i, x_j)-path in C containing $x_i, x_{i+1}, \ldots, x_j$ (and not containing x_{j+1} if $i \neq j + 1$), where subscripts are modulo n. Such path is uniquely determined when $n \geq 3$. Similarly, set

$$
\begin{aligned}
C(x_i, x_{i+1}, \ldots, x_j] &= C[x_i, x_{i+1}, \ldots, x_j] - x_i, \\
C[x_i, x_{i+1}, \ldots, x_j) &= C[x_i, x_{i+1}, \ldots, x_j] - x_j, \\
C(x_i, x_{i+1}, \ldots, x_j) &= C[x_i, x_{i+1}, \ldots, x_j] - \{x_i, x_j\}.
\end{aligned}
$$

Lemma 4.9. *Let G be a 2-connected graph and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be distinct vertices of G, with $n \geq 2$. Suppose that G contains no $(x_i - y_i, x_j - y_j)$-linkage, for all distinct $i, j \in [n]$. Then, up to a reordering of $[n]$ and swapping some x_i with y_i, $(G, (x_1, \ldots, x_n, y_1, \ldots, y_n))$ is 3-planar.*

Proof. We prove this by induction on n. When $n = 2$, the lemma follows from Theorem 4.1. So we may assume that $n \geq 3$ and the result holds for $n - 1$. Therefore, there is a collection $\mathcal{A}$ of pairwise disjoint subsets of $V(G) - \{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$ such that $(G, \mathcal{A}, (x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}))$ is 3-planar. We may choose $\mathcal{A}$ to be minimal. Since $\mathcal{A}$ is minimal, if x_n and y_n both belong to a same set $A \in \mathcal{A}$, then, by Lemma 4.3(a), G contains an $(x_1 - y_1, x_n - y_n)$-linkage. Thus we may assume that x_n and y_n do not belong to a same set $A \in \mathcal{A}$.

Let $H = \texttt{Proj}(G, \mathcal{A})$. Since G is 2-connected, so is H. Let C be the face boundary of H containing $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$ (in this circular order). Define $X_n = \{x_n\}$ if $x_n \in V(H)$ and $X_n = N_{G'}(A_{x_n})$ if $x_n \in A_{x_n}$ for some $A_{x_n} \in \mathcal{A}$. Similarly, define $Y_n = \{y_n\}$ if $y_n \in V(H)$ and $Y_n = N_{G'}(A_{y_n})$ if $y_n \in A_{y_n}$ for some $A_{y_n} \in \mathcal{A}$.

Suppose that H contains an $(x^* - y^*, x_i - y_i)$-linkage for some $x^* \in X_n$, $y^* \in Y_n$ and $i \in [n-1]$. Then by Lemma 4.4, G contains an $(x_n - y_n, x_i - y_i)$-linkage, a contradiction.

Claim 1. *We may assume that every vertex in X_n only attaches to $C[x_{n-1}, y_1]$ and every vertex in Y_n only attaches to $C[y_{n-1}, x_1]$.*

Proof of claim. By possibly swapping some x_i with y_i and changing subscripts appropriately, we may assume that some $x^* \in X_n$ attaches to $C(x_{n-1}, y_1]$.

Case 1: $x^* = y_1$. Then every vertex in Y_n attaches only to $C[y_{n-1}, x_1, x_2]$. Since G is 2-connected and $y_n \notin \{x_1, \ldots, x_n, y_1, \ldots, y_{n-1}\}$, for any $z \in \{x_1, \ldots, x_n, y_1, \ldots, y_{n-1}\}$ the set $Y_n - \{z\}$ is non-empty. Thus, if there exists a vertex $x' \in X_n$ which attaches to a vertex not in $C[x_{n-1}, y_1, y_2]$, then H contains either an $(x' - y^*, x_2 - y_2)$-linkage or an $(x' - y^*, x_{n-1} - y_{n-1})$-linkage for some $y^* \in Y_n$, a contradiction. Therefore every vertex in X_n attaches only to $C[x_{n-1}, y_1, y_2]$. Moreover, if there are vertices $x', x'' \in X_n$ such that x' attaches to $C[x_{n-1}, y_1)$ and x'' attaches to $C(y_1, y_2]$, then H contains either an $(x' - y^*, x_1 - y_1)$-linkage or an $(x'' - y^*, x_1 - y_1)$-linkage for some $y^* \in Y_n$, again a contradiction. Thus we may assume that all vertices in X_n attach only to $C[x_{n-1}, y_1]$. This, together with the fact that X_n contains a vertex other than y_1, forces all the vertices in Y_n to attach only to $C[y_{n-1}, x_1]$, and the claim holds.

Case 2: $x^* \neq y_1$. Arbitrarily choose $y^* \in Y_n$; then y^* only attaches to $C[y_{n-1}, x_1]$ (otherwise G contains either an $(x^* - y^*, x_1 - y_1)$-linkage or an $(x^* - y^*, x_{n-1} - y_{n-1})$-linkage, a contradiction). If some $y^* \in Y_n$ attaches to $C(y_{n-1}, x_1)$, then the symmetric argument shows that every vertex in X_n attaches only to $C[x_{n-1}, y_1]$. Otherwise $x_1 \in Y_n$, and we conclude by the symmetric argument to the one in Case 1. ♢

If $x_n, y_n \in V(C)$, then we are done by the claim. Now suppose that $x_n \notin V(C)$ and $y_n \in V(C)$. Let $x^* \in X_n$; since x^* only attaches to $C[x_{n-1}, y_1]$, by the planarity of H there exists a 2-vertex-cut $Z \subseteq V(C[x_{n-1}, y_1])$ such that x^* and $C(y_1, \ldots, y_n, x_1, \ldots, x_{n-1})$ are in different Z-bridges. Since the vertices in X_n are pairwise adjacent, x and the other vertices in X_n are in the same Z-bridge B^*. The 2-vertex-cut Z is also a 2-vertex-cut of G; let B be the Z-bridge in G corresponding to B^*. If $B - Z - \{x_n\} \neq \emptyset$ set

$$\mathcal{A}' = (\mathcal{A} - \{A \in \mathcal{A} |\ A \subseteq V(B)\}) \cup \{B - Z - \{x_n\}\},$$

otherwise set $\mathcal{A}' = \mathcal{A}$. Since $|N_G(A)| \leq 3$ for every $A \in \mathcal{A}'$ and $Z \subseteq V(C[x_{n-1}, y_1])$, we have that $(G, \mathcal{A}', (x_1, \ldots, x_n, y_1, \ldots, y_n))$ is 3-planar.

Now suppose that both x_n and y_n are not in C. Then we may apply a similar argument to the one above, once for x_n and once for y_n and obtain a new set $\mathcal{A}'$ such that $(G, \mathcal{A}', (x_1, \ldots, x_n, y_1, \ldots, y_n))$ is 3-planar. □

5 Small separations

In this section we will show that we can reduce our problem to the case where the tangled biased graph is 4-connected. To do so we will show that if Ω is not 4-connected then either Ω is a generalised wheel, or we can obtain Ω as a 1-, 2- or 3-sum of a balanced graph and a tangled biased graph. We first need to define summing operations on biased graphs.

Let $\Omega_1 = (G_1, \mathcal{B}_1)$ and $\Omega_2 = (G_2, \mathcal{B}_2)$ be two biased graph, where Ω_2 is balanced. Suppose that both Ω_1 and Ω_2 contain a balanced K_t subgraph, for some $t \in [3]$ and $|V(\Omega_1)|, |V(\Omega_2)| > t$. Then the graph $G_1 \oplus_t G_2$ is the graph obtained from G_1 and G_2 by identifying the common K_t and deleting the edges of K_t. We define $\mathcal{B} = \mathcal{B}_1 \oplus_t \mathcal{B}_2$ as follows. If $t = 1$, then $\mathcal{B}$ is just the union of $\mathcal{B}_1$ and $\mathcal{B}_2$. If $t = 2$, let e be the edge in the K_t. Then $\mathcal{B} = \{C \in \mathcal{B}_1 \cup \mathcal{B}_2 : e \notin C\} \cup \{(C_1 \cup C_2)\backslash e : e \in C_1 \in \mathcal{B}_1, e \in C_2 \in \mathcal{B}_2\}$. If $t = 3$, let F be the edge set of the K_t. Then $\mathcal{B}$ is the union of the set $\{C \in \mathcal{B}_1 \cup \mathcal{B}_2 : C \cap F = \emptyset\}$ and, for every $e \in F$, the sets of the form $\{(C_1 \cup C_2)\backslash e : C_1 \in \mathcal{B}_1, C_2 \in \mathcal{B}_2, C_1 \cap F = C_2 \cap F = \{e\}\}$. Finally we define $\Omega_1 \oplus_t \Omega_2$ as $(G_1 \oplus_t G_2, \mathcal{B}_1 \oplus_t \mathcal{B}_2)$. It is easy to check that, since Ω_2 and the K_t are balanced, $\Omega_1 \oplus_t \Omega_2$ is a biased graph. We say that $\Omega_1 \oplus_t \Omega_2$ is the *t-sum* of Ω_1 and Ω_2 on $V(K_t)$.

Lemma 5.1. *Let $\Omega = (G, \mathcal{B})$ be a tangled biased graph and suppose that x is a cutvertex of G. Then Ω is a 1-sum of a tangled biased graph and a balanced graph.*

Proof. Let $\Omega_1, \dots, \Omega_k$ be the bridges of $\{x\}$. Suppose that two of these bridges are unbalanced. Since two bridges have only the vertex x in common, all unbalanced cycles of Ω contain x. This is not possible, since Ω has no blocking vertex. Hence we may assume that $\Omega_2, \dots, \Omega_k$ are balanced, so Ω is the 1-sum of Ω_1 and the balanced biased graph $\Omega_2 \cup \dots \cup \Omega_k$. Clearly Ω_1 has no two disjoint unbalanced cycles. Moreover, if Ω_1 is balanced, or contains a blocking vertex, then so does Ω. It follows that Ω_1 is tangled, and Ω is a 1-sum of a tangled biased graph and a balanced graph. □

Lemma 5.2. *Let $\Omega = (G, \mathcal{B})$ be a tangled biased graph and suppose that $\{x_1, x_2\}$ is a 2-vertex-cut of G. Then Ω is a 2-sum of a tangled biased graph and a balanced graph.*

Proof. Let $\Omega_1, \dots, \Omega_k$ be the bridges of $\{x_1, x_2\}$.

Claim 1. *Exactly one of $\Omega_1, \dots, \Omega_k$ is unbalanced.*

Proof of claim. If all the $\{x_1, x_2\}$-bridges are balanced, then either Ω is balanced or all unbalanced cycles of Ω use both x_1 and x_2 (and Ω has a blocking vertex). Since Ω is tangled, this is not the case. Thus we may assume that Ω_1 is unbalanced. Now assume by way of contradiction that Ω_2 is also unbalanced. Then every unbalanced cycle of Ω_1 and Ω_2 contains x_1 or x_2.

We claim that every unbalanced cycle of Ω_1 uses both x_1 and x_2. Assume to the contrary that Ω_1 contains an unbalanced cycle C_1 with $V(C_1) \cap \{x_1, x_2\} = \{x_1\}$. Then every unbalanced cycle contained in bridges other than Ω_1 uses x_1. Since x_1 is not a blocking vertex of Ω, there exists an unbalanced cycle C_2 not using x_1. Thus, C_2 must be contained in Ω_1 and uses x_2. Let C be any unbalanced cycle contained in Ω_2. Since C must intersect both C_1 and C_2 and $V(C_1) \cap \{x_1, x_2\} = \{x_1\}, V(C_2) \cap \{x_1, x_2\} = \{x_2\}$, we have that C uses both x_1 and x_2. Let P be a path in $\Omega_2 - \{x_1, x_2\}$ connecting the two components of $C - \{x_1, x_2\}$. By the definition of bridge, such P obviously exists. The two cycles in $C \cup P$ other than C are balanced, since each one does not intersect one of C_1 or C_2. Then $C \cup P$ is a theta subgraph with exactly two balanced cycles, a contradiction. Hence, every unbalanced cycle of Ω_1 uses both x_1 and x_2.

