

N-DETACHABLE PAIRS IN 3-CONNECTED MATROIDS II: LIFE IN X

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ABSTRACT. Let M be a 3-connected matroid, and let N be a 3-connected minor of M . A pair $\{x_1, x_2\} \subseteq E(M)$ is *N-detachable* if one of the matroids $M/x_1/x_2$ or $M \setminus x_1 \setminus x_2$ is both 3-connected and has an N -minor. This is the second in a series of three papers where we describe the structures that arise when it is not possible to find an N -detachable pair in M . In the first paper in the series, we showed that, under mild assumptions, either M has an N -detachable pair, M has one of three particular 3-separators that can appear in a matroid with no N -detachable pairs, or there is a 3-separating set X with certain strong structural properties. In this paper, we analyse matroids with such a structured set X , and prove that they have either an N -detachable pair, or one of five particular 3-separators that can appear in a matroid with no N -detachable pairs.

1. INTRODUCTION

Let M be a 3-connected matroid, and let N be a 3-connected minor of M . We say that a pair $\{x_1, x_2\} \subseteq E(M)$ is *N-detachable* if one of the matroids $M/x_1/x_2$ or $M \setminus x_1 \setminus x_2$ is both 3-connected and has an isomorphic copy of N as a minor. This is the second in a series of three papers where we describe the structures that arise when it is not possible to find an N -detachable pair in M .

Our setup is as follows. Let $|E(N)| \geq 4$. We say that a triangle or triad T of M is *N-grounded* if, for all distinct $a, b \in T$, none of $M/a/b$, $M/a \setminus b$, $M \setminus a/b$, and $M \setminus a \setminus b$ have an N -minor. In this paper, we assume that every triangle or triad of M is N -grounded (due to [3, Theorem 3.2]). By Seymour's Splitter Theorem [7] and duality, we may assume that there exists some $d \in E(M)$ such that $M \setminus d$ is 3-connected and has an N -minor. Let $d' \in E(M \setminus d)$ such that $M \setminus d \setminus d'$ has an N -minor. If $M \setminus d \setminus d'$ is 3-connected, then $\{d, d'\}$ is an N -detachable pair. So suppose $M \setminus d \setminus d'$ opens up a non-trivial 2-separation (Y, Z) . Since N is 3-connected, any N -minor lies primarily on one side of the 2-separation, so we may assume, up to

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swapping Y and Z , that $|Y \cap E(N)| \leq 1$. For now, we also assume that $|Y| \geq 4$.

In the first paper of the series [3, Theorem 7.4], we showed that there is a 3-separating subset X of Y with $|X| \geq 4$ such that either for every $x \in X$:

- (a) $M \setminus d \setminus x$ is 3-connected up to series classes,
- (b) $M \setminus d / x$ is 3-connected, and
- (c) $M \setminus d \setminus x$ and $M \setminus d / x$ have N -minors,

or $X \cup \{c, d\}$ is one of three particular 3-separators that can appear in a matroid with no N -detachable pairs for some $c \in \text{cl}^*(X \cup d)$. (We defer the definition of such particular 3-separators to Section 2.)

In this paper, we analyse this structured set X further, in the case where $X \cup \{c, d\}$ is not a particular 3-separator. In Section 4, we consider when the set X contains a triad; in this case we show that M has an N -detachable pair. In Section 5, we consider when X does not contain a triad; in this case, either M has an N -detachable pair, or $X \cup d$ is contained in a particular 3-separator that can appear in a matroid with no N -detachable pairs. Combining these results, we obtain our main result, Theorem 6.1, in the final section.

Subject to Theorem 6.1 and the results in [3], it remains to consider the case when for every $d' \in E(M \setminus d)$ such that $M \setminus d \setminus d'$ has an N -minor, the pair $\{d, d'\}$ is contained in a 4-element cocircuit; and to show that when M has a particular 3-separator P and no N -detachable pairs, there is at most one element of M that is not in $E(N) \cup P$. We analyse these cases in the third paper in the series.

We denote $\{1, 2, \dots, n\}$ by $[n]$.

2. A TAXONOMY OF PARTICULAR 3-SEPARATORS

Let M be a 3-connected matroid with ground set E . We say that a 4-element set $Q \subseteq E$ is a *quad* if it is both a circuit and a cocircuit of M .

We now define five 3-separating sets with specific structure, illustrated in Figure 1. We refer to any one of these as a particular 3-separator.

Definition 2.1. Let $P \subseteq E$ be an exactly 3-separating set of M . If there exists a partition $\{L_1, \dots, L_t\}$ of P with $t \geq 3$ such that

- (a) $|L_i| = 2$ for each $i \in [t]$, and
- (b) $L_i \cup L_j$ is a quad for all distinct $i, j \in [t]$,

then P is a *spike-like 3-separator* of M .

Definition 2.2. Let $P \subseteq E$ be a 6-element exactly 3-separating set of M . If there exists a labelling $\{s_1, s_2, t_1, t_2, u_1, u_2\}$ of P such that

- (a) $\{s_1, s_2, t_2, u_1\}$, $\{s_1, t_1, t_2, u_2\}$, and $\{s_2, t_1, u_1, u_2\}$ are the circuits of M contained in P ; and
- (b) $\{s_1, s_2, t_1, t_2\}$, $\{s_1, s_2, u_1, u_2\}$, and $\{t_1, t_2, u_1, u_2\}$ are the cocircuits of M contained in P ;

then P is a *skew-whiff 3-separator* of M .

Definition 2.3. Let $P \subseteq E$ be a 6-element exactly 3-separating set such that $P = Q \cup \{p_1, p_2\}$, and Q is a quad. If there exists a labelling $\{q_1, q_2, q_3, q_4\}$ of Q such that

- (a) $\{p_1, p_2, q_1, q_2\}$, $\{p_1, p_2, q_3, q_4\}$, and Q are the circuits of M contained in P , and
- (b) $\{p_1, p_2, q_1, q_3\}$, $\{p_1, p_2, q_2, q_4\}$, and Q are the cocircuits of M contained in P ,

then P is an *elongated-quad 3-separator* of M .

Definition 2.4. Let $P \subseteq E$ be an exactly 3-separating set such that $P = Q_1 \cup Q_2$ where Q_1 and Q_2 are disjoint quads of M . If there exist labellings $\{p_1, p_2, p_3, p_4\}$ of Q_1 and $\{q_1, q_2, q_3, q_4\}$ of Q_2 such that

- (a) $\{p_1, p_2, q_1, q_2\}$, $\{p_1, p_2, q_3, q_4\}$, $\{p_3, p_4, q_1, q_2\}$, $\{p_3, p_4, q_3, q_4\}$, Q_1 , and Q_2 are the circuits of M contained in P , and
- (b) $\{p_1, p_3, q_1, q_3\}$, $\{p_1, p_3, q_2, q_4\}$, $\{p_2, p_4, q_1, q_3\}$, $\{p_2, p_4, q_2, q_4\}$, Q_1 , and Q_2 are the cocircuits of M contained in P ,

then P is a *double-quad 3-separator* with *associated partition* $\{Q_1, Q_2\}$.

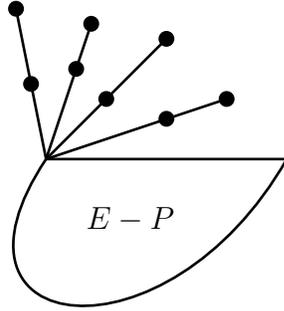
These four particular 3-separators are self-dual in the following sense: if P is a spike-like 3-separator, elongated-quad 3-separator, double-quad 3-separator, or skew-whiff 3-separator of M , then P is also a spike-like 3-separator, elongated-quad 3-separator, double-quad 3-separator, or skew-whiff 3-separator of M^* , respectively. The same is not true of the next particular 3-separator.

Definition 2.5. Let $P \subseteq E$ be an exactly 3-separating set with $P = \{p_1, p_2, q_1, q_2, s_1, s_2\}$, and let $Y = E - P$. Suppose that

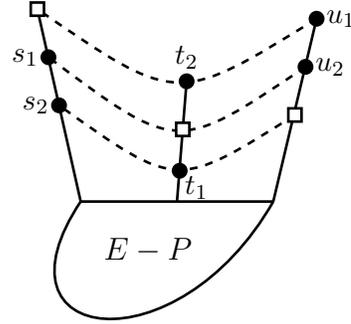
- (a) $\{p_1, p_2, s_1, s_2\}$, $\{q_1, q_2, s_1, s_2\}$, and $\{p_1, p_2, q_1, q_2\}$ are the circuits of M contained in P ; and
- (b) $\{p_1, q_1, s_1, s_2\}$, $\{p_2, q_2, s_1, s_2\}$, $\{p_1, p_2, q_1, q_2, s_1\}$, and $\{p_1, p_2, q_1, q_2, s_2\}$ are the cocircuits of M contained in P .

Then P is a *twisted cube-like 3-separator* of M .

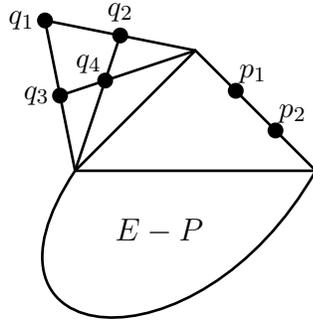
Each of these five particular 3-separators can appear in a 3-connected matroid M with a 3-connected minor N such that $E(M) - E(N) \subseteq P$ and M has no N -detachable pairs. (For a spike-like 3-separator, this is shown in [3, Section 2]. For an elongated-quad 3-separator, a skew-whiff 3-separator, or a twisted cube-like 3-separator, see the discussion in [3, Section 5]; the double-quad 3-separator is similar.) For all except the twisted cube-like 3-separator, the intrinsic problem is connectivity; that is, for such a 3-separator P in a matroid M , there is no pair of elements contained in P for which M remains 3-connected after deleting or contracting the pair. On the other hand, a twisted cube-like 3-separator P can appear in a matroid with no N -detachable pairs where P contains a pair whose deletion preserves 3-connectivity (the deletion of the pair destroys the N -minor).



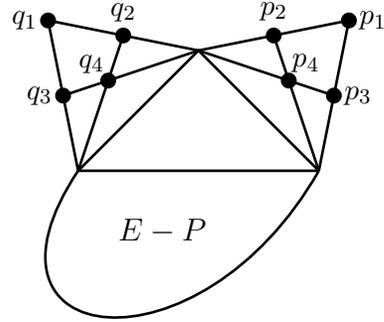
(a) An example of a spike-like 3-separator.



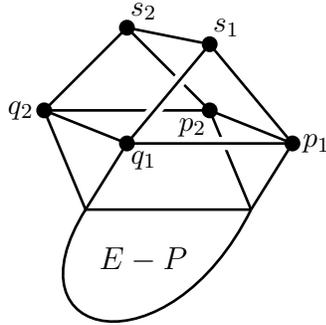
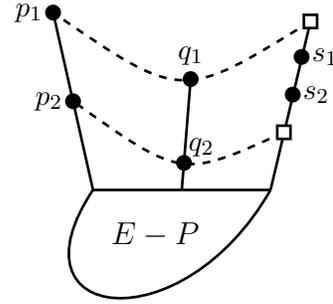
(b) A skew-whiff 3-separator.



(c) An elongated-quad 3-separator.



(d) A double-quad 3-separator.

(e) A twisted cube-like 3-separator of M .(f) A twisted cube-like 3-separator of M^* .FIGURE 1. Particular 3-separators that can appear in a matroid with no N -detachable pairs.

3. PRELIMINARIES

The notation and terminology in the paper follow Oxley [5]. For a set X and element e , we write $X \cup e$ instead of $X \cup \{e\}$, and $X - e$ instead of $X - \{e\}$. We say that X meets Y if $X \cap Y \neq \emptyset$.

The phrase “by orthogonality” refers to the fact that a circuit and a cocircuit cannot intersect in exactly one element. The following is a straightforward consequence of orthogonality, which is used freely without reference.

Lemma 3.1. *Let e be an element of a matroid M , and let X and Y be disjoint sets whose union is $E(M) - e$. Then $e \in \text{cl}(X)$ if and only if $e \notin \text{cl}^*(Y)$.*

Let M be a matroid with ground set E . The *connectivity function* of M , denoted by λ_M , is defined as follows, for a subset X of E :

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

The following is easily shown to be equivalent:

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A *k-separating* partition $(X, E - X)$ is a *k-separation* if $|X| \geq k$ and $|E - X| \geq k$. A *k-separating* set X , a *k-separating* partition $(X, E - X)$, or a *k-separation* $(X, E - X)$ is *exact* if $\lambda_M(X) = k - 1$. The matroid M is *n-connected* if, for all $k < n$, it has no *k-separations*. When a matroid is 2-connected, we simply say it is *connected*.

For subsets X and Y in a matroid M , the *local connectivity between X and Y* , denoted $\square(X, Y)$, is defined as follows:

$$\square(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

We write “by uncrossing” to refer to an application of the next lemma.

Lemma 3.2. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

- (i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
- (ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

The following connectivity lemmas are well known and used freely.

Lemma 3.3. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid, and suppose that $e \in Y$. Then $X \cup e$ is 3-separating if and only if $e \in \text{cl}(X)$ or $e \in \text{cl}^*(X)$.*

Lemma 3.4. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid. Suppose $|Y| \geq 3$ and $e \in Y$. Then $e \in \text{cl}(Y - e)$ or $e \in \text{cl}^*(Y - e)$.*

Lemma 3.5. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid. Suppose $|Y| \geq 3$ and $e \in Y$. Then $(X \cup e, Y - e)$ is exactly 3-separating if and only if e is in one of $\text{cl}(X) \cap \text{cl}(Y - e)$ and $\text{cl}^*(X) \cap \text{cl}^*(Y - e)$.*

We also freely use the next three lemmas. The first is a straightforward consequence of Lemmas 3.1 and 3.4; the second follows immediately from Lemmas 3.1 and 3.3 to 3.5; and the third is elementary (see [5, Proposition 8.2.7]).

Lemma 3.6. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid, with $|Y| \geq 3$. Then $\text{cl}(X) \cap \text{cl}^*(X) \cap Y = \emptyset$.*

Lemma 3.7. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid, with $|Y| \geq 3$. If $e \in \text{cl}(X) \cap Y$, then $e \in \text{cl}(Y - e)$ and $(X \cup e, Y - e)$ is exactly 3-separating.*

Lemma 3.8. *Let M be a matroid and let $d \in E(M)$. Suppose that $M \setminus d$ is 3-connected but M is not. Then either d is in a parallel pair of M , or d is a loop or coloop of M .*

The next two lemmas are well known. We refer to the latter as Bixby's Lemma.

Lemma 3.9. *Let M be a 3-connected matroid and let S be a rank-2 subset with at least four elements. If $s \in S$, then $M \setminus s$ is 3-connected.*

Lemma 3.10 (Bixby's Lemma [1]). *Let e be an element of a 3-connected matroid M . Then either M/e is 3-connected up to parallel pairs, or $M \setminus e$ is 3-connected up to series pairs.*

A k -separation $(X, E - X)$ of a matroid M with ground set E is *vertical* if $r(X) \geq k$ and $r(E - X) \geq k$. We also say a partition $(X, \{z\}, Y)$ of E is a *vertical 3-separation* when $(X \cup \{z\}, Y)$ and $(X, Y \cup \{z\})$ are both vertical 3-separations and $z \in \text{cl}(X) \cap \text{cl}(Y)$. Note that, given a vertical 3-separation (X, Y) and some $z \in Y$, if $z \in \text{cl}(X)$, then $(X, \{z\}, Y - z)$ is a vertical 3-separation, by Lemma 3.7.

A vertical 3-separation in M^* is known as a cyclic 3-separation in M . More specifically, a 3-separation $(X, E - X)$ of M is *cyclic* if $r^*(X) \geq 3$ and $r^*(E - X) \geq 3$; or, equivalently, if X and $E - X$ contain circuits. We also say that a partition $(X, \{z\}, Y)$ of E is a *cyclic 3-separation* if $(X, \{z\}, Y)$ is a vertical 3-separation in M^* .

