The Alon-Tarsi number of a planar graph minus a matching

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November 30, 2018

Abstract

This paper proves that every planar graph G contains a matching M such that the Alon-Tarsi number of $G - M$ is at most 4. As a consequence, $G - M$ is 4-paintable, and hence G itself is 1-defective 4-paintable. This improves a result of Cushing and Kierstead [Planar Graphs are 1-relaxed, 4-choosable, *European Journal of Combinatorics* 31(2010),1385-1397], who proved that every planar graph is 1-defective 4-choosable.

Keywords: planar graph; list colouring; on-line list colouring; Alon-Tarsi number.

1 Introduction

In this paper we study the Alon-Tarsi number of special subgraphs of planar graphs, which is motivated by list coloring problems for planar graphs. A *k-list assignment* of a graph G is a mapping L which assigns to each vertex v of G a set $L(v)$ of k permissible colors. Given a k-list assignment L of G, an *L-colouring* of G is a mapping ϕ which assigns to each vertex v a colour $\phi(v) \in L(v)$ such that $\phi(u) \neq \phi(v)$ for every edge $e = uv$ of G. A graph G is *k-choosable* if G has an L-colouring for every k-list assignment L. The *choice number* of a graph G is defined as

$$\mathrm{ch}(G) = \min\{k : G \text{ is } k\text{-choosable}\}.$$

Thomassen [13] proved that every planar graph G is 5-choosable. This bound is best possible, as proved by Voigt [14], who constructed the first non-4-choosable planar graph. Other examples of planar graphs with $\mathrm{ch}(G) = 5$ can be found in [3, 11, 18].

A natural question is "how far" a planar graph can be from being 4-choosable? One way to measure such distance is to consider defective list colouring, defined as follows. A

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d-defective colouring of a graph G is a colouring of the vertices of G such that each colour class induces a subgraph of maximum degree at most d. Thus, a 0-defective colouring of G is simply a proper colouring of G, while in a 1-defective colouring a matching is allowed as a set of non-properly coloured edges.

Defective colouring of graphs was first studied by Cowen, Cowen and Woodall in [4]. They proved that every outerplanar graph is 2-defective 2-colourable, and that every planar graph is 2-defective 3-colourable. They also found examples of an outerplanar graph that is not 1-defective 2-colourable, a planar graph that is not 1-defective 3-colourable, and for every $d \geq 2$, a planar graph that is not d-defective 2-colourable.

In a natural analogy to list colouring we may define defective list colouring of graphs. Given a k-list assignment L of G, a *d-defective L-colouring* of G is a d-defective colouring c of G with $c(v) \in L(v)$ for every vertex v of G. A graph G is *d-defective k-choosable* if for any k-list assignment L of G, there exists a d-defective L-colouring of G. Clearly, every d-defective k-choosable graph is d-defective k-colourable, however, the converse is not true. Nevertheless, the above mentioned results on defective colouring of planar graphs can be extended to defective list colouring. Eaton and Hull [6], and Škrekovski [15] proved independently that every planar graph is 2-defective 3-choosable, and every outerplanar graph is 2-defective 2-choosable. They both asked the natural question - whether every planar graph is 1-defective 4-choosable. This problem is much more difficult. Only one decade later Cushing and Kierstead [5] answered this question in the affirmative, and their proof is rather complicated.

Another way to measure distance of a graph G from being k-choosable is to consider the maximum degree of a subgraph that must be removed from G in order to get a k-choosable graph. For example, we can ask the following question.

Question 1 *Is it true that every planar graph G has a matching M such that $G - M$ is 4-choosable?*

A positive answer to this question implies a stronger property than 1-defective 4-choosability of planar graphs. Indeed, we can specify a matching of non-properly coloured edges in advance, independently of the list assignment. Cushing and Kierstead [5] proved that for any 4-list assignment L of G, there is a matching M of G such that $G - M$ is L-colourable. The matching M constructed in the proof depends on the list assignment L.