Let C be an unbalanced cycle of Ω_1. Let P be a path in $\Omega_1 - \{x_1, x_2\}$ connecting the two components of $C - \{x_1, x_2\}$. By the above claim the two cycles in $C \cup P$ other than C are balanced. Thus, $C \cup P$ is a theta subgraph with exactly two balanced cycles, a contradiction. $\diamondsuit$

By Claim 1 we may assume that only Ω_1 is unbalanced. Let G_1 be the graph obtained from $||\Omega_1|| \cup \cdots \cup ||\Omega_{k-1}||$ by adding a new edge e between x_1 and x_2, and G_2 be obtained from $||\Omega_k||$ by adding a new edge e between x_1 and x_2. Set

$$\begin{aligned}\mathcal{B}_1 =& \{C \mid C \text{ is a balanced cycle of } \Omega_1 \cup \cdots \cup \Omega_{k-1}\} \cup \{P \cup e \mid P \text{ is an } (x_1, x_2)\text{-path of } G_1 - e,\\ &\text{and } G_2 - e \text{ has an } (x_1, x_2)\text{-path } Q \text{ such that } P \cup Q \text{ is balanced in } \Omega\},\\ \mathcal{B}_2 =& \{C \mid C \text{ is a cycle of } G_2\}.\end{aligned}$$

Suppose that P is an (x_1, x_2)-path contained in $||\Omega_i||$ for some $i \in [k-1]$ and suppose that $P \cup Q$ is a balanced cycle for some (x_1, x_2)-path Q in $||\Omega_k||$. Let Q' be any other (x_1, x_2)-path in $||\Omega_k||$. Then Q may be obtained from Q' by rerouting on cycles in $||\Omega_k||$. Since Ω_k is balanced, all such cycles are balanced. Therefore $P \cup Q$ and $P \cup Q'$ have the same bias. From this, it is easy to verify that with this definition, $\Omega = (G_1, \mathcal{B}_1) \oplus_2 (G_2, \mathcal{B}_2)$. By definition, $(G_2, \mathcal{B}_2)$ is balanced. To conclude the proof of the lemma it remains to show that $(G_1, \mathcal{B}_1)$ is tangled.

If $(G_1, \mathcal{B}_1)$ contains two disjoint unbalanced cycles C_1 and C_2, then either these are disjoint unbalanced cycles of Ω (which is not possible) or one of C_1 and C_2 (say C_1) contains e. By definition of $\mathcal{B}_1$, for every (x_1, x_2)-path Q in $G_2 - e$ we have that $C_1' = C_1 - e \cup Q$ is an unbalanced cycle of Ω. Thus C_1' and C_2 are disjoint unbalanced cycles in Ω, a contradiction. We deduce that $(G_1, \mathcal{B}_1)$ has no two disjoint unbalanced cycles.

Now suppose that there is a vertex v in G_1 such that $(G_1, \mathcal{B}_1) - v$ is balanced. Since Ω has no blocking vertex, there exists an unbalanced cycle C of Ω avoiding v. Such cycle C cannot be contained in $\Omega_1 \cup \cdots \cup \Omega_{k-1}$, since v is a blocking vertex for this biased graph; moreover, C cannot be contained in Ω_k, since this biased graph is balanced. Therefore $C = P \cup Q$, where P is an (x_1, x_2)-path in $||\Omega_i||$ for some $i \in [k-1]$ and Q is an (x_1, x_2)-path in $||\Omega_k||$. By definition of $\mathcal{B}_1$, it follows that $P \cup \{e\}$ is an unbalanced cycle of $(G_1, \mathcal{B}_1)$ avoiding v, a contradiction. This also shows that $(G_1, \mathcal{B}_1)$ is not balanced (since in this case we may pick v to be any vertex). Therefore $(G_1, \mathcal{B}_1)$ is tangled and the result holds. □

Lemma 5.3. *Let $\Omega = (G, \mathcal{B})$ be a tangled biased graph. Suppose that G is the union of two connected graphs G_1 and G_2, where $V(G_1) \cap V(G_2) = \{x_1, x_2, x_3\}$, $|V(G_1)|, |V(G_2)| \geq 4$ and $\Omega[G_2]$ is balanced. Then Ω is a 3-sum of a tangled biased graph and a balanced graph.*

Proof. Let $\Omega_1, \ldots, \Omega_k$ be the bridges of $\{x_1, x_2, x_3\}$ in Ω. One of these bridges, say Ω_k, is balanced. We proceed similarly to the proof of Lemma 5.2 to construct two biased graphs

$(G_1, \mathcal{B}_1)$ and $(G_2, \mathcal{B}_2)$ such that $(G_1, \mathcal{B}_1)$ is tangled, $(G_2, \mathcal{B}_2)$ is balanced and $\Omega = (G_1, \mathcal{B}_1) \oplus_3 (G_2, \mathcal{B}_2)$. Let G_1 be the graph obtained from $||\Omega_1|| \cup \cdots \cup ||\Omega_{k-1}||$ by adding three new edges e_{12}, e_{23}, e_{13}, where the edge e_{ij} is between x_i and x_j. Let G_2 be the graph obtained from $||\Omega_k||$ by adding three new edges e_{12}, e_{23}, e_{13}, where the edge e_{ij} is between x_i and x_j. Set $\mathcal{B}_2$ to be the set of all cycles in G_2. It remains to define $\mathcal{B}_1$. The set $\mathcal{B}_1$ contains all balanced cycles of $\Omega_1 \cup \cdots \cup \Omega_{k-1}$ and the cycle $\{e_{12}, e_{13}, e_{23}\}$ plus the cycles using e_{12}, e_{13}, e_{23} which we will discuss next. Let Q be an (x_1, x_2)-path in $||\Omega_k||$. For every (x_1, x_2)-path P in $G_1 \backslash \{e_{12}, e_{13}, e_{23}\}$, we add $P \cup \{e_{12}\}$ to $\mathcal{B}_1$ if and only if $P \cup Q$ is balanced; we add $P \cup \{e_{13}, e_{23}\}$ to $\mathcal{B}_1$ if and only if $P \cup Q$ is balanced and P does not use vertex x_3. We define the bias of the other cycles using the three new edges similarly. Since we declared the cycle $\{e_{12}, e_{13}, e_{23}\}$ to be balanced, it can be checked that $(G_1, \mathcal{B}_1)$ is indeed a biased graph and that it is tangled. We leave it to the reader to check that $\Omega = (G_1, \mathcal{B}_1) \oplus_3 (G_2, \mathcal{B}_2)$. □

Lemma 5.4. *Let $\Omega = (G, \mathcal{B})$ be a simple tangled biased graph. If Ω is not 4-connected, then either*

- *Ω is a t-sum of a tangled biased graph and a balanced graph, for some $t \in [3]$, or*
- *Ω is a generalized wheel.*

Proof. Suppose that Ω is not 4-connected. By Lemma 5.1, Lemma 5.2 and Lemma 5.3, we may assume that Ω is 3-connected and contains a 3-vertex-cut $X = \{x_1, x_2, x_3\}$ and all the bridges of X are unbalanced. Let $\Omega_1, \ldots, \Omega_k$ be the bridges of X.

Claim 1. *For every bridge Ω_i there exists an unbalanced cycle in Ω_i using exactly one of x_1, x_2, x_3.*

Proof of claim. Since Ω has at least two unbalanced X-bridges, each unbalanced cycle must intersect X. Suppose that C is an unbalanced cycle contained in Ω_i using at least two vertices in X. Let P be a minimal path in $\Omega_i - X$ joining two components of $C - X$. The subgraph $C \cup P$ is a theta subgraph, and by the choice of P, no vertex of X is in P. Let C_1 and C_2 be the cycles in $C \cup P$ other than C. By the theta property at least one of C_1 and C_2 is unbalanced. Therefore one of C_1 or C_2 is the required cycle, unless one of them, say C_1, is unbalanced and contains two vertices in X, while C_2 is balanced. Thus in this case C contains all of the vertices in X. However, C_1 is an unbalanced cycle in Ω_i not using all vertices in X, so we may repeat the same argument with C_1 in place of C, and conclude that there exists an unbalanced cycle C using exactly one of x_1, x_2, x_3. ◊

The following is an immediate consequence of Claim 1.

Claim 2. *For any 3-vertex-cut X' of G, each X'-bridge has an unbalanced cycle using exactly one element of X'.*

By Claim 1, there exist unbalanced cycles C_1 and C_2 contained in Ω_1 and Ω_2 respectively, each using exactly one vertex in X. Thus C_1 and C_2 use the same vertex, say x_1, of X. Since x_1 is not a blocking vertex of Ω, there exists an unbalanced cycle C avoiding x_1. Since C intersects both C_1 and C_2, we have $C = P_1 \cup P_2$, where P_1 is an (x_2, x_3)-path of Ω_1 avoiding x_1 and P_2 is an (x_2, x_3)-path of Ω_2 avoiding x_1.

Now suppose that there is a third X-bridge Ω_3. Then Ω_3 contains an unbalanced cycle C_3 using at most one vertex in X. However, such cycle is disjoint from either C_1 or C. It follows that the only bridges of X are Ω_1 and Ω_2. Since X was chosen arbitrarily, we have

Claim 3. *For any 3-vertex-cut X' of G, there are exactly two X'-bridges.*

Because of C_1, the vertex x_1 is a blocking vertex of Ω_2. Similarly, x_1 is a blocking vertex of Ω_1. Let Q_1 be any (x_2, x_3)-path in $\Omega_1 - x_1$. All the cycles in $\Omega_1 - x_1$ are balanced, therefore $Q_1 \cup P_2$ has the same bias as $C = P_1 \cup P_2$ (i.e. $Q_1 \cup P_2$ is unbalanced). The same argument holds if we replace P_2 with some other (x_2, x_3)-path in $\Omega_2 - x_1$. It follows that every (x_2, x_3)-path in Ω_1 intersects C_1 and every (x_2, x_3)-path in Ω_2 intersects C_2. Since C_1 and C_2 were arbitrary unbalanced cycles (avoiding x_2 and x_3), we have the following result.

Claim 4. *Every unbalanced cycle in Ω_i intersects every (x_2, x_3)-path in Ω_i.*

Next we focus on the structure of Ω_1. Let $H_1, \dots, H_n$ be the blocks of $\Omega_1 - x_1$. Suppose that $n \geq 2$ and H_i is a leaf block that contains neither x_2 nor x_3 in its interior. Then $E(H_i)$ together with the edges joining H_i to x_1 forms a 2-separation of G, a contradiction. Therefore the tree of blocks of $\Omega_1 - x_1$ is a path (possibly with only one vertex) and its ends each contain one of x_2 and x_3 in the interior. We relabel the blocks so that (for every $i \in [n-1]$) H_i and H_{i+1} share a vertex, and $x_2 \in V(H_1) - V(H_2)$ and $x_3 \in V(H_n) - V(H_{n-1})$ (or, if $n = 1$, we simply have $x_2, x_3 \in V(H_1)$). For every $i \in [n-1]$, let z_i be the vertex shared by H_i and H_{i+1}. Set $z_0 := x_2$ and $z_n := x_3$.

Recall that x_1 is a blocking vertex of Ω_1. Let $U_1, \dots, U_\ell$ be the parts of the standard partition of $\delta_{\Omega_1}(x_1)$. For every $i \in [\ell]$, let Y_i be the set of vertices adjacent to x_1 with an edge in U_i. Arbitrarily choose H_t such that H_t is not a single edge. Since Ω is simple and there are no unbalanced cycles in H_t, the block H_t contains at least one vertex other than z_{t-1} and z_t. Therefore $\{z_{t-1}, z_t, x_1\}$ is a 3-vertex-cut of G; since H_t contains no unbalanced cycles, Claim 2 implies that at least two of $Y_1, \dots, Y_\ell$ intersects $H_t - \{z_{t-1}, z_t\}$. We claim that exactly two of the sets $Y_1, \dots, Y_\ell$ intersect $H_t - \{z_{t-1}, z_t\}$. Assume to the contrary that there are three distinct Y_i, Y_j, Y_k intersecting $H_t - \{z_{t-1}, z_t\}$. Arbitrarily choose $y_i \in Y_i \cap (V(H_t) - \{z_{t-1}, z_t\})$, $y_j \in Y_j \cap (V(H_t) - \{z_{t-1}, z_t\})$, and $y_k \in Y_k \cap (V(H_t) - \{z_{t-1}, z_t\})$. If, say, $y_i = y_j$, then (since H_t is a block) there is a (z_{t-1}, z_t)-path in H_t disjoint from the unbalanced cycle made of the two parallel $x_1 y_i$ edges (one from U_i and one from U_j), contradicting Claim 4. Thus the vertices y_i, y_j, y_k are all distinct. By Claim 4 for any $s, p \in \{i, j, k\}$ every (y_s, y_p)-path in H_t intersects every (z_{t-1}, z_t)-path in H_t. That is, H_t contains no $(z_{t-1} - z_t, y_s - y_p)$-linkage. Therefore, by Lemma 4.6 $\{z_{t-1}, z_t\}$ is a 2-vertex-cut of H_t with at least three $\{z_{t-1}, z_t\}$-bridges; consequently, the 3-vertex-cut $\{x_1, z_{t-1}, z_t\}$ of G has at least three $\{x_1, z_{t-1}, z_t\}$-bridges, a contradiction to Claim 3.

Assume that Y_i and Y_j intersect $H_t - \{z_{t-1}, z_t\}$. Set

$$Y_i' = Y_i \cap (V(H_t) - \{z_{t-1}, z_t\}) \text{ and } Y_j' = Y_j \cap (V(H_t) - \{z_{t-1}, z_t\}).$$

Lemma 4.8 implies that either

(a) H_t contains a 2-separation (A_1, A_2) with $z_{t-1}, z_t \in V(A_1)$ and either $Y_i' \subset V(A_1)$ or $Y_j' \subset V(A_1)$, or

(b) $(H_t, (z_{t-1}, Y_i', z_t, Y_j'))$ is 3-planar.