We say that a partition (X_1, X_2, \dots, X_m) of E is a *path of 3-separations* if $(X_1 \cup \dots \cup X_i, X_{i+1} \cup \dots \cup X_m)$ is a 3-separation for each $i \in [m - 1]$. Observe that a vertical, or cyclic, 3-separation $(X, \{z\}, Y)$ is an instance of a path of 3-separations.

The next two lemmas are also used freely. A proof of the first is in [9]; the second is a straightforward corollary of Bixby's Lemma, Lemma 3.11, and orthogonality.

Lemma 3.11. *Let M be a 3-connected matroid and let $z \in E(M)$. The following are equivalent:*

- (i) M has a vertical 3-separation $(X, \{z\}, Y)$.
- (ii) $\text{si}(M/z)$ is not 3-connected.

Lemma 3.12. *Let $(X, \{z\}, Y)$ be a vertical 3-separation of a 3-connected matroid M . Then either*

- (i) $M \setminus z$ is 3-connected, or
- (ii) z is in a triad that meets X and Y .

The following is known as Tutte's Triangle Lemma.

Lemma 3.13 (Tutte's Triangle Lemma [8]). *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M . If neither $M \setminus a$ nor $M \setminus b$ is 3-connected, then M has a triad which contains a and exactly one element from $\{b, c\}$.*

When we refer to an application of Tutte's Triangle Lemma in this paper, the following equivalent formulation is usually more pertinent. A set $X \subseteq E(M)$ is a 4-element fan if X is the union of a triangle and a triad with $|X| = 4$.

Lemma 3.14. *Let T^* be a triad in a 3-connected matroid M . If T^* is not contained in a 4-element fan, then, for any pair of distinct elements $a, b \in T^*$, either M/a or M/b is 3-connected.*

Proofs of the next two lemmas are in [9] and [2], respectively.

Lemma 3.15. *Let C^* be a rank-3 cocircuit of a 3-connected matroid M . If $x \in C^*$ has the property that $\text{cl}_M(C^*) - x$ contains a triangle of M/x , then $\text{si}(M/x)$ is 3-connected.*

Lemma 3.16. *Let M be a 3-connected matroid with $r(M) \geq 4$. Suppose that C^* is a rank-3 cocircuit of M . If there exists some $x \in C^*$ such that $x \in \text{cl}(C^* - x)$, then $\text{co}(M \setminus x)$ is 3-connected.*

A set X in a matroid M is *fully closed* if it is closed and coclosed; that is, $\text{cl}(X) = X = \text{cl}^*(X)$. The *full closure* of a set X , denoted $\text{fcl}(X)$, is the intersection of all fully closed sets that contain X . It is easily seen that the full closure is a well-defined closure operator, and that one way of obtaining the full closure of a set X is to take the closure of X , then the coclosure of the result, and repeat until neither the closure nor coclosure introduces new elements.

We use the next lemma frequently. The straightforward proof is omitted.

Lemma 3.17. *Let (X, Y) be a 2-separation in a connected matroid M where M contains no series or parallel pairs. Then $(\text{fcl}(X), Y - \text{fcl}(X))$ is also a 2-separation of M .*

We say that a 2-separation (U, V) is *trivial* if U or V is a series or parallel class.

We say that M has an N -minor if M has an isomorphic copy of N as a minor. For a matroid M with a minor N and $e \in E(M)$, we say e is N -contractible if M/e has an N -minor, we say e is N -deletable if $M \setminus e$ has an N -minor, and we say e is *doubly N -labelled* if e is both N -contractible and N -deletable.

The dual of the following is proved in [2, 4].

Lemma 3.18. *Let N be a 3-connected minor of a 3-connected matroid M . Let $(X, \{z\}, Y)$ be a cyclic 3-separation of M such that $M \setminus z$ has an N -minor with $|X \cap E(N)| \leq 1$. Let $X' = X - \text{cl}^*(Y)$ and $Y' = \text{cl}^*(Y) - z$. Then*

- (i) each element of X' is N -deletable; and
- (ii) at most one element of $\text{cl}^*(X) - z$ is not N -contractible, and if such an element x exists, then $x \in X' \cap \text{cl}(Y')$ and $z \in \text{cl}^*(X' - x)$.

Let M be a matroid with $d \in E(M)$. Suppose $X \subseteq E(M \setminus d)$ is exactly k -separating in $M \setminus d$. We say that d *blocks* X if X is not k -separating in M . If d blocks X , then it follows that $d \notin \text{cl}(E(M \setminus d) - X)$, so $d \in \text{cl}^*(X)$ by Lemma 3.1. We say that d *fully blocks* X if neither X nor $X \cup d$ is k -separating in M . It is easily shown that d fully blocks X if and only if $d \notin \text{cl}(X) \cup \text{cl}(E(M \setminus d) - X)$. Usually, when we use this terminology, we are considering elements that block a 3-separating set X ; for example, when X is a triad in a 3-connected matroid. On the other hand, if X is a series class of $M \setminus d$ of size at least two, then we say d blocks X if X is not 2-separating in M (so X is not a series class in M), and d fully blocks X if neither X nor $X \cup d$ is 2-separating in M .

Recall that we typically work under the assumption that every triangle or triad of M is N -grounded. In this setting, the following lemma shows that an N -contractible (or N -deletable) element is not in a triangle (or triad, respectively).

Lemma 3.19 ([3, Lemma 3.1]). *Let M be a 3-connected matroid with a 3-connected minor N where $|E(N)| \geq 4$. If T is an N -grounded triangle of M with $x \in T$, then x is not N -contractible.*

4. THE TRIAD CASE

In this section, we prove the following:

Theorem 4.1. *Let M be a 3-connected matroid with an element d such that $M \setminus d$ is 3-connected. Let N be a 3-connected minor of M , where every triangle or triad of M is N -grounded, and $|E(N)| \geq 4$. Suppose that $M \setminus d$ has a cyclic 3-separation $(Y, \{d'\}, Z)$ with $|Y| \geq 4$, where $M \setminus d \setminus d'$ has an N -minor with $|Y \cap E(N)| \leq 1$. Suppose Y contains a subset X that is 3-separating in $M \setminus d$, where $|X| \geq 4$ and, for each $x \in X$,*

- (a) $\text{co}(M \setminus d \setminus x)$ is 3-connected,
- (b) $M \setminus d \setminus x$ is 3-connected, and
- (c) x is doubly N -labelled in $M \setminus d$.

Let X be minimal subject to these conditions. If X contains a triad of $M \setminus d$, then M has an N -detachable pair.

Some preparatory lemmas. Let M be a 3-connected matroid and let (P_1, P_2, P_3) be a partition of $E(M)$ where P_i is 3-separating for each $i \in [3]$. If $\cap(P_i, P_j) = 2$ for all distinct $i, j \in [3]$, then we say (P_1, P_2, P_3) is a *paddle*. The following is proved in [6, Lemma 7.2].

Lemma 4.2. *Let (P_1, P_2, P_3) be a paddle in a 3-connected matroid M . Then $\text{cl}^*(P_i) = P_i$ for each $i \in [3]$.*

We first handle the following case that arises in the proof of Theorem 4.1.

Lemma 4.3. *Let M be a 3-connected matroid with a 3-connected matroid N as a minor. Suppose that $M \setminus d$ is 3-connected. Let (S, T, Z) be a paddle in $M \setminus d$ such that*

- (a) S and T are triads of $M \setminus d$ that are blocked by d ,
- (b) $|Z| \geq 3$, and
- (c) for all distinct $s, t \in S \cup T$ such that $\{s, t\} \subseteq \text{cl}((S \cup T) - \{s, t\})$, the matroid $M \setminus s \setminus t$ has an N -minor.

Then M has an N -detachable pair.

Proof. Let $M' = M \setminus d$.

4.3.1. *For $s \in S$, there is at most one element $t' \in T$ such that $(S - s) \cup (T - t')$ is a circuit in M' .*

Subproof. Let $T = \{t_1, t_2, t_3\}$ and suppose that $(S - s) \cup (T - t')$ is a circuit for each $t' \in \{t_1, t_2\}$. Then $t_1, t_2 \in \text{cl}((S - s) \cup t_3)$, so $r((S - s) \cup T) = 3$. But $r(S \cup T) = 4$, so $s \in \text{cl}^*(Z)$, contradicting Lemma 4.2. \triangleleft

Let $S = \{s, s_2, s_3\}$ and $T = \{t, t_2, t_3\}$. By 4.3.1 we may assume that $\{s_2, s_3, t_2, t_3\}$ is independent. In particular, $\{s, t\} \subseteq \text{cl}_{M'}(\{s_2, s_3, t_2, t_3\})$. This implies that $M \setminus s \setminus t$ has an N -minor, by (c). We work towards proving that $\{s, t\}$ is an N -detachable pair in M .

4.3.2. *$M' \setminus s \setminus t$ is connected.*

Subproof. Suppose that (P, Q) is a separation of $M' \setminus s \setminus t$. As $\{s_2, s_3\}$ and $\{t_2, t_3\}$ are series pairs in $M' \setminus s \setminus t$, we may assume that $\{s_2, s_3\} \subseteq P$ and $\{t_2, t_3\}$ is contained in either P or Q . If $\{t_2, t_3\} \subseteq P$, then (P, Q) is a separation in the 3-connected matroid M' , as $\{s, t\} \subseteq \text{cl}_{M'}(\{s_2, s_3, t_2, t_3\})$; a contradiction. Therefore, we may assume that $\{s_2, s_3\} \subseteq P$ and $\{t_2, t_3\} \subseteq Q$. Moreover, since $r(Z) = r(M') - 2$, it follows that $|P \cap Z|, |Q \cap Z| \geq 1$. Let $\lambda = \lambda_{M' \setminus s \setminus t}$. Since $\lambda(P) = \lambda(Q) = 0$, by the submodularity of λ we have

$$\begin{aligned} \lambda(P \cap Z) + \lambda(Q \cap Z) &\leq \lambda(P) + \lambda(Q) + 2\lambda(Z) - \lambda(P \cup Z) - \lambda(Q \cup Z) \\ &= 2\lambda(Z) - \lambda(P \cup Z) - \lambda(Q \cup Z) \\ &= 4 - \lambda(\{t_2, t_3\}) - \lambda(\{s_2, s_3\}) = 2. \end{aligned}$$

If either $\lambda(P \cap Z) = 0$ or $\lambda(Q \cap Z) = 0$, then, as $\{s, t\} \subseteq \text{cl}_{M'}(\{s_2, s_3, t_2, t_3\})$, the set $P \cap Z$ or $Q \cap Z$ is also 1-separating in M' ; a contradiction. Thus $\lambda(P \cap Z) = \lambda(Q \cap Z) = 1$. As $|Z| \geq 3$, we may assume, without loss of generality, that $|P \cap Z| \geq 2$. But it follows that $(P \cap Z, E(M') - (P \cap Z))$ is a contradictory 2-separation in M' . \triangleleft

4.3.3. *If (P, Q) is a 2-separation of $M' \setminus s \setminus t$, then, up to swapping S and T , and P and Q , we have $\{s_2, s_3\} \subseteq P$ and $\{t_2, t_3\} \subseteq \text{cl}^*(Q)$.*

Subproof. Firstly, observe that if (P, Q) is a 2-separation of $M' \setminus s \setminus t$ where $\{s_2, s_3, t_2, t_3\} \subseteq P$, then, as $\{s, t\} \subseteq \text{cl}_{M'}(\{s_2, s_3, t_2, t_3\})$, the partition $(P \cup \{s, s_2\}, Q)$ is a 2-separation in M' ; a contradiction. Thus we may assume

that no 2-separation (P, Q) of $M' \setminus s \setminus t$ has $\{s_2, s_3, t_2, t_3\}$ contained in either P or Q .

Let (P, Q) be a 2-separation of $M' \setminus s \setminus t$. As $|Z| \geq 3$, we may assume that $|P \cap Z| \geq 2$. Since $\{s_2, s_3, t_2, t_3\} \not\subseteq P$, by possibly swapping S and T , we may assume that $|Q \cap T| \geq 1$. Suppose that $|Q \cap T| = 1$; say $P \cap T = \{t'\}$ and $Q \cap T = \{t''\}$ where $\{t', t''\} = T - t$. Since $t' \in \text{cl}_{M' \setminus s \setminus t}^*(\{t''\})$, the partition $(P - \{t'\}, Q \cup \{t'\})$ is a 2-separation of $M' \setminus s \setminus t$. Since $\{s_2, s_3, t_2, t_3\} \not\subseteq Q \cup t'$, we deduce that $|Q \cap S| \leq 1$. Now, similarly, if $|Q \cap S| = \{s'\}$, then $(P - \{s', t'\}, Q \cup \{s', t'\})$ is a 2-separation of $M' \setminus s \setminus t$. But then $\{s_2, s_3, t_2, t_3\} \subseteq Q \cup \{s', t'\}$; a contradiction. So $Q \cap S = \emptyset$ and $\{t_2, t_3\} \subseteq \text{cl}^*(Q)$ when $|Q \cap T| = 1$. A similar argument gives that $Q \cap S = \emptyset$ when $|Q \cap T| = 2$. \triangleleft

Since $M' \setminus s \setminus t$ is connected, by 4.3.2, $M \setminus s \setminus t$ is also connected. Suppose that (P, Q) is a 2-separation of $M' \setminus s \setminus t$. By 4.3.3, we may assume that $\{s_2, s_3\} \subseteq P$ and $\{t_2, t_3\} \subseteq \text{cl}^*(Q)$. Thus $(P', Q') = (P - \{t_2, t_3\}, Q \cup \{t_2, t_3\})$ is also a 2-separation of $M' \setminus s \setminus t$. As $T \cup d$ is a 4-element cocircuit in M , we have that $\{t_2, t_3, d\}$ is a triad of $M \setminus s \setminus t$. Hence $d \in \text{cl}_{M \setminus s \setminus t}^*(Q') = \text{cl}_{M \setminus s \setminus t}^*(Q)$, so $d \notin \text{cl}_{M \setminus s \setminus t}(P)$. Likewise, since $d \in \text{cl}_{M \setminus s \setminus t}^*(\{s_2, s_3\})$, we have $d \notin \text{cl}_{M \setminus s \setminus t}(Q)$. We conclude that $(P \cup d, Q)$ and $(P, Q \cup d)$ are 3-separating in $M \setminus s \setminus t$. That is, d fully blocks (P, Q) for each 2-separation (P, Q) of $M' \setminus s \setminus t$. Thus $M \setminus s \setminus t$ is 3-connected. \square

Lemma 4.4. *Let M be a 3-connected matroid with a pair of disjoint triads $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$. If*

- (i) $\{s_1, s_2, t_1, t_2\}$ is a circuit of M , and
- (ii) s_3 is not in a triangle of M ,

then M/s_3 is 3-connected.

Proof. Note that $\cap(S, T) \geq 1$. Suppose that (X, Y) is a 2-separation in M/s_3 with $|X \cap T| \geq 2$. Note that M/s_3 contains no series pairs or parallel pairs. It follows, by Lemma 3.17, that $(\text{fcl}_{M/s_3}(X), Y - \text{fcl}_{M/s_3}(X))$ is also a 2-separation of M/s_3 ; so we may assume that X is fully closed, and thus $T \subseteq X$. If $\cap(S, T) = 2$, then $\{s_1, s_2\} \subseteq \text{cl}_{M/s_3}(T) \subseteq X$, implying that $(X \cup s_3, Y)$ is a 2-separation of M ; a contradiction. So assume that $\cap(S, T) = 1$. If $\{s_1, s_2\} \subseteq X$ or $\{s_1, s_2\} \subseteq Y$, then $(X \cup s_3, Y)$ or $(X, Y \cup s_3)$, respectively, is a contradictory 2-separation of M . So, without loss of generality, $s_1 \in X$ and $s_2 \in Y$. But then, due to the circuit $\{s_1, s_2, t_1, t_2\}$, we have $s_2 \in \text{cl}(X) - X$, contradicting the fact that X is fully closed. \square

Lemma 4.5. *Let M be a 3-connected matroid with distinct elements $a_1, a_2, b_1, b_2, p_1, p_2$, such that*

- (a) $\text{co}(M \setminus p_1 \setminus p_2)$ is 3-connected,
- (b) $\{a_1, a_2, p_1, p_2\}$ and $\{b_1, b_2, p_1, p_2\}$ are distinct cocircuits of M , and
- (c) $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are distinct series classes of $M \setminus p_1 \setminus p_2$.