Another way to extend the result of Cushing and Kierstead is to consider *on-line* version of list colouring of graphs, defined through the following two person game [12, 17]. A *d-defective k-painting game* on a graph G is played by two players: Lister and Painter. Initially, each vertex is uncoloured and has k tokens. In each round, Lister marks a chosen set A of uncoloured vertices and removes one token from each marked vertex. In response, Painter colours vertices in a subset X of A which induce a subgraph $G[X]$ of maximum degree at most d. Lister wins if at the end of some round there is an uncoloured vertex with no more tokens left. Otherwise, after some round, all vertices are coloured, and then Painter wins the game. We say that G is *d-defective k-paintable* if Painter has a winning strategy in this game. A 0-defective k-painting game is simply called a *k-painting game*, and a 0-defective k-paintable graph is simply called *k-paintable*. The *paint number* of G is defined as

$$\chi_{\mathrm{P}}(G) = \min\{k : G \text{ is } k\text{-paintable}\}.$$

It is not hard to see that every d-defective k-paintable graph is d-defective k-choosable. Indeed, let L be some k-list assignment of G with colours in the set $[n]$. Suppose that Lister is playing in the following way. In the i-th round, for $i \in [n]$, Lister marks the set $A_i = \{v : i \in L(v), v \notin X_1 \cup \ldots \cup X_{i-1}\}$, where X_j is the set of vertices coloured by Painter in the j-th round. But Painter has a winning strategy, so, he will eventually obtain a d-defective L-colouring of G. The converse is however not true. Though all planar graphs are 2-defective 3-choosable, an example of non-2-defective 3-paintable planar graph has been constructed in [7].

On the other hand, it is known that every planar graph is 5-paintable [12], 3-defective 3-paintable [7], and 2-defective 4-paintable [8]. For defective paintability of the family of planar graphs, Question 2 below is the only question remained open.

Question 2 *Is it true that every planar graph is* 1*-defective* 4*-paintable?*

More ambitiously, in analogy to Question 1, we may ask a similar question for paintability of planar graphs.

Question 3 *Is it true that every planar graph G has a matching M such that $G - M$ is* 4*-paintable?*

Our main result in this paper implies a positive answer to all of the stated questions. To formulate the main result, we need the following definitions. We associate to each vertex v of G a variable x_v. The *graph polynomial* $P_G(\mathtt{x})$ of G is defined as

$$P_G(\mathtt{x}) = \prod_{uv \in E(G), u<v} (x_v - x_u),$$

where $\mathtt{x} = \{x_v : v \in V(G)\}$ denotes the sequence of variables ordered accordingly to some fixed linear ordering " $<$ " of the vertices of G. It is easy to see that a mapping $\phi : V \to \mathbb{R}$ is a proper colouring of G if and only if $P_G(\phi) \neq 0$, where $P_G(\phi)$ means to evaluate the polynomial at $x_v = \phi(v)$ for $v \in V(G)$. Thus to find a proper colouring of G is equivalent to find an assignment of $\mathtt{x}$ so that the polynomial evaluated at this assignment is non-zero.

Assume now that $P(\mathtt{x})$ is any real polynomial with variable set X. Let η be a mapping which assigns to each variable x a non-negative integer $\eta(x)$. We denote by $\mathtt{x}^\eta$ the monomial $\prod_{x \in X} x^{\eta(x)}$ determined by mapping η, which we call then the *exponent* of that monomial. Let $c_{P,\eta}$ denote the coefficient of $\mathtt{x}^\eta$ in the expansion of $P(\mathtt{x})$ into the sum of monomials. The celebrated Combinatorial Nullstellensatz of Alon [1] asserts that if

$\sum_{x \in X} \eta(x) = \deg P(\mathtt{x})$ and $c_{P,\eta} \neq 0$, then for arbitrary sets A_x assigned to variables $x \in X$, each consisting of $\eta(x) + 1$ real numbers, there exists a mapping $\phi : X \to \mathbb{R}$ such that $\phi(x) \in A_x$ for each $x \in X$ and $P(\phi) \neq 0$.