To conclude our proof it remains to show that case (a) does not occur. Suppose a separation as in (a) exists, with $Y_i' \subset V(A_1)$. Set $V(A_1) \cap V(A_2) = \{p, q\}$. Let H' be the subgraph of H_t induced by the edges in A_2 together with the edges joining x_1 to $V(A_2) - \{p, q\}$. Since $Y_i' \subset V(A_1)$, the $\{x_1, p, q\}$-bridge H' of G contains no unbalanced cycle, a contradiction to Claim 2. □

6 Finding a balanced subgraph

Lemma 6.1. *Suppose that Ω is a simple 4-connected tangled biased graph with at least six vertices. Then one of the following holds:*

- *Ω has a 2-connected balanced subgraph with at least four vertices, or*
- *Ω is a tricoloured graph.*

Proof. Let Ω' be a 2-connected balanced subgraph of Ω where $|V(\Omega')|$ is maximal. By Theorem 1.1, if Ω has no balanced cycle then either it has at most 5 vertices or it is not 4-connected. Thus Ω has at least one balanced cycle, so $|V(\Omega')| \geq 3$. Assume that $|V(\Omega')| = 3$. Then

(1.1) each cycle in Ω of length at least four is unbalanced, and

(1.2) each cycle in Ω sharing some edges with a balanced triangle is unbalanced (by the theta property).

Case 1: $|V(\Omega)| = 6$.

Set $V(\Omega) = \{u_1, u_2, \dots, u_6\}$. Let K_6 be a complete graph defined on $V(\Omega)$. Since Ω is 4-connected, by symmetry we may assume that $K_6 \backslash \{u_1u_2, u_3u_4, u_5u_6\}$ is a subgraph of Ω. Since $u_1u_3u_5u_1$ and $u_2u_4u_6u_2$ are disjoint cycles, by symmetry we may assume that the former is balanced. Then every other cycle using an edge in this triangle is unbalanced by (1.2). In particular, $u_2u_3u_5u_2$ is unbalanced. Moreover, since $u_2u_3u_5u_2$ and $u_1u_4u_6u_1$ are disjoint cycles, $u_1u_4u_6u_1$ is balanced; so $u_2u_4u_6u_2$ is unbalanced by (1.2). Using this same strategy several times one can check that there are exactly four balanced cycles in $K_6 \backslash \{u_1u_2, u_3u_4, u_5u_6\}$, which are $u_1u_3u_5u_1$, $u_1u_4u_6u_1$, $u_2u_3u_6u_2$ and $u_2u_4u_5u_2$.

Now assume that u_1 and u_2 are adjacent in Ω. By (1.2) (applied to the known balanced triangles) all triangles in $K_6 \backslash \{u_3u_4, u_5u_6\}$ containing the edge u_1u_2 are unbalanced. Similar results hold when u_3u_4 or u_5u_6 are in $E(\Omega)$. Since Ω has no two disjoint unbalanced cycles, at most one edge in $\{u_1u_2, u_3u_4, u_5u_6\}$ is in Ω; so we may assume that either Ω is $K_6 \backslash \{u_1u_2, u_3u_4, u_5u_6\}$ or it is this graph together with the edge u_1u_2. This graph is depicted on the left of Figure 2. One can see that this graph is a tricoloured graph with (in the definition of tricoloured graph): $I = \{1, 2, 3\}$, $V(H_1) = \{u_1, u_3, u_5\}$, $V(H_2) = \{u_2, u_3, u_6\}$, $V(H_3) = \{u_2\}$, $V(H_4) = \{u_2, u_4\}$, $V(H_5) = \{u_1, u_4\}$, $V(H_6) = \{u_1\}$ and $Y_1 = \{u_2, u_4\}$, $Y_2 = \{u_1, u_4\}$ and Y_3 is either empty or equal to $\{u_1\}$ (depending on whether the edge u_1u_2 is present or not).

Case 2: $|V(\Omega)| \geq 7$.

Let C be an unbalanced cycle of Ω with $|V(C)|$ as small as possible. Then C is an induced subgraph of Ω. Moreover, $\Omega - V(C)$ is balanced and therefore all its blocks are isomorphic to K_1, K_2 or K_3. Note that C cannot be a loop (otherwise Ω would have a blocking vertex) and if C has exactly two vertices v_1 and v_2 then $\Omega - \{v_1, v_2\}$ is a 2-connected balanced graph with at least four vertices, against our assumptions. Therefore $|V(C)| \geq 3$.

Claim 1. $|V(C)| = 3$.

Proof of claim. Assume that there are edges f_1, f_2, f_3 joining some vertex of $\Omega - V(C)$ and $V(C)$. By (1.1) and (1.2) at most one cycle contained in $C \cup \{f_1, f_2, f_3\}$ is balanced. Then by the choice of C we have $|V(C)| = 3$. So we may assume that each vertex in $\Omega - V(C)$ has at most two neighbours in C, implying that all leaf blocks of $\Omega - V(C)$ are isomorphic to K_3.

Let B be a leaf block of $\Omega - V(C)$, and p_1, p_2 be two vertices in B not contained in other blocks of $\Omega - V(C)$. If some vertex in C has two neighbours in B, then by (1.2) and the choice of C we have $|V(C)| = 3$. Hence,

(1.3) each vertex in C has at most one neighbour in B when $|V(C)| \geq 4$.

Let y_i and z_i be two distinct neighbours of p_i in C. By (1.3) we may assume that $\{y_1, z_1\} \cap \{y_2, z_2\} = \emptyset$. Without loss of generality we may further assume that there is a (y_1, z_2)-path P on C containing neither z_1 nor y_2. Since $P \cup \{p_1y_1, p_2z_2, p_1p_2\}$ is an unbalanced cycle of length less than or equal to $|V(C)|$ by (1.1), by the choice of C we have $V(C) = V(P) \cup \{y_2, z_1\}$. Moreover, since $p_1p_2y_2z_1p_1$ is an unbalanced cycle by (1.1), we have $V(C) = \{y_1, z_1, y_2, z_2\}$.

We rename the vertices of C so that $C = v_1v_2v_3v_4v_1$. Since $|V(C)| = 4$, all triangles in Ω are balanced. By (1.3), there is a partition $(\{v_i, v_j\}, \{v_s, v_t\})$ of $V(C)$ such that $\{v_i, v_j\} = N_\Omega(p_1) \cap V(C), \{v_s, v_t\} = N_\Omega(p_2) \cap V(C)$. We say that the partition of $V(C)$ is *determined by B*. When the partitions of $V(C)$ determined by two leaf blocks B and B' are the same, Ω has two disjoint 4-cycles (which are unbalanced by our assumptions). So every two leaf blocks determine different partitions of $V(C)$.

Let w be the vertex in $V(B) - \{p_1, p_2\}$; by (1.3) w has no neighbours in C, so the component of $\Omega - V(C)$ containing B has another leaf block B'. Let q_1, q_2 be the vertices in B' not contained in other blocks of $\Omega - V(C)$.

When $\{v_1, v_3\} = N_\Omega(p_1) \cap V(C), \{v_2, v_4\} = N_\Omega(p_2) \cap V(C), \{v_1, v_2\} = N_\Omega(q_1) \cap V(C)$, and $\{v_3, v_4\} = N_\Omega(q_2) \cap V(C)$, we have that $v_2v_3v_4p_2v_2$ and $v_1p_1Pq_1v_1$ are disjoint unbalanced cycles, where P is a (p_1, q_1)-path in $\Omega - V(C)$ avoiding p_2. Hence, by symmetry we may assume that neither B nor B' determines the partition $(\{v_1, v_3\}, \{v_2, v_4\})$ of $V(C)$. Then $\Omega - V(C)$ has exactly two leaf blocks for otherwise two of them determine the same partition of $V(C)$. When $\Omega - V(C) \neq B \cup B'$, by (1.2) and the fact that each vertex in Ω has degree at least four, Ω has two disjoint unbalanced cycles. So $\Omega - V(C) = B \cup B'$. Since w has no neighbours in C, Ω is isomorphic to the graph pictured in the middle of Figure 2, where all triangles are balanced. From the picture one can see that this is a tricoloured graph. $\Diamond$

By the last claim we may assume that $V(C) = \{v_1, v_2, v_3\}$. Since Ω is 4-connected, $\Omega - V(C)$ is connected and

(1.4) for each leaf block B of $\Omega - V(C)$, we have $V(C) \subseteq N_\Omega(B - w)$, where w is the attachment of B in $\Omega - V(C)$.

Since each vertex in $V(B) - \{w\}$ is adjacent to at least two vertices in C, we have

(1.5) when B is isomorphic to K_3, there is a vertex v_i in C such that $\Omega[(V(B) - \{w\}) \cup \{v_i\}]$ is an unbalanced triangle.

Assume that $\Omega - V(C)$ has at least three leaf blocks B_1, B_2, B_3. Let w be the attachment of B_1 in $\Omega - V(C)$. By (1.2), (1.4) and (1.5), there is an unbalanced triangle C' in $C \cup (B_1 - w)$ avoiding a vertex v_i of C. Since B_1 is a leaf block, there is a path P in $\Omega - V(C)$ between B_2 and B_3 avoiding $B_1 - w$. By (1.4) there is a cycle using v_i (and no other vertex from C) together with P and vertices from B_2 and B_3. Such cycle has length at least four, so it is unbalanced (by (1.1)) and it is disjoint from C', a contradiction. Hence, $\Omega - V(C)$ has exactly two leaf blocks B_1 and B_2.

Assume that B_1 and B_2 are isomorphic to K_2. Let p_1 and p_2 be the two degree-1 vertices in $\Omega - V(C)$. Since $|V(\Omega)| \geq 7$, the graph $\Omega - V(C)$ has an internal block. When there is a vertex v_i in C such that $\Omega[(V(\Omega) - V(C) - \{p_1, p_2\}) \cup \{v_i\}]$ is unbalanced, by (1.4) the graph Ω has two disjoint unbalanced cycles (one being a 4-cycle with p_1, p_2 and the vertices in $V(C) - \{v_i\}$). We show that this implies that $\Omega - V(C)$ is a path with exactly three edges. Suppose this is not the case. First note that $\Omega - V(C)$ has at least three blocks, because $|V(\Omega)| \geq 7$. Then there is a vertex q of degree two in $\Omega - (V(C) \cup V(B_1) \cup V(B_2))$. Since Ω is 4-connected, q has two neighbours in C. Let w_1 and w_2 be the neighbours of p_1 and p_2 in $\Omega - V(C)$ respectively. Then w_1 and w_2 each have degree at most three in $\Omega - V(C)$, so they each have a neighbour in C. Let t_1 and t_2 be the neighbours of w_1 and w_2 respectively. If $t_1 = t_2$, then $\Omega[(V(\Omega) - V(C) - \{p_1, p_2\}) \cup \{t_1\}]$ is unbalanced, a contradiction. Thus $t_1 \neq t_2$ and we may assume that t_1 is a neighbour of q. Let P be a (w_1, q)-path in $\Omega - V(C)$. Then the cycle $P \cup \{w_1t_1, t_1q\}$ is balanced (else we again have that $\Omega[(V(\Omega) - V(C) - \{p_1, p_2\}) \cup \{t_1\}]$ is unbalanced), so this cycle is in fact a balanced triangle. Therefore, by (1.2) the triangle $t_1p_1w_1t_1$ is unbalanced. Let P' be a (q, p_2)-path in $\Omega - V(C)$. Then the theta induced by P' and $V(C) - \{t_1\}$ contains a cycle of length at least four (hence unbalanced by (1.1)) which is disjoint from the unbalanced triangle $t_1p_1w_1t_1$, a contradiction. So $\Omega - V(C)$ is a path with exactly three edges. Let $\Omega - V(C) = p_1p_3p_4p_2$. Since $|\delta_\Omega(p_3)|, |\delta_\Omega(p_4)| \geq 4$ and there is no v_i in C such that $v_ip_3p_4v_i$ is unbalanced, by (1.2) the graph $\Omega - \{p_1, p_2\}$ contains a spanning 4-wheel whose center v_1 is in $V(C)$ and $v_1p_3p_4v_1$ is balanced. Then $v_1p_1p_3v_1$ is unbalanced by (1.2), a contradiction as $v_1p_1p_3v_1$ is disjoint from a 4-cycle contained in $\Omega - \{v_1, p_1, p_3\}$. Hence, by symmetry we may assume that B_1 is isomorphic to K_3.