Then either

- (i) *there exists $x \in \{a_1, a_2, b_1, b_2\}$ such that M/x is 3-connected, or*
- (ii) *up to labelling, $\{a_1, b_1, p_1\}$ and $\{a_2, b_2, p_2\}$ are triangles of M .*

Proof. Assume (i) does not hold. Suppose that a_1 is not in a triangle and consider M/a_1 . Observe that any series class S of $M \setminus p_1 \setminus p_2$ with size at least two is blocked by p_1 or p_2 ; in particular, if $S \neq \{a_1, a_2\}$, then $p_i \notin \text{cl}_{M/a_1}(E(M/a_1) - (S \cup \{p_1, p_2\}))$ for some $i \in \{1, 2\}$. Since M/a_1 is not 3-connected, but $M \setminus p_1 \setminus p_2/a_1$ is 3-connected up to series classes, there is a series class S' of $M \setminus p_1 \setminus p_2/a_1$, with $|S'| \geq 2$, that is not fully blocked by both p_1 and p_2 . By the foregoing, we may assume that $p_1 \in \text{cl}_{M/a_1}(S')$. Now p_1 is in a circuit of M contained in $S' \cup a_1$. If $S' \neq \{b_1, b_2\}$, then this contradicts orthogonality with the cocircuit $\{b_1, b_2, p_1, p_2\}$. So $S' = \{b_1, b_2\}$. Let $\{i, j\} = \{1, 2\}$. Now $p_j \in \text{cl}_{M/a_1 \setminus p_i}^*(\{b_1, b_2\})$, so $p_j \notin \text{cl}_{M/a_1}(E(M/a_1) - \{b_1, b_2, p_1, p_2\})$, where $\{b_1, b_2\}$ is not fully blocked by p_j in $M \setminus p_1 \setminus p_2/a_1$. Hence $\{p_1, p_2\} \subseteq \text{cl}_M(\{b_1, b_2, a_1\})$, so $r_M(\{b_1, b_2, p_1, p_2\}) \leq 3$. Since M is 3-connected, $r_M(\{b_1, b_2, p_1, p_2\}) = 3$ and hence $a_1 \in \text{cl}(\{b_1, b_2, p_1, p_2\})$.

Suppose also that a_2 is in a triangle. Since a_1 is not, this triangle meets $\{p_1, p_2\}$, by orthogonality with the cocircuit $\{a_1, a_2, p_1, p_2\}$. Again by orthogonality, either the triangle meets $\{b_1, b_2\}$, or it is $\{a_2, p_1, p_2\}$. In either case, $r(\{a_1, a_2, b_1, b_2\}) \leq 3$. But since $\text{co}(M \setminus p_1 \setminus p_2/a_1/b_1)$ is 3-connected, $r(\{a_1, a_2, b_1, b_2\}) = 4$. We deduce that a_2 is not in a triangle of M .

Now repeating the argument in the first paragraph with a_2 in the place of a_1 , we deduce that $a_2 \in \text{cl}(\{b_1, b_2, p_1, p_2\})$, so $r(\{a_1, a_2, b_1, b_2\}) = 3$; a contradiction. Thus a_1 and a_2 are both in triangles of M .

Suppose $\{a_1, a_2, x\}$ is a triangle for some $x \in E(M) - \{a_1, a_2\}$. If $x \in \{p_1, p_2\}$, then this triangle intersects the cocircuit $\{b_1, b_2, p_1, p_2\}$ in one element; so we may assume otherwise. But then $M \setminus p_1 \setminus p_2/a_1$ contains a parallel pair; a contradiction. So the triangles containing a_1 and a_2 are distinct, and each either contains $\{p_1, p_2\}$, or meets both $\{b_1, b_2\}$ and $\{p_1, p_2\}$, by orthogonality. By symmetry, b_1 and b_2 are also in triangles of M , and each either contains $\{p_1, p_2\}$, or meets both $\{a_1, a_2\}$ and $\{p_1, p_2\}$. It now follows, by circuit elimination and up to relabelling, that $\{a_1, b_1, p_1\}$ and $\{a_2, b_2, p_2\}$ are triangles of M . \square

A key lemma. Next, we work towards proving Lemma 4.8, which we use not only in the proof of Theorem 4.1, but also in Section 5.

In the remainder of Section 4, we work under the following assumptions. Let M be a 3-connected matroid with an element d such that $M \setminus d$ is 3-connected. Let N be a 3-connected minor of M , where every triangle or triad of M is N -grounded, and $|E(N)| \geq 4$. Suppose that $M \setminus d$ has a cyclic 3-separation $(Y, \{d'\}, Z)$ with $|Y| \geq 4$, where $M \setminus d \setminus d'$ has an N -minor with $|Y \cap E(N)| \leq 1$. Note in particular that $r^*(M \setminus d) \geq 4$.

Let X be a subset of Y such that $|X| \geq 4$, the set X is 3-separating in $M \setminus d$, and, for each $x \in X$,

- (a) $\text{co}(M \setminus d \setminus x)$ is 3-connected,

- (b) $M \setminus d/x$ is 3-connected, and
- (c) x is doubly N -labelled in $M \setminus d$.

The following is proved in [3, Lemma 7.1]. A *segment* in a matroid M is a subset S of $E(M)$ such that $M|S \cong U_{2,k}$ for some $k \geq 3$, while a *cosegment* of M is a segment of M^* .

Lemma 4.6. *If Y contains a 4-element cosegment, then M has an N -detachable pair.*

In particular, Lemma 4.6 implies that if M has no N -detachable pairs, then X does not contain a 4-element cosegment.

Lemma 4.7. *Either each triad of $M \setminus d$ that meets X does so in at least two elements, or M has an N -detachable pair.*

Proof. Assume that M has no N -detachable pairs. Suppose T^* is a triad of $M \setminus d$ with $T^* \cap X = \{t\}$. Then $t \in \text{cl}^*(E(M \setminus d) - X)$. Since $|X| \geq 4$ and X does not contain a 4-element cosegment, it follows that $(X - t, \{t\}, E(M \setminus d) - X)$ is a cyclic 3-separation of $M \setminus d$, so $\text{co}(M \setminus d \setminus t)$ is not 3-connected; a contradiction. So each triad of $M \setminus d$ that meets X does so in at least two elements. \square

The next lemma is used, both in the remainder of this section and in Section 5, to find N -contractible pairs where each element in the pair is in a triad of $M \setminus d$ meeting X .

Lemma 4.8. *Let S^* and T^* be distinct triads of $M \setminus d$ meeting X , where $S^* \cup T^*$ is not a cosegment. Suppose M has no N -detachable pairs.*

- (i) *If $s \in S^*$ and $t \in T^*$ where $s \neq t$ and $d' \notin \{s, t\}$, then $M \setminus d/s/t$ has an N -minor.*
- (ii) *If $d' \in S^*$ and $t \in T^* - S^*$, then $M \setminus d/d'/t$ has an N -minor.*

Proof. By Lemma 4.7, the triads S^* and T^* each have at least two elements in X . If $t \in Z$, then, since $t \in \text{cl}_{M \setminus d}^*(Y)$, it follows that $(Y \cup t, \{d'\}, Z - t)$ is a cyclic 3-separation of $M \setminus d$. Either $|(Y \cup t) \cap E(N)| \leq 1$ or $|(Z - t) \cap E(N)| \leq 1$, but $|Y \cap E(N)| \leq 1$ and $|E(N)| \geq 4$, so $|(Y \cup t) \cap E(N)| \leq 1$. So we may assume that $t \in Y$. In case (i), we may similarly assume that $s \in Y$. Moreover, if $s \in T^*$, then $t \notin S^*$, since $S^* \cup T^*$ is not a cosegment. So we may assume, up to labels, that $t \notin S^*$, and thus $S^* \subseteq Y - t$. In case (ii), let $s = d'$. We now consider both cases together. It suffices to prove that $M \setminus d/s/t$ has an N -minor.

We first claim that t is N -contractible in $M \setminus d$. By Lemma 3.18(ii), at most one element of $\text{cl}_{M \setminus d}^*(Y) - d'$ is not N -contractible. Suppose that t is this element that is not N -contractible. Then $t \in \text{cl}(Z') - Z'$, where $Z' = \text{cl}_{M \setminus d}^*(Z) - d'$. Now t is in a circuit contained in $Z' \cup t$, so the triad T^* meets Z' , by orthogonality. Let $t_2 \in T^* \cap Z'$. If $t_2 \in X$, then $\text{co}(M \setminus d \setminus t_2)$ is 3-connected, but $t_2 \in \text{cl}_{M \setminus d}^*(Z) - Z$, which implies that $\text{co}(M \setminus d \setminus t_2)$ is not 3-connected; a contradiction. So $t_2 \notin X$, implying that $t \in X$. But $M \setminus d/t$ is

not 3-connected since $t \in \text{cl}(Z') - Z'$; a contradiction. So t is N -contractible in $M \setminus d$.

Next we claim that $M \setminus d/t$ is 3-connected. This is immediate if $t \in X$, so we assume that $t \notin X$. Let $T^* = \{t, t_2, t_3\}$; then $\{t_2, t_3\} \subseteq X$. Since $t \in \text{cl}_{M \setminus d}^*(X) - X$, the matroid $\text{co}(M \setminus d \setminus t)$ is not 3-connected, so $\text{si}(M \setminus d/t)$ is 3-connected by Bixby's Lemma. If t is in a triangle of $M \setminus d$, then, by orthogonality with T^* , the triangle also contains either t_2 or t_3 . But then either $M \setminus d/t_2$ or $M \setminus d/t_3$ is not 3-connected; a contradiction.

Now $M \setminus d/t$ is 3-connected and has an N -minor. Note that $(Y - t, \{d'\}, Z)$ is a cyclic 3-separation of $M \setminus d/t$. First, suppose that $s \neq d'$. Let $Z' = \text{cl}^*(Z) - d'$ and $Y' = (Y - t) - Z'$, so that $(Y', \{d'\}, Z')$ is a cyclic 3-separation of $M \setminus d/t$. Observe that $\text{cl}_{M \setminus d/t}^*(Y') = \text{cl}_{M \setminus d/t}^*(Y - t)$. By Lemma 3.18(ii), at most one element of $\text{cl}^*(Y - t) - d'$ is not N -contractible in $M \setminus d/t$, and if s is this exceptional element, then $s \in \text{cl}_{M \setminus d/t}(Z')$. But if $s \in \text{cl}_{M \setminus d/t}(Z')$, then $s \notin \text{cl}_{M \setminus d/t}^*(Y') = \text{cl}_{M \setminus d/t}^*(Y - t)$, implying that $s \notin \text{cl}_{M \setminus d/t}^*(S^*)$; a contradiction. So $M \setminus d/t/s$ has an N -minor if $s \neq d'$.

Finally, suppose that $s = d'$. Then d' is in a triad $\{d', s_2, s_3\}$ of $M \setminus d/t$ where $\{s_2, s_3\} \subseteq X$. If $\{s_2, s_3\} \subseteq \text{cl}_{M \setminus d}^*(Z)$, then $S^* \subseteq \text{cl}_{M \setminus d}^*(Y - S^*) \cap \text{cl}_{M \setminus d}^*(Z - S^*)$, and it follows that $\text{co}(M \setminus d \setminus s_2)$ is not 3-connected; a contradiction. So we may assume $\{s_2, s_3\} \not\subseteq \text{cl}_{M \setminus d/t}^*(Z)$, and so, by Lemma 3.18(i) and up to labels, s_2 is N -deletable in $M \setminus d/t$. Since $\{s_3, d'\}$ is a series pair in $M \setminus d \setminus s_2/t$, we deduce that $M \setminus d \setminus s_2/t/d'$ has an N -minor. In particular, $M \setminus d/s/t$ has an N -minor. \square

Towards the proof of Theorem 4.1. We now assume that X is minimal, and X contains a triad of $M \setminus d$.

More specifically, let X be a subset of Y such that $|X| \geq 4$; the set X is 3-separating in $M \setminus d$; for each $x \in X$,

- (a) $\text{co}(M \setminus d \setminus x)$ is 3-connected,
- (b) $M \setminus d/x$ is 3-connected, and
- (c) x is doubly N -labelled in $M \setminus d$;

and X is minimal subject to these conditions. Furthermore, X contains a triad of $M \setminus d$.

In practice, the following two lemmas are convenient for finding N -contractible or N -deletable pairs.

Lemma 4.9. *Let S and T be distinct triads of $M \setminus d$ that meet X , where $S \cup T$ is not a cosegment of $M \setminus d$. Suppose M has no N -detachable pairs.*

- (i) *If $s \in S$ and $t \in T$, and either $\{s, t\} \subseteq X$, $\{s, t\} \subseteq X \Delta S$, or $\{s, t\} \subseteq S \Delta T$, then $M \setminus d/s/t$ has an N -minor.*
- (ii) *If $s \in S - T$ and $t \in T$, and $M \setminus d/s/t$ does not have an N -minor, then $M \setminus d/s'/t'$ has an N -minor for any distinct $s' \in S'$ and $t' \in T$ where S' is a triad of $M \setminus d$ that meets X with $S' \neq T$ and $s \neq s'$.*

- (iii) If S and T are disjoint, and X is a corank-3 circuit contained in $S \cup T$, with $T \subseteq X$, then $M \setminus d/t/t'$ has an N -minor for all distinct $t, t' \in T$.

Proof. Consider (i). If $\{s, t\} \subseteq X$, then $M \setminus d/s/t$ has an N -minor by Lemma 4.8(i). Suppose $\{s, t\} \subseteq X \Delta S$. Then $s \in S - X$ and $t \in X - S$. If $s \neq d'$ then $M \setminus d/s/t$ has an N -minor by Lemma 4.8(i). On the other hand, if $s = d'$, then $M \setminus d/s/t$ has an N -minor by Lemma 4.8(ii). Finally, if $\{s, t\} \subseteq S \Delta T$, then $M \setminus d/s/t$ has an N -minor by Lemma 4.8(i) when $d' \notin \{s, t\}$, or by Lemma 4.8(ii) when $d' \in \{s, t\}$.

Now, for (ii), suppose $M \setminus d/s/t$ does not have an N -minor, where $s \notin T$. Then $d' \in \{s, t\}$ by Lemma 4.8(i). If $d' = t$, then, as $s \in S - T$, the matroid $M \setminus d/s/t$ has an N -minor by Lemma 4.8(ii); a contradiction. So $d' = s$, and thus $d' \notin T$. It follows, by Lemma 4.8(i), that $M \setminus d/s'/t'$ has an N -minor for any $s' \in S'$ and $t' \in T$ with $S' \neq T$ and $s \neq s'$.