Notice that a graph polynomial P_G is *homogenous*, which means that the exponents of each non-vanishing monomial sum up to the same value, which is equal the number of edges of G. Hence, condition $\sum_{x \in X} \eta(x) = \deg P_G(\mathtt{x})$ is satisfied by every non-vanishing monomial in P_G. In particular, Combinatorial Nullstellensatz implies that if $c_{P_G,\eta} \neq 0$ and $\eta(x_v) < k$ for all $v \in V$, then G is k-choosable. This result was proved earlier by Alon and Tarsi [2], who applied it, for instance, to demonstrate that planar bipartite graphs are 3-choosable. It was then strengthened by Schauz [12], who showed that under the same assumptions, a graph G is also k-paintable. Motivated by the above relations between list colourings and graph polynomials, Jensen and Toft [10] defined the *Alon-Tarsi number* $\mathrm{AT}(G)$ of a graph G as

$$\mathrm{AT}(G) = \min\{k : c_{P_G,\eta} \neq 0 \text{ for some exponent } \eta \text{ with } \eta(x_v) < k \text{ for all } v \in V(G)\}.$$

As observed in [9], $\mathrm{AT}(G)$ has some distinct features, and it is of interest to study $\mathrm{AT}(G)$ as a separate graph invariant. Summarizing, for any graph G, we have

$$\mathrm{ch}(G) \leq \chi_{\mathrm{P}}(G) \leq \mathrm{AT}(G).$$

The gaps between these three parameters can be arbitrarily large. However, upper bounds for the choice number of many natural classes of graphs are also upper bounds for their Alon-Tarsi number. For example, as a strengthening of the result of Thomassen and the result of Schauz, it was shown in [16] that every planar graph G satisfies $\mathrm{AT}(G) \leq 5$.

In this paper we prove that every planar graph G contains a matching M such that $\mathrm{AT}(G - M) \leq 4$, which implies a positive answer to all three questions we formulated above.

2 The main result

The main result of this paper is the following theorem.

Theorem 4 *Every planar graph G contains a matching M such that* $\mathrm{AT}(G - M) \leq 4$.

Before proceeding to the proof we need to fix some notation and terminology. For simplicity, we write $c_{G,\eta}$ for $c_{P_G,\eta}$. We will also say that $\mathtt{x}^\eta$ is a monomial of a graph G while formally it is a monomial in the graph polynomial P_G, and the exponent η assigns to each vertex v a non-negative integer while formally the integer is assigned to x_v. A monomial $\mathtt{x}^\eta$ of G is *non-vanishing* if $c_{G,\eta} \neq 0$. By $d_H(v)$ we denote the degree of a vertex v in a graph H. We will need the following definitions.

Definition 5 *Assume G is a plane graph, $e = v_1v_2$ is a boundary edge of G, and M is a matching in G. A monomial $\mathtt{x}^\eta$ of $G - e - M$ is* nice *for (G, e, M) if the following conditions hold:*

1. $\mathtt{x}^\eta$ *is non-vanishing.*

2. $\eta(v_1) = \eta(v_2) = 0$.

3. $\eta(v) \le 2 - d_M(v)$ *for every other boundary vertex* v.

4. $\eta(v) \le 3$ *for each interior vertex* v.

Notice that $d_M(v) = 1$ if v is covered by M, and $d_M(v) = 0$ otherwise.

Definition 6 *Assume G is a plane graph and $e = v_1v_2$ is a boundary edge of G. A matching M of G is* valid *for (G, e) if none of v_1 or v_2 is covered by M.*

If $\mathtt{x}^\eta$ is a nice monomial for (G, e, M), then let $\eta'(x) = \eta(x)$ except that $\eta'(x_{v_1}) = 1$. As $P_G(\mathtt{x}) = (x_{v_1} - x_{v_2})P_{G-e}(\mathtt{x})$ and $\eta'(v_2) = 0$, we know that $c_{G,\eta'} = c_{G-e,\eta} \ne 0$. Note that $\eta'(x_v) \le 3$ for each vertex v. Thus Theorem 4 follows from Theorem 7 below.