When B_2 is isomorphic to K_3 or B_1 and B_2 have no common vertex, by (1.4) and (1.5) the graph Ω has two disjoint unbalanced cycles. So B_2 is isomorphic to K_2 and shares a common vertex u_1 with B_1, implying that $\Omega - V(C) = B_1 \cup B_2$. When u_1 has two neighbours in C, by (1.4) and (1.5) the graph Ω has two disjoint unbalanced cycles. So by symmetry we may assume that $N_\Omega(u_1) \cap V(C) = \{v_1\}$. Set $\{u_2, u_3\} = V(B_1) - \{u_1\}$ and $\{u\} = V(B_2) - \{u_1\}$. When v_2 is adjacent with u_2 and u_3, by (1.4) we have that $v_2u_2u_3v_2$ and $v_1u_1uv_3v_1$ are disjoint unbalanced cycles. Hence, by symmetry at most one vertex of $\{u_2, u_3\}$ is adjacent with v_2 or v_3. Since Ω is 4-connected, by symmetry we may assume that $\{v_1, v_i\} = N_\Omega(u_i) \cap V(C)$ for each $1 \leq i \leq 2$. Therefore Ω is isomorphic to the graph on the right of Figure 2. Since each triangle disjoint from a 4-cycle is balanced, Ω is a tricoloured graph. □

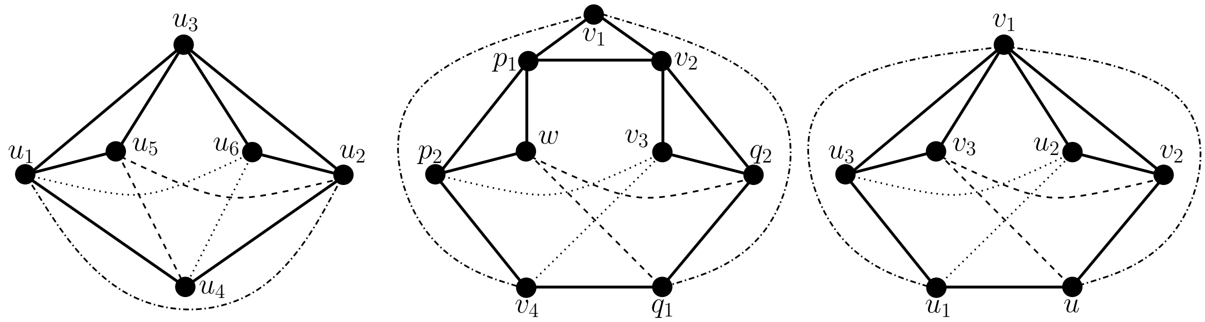

Figure 2: Graphs appearing in the proof of Lemma 6.1.

Lemma 6.2. *Suppose that Ω is a simple 4-connected tangled biased graph with at least six vertices. Then either Ω is a criss-cross, or a projective planar biased graph with a special triple, or a tricoloured graph, or there exists a subgraph Ω' of Ω that is 2-connected, spanning (i.e. $V(\Omega') = V(\Omega)$) and balanced.*

Proof. By Theorem 1.1, if Ω has no balanced cycle then it either has at most 5 vertices or it is not 4-connected. Thus Ω has at least one balanced cycle and we may choose a subgraph Ω' with the following properties:

(O1) Ω' is 2-connected;

(O2) Ω' is balanced;

(O3) subject to (O1) and (O2), $|V(\Omega')|$ is maximised;

(O4) subject to (O1), (O2) and (O3), $|E(\Omega')|$ is maximised.

By Lemma 6.1, we may assume that Ω' has at least four vertices. If Ω' is spanning, then we are done. Thus we may assume that this is not the case. Properties (O3) and (O4) imply immediately that

♢ if P is an Ω'-path (with endpoints u, v), then every cycle formed by P and a (u, v)-path in Ω' is unbalanced.

When $\Omega - V(\Omega')$ has two components G_1 and G_2, let V_i be the set of vertices in Ω' adjacent with G_i, for each $1 \leq i \leq 2$. Since Ω is 4-connected, $|V_1|, |V_2| \geq 4$. By Lemma 4.7 and Property ♢ there are two disjoint unbalanced cycles in Ω, a contradiction. So $\Omega - V(\Omega')$ is connected.

Next suppose that $\Omega - V(\Omega')$ has at least two blocks. Let B_1 be a leaf block, let w_1 the attachments of B_1 in $\Omega - V(\Omega')$ and let $H := \Omega - V(\Omega') - (V(B_1) - \{w_1\})$. Let V_1 be the subset of vertices in Ω' adjacent with $B_1 - w_1$ and let V_2 be the subset of vertices in Ω'

adjacent with H. Since Ω is 4-connected, $|V_1|, |V_2| \geq 3$ and $|V_1 \cup V_2| \geq 4$ (since Ω' has at least four vertices). Then by Lemma 4.7 and Property $\diamondsuit$ there are two disjoint unbalanced cycles in Ω, a contradiction. Therefore $\Omega - V(\Omega')$ is 2-connected.

First we consider the case that $V(\Omega) - V(\Omega')$ has at least two vertices. When $V(\Omega) - V(\Omega')$ has exactly two vertices w_1, w_2, let V_i be the neighbours of w_i in Ω' for each $1 \leq i \leq 2$. Since $|V_1|, |V_2| \geq 3$ and $|V_1 \cup V_2| \geq 4$, by Lemma 4.7 and Property $\diamondsuit$ there are disjoint unbalanced cycles in Ω, a contradiction. Now assume that $V(\Omega) - V(\Omega')$ has exactly three vertices w_1, w_2, w_3. For $i = 1, 2, 3$, let V_i be the neighbours of w_i in Ω'. Since $V(\Omega) - V(\Omega')$ is 2-connected, the graph $\Omega - V(\Omega')$ is a triangle (with possibly some edges doubled) and there are distinct vertices u_1, u_2, u_3, u_4 such that $u_1, u_3 \in V_1$ and $u_2, u_4 \in V_2$. Since Ω has no disjoint unbalanced cycles, Ω' has no $(u_1 - u_3, u_2 - u_4)$-linkage. Theorem 4.1 and Lemma 4.5 imply that $(\Omega', (u_1, u_2, u_3, u_4))$ is 3-planar and u_1, u_2, u_3, u_4 appear in a cycle C of Ω' in this order. Let u be a vertex in Ω' adjacent with w_3. By symmetry we may further assume that Ω' has a path joining u and $C[u_1, u_2)$ and disjoint from $C[u_2, u_3, u_4, u_1)$. Then, since w_3 is adjacent to w_1 in $\Omega - \Omega'$, Ω has two disjoint unbalanced cycles by Property $\diamondsuit$, a contradiction. So $\Omega - V(\Omega')$ has at least four vertices.

Let f_1, f_2, f_3, f_4 be disjoint edges joining Ω' and $\Omega - V(\Omega')$. Set $f_i = u_i v_i$ with u_i in Ω' and v_i in $\Omega - V(\Omega')$ for each $1 \leq i \leq 4$. If for each partition $(\{u_i, u_j\}, \{u_s, u_t\})$ of $\{u_1, u_2, u_3, u_4\}$ there is a $(u_i - u_j, u_s - u_t)$-linkage in Ω', then Ω obviously has two disjoint unbalanced cycles, a contradiction. So by symmetry we may assume that Ω' has no $(u_1 - u_3, u_2 - u_4)$-linkage. Theorem 4.1 and Lemma 4.5 imply that $(\Omega', (u_1, u_2, u_3, u_4))$ is 3-planar and u_1, u_2, u_3, u_4 appear in a cycle C of Ω' in this order. Since Ω has no disjoint unbalanced cycles, $\Omega - V(\Omega')$ has no $(v_1 - v_2, v_3 - v_4)$-linkage. Using Theorem 4.1 and Lemma 4.5 again, $(\Omega - V(\Omega'), (v_1, v_3, v_2, v_4))$ is 3-planar and v_1, v_3, v_2, v_4 appear in a cycle C' of $\Omega - V(\Omega')$ in this order. Then $C[u_4, u_1] \cup \{f_1, f_4\} \cup C'[v_4, v_1]$ and $C[u_2, u_3] \cup \{f_2, f_3\} \cup C'[v_2, v_3]$ are disjoint unbalanced cycles, a contradiction.

Finally consider the case that $V(\Omega) - V(\Omega')$ has only one vertex w. We will show in this case that either Ω is a criss-cross graph or it is projective planar with a special triple. Property $\diamondsuit$ implies immediately that

(1.1) every cycle formed by two edges incident with w and a path in Ω' is unbalanced.

Since Ω is 4-connected, w has at least four distinct neighbours in Ω'. Let u_1, u_2, u_3, u_4 be such neighbours and let e_i be a (w, u_i)-edge, for every $i \in [4]$. Let U be the set of edges in $E(\Omega) - E(\Omega')$ which are not incident with w. Assumption (O4) implies that, for every $f \in U$, every f-cycle for Ω' is unbalanced. Pick any $f = xy \in U$. Then Ω' does not contain a $(u_i - u_j, x - y)$-linkage, for any choice of distinct $i, j \in [4]$ with $u_i, u_j \notin \{x, y\}$. Therefore, Lemma 4.6 implies that one of the following occurs:

(a1) $x, y \in \{u_1, u_2, u_3, u_4\}$, or

(a2) $\{x, y\}$ is a 2-vertex-cut of Ω' separating u_i from u_j, for all distinct $i, j \in [4]$ with $u_i, u_j \notin \{x, y\}$.

Next we show that case (a1) holds for every choice of $f \in U$. Suppose that case (a2) occurs. If we are not also in case (a1), then we may assume that $u_1, u_2, u_3 \notin \{x, y\}$. Let

$B_1, \ldots, B_k$ be $\{x, y\}$-bridges containing $\{u_1, u_2, u_3, u_4\} - \{x, y\}$, labelled so that $u_i \in B_i$ for $i \in [k]$. Since $\{w, x, y\}$ is not a 3-vertex-cut of Ω, there exists an edge $f' \in U$ with endpoint x', y', such that $x' \in B_1 - \{x, y\}$ and $y' \in B - \{x, y\}$, for some $\{x, y\}$-bridge $B \neq B_1$. Then Ω' contains an $(x' - y', u_3 - u_4)$-linkage, a contradiction. It follows that case (a1) occurs for all $f \in U$, i.e.

(1.2) every edge in U has both endpoints in $\{u_1, u_2, u_3, u_4\}$.

Since w is not a blocking vertex, there exists an edge $f_1 \in U$ with endpoints, say, u_1 and u_3. Then Ω' does not contain a $(u_2 - u_4, u_1 - u_3)$-linkage, so $(\Omega', (u_1, u_2, u_3, u_4))$ is 3-planar. By Lemma 4.5, there exists a cycle C of Ω' which contains u_1, u_2, u_3, u_4 in this circular order.

Next we show that

(1.3) the only neighbours of w are u_1, u_2, u_3, u_4.

Suppose to the contrary that w has a fifth neighbour u_5. If u_5 attaches to C at a vertex $z \neq u_1, u_3$, then there is an unbalanced cycle (using w, u_5 and one of u_2 or u_4) which is disjoint from one of the two unbalanced cycles in $C \cup \{f_1\}$. It follows that $\{u_1, u_3\}$ is a 2-vertex-cut of Ω' separating u_5 from u_2 and u_4. Then (1.2) implies that $\{w, u_1, u_3\}$ is a 3-vertex-cut of Ω, a contradiction.

Fact (1.3) implies in particular that

(1.4) $(\Omega', (u_1, u_2, u_3, u_4))$ is planar.

Indeed, if this is not the case, then, for some $k \in [3]$, Ω' contains a k-separation with none of u_1, u_2, u_3, u_4 in the interior. Then this is also a k-separation in Ω, a contradiction. Next we show that

(1.5) there is no edge in U parallel to f_1.

Suppose to the contrary that there is an edge $f_2 \in U$ parallel to f_1. Since Ω is simple, the cycle $\{f_1, f_2\}$ is unbalanced. Therefore $\{u_1, u_3\}$ intersects every $(u_2 - u_4)$-path in Ω', i.e. $\{u_1, u_3\}$ is a 2-vertex-cut of Ω' separating u_2 from u_4. Properties (1.2) and (1.3), and the fact that Ω is 4-connected, imply that there are exactly two $\{u_1, u_3\}$-bridges in Ω', one containing u_2 and the other containing u_4. Since Ω has more than five vertices, one of these bridges (say the one containing u_2) has at least one vertex other than u_1, u_2 and u_3. Therefore $\{u_1, u_2, u_3\}$ is a 3-vertex-cut of Ω, a contradiction. Next we show that

(1.6) if f_1 is the only edge in U then Ω is projective planar with a special triple.

Suppose that $U = \{f_1\}$. First we note that there are no edges in Ω that are parallel to either e_2 or e_4. In fact, if there is an edge e_2' parallel to e_2, then $\{e_2, e_2'\}$ is an unbalanced cycle disjoint from the unbalanced cycle $C[u_3, u_4, u_1] \cup \{f_1\}$. Then Ω is projective planar with a special triple, where (following the terminology in the definition of projective planar biased graph with a special triple given in Section 3.6) we have that: $H = \Omega' \cup \{e_2\}$, $x = w$, $y_1 = u_1$, $y_2 = u_3$, $X = \{u_4\}$ and $f = f_1$.