Finally, consider (iii). Let $X' = S \cup T$. Since X is a circuit, T is a triad of $M \setminus d$, and $t \in T \subseteq X$, we have that $t \notin \text{cl}(\text{cl}_{M \setminus d}^*(Z))$. Hence, by Lemma 3.18(ii), $M \setminus d/t$ has an N -minor. Moreover, $(Y - t, \{d'\}, Z)$ is a cyclic 3-separation in the 3-connected matroid $M \setminus d/t$. As $r_{M \setminus d}^*(X - t) = 3$, we have $t' \in \text{cl}_{M \setminus d}^*(Y - \{t, t'\})$. Now $X - t$ is a circuit in $M \setminus d/t$, so Lemma 3.18(ii) implies that $M \setminus d/t/t'$ has an N -minor, as required. \square

Lemma 4.10. *Let x and x' be distinct elements in X . If $x' \in \text{cl}(X - \{x, x'\})$, then $M \setminus d \setminus x \setminus x'$ has an N -minor.*

Proof. If $x \in \text{cl}_{M \setminus d}^*(Z)$, then $(X - x, \{x\}, E(M \setminus d) - X)$ is a cyclic 3-separation in $M \setminus d$, implying $\text{co}(M \setminus d \setminus x)$ is not 3-connected; a contradiction. So $x \notin \text{cl}_{M \setminus d}^*(Z)$. Now it follows from Lemma 3.18(i) that x is N -deletable in $M \setminus d$. Let S be a set containing all but one element in each series class of $M \setminus d \setminus x$, with $x' \notin S$. Let $Y' = (Y - x) - S$ and $Z' = Z - S$. Now $(Y', \{d'\}, Z')$ is a cyclic 3-separation in $\text{co}(M \setminus d \setminus x)$. Since $x' \in \text{cl}(X - \{x, x'\})$, we have that $x' \notin \text{cl}_{\text{co}(M \setminus d \setminus x)}^*(Z')$. So x' is N -deletable in $\text{co}(M \setminus d \setminus x)$, by Lemma 3.18(i). In particular, $M \setminus d \setminus x \setminus x'$ has an N -minor, as required. \square

Lemma 4.11. *If there is an element $w \in \text{cl}_{M \setminus d}(X) - X$ that is N -deletable in $M \setminus d$, then M has an N -detachable pair.*

Proof. We may assume that $r_{M \setminus d}^*(X) \geq 3$, otherwise M has an N -detachable pair by Lemma 4.6. We work towards showing $\{d, w\}$ is an N -detachable pair. As X and $X \cup w$ are exactly 3-separating in $M \setminus d$, the matroid $\text{co}(M \setminus d \setminus w)$ is 3-connected, by Lemma 3.11 and Bixby's Lemma. So the lemma holds unless w is in a triad of $M \setminus d$. Suppose $\{x, w, y\}$ is such a triad, and let $W = E(M \setminus d) - X$. We may assume, by Lemma 3.12, that $x \in X$ and $y \in W$. Thus $x \in \text{cl}_{M \setminus d}^*(W)$. Now $X - x$ and X are exactly 3-separating in $M \setminus d$, so, by Lemma 3.5, $x \in \text{cl}_{M \setminus d}^*(X - x)$. But then

$(X - x, \{x\}, W)$ is a cyclic 3-separation of $M \setminus d$, so $\text{co}(M \setminus d \setminus x)$ is not 3-connected; a contradiction. We deduce that w is not in a triad of $M \setminus d$, so $M \setminus d \setminus w$ is 3-connected and has an N -minor. \square

Lemma 4.12. *Suppose M has no N -detachable pairs. If S and T are triads of $M \setminus d$ that meet X , the set $S \cup T$ is not a cosegment of $M \setminus d$, and $|S \cap T| = 1$, then $r_{M \setminus d}(S \Delta T) = 4$ and $S \cup T$ is not 3-separating in $M \setminus d$.*

Proof. Let $S \cap T = \{u\}$. We claim that $r_{M \setminus d}(S \Delta T) = 4$. First, suppose that $u \in X$. If $r_{M \setminus d}(S \Delta T) = 3$, then $(S \Delta T, E(M \setminus d \setminus u) - (S \Delta T))$ is a 2-separation of $M \setminus d \setminus u$. But $S \Delta T$ is not contained in a series class in $M \setminus d \setminus u$, contradicting that $\text{co}(M \setminus d \setminus u)$ is 3-connected. Now suppose that $u \notin X$. If $r_{M \setminus d}(S \Delta T) = 3$, then $X' = S \Delta T$ is a 4-element subset of X (by Lemma 4.7) that is 3-separating. Since X contains a triad, $X' \subsetneq X$, contradicting the minimality of X . Thus $r_{M \setminus d}(S \Delta T) = 4$.

Suppose that $\lambda_{M \setminus d}(S \cup T) = 2$. If $S \cup T$ contains a 4-element circuit of $M \setminus d$, then, as $r_{M \setminus d}(S \Delta T) = 4$, this circuit contains one of S or T . Without loss of generality we may assume that the circuit is $S \cup t$ for $t \in T - u$. Then $(S, \{t\}, E(M \setminus d) - (S \cup t))$ is a vertical 3-separation of $M \setminus d$, so $\text{si}(M \setminus d / t)$ is not 3-connected. This implies that $t \notin X$. Let $T = \{u, t, t'\}$; then $t' \in X$, by Lemma 4.7. Moreover, $(S \cup t, \{t'\}, E(M \setminus d) - (S \cup T))$ is a cyclic 3-separation of $M \setminus d$, so $\text{co}(M \setminus d \setminus t')$ is not 3-connected; a contradiction. So $S \cup T$ does not contain a 4-element circuit of $M \setminus d$.

By uncrossing, $(S \cup T) \cap X$ is 3-separating in $M \setminus d$. If $u \notin X$, then $S \Delta T \subseteq X$ by Lemma 4.7, so $S \Delta T$ is 3-separating, and hence $r_{M \setminus d}(S \Delta T) = 3$; a contradiction. Now, if neither S nor T is contained in X , then $|(S \cup T) \cap X| = 3$, so $(S \cup T) \cap X$ is a triangle or a triad. But this is contradictory, since $S \cup T$ is not a cosegment, and no triangle of $M \setminus d$ meets X (since $M \setminus d / x$ is 3-connected for each $x \in X$). Next, suppose $S \subseteq X$ and $T = \{t_1, t_2, u\}$ where $T - X = \{t_2\}$. Then $(S \cup T) \cap X = S \cup t_1$ is exactly 3-separating. As $t_1 \notin \text{cl}_{M \setminus d}^*(S)$, we have $t_1 \in \text{cl}_{M \setminus d}(S)$ by Lemma 3.4, so $S \cup t_1$ is a circuit; a contradiction. So $S \cup T \subseteq X$. Moreover, $S \cup T$ is a circuit, since $S \cup T$ does not contain a 4-element circuit, and $r_{M \setminus d}^*(S \cup T) = 3$, so $r_{M \setminus d}(S \cup T) = 4$.

Let s and t be distinct elements such that $s \in S$ and $t \in T$. By Lemma 4.9(i), the matroid $M \setminus d / s / t$ has an N -minor. Without loss of generality, we may assume that $s \neq u$. Since, in $M \setminus d$, the set $S \cup T$ is a corank-3 circuit, S is a triad, and s is not in a triangle, it follows from the dual of Lemma 3.16 that $M \setminus d / s$ is 3-connected. Moreover, $(S \cup T) - s$ is a corank-3 circuit in $M \setminus d / s$, and $t \in \text{cl}_{M \setminus d / s}^*((S \cup T) - \{s, t\})$, so we can apply Lemma 3.16 a second time to deduce that $\text{si}(M \setminus d / s / t)$ is 3-connected. Now, either $\{s, t\}$ is an N -detachable pair, or $\{s, t\}$ is contained in a 4-element circuit $C_{s,t}$ of M that could contain d . As $S \cup T$ does not contain a 4-element circuit in $M \setminus d$, the circuit $C_{s,t}$ either contains d or meets $E(M \setminus d) - (S \cup T)$. Suppose that $d \notin C_{s,t}$, and let $w \in C_{s,t} - (S \cup T)$. Then, by orthogonality $C_{s,t} = \{s, t, x, w\}$ for $x \in (S \cup T) - \{s, t\}$. Since $M \setminus d / s / t$ has an N -minor

and $\{x, w\}$ is a parallel pair in this matroid, w is N -deletable in $M \setminus d$, contradicting Lemma 4.11. So for all distinct $s \in S$ and $t \in T$, there is a 4-element circuit containing $\{s, t, d\}$.

Let $X' = S \cup T$. Suppose that d fully blocks X' . Since X' is a circuit, there are certainly no 4-element circuits of $M \setminus d$ contained in X' . Moreover, there are no 4-element circuits of M contained in $X' \cup d$, otherwise $d \in \text{cl}(X')$, contradicting that d fully blocks X' . Let $S = \{s_1, s_2, t_3\}$ and $T = \{t_1, t_2, t_3\}$. For each $i \in [3]$, there are elements $v_i, w_i \in \text{cl}(X' \cup d) - (X' \cup d)$ such that $\{s_1, t_i, d, v_i\}$ and $\{s_2, t_i, d, w_i\}$ are circuits.

Next, we claim that $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ is a 6-element rank-3 set, and if $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ contains a triangle, then this triangle is either $\{v_i, v_j, w_k\}$ or $\{v_i, w_j, w_k\}$ for some $\{i, j, k\} = \{1, 2, 3\}$. If $v_i = v_{i'}$ for distinct $i, i' \in [3]$, then $\{s_1, t_i, t_{i'}, d\}$ contains a circuit, by the circuit elimination axiom, contradicting the fact that d fully blocks X' . Similarly, the w_i are pairwise distinct for $i \in [3]$. Say $v_i = w_j$ for some $i, j \in [3]$. Then, again by circuit elimination, there is a circuit contained in $\{s_1, s_2, t_i, t_j, v_i\}$. If $v_i \in \text{cl}(\{s_1, s_2, t_i, t_j\})$, then $d \in \text{cl}(X')$; a contradiction. So $\{s_1, s_2, t_i, t_j\}$ is a circuit of M , but this contradicts the fact that X' is a circuit. Hence the elements $v_1, v_2, v_3, w_1, w_2, w_3$ are pairwise distinct. Now $\text{cl}(X' \cup d) - (X' \cup d)$ has rank at most 3. If $r(\{v_1, v_2, v_3\}) \leq 2$, then $\{s_1, d, v_1, v_2, v_3\}$ has rank at most four, but spans the rank-5 set $X' \cup d$; a contradiction. A similar argument applies if $r(\{w_1, w_2, w_3\}) \leq 2$, or, for some distinct $i, j \in [3]$ either $r(\{v_i, v_j, w_i\}) \leq 2$ or $r(\{v_i, w_i, w_j\}) \leq 2$. It now follows from [3, Lemma 7.2] that M has an N -detachable pair; a contradiction.

Now suppose d does not fully block X' . Then $d \in \text{cl}(X')$, and, for each of the 4-element circuits containing $\{s, t, d\}$, the fourth element is in $\text{cl}(X')$. Let $S = \{s_1, s_2, u\}$ and $T = \{t_1, t_2, u\}$, and let the 4-element circuits be $\{s_1, t_i, d, x_i\}$, $\{s_2, t_i, d, w_i\}$, $\{u, t_i, d, p_i\}$, and $\{s_i, u, d, q_i\}$, for $i \in \{1, 2\}$. Let $e \in \{p_i, q_i, w_i, x_i\}$ for some $i \in \{1, 2\}$. Since $d \in \text{cl}(X')$, we have $e \in \text{cl}_{M \setminus d}(Y - e)$, so $e \notin \text{cl}_{M \setminus d}^*(Z)$. It follows, by Lemma 3.18(i), that e is N -deletable in $M \setminus d$.

Suppose there exists some $e \in \{p_i, q_i, w_i, x_i\} - X'$ for $i \in \{1, 2\}$. Then $(Y - e, \{e\}, Z \cup d')$ is a vertical 3-separation, so $\text{co}(M \setminus d \setminus e)$ is 3-connected by Bixby's Lemma. In the case that $\{d, e\}$ is not an N -detachable pair, $\{d, e\}$ is contained in a 4-element cocircuit C^* of M . Since $e \notin \text{cl}_{M \setminus d}^*(X')$, the cocircuit contains at most one element of X' . But since X' is a circuit in M , any element $x \in X'$ is not in $\text{cl}_{M \setminus d}^*(E(M \setminus d) - X')$. So $C^* \subseteq E(M) - X'$, implying $d \notin \text{cl}(X')$; a contradiction. So $\{p_i, q_i, w_i, x_i\} \subseteq X'$ for each $i \in \{1, 2\}$.

Now, for each pair of distinct elements $s \in S$ and $t \in T$, the set $\{d, s, t\}$ is contained in a 4-element circuit that is contained in $X' \cup d$. Moreover, any two of these circuits intersect in at most two elements, otherwise, by circuit elimination, X' properly contains a circuit; a contradiction. Suppose $\{d, s, t, u\}$ is a circuit for some labelling $\{s, t, u, v, w\}$ of X' with $s \in S$ and

$t \in T$. Then, up to relabelling $\{s, t, u\}$, there is a circuit $\{d, u, v, w\}$. Up to swapping the labels on v and w , there is also a 4-element circuit containing $\{d, s, v\}$. But any such circuit intersects either $\{d, s, t, u\}$ or $\{d, u, v, w\}$ in three elements; a contradiction. \square

We now prove the main result of this section: Theorem 4.1.

Theorem 4.1. *If X contains a triad of $M \setminus d$, then M has an N -detachable pair.*

Proof. Let $x \in X$. Since $\text{co}(M \setminus d \setminus x)$ is 3-connected and $M \setminus d \setminus x$ has an N -minor, either $\{d, x\}$ is an N -detachable pair or x is in a triad of $M \setminus d$. So we may assume x is in a triad of $M \setminus d$ for every $x \in X$.

4.1.1. *Let R and S be disjoint triads of $M \setminus d$ that meet X . If M has no N -detachable pairs, then $\Pi(R, S) = 1$, and there exists some $r \in R$ such that S is not contained in a 4-element fan in $M \setminus d/r$.*

Subproof. By Lemmas 4.6 and 4.7, $\Pi(R, S) \neq 0$. Suppose that $\Pi(R, S) = 2$. Then $(R, S, E(M \setminus d) - (R \cup S))$ is a paddle. If $|R \cap X| = 2$ and $|S \cap X| = 2$, then $X \cup R$ and $X \cup R \cup S$ are 3-separating, by uncrossing, and it follows that $(X, \{r\}, \{s\}, E(M \setminus d) - (X \cup \{r, s\}))$ is a path of 3-separations where $R - X = \{r\}$ and $S - X = \{s\}$. But then $s \in \text{cl}^*(E(M \setminus d) - (R \cup S))$, contradicting Lemma 4.2. So, by Lemma 4.7, at least one of R and S is contained in X ; in fact, by a similar argument, $R \cup S \subseteq X$. Now it follows that M has an N -detachable pair by Lemmas 4.3 and 4.10. So $\Pi(R, S) = 1$.

Suppose S is contained in a 4-element fan of $M \setminus d/r'$ for some $r' \in R$. Since each $s \in S \cap X$ is N -contractible in $M \setminus d$, it follows from Lemma 4.7 and orthogonality that s is not contained in an N -grounded triangle. Thus, there is a 4-element circuit C of $M \setminus d$ with $r' \in C$ and, by orthogonality, $|C \cap R| = 2$ and $|C \cap S| = 2$. Let $R - C = \{r\}$. If S is also contained in a 4-element fan of $M \setminus d/r$, then there is a 4-element circuit C' with $r \in C'$ and $|C' \cap R| = 2$ and $|C' \cap S| = 2$, implying that $\Pi(R, S) = 2$; a contradiction. Thus, the triad S is not contained in a 4-element fan in $M \setminus d/r$ for some $r \in R$. \triangleleft

We now consider, in 4.1.2–4.1.6, each possible arrangement of three distinct triads in X , and, in each case, we prove the existence of an N -detachable pair. These configurations, as they appear in $(M \setminus d)^*$, are illustrated in Figure 2. Note that the intention is to show how these triads interact, and the illustrations are not indicative of the rank of these sets in $(M \setminus d)^*$ (in particular, the union may have rank more than three in $(M \setminus d)^*$).

Note also that every triad of $M \setminus d$ that meets X is blocked by d . To see this, let Γ be a triad of $M \setminus d$ that meets X and is not blocked by d . Then Γ is a triad of M , and Γ contains an element $x \in X$ that is N -deletable. But this implies that Γ is not N -grounded; a contradiction.