Theorem 7 *Assume G is a plane graph and $e = v_1v_2$ is a boundary edge of G. Then (G, e) has a valid matching M such that there exists a nice monomial $\mathtt{x}^\eta$ for (G, e, M).*

A variable x is *dummy* in $P(\mathtt{x})$ if x does not really occur in $P(\mathtt{x})$, or equivalently, $\eta(x) = 0$ for each non-vanishing monomial $\mathtt{x}^\eta$ in the expansion of P. We shall frequently need to consider the summation and the product of polynomials. By introducing dummy variables, we assume the involved polynomials in the sum or the product have the same set of variables. For example, we may view x_2^2 be the same as $x_1^0x_2^2x_3^0\dots x_n^0$, that is, $x_2^2 = \mathtt{x}^\eta$, where the variable set is $X = \{x_1, x_2, \dots, x_n\}$, $\eta(x_2) = 2$, and $\eta(x_i) = 0$ for $i \ne 2$. We denote by X the set of variables for polynomials in concern. For two mappings η_1, η_2, we write $\eta_1 \le \eta_2$ if $\eta_1(x) \le \eta_2(x)$ for all $x \in X$, and $\eta = \eta_2 - \eta_1$ (respectively, $\eta = \eta_2 + \eta_1$) means that $\eta(x) = \eta_2(x) - \eta_1(x)$ (respectively, $\eta(x) = \eta_2(x) + \eta_1(x)$) for all $x \in X$.

The following lemma just collects some simple observations needed for future reference. Its proof amounts to routine checking and is therefore omitted.

Lemma 8 *The following properties hold for real polynomials and their monomial coefficients.*

1. *If* $P(\mathtt{x}) = \alpha P_1(\mathtt{x}) + \beta P_2(\mathtt{x})$, *then* $c_{P,\eta} = \alpha c_{P_1,\eta} + \beta c_{P_2,\eta}$.

2. *If* $P(\mathtt{x}) = \mathtt{x}^{\eta'} P_1(\mathtt{x})$, *then* $c_{P,\eta} = c_{P_1,\eta-\eta'}$.

3. *If* $P(\mathtt{x}) = \mathtt{x}^{\eta'} P_1(\mathtt{x})$ *and* $\eta' \not\le \eta$, *then* $c_{P,\eta} = 0$.

4. *If* $P(\mathtt{x}) = P_1(\mathtt{x})P_2(\mathtt{x})$ *and for any* η' *with* $c_{P_2,\eta'} \ne 0$, *there is a dummy variable* x *of* $P_1(\mathtt{x})$ *such that* $\eta'(x) \ne \eta(x)$, *then* $c_{P,\eta} = 0$.

5. *If* G *is a graph and* $c_{G,\eta} \ne 0$, *then* $\sum_{x \in X} \eta(x) = |E(G)|$.

Proof of Theorem 7 Assume the theorem is not true and G is a minimum counterexample. It is not difficult to check that G has at least 4 vertices and is connected. It is easy to see that if (G,e,M) has a nice monomial and G' is obtained from G by deleting an edge, then (G',e,M) also has a nice monomial. Thus we may assume that the boundary of G is simple cycle.