We conclude by showing that

(1.7) if U contains at least two edges, then Ω is a criss-cross.

Suppose that U contains an edge f_2 other than f_1. By (1.2) and (1.5), we may assume by symmetry that the endpoints of f_2 are either u_2 and u_4 or u_1 and u_2. In the latter case however, $C \cup \{f_2, e_3, e_4\}$ contains two disjoint unbalanced cycles. Thus $f_2 = u_2u_4$. In the proof of (1.6) we showed that the assumption that U contains an edge $f_1 = u_1u_3$ implies that there are no edges parallel to e_2 or e_4 in Ω. The same argument applied to the case where U contains edge $f_2 = u_2u_4$ shows that there are also no edges parallel to e_1 or e_3. It remains to show that the triangles $\{e_1, e_3, f_1\}$ and $\{e_2, e_4, f_2\}$ are balanced. Suppose that the triangle $\{e_1, e_3, f_1\}$ is unbalanced. Because of edge f_2, this implies that $\{u_1, u_3\}$ intersect every (u_2, u_4)-path in Ω'. Thus $\{u_1, u_3\}$ is a 2-vertex-cut of Ω'. Since $|V(\Omega')| = |V(\Omega)| - 1 \geq 5$, this implies that at least one of $\{u_1, u_2, u_3\}$ and $\{u_1, u_3, u_4\}$ is a 3-vertex-cut of Ω, a contradiction. We conclude that Ω' is indeed a criss-cross. □

Lemma 6.3. *Suppose that Ω is a simple 4-connected tangled signed graph with at least six vertices. Suppose moreover that Ω is not a criss-cross, a tricoloured graph, a projective planar biased graph with a special triple, or a fat triangle. Then there is a maximal 2-connected spanning balanced subgraph Ω' of Ω such that $(\Omega', (x_1, \ldots, x_m, y_1, \ldots, y_m))$ is planar (where consecutive vertices may be repeated), where $E(\Omega) - E(\Omega') = \{x_iy_i \mid i \in [m]\}$.*

Proof. Let $\Omega = (G, \mathcal{B})$ be a simple 4-connected tangled biased graph. By Lemma 6.2, we have that

(A1) Ω contains a 2-connected spanning balanced subgraph Ω'.

For the remainder of the proof we choose Ω' as in (A1) to be edge-maximal and we set $U := E(\Omega) - E(\Omega')$. Thus

(A2) For every $e \in U$, every e-cycle for Ω' is unbalanced.

First we show that

(A3) U contains at least two independent edges.

If (A3) does not hold, then either $G[U]$ is a star or there exist vertices x_1, x_2, x_3 in G such that every edge in U has both endpoints in $\{x_1, x_2, x_3\}$. In the first case Ω has a blocking vertex, in the second it is a fat triangle.

For the remainder of the proof we let $U' = \{f_i = x_iy_i \mid i \in [n]\}$ be a maximum-sized subset of U of pairwise independent edges. The next result follows immediately from Lemma 4.9.

Claim 1. *Up to a reordering of $[n]$, $(\Omega', (x_1, \ldots, x_n, y_1, \ldots, y_n))$ is 3-planar.*

For every $i \in [n]$, define sets:

$$Y_i = \{y \in V(\Omega) \mid x_iy \in U\}, \text{and}$$
$$X_i = \{x \in V(\Omega) \mid xy_i \in U\}.$$

Note that $x_i \in X_i$ and $y_i \in Y_i$ for every $i \in [n]$.

Claim 2. *Up to a reordering of* $[n]$*,* $(\Omega', (X_1, \dots, X_n, Y_1, \dots, Y_n))$ *is 3-planar.*

Proof of claim. By Claim 1, $(\Omega', \mathcal{A}, (x_1, \dots, x_n, y_1, \dots, y_n))$ is 3-planar for some $\mathcal{A}$. Let $H = \texttt{Proj}(\Omega', \mathcal{A})$. For any vertex w, let $w^* = w$ if $w \in V(H)$ and w^* be an arbitrary vertex in $N_G(A)$ if $w \in A$ for some $A \in \mathcal{A}$. Let C be the face boundary of H containing $x_1, \dots, x_n, y_1, \dots, y_n$. For every $i \in [n]$, let $X'_i = X_i \cap V(C)$ and $Y'_i = Y_i \cap V(C)$. Among all possible choices for $\mathcal{A}$, pick one such that $|X'_1 \cup \dots \cup X'_n \cup Y'_1 \cup \dots \cup Y'_n|$ is maximized.

Since U' is maximal, every edge in U has at least one end in $\{x_1, \dots, x_n, y_1, \dots, y_n\}$. We easily see from Lemma 4.4 that

(A4) For every $y \in Y_i$, y^* only attaches to vertices in $V(C[y_{i-1}, y_i, y_{i+1}])$ (where we take $y_0 = x_n$ and $y_{n+1} = x_1$). A symmetric statement holds for vertices in X_i.

Therefore, the claim is easily seen to hold when $X_i, Y_i \subseteq V(C)$ for every $i \in [n]$. Suppose that $x_1 y \in U$ for some $y \notin V(C)$.

First assume that $y^* \notin V(C)$. Since y^* only attaches to vertices in $C[x_n, y_1, y_2]$ in C, by planarity there exists a 2-vertex-cut $\{a, b\} \subseteq V(C[x_n, y_1, y_2])$ of H such that y^* and $C[y_3, \dots, y_n, x_1, \dots, x_{n-1}]$ are in different $\{a, b\}$-bridges. Let B be the $\{a, b\}$-bridge containing y^*. Since $\{x_1, a, b\}$ is not a 3-vertex-cut of Ω, there exists some vertex $z^* \in B - \{a, b\}$ such that $z \in X_2 \cup \dots \cup X_n \cup Y_1 \cup \dots \cup Y_n$. Then (A4) implies that either $z \in X_n$ (and $a, b \in V(C[x_n, y_1])$) or $z \in Y_2$ (and $a, b \in V(C[y_1, y_2])$). By symmetry we may assume the latter. If, for some i, a vertex $w \in Y_i$ is in $C(a, b)$, then Ω' contains an $(x_i - w, y - x_1)$-linkage (or an $(x_i - w, z - x_2)$-linkage when $i = 1$), a contradiction. By a similar argument we conclude that $C(a, b)$ does not contain any vertex in $X_1 \cup \dots \cup X_n \cup Y_1 \cup \dots \cup Y_n$. Let B' be the $\{a, b\}$-bridge in Ω' corresponding to B. Define a new set $\mathcal{A}'$ from $\mathcal{A}$ by

$$\mathcal{A}' = (\mathcal{A} - \{A \in \mathcal{A} \mid N_{\Omega'}(A) \subseteq B'\}) \cup \{B' - \{a, b, y\}\}.$$

Then $(\Omega', \mathcal{A}', (x_1, \dots, x_n, y_1, y, y_2, \dots, y_n))$ is 3-planar and $\mathcal{A}'$ contradicts the choice of $\mathcal{A}$.

Now suppose that $y^* \in V(C)$. Since $y \notin V(C)$, this implies that $y \in A$ for some $A \in \mathcal{A}$. If some vertex in $N_G(A)$ is not in $V(C)$, then we may replace y^* with this vertex, and reduce to the previous case. Now suppose that $N_G(A) \subset V(C)$. Since y^* is chosen arbitrarily in $N_G(A)$, property (A4) implies that $N_G(A) \subset V(C[x_n, y_1, y_2])$. Let B be the $N_G(A)$-bridge in Ω' containing y. Since Ω' is 2-connected, $N_G(A)$ contains at least two vertices. Let $N_G(A) = \{a, b\}$ if $N_G(A)$ has size two and $N_G(A) = \{a, b, c\}$ otherwise, chosen so that c is in the (a, b)-path of C avoiding x_1. Since a and b are adjacent in H, the planarity of H implies that $\{a, b\}$ is a 2-vertex-cut of Ω'. Since $\{a, b, x_1\}$ is not a 3-vertex-cut of Ω, there exists some vertex $z \in B - \{a, b\}$ such that $z \in X_2 \cup \dots \cup X_n \cup Y_1 \cup \dots \cup Y_n$. Then (A4) implies that either $z \in X_n$ (and $a, b \in V(C[x_n, y_1])$) or $z \in Y_2$ (and $a, b \in V(C[y_1, y_2])$). By symmetry we may assume the latter. From here we may proceed in a similar manner to the previous case. $\Diamond$

Since Ω is 4-connected and each edge in U is incident with $X_1 \cup \dots \cup X_n \cup Y_1 \cup \dots \cup Y_n$, Claim 2 implies that Ω' is planar and all the vertices incident with edges in U are on a same face boundary C. Since Ω' is 2-connected, C is a cycle. Therefore, if $U = \{f_1, \dots, f_m\}$, where $f_i = x_i y_i$ for every $i \in [m]$, then (by relabelling) we may assume that $(\Omega', (x_1, \dots, x_m, y_1, \dots, y_m))$ is planar (where some consecutive vertices may be repeated). $\square$

7 Proof of Theorem 1.3

First we show that Theorem 1.3 holds when Ω has at most five vertices.

Lemma 7.1. *Let Ω be a simple tangled biased graph with at most five vertices. Then Theorem 1.3 holds.*

Proof. Since Ω has no blocking vertex, it has at least three vertices. Moreover, by Lemmas 5.1 and 5.2 we may assume that Ω is 2-connected. If Ω has exactly three vertices, then it is a fat triangle. Assume that Ω has exactly four vertices. If some vertex v of Ω is not adjacent any parallel edges, define H to be a maximal balanced subgraph of Ω containing all the edges incident with v. Let v_1, v_2 and v_3 be the vertices of Ω other than v. Then there is at least one v_1v_2-edge not in H (and forming an unbalanced cycle with H), otherwise v_3 would be a blocking vertex of Ω. Similarly for the other two pairs of vertices v_i and v_j with $i \neq j$. It follows that Ω is a fat triangle. So we may assume that every vertex of Ω is adjacent with some parallel edges. Moreover, since Ω is simple and has no two disjoint unbalanced cycles, there is a vertex v of Ω such that all parallel edges are incident with v and for any other vertex w of Ω there is a pair of parallel edges between v and w. Then Ω is a generalized wheel (with center v).

Finally suppose that Ω has exactly five vertices v_1, v_2, v_3, v_4, v_5. Lemma 5.4 implies that either Ω satisfies cases (T1) or (T3) or all the vertices of Ω are pairwise adjacent, i.e. $||\Omega||$ is obtained from K_5 by possibly adding parallel edges.

Assume that (T2) does not hold. By symmetry we may assume that there are two unbalanced 2-cycles in Ω between vertex v_1 and vertices v_2 and v_3. Then triangles $v_2v_4v_5v_2$ and $v_3v_4v_5v_3$ are balanced. Thus, the 4-cycle $v_5v_2v_4v_3v_5$ is balanced. If there is also an unbalanced 2-cycle between v_2 and v_3, then Ω is a fat triangle. So now suppose that there is no such unbalanced 2-cycle. Since the 4-cycle $v_5v_2v_4v_3v_5$ is balanced, and it forms a theta subgraph with the edge v_2v_3, triangles $v_2v_3v_4v_2$ and $v_2v_3v_5v_2$ are either both balanced or both unbalanced. Since v_1 is not a blocking vertex, they are both unbalanced. It follows that there are no other unbalanced 2-cycles in Ω. Now if the triangle $v_1v_4v_5v_1$ is balanced, then Ω is a fat triangle. Otherwise Ω is a generalized wheel (with center v_1 and, in the definition of generalized wheel, vertices $z_1 = v_2$ and $z_2 = v_3$.) □

Next we prove a useful lemma to identify when Ω is signed-graphic.

Lemma 7.2. *Let Ω' be a maximal balanced subgraph of a biased graph Ω. Suppose that Ω contains no two disjoint unbalanced cycles. If F is 2-balanced with respect to Ω' then $\Omega' \cup F$ is a signed graph with signature F.*

Proof. It suffices to prove the statement for each connected component of Ω. Thus we assume that Ω is connected. Let C be any cycle in $\Omega' \cup F$. To prove the result it suffices to show that C is balanced if and only if $|F \cap E(C)|$ is even. We proceed by induction on $k = |F \cap E(C)|$. If $k = 0$, then C is a cycle of Ω', which is balanced, so C itself is balanced. If $k = 1$ then C is unbalanced by the maximality of Ω'. If $k = 2$, then C is balanced since F is 2-balanced. Now suppose that $k \geq 3$. Let $P_1, \ldots, P_k$ be the components of $C \backslash F$. Each P_i is a path of Ω' (possibly comprising a single vertex) and by relabelling we may assume that $P_1, \ldots, P_k$

appear in this order along C. For the remainder of the proof, for distinct $i, j \in [k]$, we say that P_i connects to P_j if there exists an $(x_i - x_j)$-path Q in Ω' with $x_i \in V(P_i)$ and $x_j \in V(P_j)$ and Q is internally disjoint from all of $P_1, \dots, P_k$. In this case the path Q *connects* P_i to P_j.

