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Cameron, Kathie; Thomassen, Carsten

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Notes

Cycles containing all the odd-degree vertices

Kathie Cameron [a,1], Carsten Thomassen [b,2]

[a] *Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada, N2L 3C5*
[b] *Department of Applied Mathematics and Computer Science, Technical University of Denmark, DK-2800 Lyngby, Denmark*

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ABSTRACT

The number of cycles in a graph containing any fixed edge and also containing all vertices of odd degree is odd if and only if all vertices have even degree. If all vertices have even degree this is a theorem of Shunichi Toida. If all vertices have odd degree it is Andrew Thomason's extension of Smith's theorem.

1. Introduction

All graphs in this paper are finite, and may contain loops and parallel edges. A *walk* is a sequence $v_1, e_1, v_2, ..., v_k, e_k, v_{k+1}$ of vertices $v_i, 1 \leq i \leq k+1$ and distinct edges $e_i = v_i v_{i+1}, 1 \leq i \leq k$ and is sometimes called a $v_1 - v_{k+1}$-*walk*; k is the *length* of the walk; if $v_1 = v_{k+1}$, then the walk is called a *closed walk*. We sometimes refer to a walk by listing its vertices. A *path* is a walk where the vertices are distinct. A *cycle* is a walk where the first and last vertex are the same and all other vertices are distinct. A cycle of length 1 is called a *loop*. A loop contributes 2 to the degree of its vertex. Two non-loop

E-mail addresses: kcameron@wlu.ca (K. Cameron), ctho@dtu.dk (C. Thomassen).
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edges with the same end-vertices are called *parallel edges*. A path or cycle in a graph G is called *hamiltonian* if it contains each vertex of G. A graph is *eulerian* if every vertex has even degree. For definitions not stated, we follow [2].

Our main result is an extension and unification of the following two theorems:

Theorem 1.1. (Shunichi Toida, [10]) *Let G be an eulerian graph. Then for any edge e, the number of cycles of G containing e is odd.*

Theorem 1.2. (Andrew Thomason, [8]) *Let G be a graph with at least 3 vertices and let $e = xy$ be an edge of G. Assume that all vertices except possibly x and y have odd degree. Then the number of hamiltonian cycles of G containing e is even.*

These theorems say that for a given edge e in a graph G, the number of cycles containing e and all the odd-degree vertices is even if all vertices of G have odd degree and is odd if all vertices of G have even degree. We show that the parity of the number of cycles containing e and all the odd-degree vertices is even as soon as G has an odd-degree vertex:

Theorem 1.3. *Let G be a graph and let $e = xy$ be an edge of G. The number of cycles of G containing e and all the odd-degree vertices is odd if and only if G is eulerian.*

Our proof uses Thomason's Theorem but not Toida's Theorem. It provides an alternative proof of Toida's Theorem.

The first result in this area was Smith's Theorem (Tutte [11]) which is the instance of Thomason's Theorem when all vertices have degree 3. The second author [9] recently proved a partial generalization of Thomason's Theorem; he showed that in a graph with an odd-degree vertex and in which no two even-degree vertices are adjacent, if there is one cycle containing all the odd-degree vertices, then there is another. The first author [3] extended this to include Thomason's Theorem, proving that in a graph with an odd-degree vertex and in which no two even-degree vertices are adjacent, for any edge e, the number of cycles containing e and all the odd-degree vertices is even (also see [5]). In both cases a variation of Thomason's lollipop proof was used. The purpose of this note is give a simple inductive proof of a further extension without refining the lollipop method in the proof but instead using the lollipop theorem as the basis of the induction.

2. Proof of the main result and some consequences

Let u be a vertex incident with no loop and with positive even degree in a graph G. A *lifting* of G with respect to u, or more briefly, a *lifting of* u, is obtained from G by partitioning the edges meeting u into pairs, removing vertex u, and replacing each pair of the partition by a single edge joining the end-vertices different from u (so uv and uw are replaced by vw). If u has degree q in G, then there are $(q-1)!! = (q-1)(q-3)\dots(3)(1)$

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liftings of G with respect to u. Where uv and uw are edges incident with u in G, the edge vw (corresponding to uv, uw) appears in $(q-3)!! = (q-3)(q-5)\dots(3)(1)$ liftings of G with respect to u. A cycle $C = v, u, w \dots, v$ of G containing vertex u and edges uv and uw corresponds to the cycle $C_{vw} = v, w, \cdots, v$ in each of the liftings containing vw. We will call C_{vw} an *image* of C. A cycle C of G which does not contain vertex u appears in every lifting of G with respect to u; we will call each of the copies of C an *image* of C.