4.1.2. *Let R, S , and T be distinct triads of $M \setminus d$ that meet X . If these three triads are pairwise disjoint, then M has an N -detachable pair.*

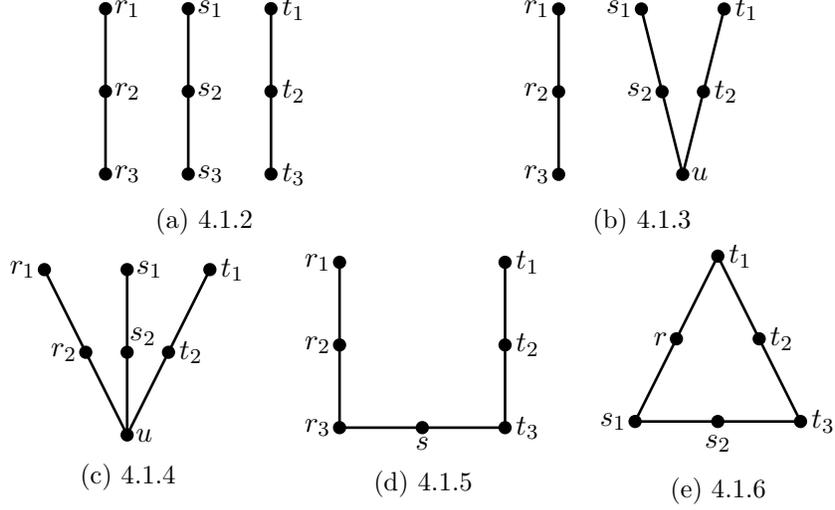


FIGURE 2. Each configuration of three distinct triads in X as they appear in $(M \setminus d)^*$, and the claim in which the configuration is considered.

Subproof. Suppose R , S , and T are pairwise disjoint. Let $R = \{r_1, r_2, r_3\}$, $S = \{s_1, s_2, s_3\}$, and $T = \{t_1, t_2, t_3\}$ (see Figure 2a). By 4.1.1, we may assume that the triad S is not contained in a 4-element fan in $M \setminus d/r_1$, say. Now, by Tutte's Triangle Lemma, both $M \setminus d/r_1/s_1$ and $M \setminus d/r_1/s_2$ are 3-connected, up to relabelling the elements of S . By Lemma 4.9(i), either M has an N -detachable pair, or there are elements α and β such that $\{d, r_1, s_1, \alpha\}$ and $\{d, r_1, s_2, \beta\}$ are circuits of M . Moreover, $\alpha, \beta \in T$, as otherwise d does not block the triad T of $M \setminus d$. So we may assume that $\{d, s_1, t_1\}$ and $\{d, s_2, t_2\}$ are triangles in M/r_1 , and it follows, by circuit elimination, that $\{s_1, s_2, t_1, t_2\}$ contains a circuit of M/r_1 . Since $r_1 \in \text{cl}^*(\{r_2, r_3, d\})$, this circuit is also a circuit of M . As each element of $S \cup T$ is not in a triangle, $\{s_1, s_2, t_1, t_2\}$ is a circuit of M/r_1 and M .

If $\{d, t_3\}$ is also contained in a triangle of M/r_1 , then the triangle must contain an element s in $\{s_1, s_2, s_3\}$, by orthogonality. Thus, by circuit elimination with the triangle $\{d, s_1, t_1\}$, we see that $\{s, s_1, t_1, t_3\}$ contains a circuit of M/r_1 , and $r_1 \notin \text{cl}(\{s, s_1, t_1, t_3\})$, so it follows that $\{s_1, s, t_1, t_3\}$ is a circuit of M . But then S is contained in a 4-element fan in $M \setminus d/t$ for each $t \in T$, which, by 4.1.1, implies that M has an N -detachable pair.

Recall that $\{s_1, s_2, t_1, t_2\}$ is a circuit of M/r_1 , and S is not in a 4-element fan of $M \setminus d/r_1$, so, in particular, s_3 is not in a triangle of $M \setminus d/r_1$. Suppose s_3 is in a triangle of M/r_1 . By orthogonality, and the previous paragraph, this triangle is $\{d, s_3, t_i\}$ for some $i \in \{1, 2\}$. By circuit elimination with the triangle $\{d, s_i, t_i\}$, we deduce that $\{s_3, t_i, s_i\}$ contains a circuit in M/r_1 ; a contradiction. So s_3 is not in a triangle of M/r_1 . By Lemma 4.4, $M/r_1/s_3$ is 3-connected, so M has an N -detachable pair by Lemma 4.9(i). \triangleleft

4.1.3. *Let R , S , and T be distinct triads of $M \setminus d$ that meet X . If $|S \cap T| = 1$, the set $S \cup T$ is not a cosegment of $M \setminus d$, and $R \cap (S \cup T) = \emptyset$, then M has an N -detachable pair.*

Subproof. Let $R = \{r_1, r_2, r_3\}$, $S = \{s_1, s_2, u\}$ and $T = \{t_1, t_2, u\}$, where the elements $r_1, r_2, r_3, s_1, s_2, t_1, t_2, u$ are pairwise distinct (see Figure 2b). Suppose that M has no N -detachable pairs. By 4.1.1, S is not contained in a 4-element fan in $M \setminus d/r_1$, say. By Tutte's Triangle Lemma, there are distinct elements $s, s' \in \{s_1, s_2, u\}$ such that $M \setminus d/r_1/s$ and $M \setminus d/r_1/s'$ are 3-connected. Since R and $S \cup T$ are disjoint, $M \setminus d/r_1/x$ has an N -minor for each $x \in S \cup T$, by Lemma 4.9(i).

First, suppose that $M \setminus d/r_1/s_1$ and $M \setminus d/r_1/s_2$ are 3-connected. Since $M/r_1/s_1$ and $M/r_1/s_2$ have N -minors, either M has an N -detachable pair, or M/r_1 has a triangle containing $\{s_1, d\}$ and a triangle containing $\{s_2, d\}$. As $T \cup d$ is a cocircuit of M/r_1 , each of these triangles meets T . Suppose these triangles are $\{s_1, d, t\}$ and $\{s_2, d, t'\}$, for $t, t' \in T$. Then $\{s_1, s_2, t, t'\}$ contains a circuit C of M/r_1 , by circuit elimination. Since $r_1 \in \text{cl}_M^*(\{r_2, r_3, d\})$, the circuit C is also a circuit of M , so $C = \{s_1, s_2, t, t'\}$. Now $r_{M \setminus d}(S \cup T) \leq 4$, and $r_{M \setminus d}^*(S \cup T) = 3$, so $S \cup T$ is 3-separating in $M \setminus d$, contradicting Lemma 4.12.

Now suppose that $M \setminus d/r_1/s_1$ and $M \setminus d/r_1/u$ are 3-connected. In this case, either M has an N -detachable pair, or M/r_1 has triangles containing $\{d, s_1\}$ and $\{d, u\}$. By orthogonality, the first of these triangles meets $\{t_1, t_2, u\}$. Suppose $\{d, s_1, u\}$ is a triangle of M/r_1 . It follows that $S \cup d$ is 3-separating, and $(\{d, s_1, u\}, \{s_2\}, E(M/r_1) - (S \cup d))$ is a cyclic 3-separation of M/r_1 , implying that $\text{si}(M/r_1/s_2)$ is 3-connected. Since s_2 is not in an N -grounded triangle, and S is not contained in a 4-element fan in $M \setminus d/r_1$, either $\{r_1, s_2\}$ is an N -detachable pair, or $\{d, s_2\}$ is contained in a triangle of M/r_1 that, by orthogonality, meets T . If this triangle is $\{d, s_2, u\}$, then $\{s_1, s_2, u\}$ contains a triangle by circuit elimination; a contradiction. So $\{d, s_2, t\}$ is a triangle of M/r_1 for some $t \in \{t_1, t_2\}$.

Now we may assume, up to swapping s_1 and s_2 , and t_1 and t_2 , that $\{d, s_1, t_1\}$ is a triangle of M/r_1 . If $M/r_1/s_i$ or $M/r_1/t_i$ is 3-connected for some $i \in \{1, 2\}$, then M has a contradictory N -detachable pair, so we may assume otherwise. Observe that $\{s_1, s_2\}$ and $\{t_1, t_2\}$ are distinct series classes of $M/r_1 \setminus d \setminus u$, otherwise X contains a 4-element cosegment of $M \setminus d$, contradicting Lemma 4.6. Applying Lemma 4.5 on M/r_1 , the element s_2 is also in a triangle that meets both $\{d, u\}$ and $\{t_1, t_2\}$. If this triangle contains u , then S is contained in a 4-element fan of M/r_1 ; a contradiction. But otherwise we have that $r_{M/r_1}(\{s_1, s_2, u, t_1, t_2\}) = 4$. Since $r_1 \notin \text{cl}(E(M) - \{r_2, r_3, d\})$, the set $\{s_1, s_2, u, t_1, t_2\}$ also has rank four in M , implying that $S \cup T$ is 3-separating in $M \setminus d$, which contradicts Lemma 4.12. \triangleleft

4.1.4. *Let R , S , and T be distinct triads of $M \setminus d$ that meet X , where the union of any two of these triads is not a cosegment of $M \setminus d$. If $|R \cap S \cap T| = 1$, then M has an N -detachable pair.*

Subproof. Suppose that $R = \{r_1, r_2, u\}$, $S = \{s_1, s_2, u\}$ and $T = \{t_1, t_2, u\}$ (see Figure 2c). Consider $M/t_1 \setminus d \setminus u$. Since $\{t_1, t_2\}$ is a series pair in $M \setminus d \setminus u$, the matroid $\text{co}(M/t_1 \setminus d \setminus u)$ is 3-connected. Observe that $\{r_1, r_2\}$ and $\{s_1, s_2\}$ are distinct series classes of $M/t_1 \setminus d \setminus u$. Now applying Lemma 4.5 to M/t_1 , either M has an N -detachable pair by Lemma 4.9(i), or M/t_1 has triangles $\{r_1, s_1, u\}$ and $\{r_2, s_2, d\}$, up to relabelling. So $\{r_1, s_1, t_1, u\}$ and $\{r_2, s_2, t_1, d\}$ are circuits in M . By Lemma 4.12, we may assume that $r(\{r_1, r_2, t_1, t_2\}) = 4$. Since $s_1 \in \text{cl}^*(\{s_2, u, d\})$, it follows that $r(\{r_1, r_2, s_1, t_1, t_2\}) = 5$. By symmetry, any 5-element subset of $\{r_1, r_2, s_1, s_2, t_1, t_2\}$ is independent.

Consider now M/t_2 . Applying Lemma 4.5, we see that M contains circuits $\{r_i, s_j, t_2, u\}$ and $\{r_{i'}, s_{j'}, t_2, d\}$, where $\{i, i'\} = \{1, 2\}$ and $\{j, j'\} = \{1, 2\}$. Suppose $j = 1$. Then $\{r_1, s_1, t_2, u\}$ or $\{r_2, s_1, t_2, u\}$ is a circuit. In either of these cases $\{t_1, t_2\} \subseteq \text{cl}(\{r_1, r_2, s_1, u\})$, thus $r(\{r_1, r_2, s_1, t_1, t_2\}) \leq 4$; a contradiction. Similarly, if $i = 1$, then $\{t_1, t_2\} \subseteq \text{cl}(\{s_1, s_2, r_1, u\})$; a contradiction. So $\{r_2, s_2, t_2, u\}$ and $\{r_1, s_1, t_2, d\}$ are circuits.

Now consider M/r_1 . Applying Lemma 4.5 once more, we arrive at the conclusion that $\{r_1, s_2\}$ is an N -detachable pair unless s_2 is in a triangle with exactly one element from each of $\{d, u\}$ and $\{t_1, t_2\}$. But the existence of this triangle implies that $s_2 \in \text{cl}_{M/r_1}(\{s_1, d, u, t_1, t_2\})$, and, due to the circuits $\{r_1, s_1, t_1, u\}$ and $\{r_1, s_1, t_2, d\}$ of M , that $r_{M/r_1}(\{s_1, s_2, t_1, t_2\}) = 3$. Hence, by Lemma 4.12, M has an N -detachable pair. \triangleleft

4.1.5. *Let R , S , and T be distinct triads of $M \setminus d$ that meet X , where the union of any two of these triads is not a cosegment of $M \setminus d$. If $|S \cap T| = 1$, and $|R \cap (S \triangle T)| = 1$, then M has an N -detachable pair.*

Subproof. We may assume that $T \cap R = \emptyset$ and $|S \cap R| = 1$. Thus, we have triads $R = \{r_1, r_2, r_3\}$, $T = \{t_1, t_2, t_3\}$ and $S = \{r_3, s, t_3\}$ (see Figure 2d). We begin by handling the case where X consists of more than these three triads.

4.1.5.1. *If $X - (R \cup S \cup T) \neq \emptyset$, then M has an N -detachable pair.*

Subproof. Suppose that $q \in X - (R \cup S \cup T)$. Recall that every $x \in X$ is in a triad of $M \setminus d$, so q is in a triad Q . Suppose Q intersects one of the triads R , S , or T , in two elements. Then $R \cup S \cup T \cup q$ is the union of three triads that meet X : two triads that intersect in one element, and a third triad disjoint from the other two. In this case 4.1.3 implies that M has an N -detachable pair. So we may assume that Q intersects each of the triads R , S and T in no more than a single element. By Lemmas 4.6 and 4.7, we may also assume that Q is not in a cosegment with R , S , or T , otherwise M has an N -detachable pair.

By 4.1.4, we have that $\{r_3, t_3\} \cap Q = \emptyset$. If $Q \cap (R \cup S \cup T) = \{s\}$, or $Q \cap (R \cup S \cup T) = \emptyset$, then Q , R and T are disjoint, so M has an N -detachable pair by 4.1.2. So we may assume, without loss of generality, that $Q \cap \{t_1, t_2\} \neq \emptyset$. Now, if $Q \cap R = \emptyset$, then R is disjoint from $Q \cup T$, so M has an N -detachable pair by 4.1.3. Thus we may assume, without loss of generality, that $Q = \{q, r_2, t_2\}$. If there exists some other $q' \in X - (R \cup S \cup T \cup q)$, with corresponding triad Q' , then again we deduce that $Q' = \{q', r, t\}$ for $r \in \{r_1, r_2\}$ and $t \in \{t_1, t_2\}$. Suppose $\{r, t\} = \{r_2, t_2\}$. Then $\{q, q', t_2\}$ is a triad, so, by applying 4.1.3 using the triads R , $\{q, q', t_2\}$, and T , we deduce that M has an N -detachable pair. So we may assume that $\{r, t\} \neq \{r_2, t_2\}$. But now, due to the disjoint triad S , we obtain an N -detachable pair at the hands of either 4.1.2 or 4.1.3. Therefore $X \subseteq R \cup S \cup T \cup q$.

Next, we claim that, up to a cyclic shift on the labels given to R , S , T , and Q , we may assume that S is not in a 4-element fan in $M \setminus d / q$. Observe that R and T are disjoint triads, as are S and Q . By 4.1.1, we may assume that $\square(R, T) = \square(S, Q) = 1$. If S is in a 4-element fan in $M \setminus d / q$, then, by orthogonality, there is a 4-element circuit $C = \{q, q_2, s', s''\}$ in $M \setminus d$ for $q_2 \in Q - q$ and $s', s'' \in S$. Moreover, $s \in \{s', s''\}$, for otherwise C intersects either R or T in a single element, contradicting orthogonality. Again by orthogonality, C contains either $\{t_2, t_3\}$ or $\{r_2, r_3\}$.

Due to symmetry, if R is not in a 4-element fan in $M \setminus d / t_1$, then, after a cyclic shift on the labels R , S , T , and Q , the claim holds. So, repeating the argument used on S and Q , but this time for R and T , we reveal a circuit containing $\{r_1, t_1\}$ and either $\{r_2, t_2\}$ or $\{r_3, t_3\}$. Without loss of generality, we may assume that $M \setminus d$ has circuits $\{s, q, t_2, t_3\}$ and $\{t_1, t_2, r_1, r_2\}$.