First suppose that k is odd. Since Ω' is maximal and Ω is connected, Ω' is also connected. Thus there exists a path Q connecting P_i and P_j for some distinct $i, j \in [k]$. Then $C \cup Q$ is a theta graph. Let C_1 and C_2 be the two cycles in $C \cup Q$ containing Q. Note that $C_1 \cap F$ and $C_2 \cap F$ are both nonempty. Since Q does not contain any edge in F, we have that $k_1 = |C_1 \cap F| < k$, $k_2 = |C_2 \cap F| < k$ and one of k_1 and k_2 is odd, and the other is even. By induction, one of C_1 and C_2 is unbalanced and the other is balanced. Therefore, by the theta property C is unbalanced.

Now suppose that k is even. Suppose that some P_i connects to some $P_{i+2\ell}$ for some number ℓ, through a path Q. Then we may apply a similar argument to the one above, where now k_1 and k_2 are even. Thus the two cycles in $C \cup Q$ using Q are balanced and the theta property implies that C is balanced as well. To complete the proof it remains to show that we may always find such P_i and $P_{i+2\ell}$.

If every P_i connects to only one other P_j, then Ω' is disconnected (since $k \geq 4$). So without loss of generality we may assume that P_1 connects to P_{j_1} and to P_{j_2}, where $j_1 < j_2$. If one of j_1 or j_2 is odd then we are done. So assume that j_1 and j_2 are even. Choose an odd j_3 with $j_1 < j_3 < j_2$. Now P_{j_3} connects to some other P_{j_4}. If j_4 is odd then we are done. Let Q_i be a path connecting P_1 to P_{j_i}, for $i = \{1, 2\}$. Let Q_3 be a path connecting P_{j_3} to P_{j_4}. For $i \in [3]$, both cycles in $C \cup Q_i$ using Q_i are unbalanced, hence $C \cup Q_1 \cup Q_2 \cup Q_3$ contains two disjoint unbalanced cycles. □

Suppose that Ω is a tangled biased graph with a maximal 2-connected spanning balanced subgraph Ω' as in Lemma 6.3, i.e. $E(\Omega) - E(\Omega') = \{x_i y_i \mid i \in [m]\}$ and $(\Omega', (x_1, \dots, x_m, y_i, \dots, y_m))$ is planar (where consecutive vertices may be repeated). We say that such an Ω is a *projective planar* biased graph *based on* Ω'. Every time we refer to such a graph we will indicate by $\{f_i = x_i y_i \mid i \in [m]\}$ the set of edges in $E(\Omega) - E(\Omega')$.

Lemma 7.3. *Let Ω be a simple 4-connected projective planar tangled biased graph. Then either*

(1) Ω has a blocking pair, or

(2) Ω is a projective planar signed graph, or

(3) Ω is projective planar with a special vertex, or

(4) Ω is a tricoloured graph, or

(5) Ω is a criss-cross.

Proof. Suppose that Ω is projective planar based on Ω'. Thus $(\Omega', (x_1, \dots, x_m, y_i, \dots, y_m))$ is planar and $U := \{f_i = x_i y_i \mid i \in [m]\} = E(\Omega) - E(\Omega')$. Unless otherwise specified, throughout the proof we will refer to A-cycles (for some edge or some set A of edges), meaning an A-cycle with respect to Ω'. By the maximality of Ω', every f_i-cycle is unbalanced. Denote by C the face boundary of Ω' containing $x_1, \dots, x_m, y_1, \dots, y_m$.

We assume that we are not in case (1) (i.e. Ω does not have a blocking pair).

Claim 1. *For every $f_i \in U$ there exists an edge $f_j \in U$ that is independent from f_i and f_{i+1}.*

Proof of claim. For simplicity, assume that $i = 1$. Since x_1, x_2 is not a blocking pair, there is an edge f_k not incident with x_1, x_2. If f_k is not incident with y_1 and y_2 then we are done. Otherwise we may assume that $y_k = y_2$. Similarly, y_1 and y_2 do not form a blocking pair, so there is an edge f_ℓ with $y_\ell = x_1$ (f_ℓ has to be incident with x_1 because of the position of x_k). Now one can check that x_1, y_2 form a blocking pair, a contradiction. ♢

When Ω contains two parallel edges f and f', since Ω is simple, $\{f, f'\}$ is an unbalanced cycle of Ω and the endpoints of f intersect all unbalanced cycles of Ω, thus forming a blocking pair. Therefore there are no parallel edges in U.

Because of Lemma 7.2, if U is 2-balanced with respect to Ω' then Ω is a projective planar signed graph (i.e. (2) holds). Thus for the remainder of the proof we assume that U is not 2-balanced with respect to Ω'. Hence there exist $i, j \in [m]$, with $i < j$, such that some $\{f_i, f_j\}$-cycle is unbalanced. We may assume that, aside from f_i and f_j, such cycle comprises an (x_i, x_j)-path P_x and a (y_i, y_j)-path P_y in Ω'. Since Ω' is planar, $C[x_i, x_{i+1}, \dots, x_j]$ may be obtained from P_x by rerouting along balanced cycles (disjoint from P_y) and $C[y_i, y_{i+1}, \dots, y_j]$ may be obtained from P_y by rerouting along balanced cycles (disjoint from P_x). Therefore, $C[x_i, x_{i+1}, \dots, x_j] \cup C[y_i, y_{i+1}, \dots, y_j] \cup \{f_i, f_j\}$ is an unbalanced cycle.

Next we show that we may choose i and j so that $j = i+1$. Choose i and j so that $j-i$ is minimised. If $j \neq i+1$ (so $f_{i+1} \neq f_j$), then $C[x_i, x_{i+1}, \dots, x_j] \cup C[y_i, y_{i+1}, \dots, y_j] \cup \{f_i, f_{i+1}, f_j\}$ is a theta subgraph, thus either the cycle using f_i and f_{i+1} or the cycle using f_{i+1} and f_j is unbalanced, contradicting the choice of i and j. It follows that U contains edges f_i and f_{i+1} such that $C[x_i, x_{i+1}] \cup C[y_i, y_{i+1}] \cup \{f_i, f_{i+1}\}$ is unbalanced. We call such a pair $\{f_i, f_{i+1}\}$ a *close unbalanced pair*. In this definition we choose indices modulo m, so it may be that f_1 and f_m form a close unbalanced pair. If $\{f_i, f_{i+1}\}$ is a close unbalanced pair, we denote by $C_{i,i+1}$ the unbalanced cycle $C[x_i, x_{i+1}] \cup C[y_i, y_{i+1}] \cup \{f_i, f_{i+1}\}$.

Next we introduce a few more definitions that will come in handy for the remainder of the proof. If a vertex c is in a 2-vertex-cut of Ω', then we say that c is an *intermediate* vertex of Ω'. If a 2-vertex-cut $\{c, d\}$ of Ω' intersects all (x_i, y_i)-paths for every $i \in [m]$, then we say that $\{c, d\}$ is a *diagonal* 2-vertex-cut of Ω'. Let $\{f_i, f_{i+1}\}$ be a close unbalanced pair and let f_k be an edge in U independent from f_i and f_{i+1} (as in Claim 1). Since $C_{i,i+1}$ intersects every f_k-cycle, there exist vertices $c \in C[x_i, x_{i+1}]$ and $d \in C[y_i, y_{i+1}]$ such that $\{c, d\}$ is a 2-vertex-cut separating x_k from y_k. Then $\{c, d\}$ is a diagonal 2-vertex-cut of Ω'. We say that $\{c, d\}$ is a diagonal 2-vertex-cut *associated* with the close unbalanced pair $\{f_i, f_{i+1}\}$.

Claim 2. *There exists at least two close unbalanced pairs.*

Proof of claim. We already showed that there exists a close unbalanced pair. To simplify notation suppose that $\{f_1, f_2\}$ is such a pair. Let $\{c, d\}$ be a diagonal 2-vertex-cut of Ω' associated with $\{f_1, f_2\}$. Since $\{c, d\}$ is not a blocking pair, there exists an unbalanced cycle C' avoiding c and d. Thus $|E(C') \cap U| \geq 2$ and by the theta property we may in fact choose C' so that $|E(C') \cap U| = 2$. Then again the theta property (and Claim 1, if $E(C') \cap U = \{f_1, f_2\}$) implies that U contains another close unbalanced pair. ♢

Claim 3. *Any two close unbalanced pairs share at least one vertex and at most two. If they share two, then those are the endpoint of an edge common to the two pairs.*

Proof of claim. Let $\{f_i, f_{i+1}\}$ and $\{f_j, f_{j+1}\}$ be two distinct close unbalanced pairs. Since $C_{i,i+1}$ and $C_{j,j+1}$ intersect, at least one of the edges f_i and f_{i+1} shares an endpoint with one of f_j and f_{j+1}. Thus the two pairs share at least one vertex. Now it is easy to see that the second part of the claim holds since Ω is projective planar. ◊

Claim 4. *Let $\{f_i, f_{i+1}\}$ and $\{f_{i+1}, f_{i+2}\}$ be two close unbalanced pairs. Then the endpoints of f_{i+1} form a diagonal 2-vertex-cut of Ω'.*

Proof of claim. To simplify notation we assume that $i = 1$, i.e. the two close unbalanced pairs are $\{f_1, f_2\}$ and $\{f_2, f_3\}$. Assume that x_2 is not an intermediate vertex of Ω'. Then there are vertices $c \in C[x_1, x_2)$ and $c' \in C(x_2, x_3]$ such that c and c' are intermediate vertices of Ω'. In particular, $x_1 \neq x_2$ and $x_2 \neq x_3$, so the only edge in U that is incident with x_2 is f_2. Then $\{c, c', y_2\}$ is a 3-vertex-cut of Ω, a contradiction. So x_2 is an intermediate vertex. By symmetry y_2 is also an intermediate vertex. It follows that $\{x_2, y_2\}$ is a diagonal 2-vertex-cut. ◊

Claim 5. *Let $\{f_i, f_{i+1}\}$ and $\{f_j, f_{j+1}\}$ be two close unbalanced pairs sharing exactly one vertex (say $x_{i+1} = x_j$). Then $j = i + 2$ and y_{i+1} and y_j are intermediate vertices of Ω'. Moreover, either x_{i+1} is an intermediate vertex, or there exist $c \in C[x_i, x_{i+1})$ and $c' \in C(x_j, x_{j+1}]$ such that $\{c, y_{i+1}\}$ and $\{c', y_j\}$ are diagonal 2-vertex-cuts of Ω'.*

Proof of claim. To simplify notation we assume that $i = 1$, and the pairs share the vertex $x_2 = x_j$. Then $y_2 \neq y_j$ and there are vertices $c \in C[x_1, x_2]$ and $d \in C[y_1, y_2]$ such that $\{c, d\}$ is a diagonal 2-vertex-cut and vertices $c' \in C[x_j, x_{j+1}]$ and $d' \in C[y_j, y_{j+1}]$ such that $\{c', d'\}$ is a diagonal 2-vertex-cut. If y_2 or y_j is not an intermediate vertex, then $\{x_2, d, d'\}$ is a 3-vertex-cut of Ω'. Thus both y_2 and y_j are intermediate vertices; moreover $y_2 y_j$ is an edge of Ω' (so $j = 3$ and the close unbalanced pair $\{f_j, f_{j+1}\}$ is in fact $\{f_3, f_4\}$). If x_2 is an intermediate vertex then we are done. Else, c and c' are distinct from x_2 and the claim follows. ◊

Claim 6. *There exist two close unbalanced pairs with a common vertex x which is not an intermediate vertex of Ω'.*

Proof of claim. Suppose this is not the case. Claim 4 and Claim 5 imply that for every two close unbalanced pairs there exists a diagonal 2-vertex-cut of Ω' containing all vertices common to the two pairs. Suppose that $\{f_i, f_{i+1}\}$ and $\{f_j, f_{j+1}\}$ are close unbalanced pairs and let $\{c, d\}$ be a diagonal 2-vertex-cut containing the vertices common to the two pairs. Since $\{c, d\}$ is not a blocking pair, there exists a third close unbalanced pair $\{f_k, f_{k+1}\}$ such that $\{c, d\}$ does not intersect $C_{k,k+1}$. Therefore, no vertex of Ω' belongs to all three close unbalanced pairs $\{f_i, f_{i+1}\}$, $\{f_j, f_{j+1}\}$ and $\{f_k, f_{k+1}\}$. Claim 4 and Claim 5 imply that $U = \{f_i, f_{i+1}, f_j, f_{j+1}, f_k, f_{k+1}\}$ and by symmetry we may assume that $i < j < k$. Thus f_{i+1} shares an intermediate vertex w_1 with f_j, f_{j+1} shares an intermediate vertex w_2 with f_k, and f_{k+1} shares an intermediate vertex w_3 with f_i. Then w_1, w_2, w_3 are three distinct intermediate vertices of Ω'. Since $\Omega - \{w_1, w_2, w_3\}$ is connected, we have that Ω' comprises only a, say, $\{w_1, w_2\}$-bridge and two edges w_1w_3 and w_2w_3. It follows that either $\{w_1, w_2\}$ is a blocking pair of Ω or $\{f_{k+1}, f_i\}$ is a close unbalanced pair. In this case either Ω has two disjoint unbalanced cycles or a 3-vertex-cut, a contradiction. $\Diamond$

To complete the proof it remains to show the following.