Case 1 (**G has a chord.**) Let $f = xy$ be a chord in G. Let G_1, G_2 be the two f-components, that is, G_1, G_2 are induced subgraphs of G, where $V(G) = V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) = \{x,y\}$), with $e \in G_1$. By the minimality of G, (G_1, e) has a valid matching M_1 such that (G_1, e, M_1) has a nice monomial $\mathtt{x}^{\eta_1}$, and (G_2, f) has a valid matching M_2 such that (G_2, f, M_2) has a nice monomial $\mathtt{x}^{\eta_2}$. Let $M = M_1 \cup M_2$. It is easy to see that M is a valid matching in (G,e). Let $\eta = \eta_1 + \eta_2$. We shall show that $\mathtt{x}^{\eta}$ is a nice monomial for (G,e,M). It is obvious that conditions (2)-(4) of a Definition 5 are satisfied by η. It remains to show that $c_{G-e-M,\eta} \neq 0$. It suffices to show that $c_{G-e-M,\eta} = c_{G_1-e-M_1,\eta_1} c_{G_2-f-M_2,\eta_2}$.

Assume $\mathtt{x}^{\eta_1'}$ is a non-vanishing monomial of $G_1 - e - M_1$ and $\mathtt{x}^{\eta_2'}$ is a non-vanishing monomial of $G_1 - f - M_2$. Let $\eta' = \eta_1' + \eta_2'$. We shall show that $\eta' = \eta$ only if $\eta_1' = \eta_1$ and $\eta_2' = \eta_2$. Assume $\eta' = \eta$. Since $\eta_1'(v) = 0$ for each vertex $v \in V(G_2) - \{x,y\}$, we have $\eta_2'(v) = \eta'(v) = \eta(v) = \eta_2(v)$ for each $v \in V(G_2) - \{x,y\}$. As $\sum_{v \in V(G_2)} \eta_2'(v) = \sum_{v \in V(G_2)} \eta_2(v)$, we conclude that $\eta_2'(x) = \eta_2'(y) = 0$. Hence $\eta_2' = \eta_2$. This implies that $\eta_1' = \eta - \eta_2 = \eta_1$. This completes the proof of Case 1.

Case 2 (**G has no chord.**) Assume G has no chord and let $B(G) = (v_1, v_2, \ldots, v_n)$ be the vertices of the boundary of G in a cyclic order. Let $G' = G - v_n$. By the minimality of G, G' has a valid matching M such that (G', e, M) has a nice monomial $\mathtt{x}^{\eta'}$.

Let $v_1, u_1, u_2, \ldots, u_k, v_{n-1}$ be the neighbours of v_n. Since G has no chord, each u_i is an interior vertex of G. Let

$$S(\mathtt{x}) = (x_{v_n} - x_{v_1})(x_{v_{n-1}} - x_{v_n})(x_{u_1} - x_{v_n}) \ldots (x_{u_k} - x_{v_n}).$$

Then $P_{G-e-M}(\mathtt{x}) = S(\mathtt{x}) P_{G'-e-M}(\mathtt{x})$.

Assume first that $n = 3$. Let then $\eta(x_v) = \eta'(x_v)$ for $v \notin \{u_1, u_2, \ldots, u_k\}$, $\eta(x_v) = \eta'(x_v) + 1$ for $v \in \{u_1, u_2, \ldots, u_k\}$, and $\eta(x_{v_n}) = 2$. Let $\eta''(x_{v_n}) = 2$, $\eta''(x_{u_i}) = 1$ for $i = 1, 2, \ldots, k$, and $\eta''(x) = 0$ for other x. Then we may write

$$S(\mathtt{x}) = -\mathtt{x}^{\eta''} + x_{v_1} A(\mathtt{x}) + x_{v_2} B(\mathtt{x}) + x_{v_n}^3 C(\mathtt{x})$$

for some polynimals $A(\mathtt{x})$, $B(\mathtt{x})$ and $C(\mathtt{x})$. Let

$$\begin{aligned} P_1(\mathtt{x}) &= x_{v_1} A(\mathtt{x}) P_{G'-e-M}(\mathtt{x}), \\ P_2(\mathtt{x}) &= x_{v_2} B(\mathtt{x}) P_{G'-e-M}(\mathtt{x}), \\ P_3(\mathtt{x}) &= x_{v_n}^3 C(\mathtt{x}) P_{G'-e-M}(\mathtt{x}). \end{aligned}$$

As $\eta(x_{v_1}) = \eta(x_{v_2}) = 0$ and $\eta(x_{v_n}) = 2$, it follows from (3) of Lemma 8 that $c_{P_1,\eta} = c_{P_2,\eta} = c_{P_3,\eta} = 0$. By (1) and (2) of Lemma 8, $c_{G-e,\eta} = -c_{G'-e,\eta'} \neq 0$. Hence η is nice for $(G, e.M)$.