By Lemma 4.12 we may assume that $r_{M \setminus d}(Q \triangle T) = 4$. So $r_{M \setminus d}(Q \cup T) \leq 5$, and it follows, due to the existence of the circuits $\{s, q, t_2, t_3\}$ and $\{t_1, t_2, r_1, r_2\}$, that $r_{M \setminus d}(X' - r_3) \leq 5$ where $X' = R \cup S \cup T \cup Q$. But since each element of $\{s, t_1, r_1, q\}$ is in a triad of $M \setminus d$ where the other elements are in $X' - \{s, t_1, r_1, q\}$, we have $r(E(M \setminus d) - X') \leq r(M \setminus d) - 4$. Since $\lambda_{M \setminus d}(X) = 2$, it follows, by uncrossing, that $\lambda_{M \setminus d}(X') = 2$, and we deduce that $(X' - r_3, \{r_3\}, E(M \setminus d) - X')$ is a cyclic 3-separation of $M \setminus d$. Thus $\text{co}(M \setminus d \setminus r_3)$ is not 3-connected, implying that $r_3 \notin X$. Now, for every $x \in X' - r_3$, we have $x \in \text{cl}(X' - \{r_3, x\}) \cap \text{cl}_{M \setminus d}^*(X' - \{r_3, x\})$, so $X' - \{r_3, x\}$ is not 3-separating. As X is 3-separating, $|X| < |X' - r_3| - 1 = 6$. By Lemma 4.7, $X = \{r_1, r_2, s, t_3, t_i\}$ for some $i \in \{1, 2\}$. But then X does not contain a triad; a contradiction. This proves the claim, so henceforth we assume that S is not in a 4-element fan in $M \setminus d / q$.

Now we are in a position where we can apply Tutte's Triangle Lemma on the 3-connected matroid $M \setminus d / q$. We have two possible scenarios: either $M \setminus d / q / r_3$ and $M \setminus d / q / t_3$ are 3-connected, or $M \setminus d / q / s$ is 3-connected. Assume that the first of these possibilities holds. Then M / q contains triangles $\{d, r', r_3\}$ and $\{d, t_3, t'\}$, say.

Suppose that the triangles $\{d, r', r_3\}$ and $\{d, t_3, t'\}$ coincide; that is, $r' = t_3$ and $t' = r_3$. Now $\{d, r_3, t_3\}$ is a triangle and a triad of $M/q \setminus s$, so it is 2-separating and, by Bixby's Lemma, $M/q \setminus s$ is 3-connected up to parallel pairs. However, s is not in a triangle of M/q and thus $M/q \setminus s$ is 3-connected. Hence, by Lemma 4.9(i), M has an N -detachable pair. Therefore we may assume that $\{d, r', r_3\}$ and $\{d, t_3, t'\}$ are distinct.

Now, by the circuit elimination axiom, there is a circuit of M/q contained in $\{r_3, t_3, r', t'\}$. By orthogonality with R and T , we deduce that $\{r_3, t_3, r', t'\}$ is a circuit of $M \setminus d/q$ where $r' \in \{r_1, r_2\}$ and $t' \in \{t_1, t_2\}$.

We now switch to the dual: let $M' = M^*/d$ and consider $M' \setminus q \setminus s$. Since the triangle S is not in a 4-element fan in $M' \setminus q$, the matroid $M' \setminus q \setminus s$ does not have any series pairs. Suppose that (A, B) is a 2-separation of $M' \setminus q \setminus s$. We may assume that $r_3 \in A$ and $t_3 \in B$, for otherwise we would have $s \in \text{cl}_{M' \setminus q}(A)$ or $s \in \text{cl}_{M' \setminus q}(B)$, which would imply that $M' \setminus q$ has a 2-separation; a contradiction. So $|R \cap A| \geq 2$ and $|T \cap B| \geq 2$, for otherwise $r_3 \in \text{cl}_{M' \setminus q}(B)$ or $t_3 \in \text{cl}_{M' \setminus q}(A)$ in which case again $M' \setminus q$ has a 2-separation. By Lemma 3.17, $(\text{fcl}(A), B - \text{fcl}(A))$ and $(A - \text{fcl}(B), \text{fcl}(B))$ are also 2-separations. Hence we may assume that $R \subseteq A$ and $T \subseteq B$.

Let $X' = X \cup (R \cup S \cup T)$, and observe that X' is 3-separating in M' by uncrossing. Let $W = E(M' \setminus q \setminus s) - X'$. As $|W| \geq 3$, we may assume that $|A \cap W| \geq 2$. Denote $\lambda_{M' \setminus q \setminus s}$ by λ . By the submodularity of λ , we have

$$\lambda(A \cap W) + \lambda(A \cup W) \leq \lambda(A) + \lambda(W) = 3.$$

So either $A \cap W$ or $A \cup W$ is 2-separating in $M' \setminus q \setminus s$. The first possibility implies that $(X' - \{q, s\}) \cup B$ is 2-separating, but since $s \in \text{cl}_{M' \setminus q}(X' - \{q, s\})$, this implies that $A \cap W$ is a 2-separation in $M' \setminus q$; a contradiction. So $A \cup W$ is 2-separating in $M' \setminus q \setminus s$.

Now $B \cap X'$ is a triangle in $M' \setminus q \setminus s$, implying that $r_{M'}(A \cup W) = r(M') - 1$. Note that since $X' - q$ is 3-separating in the 3-connected matroid $M' \setminus q$, and $s \in \text{cl}_{M'}(X' - \{q, s\})$, we have that $X' - \{q, s\}$ is exactly 3-separating in $M' \setminus q \setminus s$, otherwise $M' \setminus q$ is not 3-connected. If $X' - \{q, s\}$ has rank three in M' , then $r_{M'}(R \cup S) = 3$, contradicting Lemma 4.12. So $X' - \{q, s\}$ has rank four, and $r_{M'}(W) = r(M') - 2$. Pick r so that $\{r, r'\} = \{r_1, r_2\}$. If $r \notin X$, then $r \in \text{cl}_{M'}^*(X)$, so $r \notin \text{cl}_{M'}(W)$. On the other hand, if $r \in X$, then $r \notin \text{cl}_{M'}(W)$, for otherwise $\text{si}(M'/r)$ fails to be 3-connected; a contradiction. So $r_{M'}(W \cup r) = r(M') - 1$. As r_3 is in the cocircuit $\{r', t', r_3, t_3\}$, it then follows that $r_{M'}(W \cup \{r, r_3\}) = r(M')$. But this is contradictory, since $r_{M'}(A \cup W) = r(M') - 1$. We are left to conclude that $M' \setminus q \setminus s$ is 3-connected.

Returning to the application of Tutte's Triangle Lemma, we now have that $M \setminus d/q \setminus s$ is 3-connected. So either $M/q \setminus s$ is 3-connected, in which case M has an N -detachable pair by Lemma 4.9(i), or $\{d, s, r'\}$ is a triangle in M/q . But since $\{d, r_1, r_2, r_3\}$ and $\{d, t_1, t_2, t_3\}$ are cocircuits of M/q , orthogonality implies that $r' \in R \cap T$; a contradiction. \blacksquare

Let $X' = R \cup S \cup T$ and $W' = E(M \setminus d) - X'$. With 4.1.5.1 in hand, we henceforth assume that $X \subseteq X'$. The next step is to show the following:

4.1.5.2. *Suppose that $M \setminus d/t_2/t_3$ has an N -minor. If the triad S is closed in $M \setminus d/t_2$, then M has an N -detachable pair.*

Subproof. Suppose that S is closed in $M \setminus d/t_2$. Clearly $M \setminus d/t_2$ is 3-connected when $t_2 \in X$. Suppose $t_2 \notin X$. We may assume that $T - t_2 \subseteq X$, by Lemma 4.7. Then $(X, \{t_2\}, E(M \setminus d) - (X \cup t_2))$ is a cyclic 3-separation of $M \setminus d$, so $\text{si}(M \setminus d/t_2)$ is 3-connected, by Bixby's Lemma. But t_2 is N -contractible in $M \setminus d$, so it is not contained in an N -grounded triangle. Since every triangle of M is N -grounded, we deduce that $M \setminus d/t_2$ is 3-connected.

Now, by Tutte's Triangle Lemma, either

- (I) $M \setminus d/t_2/t_3$ and $M \setminus d/t_2/s$ are 3-connected, or
- (II) $M \setminus d/t_2/r_3$ and $M \setminus d/t_2/s'$ are 3-connected, but $M \setminus d/t_2/s''$ is not 3-connected, for some $\{s', s''\} = \{s, t_3\}$.

We first establish some properties that hold in either case. Observe that M/t_2 is 3-connected, since $M \setminus d/t_2$ is 3-connected and $\{t_2, d\}$ is not contained in a triangle of M . We may assume that $r_{M \setminus d}^*(X) = r_{M \setminus d}^*(X')$, by Lemma 4.7. Hence $r_{M \setminus d}^*(W') \geq 3$.

Our first claim is that we may assume that d is not in a triangle of M/t_2 with two elements from S . Suppose d is in such a triangle U . By orthogonality, $r_3 \in U$. So let $S - U = \{s'\}$ and let $U = \{d, r_3, s''\}$, where $\{s', s''\} = \{s, t_3\}$. Since $E(M/t_2) - (S \cup d)$ is a hyperplane of $M/t_2 \setminus s'$, we have that $(U, E(M/t_2 \setminus s') - U)$ is a 2-separation in $M/t_2 \setminus s'$, so $\text{co}(M/t_2 \setminus s')$ is not 3-connected. By Bixby's Lemma, $\text{si}(M/t_2/s')$ is 3-connected. If $M/t_2/s'$ is 3-connected, then M has an N -detachable pair as required (using Lemma 4.9(i) when $s' \neq t_3$). Otherwise, if $M/t_2/s'$ has a parallel pair that does not contain d , then, by orthogonality, S is not closed in $M \setminus d/t_2$; a contradiction. Suppose $M/t_2/s'$ has a parallel pair $\{d, q\}$. Then M/t_2 has triangles U and $\{s', d, q\}$, so, by circuit elimination, $S \cup q$ contains a circuit of M/t_2 . Since S is closed in $M \setminus d/t_2$, the set S contains a circuit of M/t_2 . But then S is a triangle and a triad of $M \setminus d/t_2$, so this matroid is not 3-connected; a contradiction. This proves the first claim.

Let $s' \in \{t_3, s\}$ such that $M \setminus d/t_2/s'$ is 3-connected. Then either $M/t_2/s'$ is 3-connected, in which case M has an N -detachable pair (using Lemma 4.9(i) when $s' = s$), or there exists a triangle $\{s', d, \alpha\}$ in M/t_2 . As d blocks the triad R of $M \setminus d$, the set $R \cup d$ is a cocircuit in M/t_2 , and so, by orthogonality, $\alpha \in R$. Since d is not in a triangle of M/t_2 with $\{s', r_3\} \subseteq S$, we have that $\alpha \neq r_3$. Thus $\{d, s', \alpha\}$ is a triangle of M/t_2 for some $\alpha \in \{r_1, r_2\}$.

Suppose (I) holds. Since $M \setminus d/t_2/t_3$ and $M \setminus d/t_2/s$ are 3-connected, we may assume that M/t_2 has triangles $\{d, t_3, \alpha\}$ and $\{d, s, \beta\}$ for some $\alpha, \beta \in \{r_1, r_2\}$. By circuit elimination, $\{\alpha, \beta, s, t_3\}$ contains a circuit of M/t_2 . It

follows by orthogonality that $\{\alpha, \beta\} = \{r_1, r_2\}$, so we may assume that $\{d, t_3, r_1\}$, $\{d, s, r_2\}$, and $\{r_1, r_2, s, t_3\}$ are circuits in M/t_2 . As S and R are triads of $M/t_2 \setminus d$, we have that $r_{M/t_2 \setminus d}(W' \cup t_1) \leq r(M/t_2 \setminus d) - 2$. But now $r_{M/t_2}(W' \cup t_1) \leq r(M/t_2 \setminus r_3) - 2$ and $r_{M/t_2}(\{r_1, r_2, s, t_3, d\}) = 3$, so that $\lambda_{M/t_2 \setminus r_3}(\{r_1, r_2, s, t_3, d\}) \leq 1$.

By Bixby's Lemma, $\text{si}(M/t_2/r_3)$ is 3-connected, hence either $M/t_2/r_3$ is 3-connected, in which case M has an N -detachable pair by Lemma 4.9(i), or r_3 is contained in some triangle U of M/t_2 . Since $R \cup d$ and $S \cup d$ are both cocircuits of M/t_2 containing r_3 , orthogonality and the fact that S is closed in $M \setminus d/t_2$ implies that U also contains d . The final element of U cannot be in $\{s, t_3\}$, and also cannot be in $\{r_1, r_2\}$, otherwise $(R \cup S \cup d, W' \cup t_1)$ is a 2-separation of M/t_2 . So either U contains t_1 , or U meets W' .

We first consider the latter case. Let $U = \{d, r_3, w\}$ for $w \in W'$. Then $\{t_2, d, r_3, w\}$ is a circuit of M , and, by circuit elimination with $\{t_2, d, s, r_2\}$, we have that $\{r_3, r_2, d, s, w\}$ contains a circuit. But $d \in \text{cl}^*(T)$, so $\{r_3, r_2, s, w\}$ is a circuit. Since $\{w, r_3\}$ is a parallel pair in $M \setminus d/s/r_2$, which has an N -minor by Lemma 4.9(i), w is N -deletable in $M \setminus d$. As $\text{co}(M \setminus d \setminus w)$ is 3-connected and M has no N -detachable pairs, w is in a triad of $M \setminus d$ that contains an element $y \in \{s, r_2, r_3\}$ and an element $w' \in W' - w$. If $y \in X$, then $\text{co}(M \setminus d \setminus y)$ is 3-connected; a contradiction. So $y \in \{s, r_2, r_3\} - X$. By orthogonality between the triad $\{y, w, w'\}$ and the circuit $\{r_1, r_2, s, t_3\}$ of M/t_2 , we deduce that $y = r_3$. Now $(R \cup S) - r_3 \subseteq X$. If $R \cup T$ contains a 4-element cosegment, then by orthogonality with the circuit $\{r_3, r_2, s, w\}$, the cosegment is $T \cup r_1$. But then $\{d, r_1, t_2, t_3\}$ is a cocircuit of M that intersects the triangle $\{d, s, r_2\}$ in a single element; a contradiction. As X contains a triad, it now follows that $T \subseteq X$, and hence $X = X' - r_3$.

Now $M/r_1/r_3$ has an N -minor by Lemma 4.9(i). In what follows, we frequently use the fact that no triangle of M meets X , and $r_3 \notin \text{cl}(X)$, for otherwise $\lambda_{M \setminus d}(X \cup r_3) = 1$. Observe that $(X - r_1, \{r_3\}, W')$ is a cyclic 3-separation in the 3-connected matroid $M \setminus d/r_1$, so $\text{si}(M \setminus d/r_1/r_3)$ is 3-connected. If $\{r_1, r_3\}$ is contained in a 4-element circuit C of M , then, by orthogonality, either $d \in C$, or C contains an element in $\{w, w'\}$ and an element in $\{s, t_3\}$. If $w \in C$, then by circuit elimination with $\{r_2, r_3, s, w\}$, the set $R \cup S$ contains a circuit, in which case, by orthogonality, $R \cup s$ is a circuit; a contradiction. If $w' \in C$, then $w' \in \text{cl}(X' \cup w) \cap \text{cl}_{M \setminus d}^*(X' \cup w)$, where $X' \cup w$ is 3-separating since $w \in \text{cl}(X')$; a contradiction. So $d \in C$. Now $C = \{r_1, r_3, d, t_i\}$ for some $i \in [3]$. If $i = 2$, then $R \cup \{s, t_2\}$ contains a circuit, by circuit elimination with $\{d, s, r_2, t_2\}$, in which case, by orthogonality, this circuit is $R \cup s$; a contradiction. On the other hand, if $i = 3$, then, by circuit elimination with $\{d, t_3, r_1, t_2\}$, the set $\{r_1, r_3, t_2, t_3\}$ is a circuit, which is again contradictory. So $C = \{r_1, r_3, d, t_1\}$, in which case $T \cup \{r_1, r_3\}$ is a circuit, by circuit elimination with $\{d, t_3, r_1, t_2\}$. This circuit cannot contain r_3 , so $T \cup r_1$ is a circuit, contradicting orthogonality. We deduce that $\{r_1, r_3\}$

is not contained in a 4-element circuit of M , so $M/r_1/r_3$ is 3-connected, and hence M has an N -detachable pair.