For a specified edge e in a graph G, a cycle containing e and all the odd-degree vertices of G will be called a *good cycle*.

Proof of Theorem 1.3. Let $G = (V, E)$ be a graph and let $e = xy$ be an edge of G. The theorem is easy to verify if e is a loop, so assume e is not a loop.

Our proof is by induction on the number of even-degree vertices in $V \setminus \{x, y\}$.

If $V \setminus \{x, y\} = \emptyset$, then G consists of vertices x and y, edge $e = xy$, possibly several edges parallel to e and possibly some loops. The parity of the degrees of x and y are the same, and clearly, if this is even, then e is in an odd number of cycles and if it is odd, then e is in an even number of cycles; in either case, the cycles contain all the odd-degree vertices.

Now we can assume that $V \setminus \{x, y\} \neq \emptyset$. If no vertices of $V \setminus \{x, y\}$ have even degree, the result follows by Andrew Thomasson's Theorem 1.2.

Let u be an even-degree vertex in $V \setminus \{x, y\}$. We delete all loops incident with u and let $\mathcal{H}$ be the set of liftings of G with respect to u. Each graph $H \in \mathcal{H}$ has one fewer even-degree vertex than G and contains all the odd-degree vertices of G. By induction, the number of good cycles in each $H \in \mathcal{H}$ is even if H has an odd-degree vertex and is odd otherwise. Also, $H \in \mathcal{H}$ has an odd-degree vertex if and only if G does.

Each good cycle of G containing u has an image in an odd number of the liftings, and the image is a good cycle. Each good cycle of G not containing u is a good cycle in each of the liftings, and the number of liftings is odd. Thus each good cycle of G has an odd number of images in $\mathcal{H}$.

Note that not all good cycles in $H \in \mathcal{H}$ are images of good cycles in G; it is possible that a good cycle in H is the image of a closed walk in G which repeats the lifted vertex u. We claim that the total number of such cycles in graphs of $\mathcal{H}$ is even.

Consider therefore a non-cycle walk W in G that has an image which is a good cycle $C(W)$ in H. The walk W is of the form $...x, y, ..., v, u, s, ..., w, u, t, ...$ (with no occurrence of u between $y, ..., v$ and $s, ..., w$). Then also $C(W')$ is a good cycle in some $H' \in \mathcal{H}$ where W' is the walk $..., x, y, ..., v, u, w, ..., s, u, t, ...$ Thus the good cycles in graphs of $\mathcal{H}$ having non-cycle walks of G as pre-image can be paired.

A good cycle in a graph of $\mathcal{H}$ is either an image of a good cycle of G or an image of a closed non-cycle walk of G. Since the latter images can be paired, the parity of the total number of good cycles in graphs of $\mathcal{H}$ and the parity of the total number of images in $\mathcal{H}$ of good cycles of G are the same.

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As mentioned above, each good cycle of G has an odd number of images in graphs of $\mathcal{H}$. Thus the parity of the number of good cycles of G and the parity of the total number of good cycles in graphs of $\mathcal{H}$ are the same.

If G contains an odd-degree vertex, then so does each $H \in \mathcal{H}$, and by induction, the number of good cycles in each $H \in \mathcal{H}$ is even, so the total number of good cycles in graphs of $\mathcal{H}$ is even, and so the total number of good cycles of G is even.

If G does not contain an odd-degree vertex, then neither does any $H \in \mathcal{H}$, and by induction, the number of good cycles in each $H \in \mathcal{H}$ is odd, and then since $|\mathcal{H}|$ is odd, the total number of good cycles in graphs of $\mathcal{H}$ is odd, and so the total number of good cycles of G is odd. $\square$

We mention two consequences of Theorem 1.3. The equivalence of (i) and (iii) below is due to Herbert Fleischner, Corollary 5 in [6]. Shunichi Toida [10] proved that (iii) implies (ii) and Terry McKee [7] proved that (ii) implies (iii).

Theorem 2.1. *Let G be a graph. The following three statements are equivalent.*

(i) For any two distinct vertices x, y, the number of paths between x and y is even.