Assume now that $n \geq 4$. We say a monomial $\mathtt{x}^{\tau}$ for $G' - e - M$ is *special* if

- $\tau(v_1) = \tau(v_2) = 0$.
- $\tau(v_{n-1}) \le 1 - d_M(v_{n-1})$.
- $\tau(v) \le 2 - d_M(v)$ for every other boundary vertex v, except that there may be at most one index $i \in \{1, 2, \ldots, k\}$ such that $\tau(u_i) = 3 - d_M(u_i)$.
- $\tau(v) \le 3$ for each interior vertex v.

Subcase 2(i) (There is a non-vanishing special monomial in $G' - e - M$.) Assume that $c_{G'-e-M,\tau} \neq 0$ for some special monomial $\mathtt{x}^\tau$ in $G' - e - M$. For $i \in \{1, 2, \ldots, k\}$, we say u_i is *saturated* if $\tau(u_i) = 3$. By the definition of special monomial, we know that there is at most one u_i that is saturated. Moreover, if u_i is saturated, then $d_M(u_i) = 0$. Let

$$M' = \begin{cases} M, & \text{if no } u_i \text{ is saturated,} \\ M \cup \{u_i v_n\}, & \text{if } u_i \text{ is saturated.} \end{cases}$$

It follows from the definition that M' is a valid matching in (G, e). Let

$$\eta(v) = \begin{cases} \tau(v), & \text{if } v \notin \{u_1, u_2, \ldots, u_k, v_{n-1}\} \text{ or } v = u_i \text{ is satuarated} \\ \tau(v) + 1, & \text{if } v \in \{u_1, u_2, \ldots, u_k, v_{n-1}\} \text{ is not saturated,} \\ 1, & \text{if } v = v_n. \end{cases}$$

Let τ' be a mapping defined as $\tau'(v_n) = \tau'(v_{n-1}) = 1$, $\tau'(u_j) = 1$ if u_j is not saturated, and $\tau'(v) = 0$ for other vertices v. Let

$$\tilde{S}(\mathtt{x}) = (x_{v_n} - x_{v_1})(x_{v_{n-1}} - x_{v_n}) \prod_{u_i \text{ is not saturated}} (x_{u_i} - x_{v_n}).$$

Then

$$P_{G-e-M',\eta} = \tilde{S}(\mathtt{x}) P_{G'-e-M,\tau}$$

and

$$\tilde{S}(\mathtt{x}) = \mathtt{x}^{\tau'} + x_{v_1} A(\mathtt{x}) + x_{v_n}^2 B(\mathtt{x})$$

for some polynomials $A(\mathtt{x})$ and $B(\mathtt{x})$.

Let $P(\mathtt{x}) = \mathtt{x}^{\tau'} P_{G'-e-M}(\mathtt{x})$, $P_1(\mathtt{x}) = x_{v_1} A(\mathtt{x}) P_{G'-e-M}(\mathtt{x})$, and $P_2(\mathtt{x}) = x_{v_n}^2 B(\mathtt{x}) P_{G'-e-M}(\mathtt{x})$. Then

$$P_{G-e-M'}(\mathtt{x}) = P(\mathtt{x}) + P_1(\mathtt{x}) + P_2(\mathtt{x}).$$

As $\eta(v_1) = 0$ and $\eta(v_n) = 1$, it follows from (3) of Lemma 8 that $c_{P_1,\eta} = c_{P_2,\eta} = 0$. By (1) and (2) of Observation 8, we have $c_{G-e-M',\eta} = c_{P,\eta} = c_{G'-e-M,\tau} \neq 0$. Hence $\mathtt{x}^\eta$ is a nice monomial for (G, e, M').