Now we may assume that the triangle U is $\{d, t_1, r_3\}$. Note that W' and $W' \cup t_1$ are each exactly 3-separating in M/t_2 , and thus $t_1 \in \text{cl}_{M/t_2}(R \cup S \cup d) \cap \text{cl}_{M/t_2}(W')$. Suppose that $M/t_2 \setminus r_2$ is not 3-connected. Then, as M/t_2 has no series pairs, $M/t_2 \setminus r_2$ has a non-trivial 2-separation (P, Q) . By Lemma 3.17, we may assume that the triad $\{d, r_1, r_3\}$ is contained in P . Likewise, as $\{d, t_3, r_1\}$ is a triangle in $M/t_2 \setminus r_2$, we may assume that $t_3 \in P$, and, as $S \cup d$ is a cocircuit, that $s \in P$. But $r_2 \in \text{cl}_{M/t_2}(\{s, d\})$, so $(P \cup r_2, Q)$ is a 2-separation in the 3-connected matroid M/t_2 ; a contradiction. So $M/t_2 \setminus r_2$ is 3-connected.

By Bixby's Lemma, $M/t_2 \setminus r_2 \setminus t_1$ is now 3-connected unless t_1 is in a triad Γ of $M/t_2 \setminus r_2$ that meets both W' and $\{r_1, r_3, d, s, t_3\}$. Let $\Gamma \cap W' = \{w\}$. If $d \notin \Gamma$, then, as $\Gamma \cup r_2$ is a cocircuit of M/t_2 , orthogonality implies that this cocircuit intersects the triangles $\{d, s, r_2\}$ and $\{d, t_1, r_3\}$ in at least two elements; a contradiction. So we may assume that $\Gamma = \{w, d, t_1\}$. But, recalling that $\{d, t_3, r_1\}$ is a triangle of M/t_2 , this also contradicts orthogonality. Therefore $M/t_2 \setminus r_2 \setminus t_1$ is 3-connected.

Suppose that $M \setminus r_2 \setminus t_1$ is not 3-connected. Then M has a cocircuit $C^* = \{t_1, t_2, r_2, \delta\}$ where $\delta \in \{r_1, t_3, d\}$, by orthogonality. Recall that M has the following cocircuits: $C_T = T \cup d$, $C_R = R \cup d$, and $C_S = S \cup d$. If $\delta \in C_S$, then $r_1 \notin C^* \cup C_S \cup C_T$ so that $W' = E(M) - (C^* \cup C_S \cup C_T \cup C_R)$ is a flat of rank at most $r(M) - 4$. If $\delta \notin C_S$, then $\delta = r_1$, so that $s \notin C^* \cup C_R \cup C_T$, again implying that W' is a flat of rank at most $r(M) - 4$. But W' is exactly 3-separating in M , and, due to the triangles $\{d, t_3, r_1\}$, $\{d, s, r_2\}$, and $\{d, t_1, r_3\}$ of M/t_2 , we have $r_M(X') \leq 5$, contradicting the fact that M is 3-connected. So $M \setminus r_2 \setminus t_1$ is 3-connected.

It remains to show that $M \setminus r_2 \setminus t_1$ has an N -minor. First we show that $X = X'$. Recall that $\{r_1, r_2, s, t_3\}$ is a circuit of M/t_2 . By orthogonality, $\{r_1, r_2, s, t_3, t_2\}$ is a circuit of M . Recall also that $\{d, t_1, r_3\}$ and $\{d, s, r_2\}$ are triangles of M/t_2 . By circuit elimination, M has a circuit contained in $\{t_1, t_2, s, r_2, r_3\}$. By orthogonality, and since each triangle of M is N -grounded, $\{t_1, t_2, s, r_2, r_3\}$ is a circuit of M . So $r(X') \leq 5$. Now

$$\begin{aligned} \lambda_{M \setminus d}(X') &= r(X') + r_{M \setminus d}^*(X') - |X'| \\ &\leq 5 + 4 - 7 = 2. \end{aligned}$$

Hence $r(X') = 5$ and $r_{M \setminus d}^*(X') = 4$. Moreover, for each $x \in X'$, we have $x \in \text{cl}(X' - x) \cap \text{cl}_{M \setminus d}^*(X' - x)$, so $X' - x$ is not 3-separating. Suppose $X \subsetneq X'$. Then, by Lemma 4.7 and since X contains a triad, we may assume that $|X| = 5$, and hence $r(X) \geq 4$. But $r^*(X) = r^*(X') = 4$, so $\lambda_{M \setminus d}(X) \geq 3$; a contradiction. We deduce that $X = X'$.

Now $r_2, t_1 \in X$ and $r_2 \in \text{cl}(X - \{r_2, t_1\})$. Hence $\{r_2, t_1\}$ is an N -detachable pair by Lemma 4.10. This proves 4.1.5.2 when (I) holds.

Now suppose (II) holds. Since $M \setminus d/t_2/r_3$ is 3-connected, either M has an N -detachable pair, or there exists a triangle $\{d, r_3, \gamma\}$ in M/t_2 . For some $\{s', s''\} = \{s, t_3\}$, the matroid $M \setminus d/t_2/s'$ is 3-connected but $M \setminus d/t_2/s''$ is not 3-connected. Since $M \setminus d/t_2/s'$ is 3-connected we may assume that M/t_2 has a triangle $\{d, s', \alpha\}$ for some $\alpha \in \{r_1, r_2\}$. Without loss of generality let $\alpha = r_1$.

Consider $M/t_2/s''$. If this matroid is 3-connected, then M has an N -detachable pair (by Lemma 4.9(i) when $s'' = s$). So we may assume that $M/t_2/s''$ is not 3-connected. Note that there are no triangles in M/t_2 that contain s'' but not d , by orthogonality and since S is closed in $M \setminus d/t_2$. Thus, if $M/t_2/s''$ is 3-connected up to parallel classes, then this matroid has only a single parallel pair, which contains d , so $M \setminus d/t_2/s''$ is 3-connected; a contradiction. So $M/t_2/s''$ has a 2-separation (P, Q) where we may assume that either P or Q is fully closed, by Lemma 3.17. Thus, to begin with, we may assume that the triangle $\{d, s', r_1\} \subseteq P$ and P is fully closed. Now $r_3 \notin P$, as otherwise $s'' \in \text{cl}_{M/t_2}^*(P)$, which would result in a 2-separation $(P \cup s'', Q)$ in M/t_2 . So $r_3 \in Q$ and thus $\{\gamma, r_2\} \subseteq Q$ as well, since $\{d, r_3, \gamma\}$ is a triangle and $R \cup d$ is a cocircuit. But now $d \in \text{cl}_{M/t_2/s''}(Q)$, and $r_1 \in \text{cl}_{M/t_2/s''}^*(Q \cup d)$, so that $(P', Q') = (P - \text{fcl}_{M/t_2/s''}(Q), \text{fcl}_{M/t_2/s''}(Q))$ is a 2-separation of $M/t_2/s''$ in which $R \cup \{s', \gamma, d\} \subseteq Q'$. But now $s'' \in \text{cl}_{M/t_2}^*(Q')$ and $(P', Q' \cup s'')$ is a 2-separation in the 3-connected matroid M/t_2 ; a contradiction. This completes the proof of 4.1.5.2. \blacksquare

4.1.5.3. *Suppose that $M/t_1/t_3$ and $M/t_2/t_3$ have N -minors, and M has no N -detachable pairs. Then, for each $t \in \{t_1, t_2\}$, there exists $w_t \in E(M \setminus d)$ such that $S \cup \{t, w_t\}$ contains a circuit. Moreover, if $w_{t_2} \notin X'$, then*

- $\{s, t_2, t_3, w_{t_2}\}$ is a circuit,
- $\{y, w_{t_2}, w'\}$ is a triad of $M \setminus d$ for some $y \in \{s, t_2, t_3\} - X$ and $w' \in W' - w_{t_2}$,
- $w_{t_1} \in X'$, and
- either $R \cup t_1$ is a cosegment of $M \setminus d$, or $X = X' - y$.

Subproof. By 4.1.5.2 and symmetry, there exists $w_t \in E(M \setminus d) - (S \cup t)$ such that $S \cup \{w_t, t\}$ contains a circuit, for each $t \in \{t_1, t_2\}$. Suppose that $w_{t_2} \notin X'$. Then, by orthogonality, $\{s, t_2, t_3, w_{t_2}\}$ is a circuit. Note that $M \setminus d/s/t_2$ has an N -minor by Lemma 4.9(i), so w_{t_2} is N -deletable in $M \setminus d$. Since $\text{co}(M \setminus d \setminus w_{t_2})$ is 3-connected and M has no N -detachable pairs, w_{t_2} is in a triad of $M \setminus d$ that contains an element $y \in \{s, t_2, t_3\}$ and an element $w' \in W' - w_{t_2}$. If $y \in X$, then $\text{co}(M \setminus d \setminus y)$ is 3-connected; a contradiction. So $y \in \{s, t_2, t_3\} - X$.

Suppose that $w_{t_1} \notin X'$. Then, by symmetry, $\{s, t_1, t_3, w_{t_1}\}$ is a circuit, and w_{t_1} is in a triad $\{y'', w_{t_1}, w''\}$ for some $y'' \in \{s, t_1, t_3\}$ and $w'' \in W' - w_{t_1}$. If $w_{t_1} = w_{t_2}$, then $T \cup s$ contains a circuit, by circuit elimination. By orthogonality with the triad $\{y, w_{t_2}, w'\}$, this circuit is a triangle. But this triangle meets X ; a contradiction. So $w_{t_1} \neq w_{t_2}$. If $y \neq y''$, then, as

$y, y'' \notin X$ and by Lemma 4.7, we may assume that $\{y, y''\} = \{s, t_2\}$, so $y = t_2$ and $y'' = s$. But then the triad $\{s, w_{t_1}, w''\}$ meets the circuit $\{s, t_2, t_3, w_{t_2}\}$, so $w_{t_2} = w''$ by orthogonality. Now $w_{t_1}, w'' \in \text{cl}(X')$. Hence $X' \cup w_{t_1}$ is 3-separating, but then, due to the triad $\{y'', w_{t_1}, w''\}$, we have that $w'' \in \text{cl}(X' \cup w_{t_1}) \cap \text{cl}_{M \setminus d}^*(X' \cup w_{t_1})$. Since $M \setminus d$ is 3-connected, this implies $|W'| = 3$, but then W' is a triangle, contradicting that $r^*(W') \geq 3$. We deduce that $y = y''$, and hence $y \in \{s, t_3\}$. Now, by orthogonality between the circuit $\{s, t_1, t_3, w_{t_1}\}$ and the triad $\{y, w_{t_2}, w'\}$, we have $w_{t_1} = w'$. As before, since $w_{t_1}, w_{t_2} \in \text{cl}(X')$ and $w_{t_1} \in \text{cl}_{M \setminus d}^*(X' \cup w_{t_2})$, this is contradictory. This proves that $w_{t_1} \in X'$.

Recall that $r_{M \setminus d}^*(X) = r_{M \setminus d}^*(X')$. Suppose that $r_{M \setminus d}^*(X) \geq 4$. Then the triad Γ contained in X is either R , S , or T . Suppose also that $|X| \leq 5$. Then $r(X) \leq 3$, since $\lambda_{M \setminus d}(X) = 2$. Since X does not contain any triangles, $\Gamma \cup x$ is a circuit for every $x \in X - \Gamma$; but this contradicts orthogonality. We deduce that $|X| = 6$, and hence $X = X' - y$.

Now suppose $r_{M \setminus d}^*(X) = 3$. Then $\{t_3, r_3, r_1\}$ cospans X' in $M \setminus d$, so $\{t_3, r_3, r_1, t_1\}$ contains a cocircuit. By orthogonality, this cocircuit does not meet the circuit $\{s, t_2, t_3, w_{t_2}\}$, so $\{r_1, r_3, t_1\}$ is a triad, and hence $R \cup t_1$ is a cosegment of $M \setminus d$ as required. \blacksquare

4.1.5.4. *Either M has an N -detachable pair, or, up to swapping R and T , for each $i \in \{1, 2\}$ there exists $w_{t_i} \in X'$ such that $S \cup \{t_i, w_{t_i}\}$ contains a circuit.*

Subproof. Suppose that $M/t_i/t_3$ does not have an N -minor for some $i \in \{1, 2\}$. Then, by Lemma 4.9(ii), we may assume that $M/r_1/r_3$ and $M/r_2/r_3$ have N -minors, for otherwise M has an N -detachable pair. So, up to swapping R and T , we may assume that $M/t_1/t_3$ and $M/t_2/t_3$ have N -minors.

Now, by 4.1.5.2, we may assume that for each $i \in \{1, 2\}$ there exists w_{t_i} such that $S \cup \{t_i, w_{t_i}\}$ contains a circuit. If $\{w_{t_1}, w_{t_2}\} \subseteq X'$, then 4.1.5.4 holds; so assume that $w_{t_2} \notin X'$. Then, by 4.1.5.3, $w_{t_1} \in X'$, and either $R \subseteq X$, or $R \cup t_1$ is a cosegment of $M \setminus d$. In either case, Lemma 4.9(i) implies that $M/r_i/r_3$ has an N -minor for each $i \in \{1, 2\}$. By 4.1.5.2 and symmetry, for each $i \in \{1, 2\}$ there exists w_{r_i} such that $S \cup \{r_i, w_{r_i}\}$ contains a circuit. If $\{w_{r_1}, w_{r_2}\} \subseteq X'$, then 4.1.5.4 holds, after swapping R and T . So we may assume, without loss of generality, that $w_{r_2} \notin X'$.

We can now apply 4.1.5.3 a second time, with R in the role of T . Then $\{s, r_2, r_3, w_{r_2}\}$ is a circuit and $\{y'', w_{r_2}, w''\}$ is a triad of $M \setminus d$ for some $y'' \in \{s, r_2, r_3\} - X$ and $w'' \in W' - w_{r_2}$. By orthogonality, $R \cup t_1$ is not a cosegment of $M \setminus d$, so $R \subseteq X$, and hence $y'' = s$. If $w_{r_2} \neq w_{t_2}$, then, as $\{s, t_2, t_3, w_{t_2}\}$ is a circuit, $w'' = w_{t_2}$, so $\{s, w_{r_2}, w_{t_2}\}$ is a triad. Then $X' \cup w_{r_2}$ is 3-separating, and $w_{t_2} \in \text{cl}(X' \cup w_{r_2}) \cap \text{cl}_{M \setminus d}^*(X' \cup w_{r_2})$; a contradiction. So $w_{r_2} = w_{t_2}$. Now, by circuit elimination, $S \cup \{t_2, r_2\}$ contains a circuit. Since $w_{t_1} \in X'$, this completes the proof of 4.1.5.4. \blacksquare

4.1.5.5. *We may assume that both $S \cup \{r_1, t_1\}$ and $S \cup \{r_2, t_2\}$ contain circuits of $M \setminus d$.*

Subproof. Suppose that $S \cup t_1$ contains a circuit in $M \setminus d/t_2$. As $r_3 \in \text{cl}_{M \setminus d/t_2}^* (\{r_1, r_2\})$, the set $\{t_1, t_3, s\}$ is a triangle in $M \setminus d/t_2$. Hence $\{t_1, t_2, t_3, s\}$ is a circuit of $M \setminus d$. But now $(T, \{s\}, E(M \setminus d) - (T \cup s))$ is a vertical 3-separation, and $M \setminus d/s$ is not 3-connected; so $s \notin X$. By Lemma 4.7, we may assume that $S - s \subseteq X$ and $|T \cap X| \geq 2$. Then, by uncrossing, $X \cup T$ is 3-separating. But $s \in \text{cl}(X \cup T) \cap \text{cl}_{M \setminus d}^*(X \cup T)$; a contradiction.