(ii) For any two distinct adjacent vertices x, y, the number of paths between x and y is even.

(iii) G is eulerian.

Proof of Theorem 2.1. To see that (iii) implies (i), just add an edge between x and y and use Theorem 1.3. Clearly, (i) implies (ii). To prove that (ii) implies (iii), let v be any vertex of degree k say. Let c be the number cycles of length > 2 through v. Then G has $2c$ paths of length ≥ 2 from v to one of its neighbors, and k paths of length 1 from v to one of its neighbors. Hence $k + 2c$ is even, and so k is even. $\square$

We proved that (iii) implies (i) by adding an edge between x and y and using Theorem 1.3. We could equally well add two edges between x and y and use Theorem 1.1. In the next result, however, we use Theorem 1.3 in its full strength.

Theorem 2.2. *Let G be a non-eulerian graph having a cycle C which contains all vertices of odd degree. Then G has at least three cycles containing all vertices of odd degree.*

Proof of Theorem 2.2. The proof is a repetition of the well-known proof that a cubic hamiltonian graph has at least three hamiltonian cycles [11]. Let e be any edge of C. By Theorem 1.3, G has a cycle C' distinct from C such that C' contains e and all vertices of odd degree. Let e' be an edge in C' but not in C. Then G has a cycle C'' distinct from C' and containing e' and all vertices of odd degree. As C'' contains e', C'' is also distinct from C. $\square$

Our method also gives short proofs of two results below. The first is due to Adrian Bondy and Faye Halberstam [1]. An algorithmic proof was given in [4].

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Theorem 2.3. *[1] Let r be a vertex in an eulerian graph G, and let k be an integer ≥ 1. Then the number of paths of length k starting at r is even.*

The proof is a repetition of the proof of Theorem 1.3 with a minor modification: When we lift with respect to a vertex u, two edges uv and uw are not replaced by one edge vw but by a path of length 2 whose intermediate vertex, call it $[v, w]$, has degree 2 in the resulting graph. So the resulting graph has the same number of edges. We call this an *edge-preserving lifting.*

Proof of Theorem 2.3. Without loss of generality, we assume the eulerian graph G has no loops and is connected.

We do induction with respect to the number of vertices distinct from r of degree > 2. The basis of the induction is when r is the only vertex of degree > 2, that is, the graph is a collection of cycles having r in common. Then the number of paths of length k starting at r is twice the number of cycles of length at least $k + 1$.

So now assume G has a vertex u of degree > 2 different from r. Let $\mathcal{H}$ be the set of edge-preserving liftings of G with respect to u. Each graph $H \in \mathcal{H}$ has one fewer vertex of degree > 2 than G. By induction, for each $H \in \mathcal{H}$, the number of paths in H of length k starting at r is even.

Each path of length k in G starting with r has an image in an odd number of liftings: A path P of length k in G not containing u appears in each of the liftings; we call each of these paths an image of P. Let P be a path of length k in G starting with r and containing u; if u is preceded by t and followed by v in P, path P corresponds to a path in each lifting in which t is paired with v, namely the path with u replaced by vertex $[t, v]$; if u is the last vertex of P and t is the vertex preceding u in P, then P corresponds to a path in each lifting, namely the path where u is replaced by the vertex $[t, v]$, where v is the vertex t is paired with in the lifting. We use the term images of P for the paths corresponding to P in the liftings. Since the total number of liftings is odd and the total number of liftings in which neighbors t and v of u are paired is odd, it follows that each path of length k in G starting with r has an image in an odd number of liftings.

The number of paths of length k starting at r in graphs of $\mathcal{H}$ with non-path preimage is even: Similar to the proof of Theorem 1.3, not all paths of length k starting at r in $H \in \mathcal{H}$ are images of paths of length k starting at r in G; it is possible that a path of length k starting at r in H is the image of a walk in G which repeats the lifted vertex u. An argument similar to that used before establishes that the paths of length k starting at r in graphs in $\mathcal{H}$ with non-path preimage can be paired, and thus the number of them is even.