Subcase 2(ii) (There is no non-vanishing special monomial in $G' - e - M$.) We assume now that $c_{G'-e-M,\tau} = 0$ for every special monomial $\mathtt{x}^\tau$ of $G' - e - M$. Recall that $\mathtt{x}^{\eta'}$ is a nice monomial for (G', e, M). Let $\eta(x_v) = \eta'(x_v)$ for $v \notin \{u_1, u_2, \ldots, u_k\}$, $\eta(x_v) = \eta'(x_v) + 1$ for $v \in \{u_1, u_2, \ldots, u_k\}$, and $\eta(x_{v_n}) = 2$. We shall show that $\mathtt{x}^{\eta}$ is a nice monomial for $G - e - M$.

It is obvious that Conditions (2)-(4) of Definition 5 are satisfied by $\mathtt{x}^{\eta}$. It remains to show that $\mathtt{x}^{\eta}$ is non-vanishing. Note that $P_{G-e-M}(\mathtt{x}) = S(\mathtt{x}) P_{G'-e-M}(\mathtt{x})$. Let $\tau'(x_{v_n}) = 2$, $\tau'(x_{u_i}) = 1$ for $i = 1, 2, \ldots, k$, and $\tau'(x) = 0$ for other x. Then $\eta = \tau' + \eta'$ and

$$S(\mathtt{x}) = \mathtt{x}^{\tau'} - \sum_{i=1}^{k} x_{v_n}^2 x_{v_{n-1}} \prod_{j \neq i} x_{u_j} + x_{v_1} A(\mathtt{x}) + x_{v_n}^3 B(\mathtt{x})$$

for some polynimals $A(\mathtt{x})$ and $B(\mathtt{x})$. Let

$$\begin{aligned} P(\mathtt{x}) &= \mathtt{x}^{\tau'} P_{G'-e-M}(\mathtt{x}), \\ P_1(\mathtt{x}) &= \sum_{i=1}^{k} x_{v_n}^2 x_{v_{n-1}} \prod_{j \neq i} x_{u_j} P_{G'-e-M}(\mathtt{x}), \\ P_2(\mathtt{x}) &= x_{v_1} A(\mathtt{x}) P_{G'-e-M}(\mathtt{x}), \\ P_3(\mathtt{x}) &= x_{v_n}^3 B(\mathtt{x}) P_{G'-e-M}(\mathtt{x}). \end{aligned}$$

As $\eta(x_{v_1}) = 0$ and $\eta(x_{v_n}) = 2$, it follows from (3) of Lemma 8 that $c_{P_2,\eta} = c_{P_3,\eta} = 0$.

For $i = 1, 2, \ldots, k$, let

$$P_{1,i} = x_{v_n}^2 x_{v_{n-1}} \prod_{j \neq i} x_{u_j} P_{G'-e-M}(\mathtt{x})$$

and let

$$\tau_i(v) = \begin{cases} \eta'(v) - 1, & \text{if } v = v_{n-1}, \\ \eta'(v) + 1, & \text{if } v = u_i, \\ \eta'(v), & \text{otherwise.} \end{cases}$$

Then

$$c_{P_{1,i},\eta} = -c_{G'-e-M,\tau_i}$$

and hence

$$c_{P_1,\eta} = -\sum_{i=1}^{k} c_{G'-e-M,\tau_i}.$$

For each $i \in \{1, 2, \ldots, k\}$, $\mathtt{x}^{\tau_i}$ is a special monomial for $G' - e - M$. Hence $c_{G'-e-M,\tau_i} = 0$, and consequently, $c_{P_1,\eta} = 0$. As

$$P_{G-e-M}(\mathtt{x}) = P(\mathtt{x}) + P_1(\mathtt{x}) + P_2(\mathtt{x}) + P_3(\mathtt{x}),$$

by (1) and (2) of Lemma 8 we get $c_{G-e-M,\eta} = c_{P,\eta} = c_{G'-e-M,\eta'} \neq 0$. This finishes the proof of Case 2, and completes the proof of the theorem. ∎