Claim 7. *Either Ω is projective planar with a special vertex or it is a tricoloured graph.*

Proof of claim. Claim 6 and Claim 5 imply that there exist close unbalanced pairs $\{f_i, f_{i+1}\}$ and $\{f_j, f_{j+1}\}$ sharing vertex $x_{i+1} = x_j$ such that x_{i+1} is not an intermediate vertex of Ω'. To simplify notation assume that $i = 1$. That is, the two close unbalanced pairs are $\{f_1, f_2\}$ and $\{f_3, f_4\}$ and $x_2 = x_3$ is not an intermediate vertex. By Claim 5, there exist vertices $c \in C[x_1, x_2)$ and $c' \in C(x_3, x_4]$ such that $\{c, y_2\}$ and $\{c', y_3\}$ are diagonal 2-vertex-cuts of Ω'. In particular, this implies that $x_1 \neq x_2$ and $x_3 \neq x_4$. This also implies that x_2 is a degree-2 vertex in Ω' (adjacent to c and c'), for otherwise $\{x_2, c, c'\}$ is a 3-vertex-cut of Ω. Moreover, cc' is an edge of Ω' as x_2 is not an intermediate vertex of Ω'. Since $\{x_2, y_2, y_3\}$ is not a 3-vertex-cut of Ω, there is only one $\{y_2, y_3\}$-bridge in Ω' (the one containing x_2) and y_2y_3 is an edge of Ω'. Since we will frequently refer to this edge, we denote it as $g = y_2y_3$. For any two $f_i, f_j \in U$ (with $i < j$), denote by $C_{i,j}$ the cycle $C[x_i, x_{i+1}, \ldots, x_j] \cup C[y_i, y_{i+1}, \ldots, y_j] \cup \{f_i, f_j\}$. We also denote as $C_{i,1}$ the cycle $C[y_i, y_{i+1}, \ldots, y_m, x_1] \cup C[x_i, x_{i+1}, \ldots, x_m, y_1] \cup \{f_1, f_i\}$, for every $1 < i \leq m$. Since any $\{f_i, f_j\}$-cycle C' for $\Omega' - g$ may be obtained by rerouting $C_{i,j}$ along balanced cycles in $\Omega' - g$,

(P1) for any two $f_i, f_j \in U$ (with $i < j$), every $\{f_i, f_j\}$-cycle for $\Omega' - g$ has the same bias as $C_{i,j}$.

Define a relation $\sim$ on U as $f_i \sim f_j$ if either $i = j$ or $\{f_i, f_j\}$ is 2-balanced for $\Omega' - g$. We claim that

(P2) the relation $\sim$ is an equivalence relation on U.

Clearly $\sim$ is reflexive and symmetric. Now suppose that $f_i \sim f_j$ and $f_j \sim f_k$. To simplify notation assume that $i < j < k$. Then $C_{i,j}$ and $C_{j,k}$ are both balanced, and they form a theta subgraph, whose third cycle is $C_{i,k}$. It follows that $C_{i,k}$ is balanced and (P1) implies that $\{f_i, f_k\}$ is 2-balanced for $\Omega' - g$.

We assign to each equivalence class for $\sim$ a colour, so that distinct classes get different colours. Then (P1) implies that if f_i and f_j have distinct colours, every $\{f_i, f_j\}$-cycle for $\Omega' - g$ is unbalanced.

Applying Claim 1 to f_2 and f_3 shows that there exists an edge $f_k \in U$ not incident with y_2, y_3 and x_2. Thus, since cc' is an edge of Ω', the cycle $\{f_2, f_3, g\}$ is disjoint from some f_k-cycle. It follows that $\{f_2, f_3, g\}$ is balanced. Moreover, since f_2 and f_3 share a vertex, $\{f_2, f_3\}$ is 2-balanced for Ω'. In particular, f_2 and f_3 have the same colour.

Define the set $U' = \{f_k \in U \mid x_k, y_k \notin \{x_1, y_1, x_4, y_4\}\} - \{f_2, f_3\}$ (i.e. U' is the set of edges in U that are independent from f_1, f_2, f_3 and f_4). First suppose that U' is nonempty. We claim that in this case Ω is projective planar with a special vertex. First we show that all the edges in $U - \{f_2, f_3\}$ have the same colour. Pick any $f_k \in U'$; then $C_{k,1}$ is disjoint from $C_{3,4}$, thus $C_{k,1}$ is balanced. Similarly, $C_{4,k}$ is disjoint from $C_{1,2}$, thus $C_{4,k}$ is also balanced. It follows that every edge in U' has the same colour as f_1 and f_4. A similar argument shows that any other edge in U incident with x_1, x_4, y_1 or y_4 has also the same colour. Thus all the edges in $U - \{f_2, f_3\}$ have the same colour. Such colour is different from the colour of f_2 and f_3, since $\{f_1, f_2\}$ is a close unbalanced pair. It follows that in this case Ω is projective planar with a special vertex. We therefore assume for the remainder of the proof that U' is empty. This implies that all the edges in $U - \{f_2, f_3\}$ are incident to a vertex in $\{x_1, x_4, y_1, y_4\}$ (because Ω is projective planar on Ω'). Up to symmetry, this leaves three possibilities.

Case 1: $U - \{f_2, f_3\}$ partitions into two sets U_1 and U_4, where all the edges in U_1 are incident to x_1 and all the edges in U_4 are incident to x_4. In this case at most one of x_1 and x_4 is an intermediate vertex, for otherwise $\{x_1, x_4\}$ is a blocking pair (since $\{f_2, f_3\}$ is 2-balanced). Moreover, $x_1 \neq y_4$ (and symmetrically, $x_4 \neq y_1$), for otherwise $U_4 = \{f_4\}$ and $\{x_1, x_2\}$ is a blocking pair. If U_1 and U_4 contain no close unbalanced pair, then all edges in U_1 have the same colour and all edges in U_4 have the same colour. Else we may assume (by symmetry) that there is a close unbalanced pair $\{f_i, f_{i+1}\}$ in U_1. Since every f_3-cycle intersects $C_{i,i+1}$, we have that x_1 is an intermediate vertex. Since the close unbalanced pairs $\{f_3, f_4\}$ and $\{f_i, f_{i+1}\}$ share a vertex, we have $x_i = x_4$ (i.e. f_i is an edge between x_1 and x_4). Thus the edges in $U_1 - \{f_i\}$ have all the same colour. Now if $U_4 \cup \{f_i\}$ is not 2-balanced for $\Omega' - g$, then x_4 is also an intermediate vertex, a contradiction to the fact that at most one of x_1 and x_4 is an intermediate vertex. It follows that $U_1 - \{f_i\}$ and $U_4 \cup \{f_i\}$ are both equivalence classes for $\sim$. Thus we may assume that we started with U_1 and U_4 where all the edges in U_1 have the same colour and all the edges in U_4 have the same colour.

Now if $U_1 \cup U_4$ have the same colour, then Ω is again projective planar with a special vertex. So we may assume that U_1 and U_4 have different colours. Such colours are both different from the colour of f_2 and f_3 (since $\{f_1, f_2\}$ and $\{f_3, f_4\}$ are close unbalanced pairs). We show that in this case Ω is a tricoloured graph. Denote as N_1 the neighbour of x_1 via edges in U_1 and as N_4 the neighbours of x_4 via edges in U_4. Since U_1 and U_4 have different colours, for every $f_i \in U_1$ and every $f_j \in U_4$, every $\{f_i, f_j\}$-cycle for $\Omega' - g$ intersects every f_2-cycle and every f_3-cycle. Therefore there is a 2-vertex-cut $\{d, d'\}$ of Ω' with $d \in C[y_4, \ldots, y_m, x_1]$ and $d' \in C[x_4, \ldots, x_m, y_1]$, where $\{d, d'\}$ separates x_1 from N_1 and x_4 from N_4. Now Ω is a tricoloured graph where (following the notation in the definition of tricoloured graphs given in Section 3.7) we have $I = \{1, 2, 3\}$ and:

- $H_1 = \{c\}$ if $c = d$, otherwise H_1 is the $\{d, c\}$-bridge of Ω' not containing x_2 (or $H_1 = \{dc\}$ if there is only one $\{d, c\}$-bridge);
- H_2 is the $\{c, c'\}$-bridge containing x_2;
- $H_3 = \{c'\}$ if $c' = d'$, otherwise H_3 is the $\{c', d'\}$-bridge of Ω' not containing x_2 (or $H_3 = \{c'd'\}$ if there is only one $\{c', d'\}$-bridge);
- $H_4 = \{d'\}$ if $y_2 = d'$, otherwise H_4 is the $\{d', y_2\}$-bridge of Ω' not containing x_2 (or $H_4 = \{d'y_2\}$ if there is only one $\{d', y_2\}$-bridge);
- $H_5 = \{g\}$;
- $H_6 = \{d\}$ if $y_3 = d$, otherwise H_6 is the $\{y_3, d\}$-bridge of Ω' not containing x_2 (or $H_6 = \{y_3d\}$ if there is only one $\{y_3, d\}$-bridge).

Case 2: $U - \{f_2, f_3\}$ partitions into two sets U_1 and U_4, where all the edges in U_1 are incident to y_1 and all the edges in U_4 are incident to y_4. This case is very similar to Case 1; we include the proof for completeness.

At most one of y_1 and y_4 is an intermediate vertex, for otherwise $\{y_1, y_4\}$ is a blocking pair (since $\{f_2, f_3\}$ is 2-balanced). Moreover, $x_1 \neq y_4$ (and symmetrically, $x_4 \neq y_1$), for otherwise $U_1 = \{f_1\}$ and $\{y_4, x_2\}$ is a blocking pair. If U_1 and U_4 contain no close unbalanced pair, then all edges in U_1 have the same colour and all edges in U_4 have the same colour. Else we may assume (by symmetry) that there is a close unbalanced pair $\{f_i, f_{i+1}\}$ in U_1. Since every f_2-cycle intersects $C_{i,i+1}$, we have that y_1 is an intermediate vertex. Since the close unbalanced pairs $\{f_3, f_4\}$ and $\{f_i, f_{i+1}\}$ share a vertex, we have $y_i = y_4$ (i.e. f_i is an edge between y_1 and y_4). Thus the edges in $U_1 - \{f_i\}$ have all the same colour. Now if $U_4 \cup \{f_i\}$ is not 2-balanced for $\Omega' - g$, then y_4 is also an intermediate vertex, a contradiction to the fact that at most one of y_1 and y_4 is an intermediate vertex. It follows that $U_1 - \{f_i\}$ and $U_4 \cup \{f_i\}$ are both equivalence classes for $\sim$. Thus we may assume that we started with U_1 and U_4 where all the edges in U_1 have the same colour and all the edges in U_4 have the same colour.

Now if $U_1 \cup U_4$ have the same colour, then Ω is projective planar with a special vertex. So we may assume that U_1 and U_4 have different colours. Such colours are both different from the colour of f_2 and f_3 (since $\{f_1, f_2\}$ and $\{f_3, f_4\}$ are close unbalanced pairs). We show that in this case Ω is a tricoloured graph. Denote as N_1 the neighbour of y_1 via edges in U_1 and as N_4 the neighbours of y_4 via edges in U_4. Since U_1 and U_4 have different colours, there is a 2-vertex-cut $\{d, d'\}$ of Ω' with $d \in C[y_4, \ldots, y_m, x_1]$ and $d' \in C[x_4, \ldots, x_m, y_1]$, where $\{d, d'\}$ separates y_1 from N_1 and y_4 from N_4. Now Ω is a tricoloured graph where (following the notation in the definition of tricoloured graphs given in Section 3.7) we have $I = \{1, 3, 5\}$ and:

- H_1 is the $\{c, c'\}$-bridge containing x_2;
- $H_2 = \{c'\}$ if $c' = d'$, otherwise H_2 is the $\{c', d'\}$-bridge of Ω' not containing x_2 (or $H_2 = \{c'd'\}$ if there is only one $\{c', d'\}$-bridge);
- $H_3 = \{d'\}$ if $y_2 = d'$, otherwise H_3 is the $\{d', y_2\}$-bridge of Ω' not containing x_2 (or $H_3 = \{d'y_2\}$ if there is only one $\{d', y_2\}$-bridge);
- $H_4 = \{g\}$;
- $H_5 = \{d\}$ if $y_3 = d$, otherwise H_5 is the $\{y_3, d\}$-bridge of Ω' not containing x_2 (or $H_5 = \{y_3d\}$ if there is only one $\{y_3, d\}$-bridge);
- $H_6 = \{c\}$ if $c = d$, otherwise H_6 is the $\{d, c\}$-bridge of Ω' not containing x_2 (or $H_6 = \{dc\}$ if there is only one $\{d, c\}$-bridge).