A path of length k starting at r in a graph of $\mathcal{H}$ is either an image of a such a path in G or an image of a non-path walk of G. Since the number of the latter type of image is even, the parity of the total number of paths of length k starting at r in graphs of $\mathcal{H}$ and the parity of the total number of images in graphs of $\mathcal{H}$ of paths of G of length k starting at r are the same. Since each path of length k starting at r in G has an odd number of images in graphs of $\mathcal{H}$, it follows that the parity of the number of paths in G of length k starting at r and the parity of the total number of paths of length k starting at r in graphs of $\mathcal{H}$ are the same. Since for each $H \in \mathcal{H}$, the number of paths in H of length k starting at r is even, the total number of paths of length k starting at r in graphs of $\mathcal{H}$ is even, and thus the number of paths in G of length k starting at r is even. □

Jack Edmonds and the first author [4] gave an algorithmic proof of the following theorem which extends a result in [1].

Theorem 2.4. *[4] For any graph G and vertex r in G such that the degree of each vertex except possibly r is odd, and for each integer $k \geq 2$, each subset W of k vertices of G including r is in an even number of length k paths starting at r with the rest of W as the interior vertices of the path.*

To keep the paper self-contained we point out how to derive Theorem 2.4 from Theorem 1.2: If $W = V(G)$, the number of paths of length k is 0 and hence even. Otherwise, let G' be the graph obtained by identifying all vertices of $V(G) \setminus W$ into a vertex y and adding an edge ry. Now we apply Theorem 1.2 to G' with r playing the role of x.

Theorem 2.5. *Let r be a vertex in a graph G, and let W be a set of at least 2 vertices including r such that all vertices in W (except possibly r) have odd degree in G. Then the number of paths starting at r, with interior vertices the rest of W and possibly some even-degree vertices, and ending at a vertex of odd degree not in W is even.*

The proof is a repetition of the proof of Theorem 1.3. A path starting at r and ending at a vertex of odd degree not in W and whose interior vertices are the rest of W and possibly some even-degree vertices will be called (r, W)-*good*.

Proof of Theorem 2.5. Let r be a vertex in graph G. Without loss of generality, we assume that G has no loops. Let W be a set of at least 2 vertices including r such that all vertices in W (except possibly r) have odd degree in G.

We do induction with respect to the number of vertices of even degree distinct from r. The basis of the induction is when r is the only vertex that may have even degree. This follows from Theorem 2.4 which we have shown above follows from Theorem 1.2.

So now assume G has a vertex u of even degree different from r. Let $\mathcal{H}$ be the set of liftings of G with respect to u. Each graph $H \in \mathcal{H}$ has one fewer even-degree vertex than G. By induction, for each graph $H \in \mathcal{H}$, the number of (r, W)-good paths is even.

Each (r, W)-good path in G has an image in an odd number of the liftings: A path P in G not containing u appears in each of the liftings; we call each of these paths an image of P. A path P in G containing u as an interior vertex where u is preceded by t and followed by v corresponds to a path in each lifting in which t is paired with v; we use the term images of P for the paths corresponding to P in the liftings. Since the total number of liftings is odd and the total number of liftings in which neighbors t and v of

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u are paired is odd, it follows that each (r, W)-good path in G has an image in an odd number of liftings.

The number of (r, W)-good paths in graphs of $\mathcal{H}$ with non-path preimage is even: Similar to the proof of Theorem 1.3, not all (r, W)-good paths in $H \in \mathcal{H}$ are images of paths in G; it is possible that a path in H is the image of a walk in G which repeats the lifted vertex u. An argument similar to that used before establishes that (r, W)-good paths in graphs of $\mathcal{H}$ can be paired, and thus the number of them is even.

A (r, W)-good path in a graph of $\mathcal{H}$ is either an image of a (r, W)-good path in G or the image of a non-path walk in G. Since number of (r, W)-good paths in graphs of $\mathcal{H}$ with non-path preimage is even, it follows that the parity of the total number of (r, W)-good paths in graphs of $\mathcal{H}$ and the parity the of total number of images in graphs of $\mathcal{H}$ of (r, W)-good paths are the same. Since each (r, W)-good path in G has an odd number of images in graphs of $\mathcal{H}$, the parity of the number of (r, W)-good paths in G and the parity of the total number of (r, W)-good paths in graphs of $\mathcal{H}$ are the same. For each $H \in \mathcal{H}$, the number of (r, W)-good paths in H is even. Thus the total number of (r, W)-good paths in graphs of $\mathcal{H}$ is even, and thus the number of (r, W)-good paths in G is even. $\square$

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