3 Some remarks

Assume $f : V(G) \to \{1, 2, \ldots, \}$ is a function which assigns to each vertex v of G a positive integer. We say G is *f-choosable* if for any list assignment L with $|L(v)| = f(v)$ for every vertex v, G is L-colourable. The *f-painting game* is defined in the same way as the k-painting game, except that initially, instead of k tokens, each vertex v has $f(v)$ tokens. We say G is *f-paintable* if Painter has a winning strategy in the f-painting game on G. We say G *f-Alon-Tarsi*, or f-AT for short, if P_G has a non-vanishing monomial $\mathtt{x}^\eta$ with $\eta(v) < f(v)$ for each vertex v.

In the proof of Theorem 7, in Case 1, instead of adding the edge $u_i v_n$ to the matching M', we may increase the power of x_{u_i} by 1. Then the resulting monomial is non-vanishing in P_{G-e}. Thus a slight modification of the proof of Theorem 7 proves the following theorem, which is a strengthening of the result that every planar graph G has $\mathrm{AT}(G) \le 5$.

Theorem 9 *Assume G is a planar graph. Then G has a matching $M = \{(x_i, y_i) : i = 1, 2, \ldots, p\}$ (note that edges in M are oriented) such that G is f-AT, where $f : V(G) \to \{4, 5\}$ is defined as $f(x_i) = 5$ and $f(v) = 4$ for $v \in V - \{x_1, x_2, \ldots, x_p\}$. Consequently, G is f-paintable, and hence f-choosable.*

Note that $|V(G)| > 2p$. Thus we have the following corollary, which is the strengthening of the 5-choosability of planar graphs.

Corollary 10 *Every planar graph G has a subset X of vertices with $|X| < |V(G)|/2$ such that if L is a list assignment which assigns to each vertex in X five permissible colours and assigns to each other vertex four permissible colours, then G is L-colourable.*

A *signed graph* is a pair (G, σ), where G is a graph and σ is a *signature* of G which assigns to each edge e of G a sign $\sigma_e \in \{1, -1\}$. A *proper colouring* of (G, σ) is a mapping f which assigns to each vertex v an integer $f(v)$ so that for each edge $e = xy$, $f(x) \neq \sigma_e f(y)$. The chromatic number $\chi(G, \sigma)$ of (G, σ) is the minimum integer k such that for any set S of k integers, (G, σ) has a proper colouring using colours from S. The choice number of (G, σ) is the minimum integer k such that for every k-list assignment L of G, there is a proper L-colouring of (G, σ). The polynomial associated to (G, σ) is defined as

$$P_{G,\sigma}(\mathtt{x}) = \prod_{u \sim v, u < v} (x_v - \sigma(uv) x_u).$$

The Alon-Tarsi number of signed graphs is defined similarly. Then all the arguments in the proof of Theorem 7 works. Hence we have the following result.

Theorem 11 *If (G, σ) is a signed planar graph, then G has a matching M such that $AT(G - M, \sigma) \le 4$. Consequently, $(G - M, \sigma)$ is 4-choosable, and (G, σ) itself is 1-defective 4-choosable.*

Corollary 10 also works for signed planar graphs.

Corollary 12 *Every signed planar graph (G, σ) has a subset X of vertices with $|X| < |V(G)|/2$ such that if L is a list assignment which assigns to each vertex in X five permissible colours and asigns to each other vertex four permissible colours, then (G, σ) is L-colourable.*

Acknowledgement Jarosław Grytczuk would like to thank Zhejiang Normal University for great hospitality, and to all people from the Center for Discrete Mathematics for a wonderful atmosphere during his visit in Jinhua, where this research was carried out. His visit was supported by the 111 project of the Ministry of Education of China on "Graphs and Network Optimization".

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