Case 3: $U - \{f_2, f_3\}$ partitions into two sets U_1 and U_4, where all the edges in U_1 are incident to y_1 and all the edges in U_4 are incident to x_4. Since $\{x_2, x_4\}$ is not a blocking pair, $y_1 \neq x_4$. First suppose that U_4 contains a close unbalanced pair $\{f_i, f_{i+1}\}$. Since $C_{i,i+1}$ and $C_{1,2}$ intersect, this implies that f_{i+1} is an edge between x_4 and x_1. Thus $U_1 = \{f_1\}$ and we are back to Case 1. So we may assume that all the edges in U_4 have the same colour. Similarly, we either reduce to Case 2 or we may assume that all the edges in U_1 have the same colour.

Now if U_1 and U_4 have the same colour, then Ω is projective planar with a special vertex. Otherwise, U_1 and U_4 have distinct colours (which are also distinct from the colour of $\{f_2, f_3\}$). Denote as N_1 the neighbour of y_1 via edges in U_1 and as N_4 the neighbours of x_4 via edges in U_4. Since U_1 and U_4 have different colours, there is a 2-vertex-cut $\{d, d'\}$ of Ω' with $d \in C[y_4, \ldots, y_m, x_1]$ and $d' \in C[x_4, \ldots, x_m, y_1]$, where $\{d, d'\}$ separates y_1 from N_1 and x_4 from N_4. Now we can see (similarly to the previous cases) that Ω is a tricoloured graph. $\Diamond$

$\square$

Proof of Theorem 1.3. Let $\Omega = (G, \mathcal{B})$ be a simple tangled biased graph. By Lemma 7.1 we may assume that Ω has at least six vertices. If Ω is not 4-connected, then the result holds by Lemma 5.4. Thus for the remainder of the proof we will assume that Ω is 4-connected. By Lemma 6.3 we only need to consider the case that Ω is projective planar, that is, Ω contains a 2-connected spanning balanced subgraph Ω' such that $(\Omega', (x_1, \ldots, x_m, y_1, \ldots, y_m))$ is planar (where consecutive vertices may be repeated), where $E(\Omega) - E(\Omega') = \{x_i y_i \mid i \in [m]\}$. By Lemma 7.3 we may assume that Ω has a blocking pair w_1, w_2.

Next we show that there is a balanced spanning subgraph Ω'' of Ω such that all edges in $E(\Omega) - E(\Omega'')$ are incident to $\{w_1, w_2\}$ and such that, up to swapping w_1 with w_2, Ω'' satisfies one of the following:

- Ω'' is 2-connected, or
- $\delta_{\Omega''}(w_1) = \{w_1 q\}$ for $q \neq w_2$ and $\Omega''/w_1 q$ is 2-connected, or
- $\delta_{\Omega''}(w_1) = \{w_1 q_1\}$, $\delta_{\Omega''}(w_2) = \{w_2 q_2\}$, where w_1, w_2, q_1, q_2 are all distinct and $\Omega''/\{w_1 q_1, w_2 q_2\}$ is 2-connected.

Set $E(\Omega) - E(\Omega') = U$. Let U' be the set of edges in U not incident with $\{w_1, w_2\}$. If U' is empty then we are done, choosing $\Omega'' = \Omega'$.

Let C be the facial boundary cycle containing $x_1, \ldots, x_m, y_1, \ldots, y_m$. For every $f = xy \in U'$, every (x, y)-path in Ω' intersects $\{w_1, w_2\}$. Thus (up to swapping w_1 with w_2) either for some $i \in [m]$, $w_1 \in C[x_i, x_{i+1}]$ and $w_2 \in C[y_i, y_{i+1}]$ or $w_1 \in C[y_m, x_1]$ and $w_2 \in C[x_m, y_1]$. By possibly relabelling $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ we may assume that $w_1 \in C[y_m, x_1]$ and $w_2 \in C[x_m, y_1]$. It follows that Ω' contains a 2-separation (A, B), where the boundary of A is $\{w_1, w_2\}$ and $C[x_1, \ldots, x_m]$ is contained in $\Omega'[A]$ and $C[y_1, \ldots, y_m]$ is contained in $\Omega'[B]$. For $i = 1, 2$, let A_i be the set of edges in A incident with w_i. Note that $\Omega - \{w_1, w_2\} = (\Omega' \cup U') - \{w_1, w_2\}$ is 2-connected, since Ω is 4-connected. Moreover, $\Omega' \cup U' - (A_1 \cup A_2)$ is balanced, since w_1, w_2 is a blocking pair. Now we may choose $\Omega'' = \Omega' \cup U' - (A_1 \cup A_2)$. The desired connectivity properties for Ω'' hold, since Ω' is 2-connected (with different cases occurring when there is only one edge in B incident with w_i or not). By our choice of Ω'' we have that all edges in $E(\Omega) - E(\Omega'')$ are incident with $\{w_1, w_2\}$.

With this structure in place, we show that Ω is projective planar with either a special pair or a special triple.

Set $U'' = E(\Omega) - E(\Omega'')$. Let U_1 be the set of edges in U'' incident with w_1 but not with w_2 and U_2 be the set of edges in U'' incident with w_2 but not with w_1. Let W_1 be the set of neighbours of w_1 via edges in U_1 and W_2 be the set of neighbours of w_2 via edges in U_2. Then $(\Omega'', (w_1, w_2, W_1, W_2))$ is planar.

Therefore, either $\Omega'' \cup U_1$ and $\Omega'' \cup U_2$ are signed graphs (and Ω is projective planar with a special pair), or we may assume that there exist edges $e_1 = w_1z_1$ and $e_2 = w_1z_2$ in U_1 such that every $\{e_1, e_2\}$-cycle for Ω'' is unbalanced. We may choose e_1 and e_2 so that w_1, w_2, z_1, z_2 appear in this order in the outer face of Ω'' and let z_2 be in $W_1 - W_2$ if possible. Note that $z_1 \neq z_2$ for otherwise $\{z_1, z_2\}$ is a cut of Ω.

Suppose that there exists a vertex $z \in W_2 - \{z_2\}$. Then one of the following occurs: Ω'' contains a $(z_1 - z_2, z - w_2)$-linkage avoiding w_1, or there is a 2-vertex-cut $\{w_1, c\}$ in Ω'' separating z from w_2 and z_1 from z_2. The former case does not occur, else Ω contains two disjoint unbalanced cycles. Suppose that the latter occurs. Since $\{w_1, w_2, c\}$ is not a 3-vertex-cut of Ω, we have that $c = z_1$ and w_2 is only incident with w_1 and z_1 in Ω''. Hence, $U_1 - \{e_1\}$ is 2-balanced. If U_2 is not 2-balanced, then $w_2z_2 \in U_2$, and there is a vertex $z_2' \in W_2$ such that every $\{w_2z_2, w_2z_2'\}$-cycle for Ω'' is unbalanced and w_1, w_2, z_1, z_2, z_2' appear in this order in the outer face of Ω''. By symmetry w_1 is only incident with w_2 and z_2' in Ω'', and $U_2 - \{w_2z_2'\}$ is 2-balanced. Moreover, since Ω has no two disjoint unbalanced cycles, $W_1 = \{z_1, z_2\}$ and $W_2 = \{z_2, z_2'\}$; so either $\{z_1, z_2, z_2'\}$ is a cut of Ω or $V(\Omega) = \{w_1, w_2, z_1, z_2, z_2'\}$, a contradiction. So U_2 is 2-balanced, implying that $\Omega'' \cup U_2 - \{w_1w_2, w_2z_1\}$ is balanced. In this case Ω is projective planar with a special triple, by setting (in the definition of projective planar graph with a special triple), $H = \Omega'' \cup U_2 - \{w_1w_2, w_2z_1\}$ and $x = w_1, y_1 = w_2$ and $y_2 = z_1$.

The remaining case is when $W_2 = \{z_2\}$ and, if there exists any other edge $e \in U_1$ such that the $\{e, e_1\}$-cycles for Ω'' are unbalanced, then e is also incident with z_2 by the choice of z_2. In this case Ω is projective planar with a special triple, by setting (in the definition of projective planar graph with a special triple) $x = w_1$, $y_1 = z_2$ and $y_2 = w_2$. □

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A Pictures of tangled structures

Following are pictures of the structures described in Section 3. The grey subgraphs represent balanced graphs; these balanced graphs are all planar, with the exception of Figure 5.

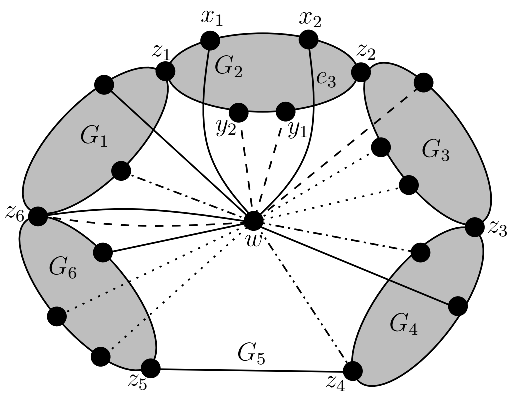

Figure 3: A generalised wheel.

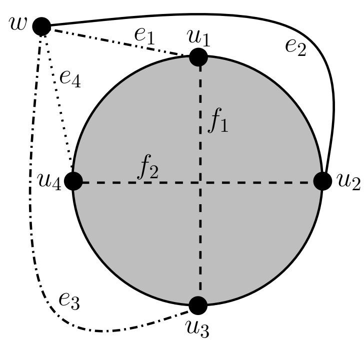

Figure 4: A criss-cross.

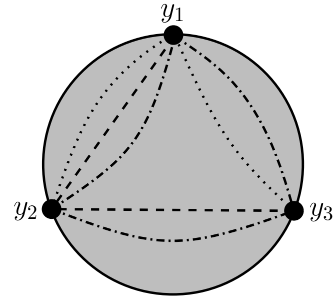

Figure 5: A fat triangle.

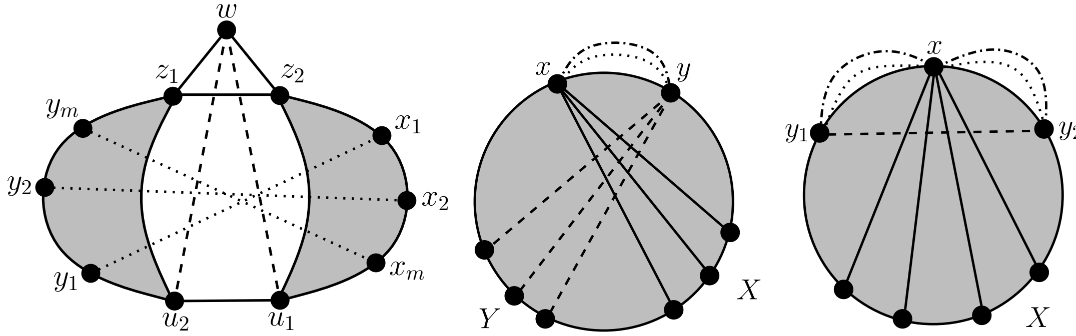

Figure 6: A projective planar biased graph with, from left to right, a special vertex, a special pair and a special triple.

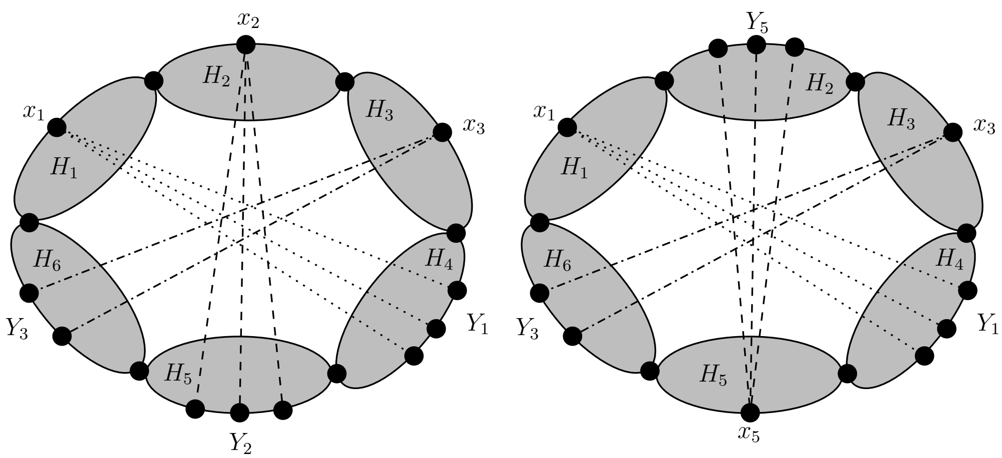

Figure 7: Tricoloured biased graphs.

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