By 4.1.5.4, we may assume that, for each $i \in \{1, 2\}$, there exists $w_{t_i} \in X'$ such that $S \cup \{t_i, w_{t_i}\}$ contains a circuit. From the previous paragraph, and symmetry, $w_{t_1}, w_{t_2} \in \{r_1, r_2\}$. Now, up to swapping the labels on r_1 and r_2 , 4.1.5.5 holds unless r_1 is in a circuit C_1 contained in $S \cup \{t_1, r_1\}$ and a circuit C_2 contained in $S \cup \{t_2, r_1\}$. Suppose we are in the exceptional case. Note that $S \cup r_1$ does not contain a circuit, by orthogonality and since no element of X is in a triangle. So $C_1 \neq C_2$. By circuit elimination, there is a circuit contained in $S \cup \{t_1, t_2\}$; a contradiction. This completes the proof. ■

By 4.1.5.5, $\{r_1, r_3, t_1, t_3\}$ and $\{r_2, r_3, t_2, t_3\}$ each contain circuits in $M \setminus d/s$. In fact, by orthogonality and since no triangles meet X , $\{r_1, r_3, t_1, t_3\}$ and $\{r_2, r_3, t_2, t_3\}$ are circuits of $M \setminus d/s$. If $\{r_1, r_3, t_1, t_3\}$ and $\{r_2, r_3, t_2, t_3\}$ are circuits of $M \setminus d$, then R and T are disjoint triads of $M \setminus d$ with $\square(R, T) = 2$, so M has an N -detachable pair by 4.1.1. So we may also assume that $s \in \text{cl}(X' - s)$.

Observe now that $x \in \text{cl}(X' - x)$ for each $x \in X'$. If $X \neq X'$, then by Lemma 4.7 and uncrossing, there exists some element $x \in X' - X$ for which $X' - x$ is 3-separating. But X' is 3-separating and $x \in \text{cl}(X' - x) \cap \text{cl}_{M \setminus d}^*(X' - x)$ for each $x \in X'$; a contradiction. We deduce that $X = X'$.

4.1.5.6. *We may assume that $\{r_1, t_1, s, d\}$, $\{r_2, t_2, s, d\}$ and $\{r_3, t_3, s, d\}$ are circuits of M .*

Subproof. Suppose C is a 4-element circuit of $M \setminus d$ containing s . By orthogonality, $|C \cap X| \geq 3$. Suppose $C - X = \{w\}$. Then $w \in \text{cl}_{M \setminus d}(X) - X$. By Lemma 4.9(i), w is N -deletable in $M \setminus d$, so M has an N -detachable pair by Lemma 4.11. So we may assume that $C \subseteq X$. If $T \subseteq C$, then T is a triangle and a triad of $M \setminus d/s$, contradicting that this matroid is 3-connected. So $T \not\subseteq C$ and, similarly, $R \not\subseteq C$. But now either $|T \cap C| = 1$ or $|R \cap C| = 1$, contradicting orthogonality. Hence, for each $x \in X - s$, the matroid $M \setminus d/s/x$ has no parallel pairs.

Suppose $M \setminus d/s/t_2$ is not 3-connected. Then it has a non-trivial 2-separation (P, Q) . In what follows, Lemma 3.17 will be used freely. We may assume that $R \subseteq P$ and P is fully closed in $M \setminus d/s/t_2$. Now $t_3 \in Q$, as otherwise $(P \cup s, Q)$ is 2-separating in $M \setminus d/t_2$. Moreover, $t_1 \in Q$ as $\{t_1, t_3, r_1, r_3\}$ is a circuit in $M \setminus d/s/t_2$. But now $t_2 \in \text{cl}_{M \setminus d/s}(Q)$ and $(P, Q \cup t_2)$ is 2-separating in $M \setminus d/s$. Therefore $M \setminus d/s/t_2$ is 3-connected.

By symmetry, $M \setminus d/s/t_1$, $M \setminus d/s/r_1$ and $M \setminus d/s/r_2$ are 3-connected. A similar argument also gives that both $M \setminus d/s/r_3$ and $M \setminus d/s/t_3$ are 3-connected. Thus, by Lemma 4.9(i) and since $X = X'$, the element d is in some triangle with every element from $X - s$ in the matroid M/s . By orthogonality, these triangles intersect R and T in a single element each. As $\{r_1, t_1, r_3, t_3\}$ and $\{r_2, t_2, r_3, t_3\}$ are circuits in M/s , the only possible arrangement is that $\{r_1, t_1, d\}$, $\{r_2, t_2, d\}$ and $\{r_3, t_3, d\}$ are triangles of M/s . ■

We now work towards showing that $M \setminus t_1 \setminus r_2$ is an N -detachable pair. First, suppose that $M \setminus t_1 \setminus r_2$ has a series pair. Then there is a 4-element cocircuit of M containing $\{t_1, r_2\}$. By orthogonality, either this cocircuit meets $\{d, s\}$, in which case the other two elements are from the circuit $\{d, s, t_3, r_3\}$, or the cocircuit is $\{t_1, t_2, r_1, r_2\}$. In the latter case, $X \subseteq \text{cl}_{M \setminus d}^*(\{t_1, t_2, r_1\})$, so $r_{M \setminus d}^*(X) = 3$. But as $r(X) = 5$ and $|X| = 7$, the set X is 2-separating in M ; a contradiction. Similarly, if $\{t_1, r_2, z_1, z_2\}$ is a cocircuit for distinct $z_1, z_2 \in \{s, t_3, r_3\}$, then again $r_{M \setminus d}^*(X) = 3$; a contradiction. So we may assume that $\{t_1, r_2, d, z\}$ is a cocircuit for $z \in \{s, t_3, r_3\}$. By orthogonality with the circuits $\{r_1, r_3, t_1, t_3\}$ and $\{r_2, r_3, t_2, t_3\}$, we see that $z \neq s$. Now $\{s, r_1, t_1, t_2\} \subseteq \text{cl}_{M \setminus d}^*(\{r_2, r_3, t_3\})$, so $r_{M \setminus d}^*(X) = 3$; a contradiction.

Now we may assume that if $M \setminus t_1 \setminus r_2$ is not 3-connected, it has a non-trivial 2-separation (P, Q) . By Lemma 3.17, we may assume that $\{r_1, r_3, d\} \subseteq P$. If $t_3 \in P$, then $\{s, t_2\} \subseteq P$, and it follows that $(P \cup \{t_1, r_2\}, Q)$ is a 2-separation in M ; a contradiction. So $t_3 \in Q$, and, similarly, $\{s, t_2\} \subseteq Q$. But now, $(Q', P') = (\text{fcl}(Q), P - \text{fcl}(Q))$ is also a 2-separation, where $r_3 \in Q'$, hence $r_1 \in Q'$, so $(Q' \cup \{t_1, r_2\}, P')$ is a 2-separation of M ; a contradiction. We deduce that $M \setminus t_1 \setminus r_2$ is 3-connected. By Lemma 4.10, $\{t_1, r_2\}$ is an N -detachable pair. This completes the proof of 4.1.5. ◁

4.1.6. *Let R , S , and T be distinct triads of $M \setminus d$ that meet X , where the union of any two of these triads is not a cosegment of $M \setminus d$. If $|S \cap T| = 1$, and $|R \cap (S \cup T)| = 2$, then M has an N -detachable pair.*

Subproof. Since the union of any two of R , S , and T is not a cosegment, $|R \cap S| = |S \cap T| = |R \cap T| = 1$ and $R \cap S \cap T = \emptyset$. Let $S = \{s_1, s_2, t_3\}$, $T = \{t_1, t_2, t_3\}$ and $R = \{s_1, t_1, r\}$ (see Figure 2e).

4.1.6.1. *If $X - (R \cup S \cup T) \neq \emptyset$, then M has an N -detachable pair.*

Subproof. Suppose $x \in X - (R \cup S \cup T)$. As $\text{co}(M \setminus d \setminus x)$ is 3-connected, we may assume x is in a triad $\Gamma \subseteq X$, otherwise $\{d, x\}$ is an N -detachable pair. By 4.1.3 and 4.1.5, we may assume that Γ intersects each of R , S and T . If Γ intersects R , S , or T in two elements, then M has an N -detachable pair by 4.1.5. Now, $\Gamma \in \{\{x, t_3, r\}, \{x, s_1, t_2\}, \{x, t_1, s_2\}\}$. By 4.1.4, M has an N -detachable pair in each case. ■

Now suppose that M has no N -detachable pairs.

4.1.6.2. *Either $X = RUSUT$ or $X = (RUSUT) - z$ for some $z \in \{t_1, t_3, s_1\}$.*

Subproof. By 4.1.6.1, $X \subseteq R \cup S \cup T$. Suppose $X \subsetneq R \cup S \cup T$, and $r \notin X$. Then $X \cup S \cup T$ is 3-separating, by Lemma 4.7 and uncrossing, but this 3-separating set is just $S \cup T$, contradicting Lemma 4.12. By symmetry, we deduce $\{r, s_2, t_2\} \subseteq X$. Now, if $z \notin X$ for some $z \in \{t_1, t_3, s_1\}$, then $X = (R \cup S \cup T) - z$ by Lemma 4.7, thus proving 4.1.6.2 \triangleleft

Let $X' = R \cup S \cup T$, and observe that $\lambda_{M \setminus d}(X') = 2$.

4.1.6.3. *For each $x \in X$, if $M \setminus d/x$ contains a triangle that meets $X - x$, then this triangle is $\{x', z, w\}$ where $x' \in X - x$, $z \in X' - X$, and $w \notin X'$.*

Subproof. Suppose that for some $x \in X$, there is a triangle U of $M \setminus d/x$ that meets $X - x$. Then $U \cup x$ is a 4-element circuit C of $M \setminus d$. By orthogonality with R , S , and T , we have $|C \cap X'| \geq 3$. Suppose $|C \cap X'| = 3$ and let $C - X' = \{w\}$. If $C \cap X' \subseteq X$, then $w \in \text{cl}(X) - X$, and w is N -deletable in $M \setminus d$ by Lemma 4.9(i), contradicting Lemma 4.11. By 4.1.6.2, $C \cap X' = \{x', z, x\}$ where $x' \in X - x$ and $z \in X' - X$, as required.

Now suppose that $C \subseteq X'$. If $C \not\subseteq X$, then $z \in C$ where $z \in X' - X$, and $z \in \text{cl}_{M \setminus d}(X) - X$. But $z \in \text{cl}_{M \setminus d}^*(X) - X$, contradicting Lemma 3.6. So $C \subseteq X$. By orthogonality and Lemma 4.12, C contains one of R , S , or T . Let $y \in X - C$. Since $\lambda_{M \setminus d}(X') = 2$, we have $r_{M \setminus d}(X') = 5$. So $y \notin \text{cl}_{M \setminus d}(X' - y)$. As $y \in \text{cl}_{M \setminus d}^*(X' - z) \cap \text{cl}_{M \setminus d}^*(E(M \setminus d) - X')$, we see that $\text{co}(M \setminus d/y)$ is not 3-connected; a contradiction. So $C \not\subseteq X'$. \blacksquare

Since $r_{M \setminus d}^*(X) = 3$ and $\lambda_{M \setminus d}(X) = 2$, the set X contains a circuit of $M \setminus d$. Suppose that X properly contains a circuit C . By 4.1.6.3, $|C| \geq 5$, so $|X| = 6$ and $|C| = 5$. Let $X - C = \{y\}$. Then $y \in \text{cl}^*(C)$ and $y \notin \text{cl}(C)$, so $(C, \{y\}, E(M \setminus d) - X)$ is a cyclic 3-separation of $M \setminus d$. Hence $\text{co}(M \setminus d/y)$ is not 3-connected; a contradiction. We deduce that X is a corank-3 circuit.

Combining 4.1.6.3 and two applications of the dual of Lemma 3.16, it now follows that, for all distinct $x, x' \in X$, either $M \setminus d/x/x'$ is 3-connected, or there is a 4-element circuit $\{x, x', z, w\}$ of $M \setminus d$, where $z \in X' - X$ and $w \in E(M \setminus d) - X'$. By symmetry, we may assume that $X = X' - s_1$, so $z = s_1$. By orthogonality, $\{t_1, t_2, z, w\}$ is not a circuit of $M \setminus d$ for any $w \in E(M \setminus d) - X'$. Similarly, neither $\{t_1, r, z, w\}$ nor $\{t_2, r, z, w\}$ is a circuit of $M \setminus d$ for any $w \in E(M \setminus d) - X'$. Since neither $\{t_1, t_2\}$, $\{t_1, r\}$, nor $\{t_2, r\}$ is an N -detachable pair, there are distinct 4-element circuits C_1 , C_2 , and C_3 of M containing $\{d, t_1, t_2\}$, $\{d, t_1, r\}$, and $\{d, t_2, r\}$, respectively. By orthogonality with the cocircuit $\{s_1, s_2, t_3, d\}$ of M , the circuits C_1 , C_2 , and C_3 each meet $\{s_1, s_2, t_3\}$. There exists an element $y \in \{s_2, t_3\}$ that is in at most one of these three circuits. By circuit elimination on two circuits not containing y , the set $X' - y$ contains a circuit of $M \setminus d$, so $r_{M \setminus d}(X' - y) \leq 4$. As $r_{M \setminus d}(X') = 5$, it follows that $y \notin \text{cl}_{M \setminus d}(X' - y)$, so $y \in \text{cl}_{M \setminus d}^*(X' - y)$ by Lemma 3.4. Now $(X' - y, \{y\}, E(M \setminus d) - X')$ is a cyclic 3-separation of $M \setminus d$. Hence $\text{co}(M \setminus d/y)$ is not 3-connected, where $y \in X$; a contradiction. \triangleleft

We now return to the proof of Theorem 4.1. Suppose that M has no N -detachable pairs. Then every $x \in X$ is in a triad of $M \setminus d$. As $|X| \geq 4$, there are distinct triads S and T that meet X , and $S \cup T$ is not a cosegment, by Lemma 4.6. Suppose S and T meet at an element in X . As $|S \cap T| = 1$, Lemma 4.12 implies that $\lambda_{M \setminus d}(S \cup T) > 2$. By uncrossing and Lemma 4.7, the set $X \cup S \cup T$ is 3-separating. Thus, there exists some $r \in X - (S \cup T)$, where r is in a triad R .

First, suppose that every such r is such that either $S \cup r$ or $T \cup r$ is a cosegment. Without loss of generality, let $S \cup r$ be a cosegment. Now $S \cup r \not\subseteq X$, by Lemma 4.6, so S contains an element z not in X . Since $(S - z) \cup r$ and T are triads that intersect in one element, $T \cup (S - z) \cup r$ is not 3-separating by Lemma 4.12. As the union of this set and X is 3-separating, by uncrossing, there exists some $r' \in X - (S \cup T \cup r)$, where either $S \cup r'$ or $T \cup r'$ is a cosegment. If $S \cup r'$ is a cosegment, then $(S - z) \cup \{r, r'\}$ is a 4-element cosegment contained in X , contradicting Lemma 4.6. So $T \cup r'$ is a cosegment. Now $T \cup r'$ is not contained in X , by Lemma 4.6, so there is an element $z' \in T - X$. As $(T - z') \cup r'$ and $(S - z) \cup r$ are triads that intersect in one element, repeating the argument above we deduce an element $r'' \in X$ such that either $(T - z') \cup \{r', r''\}$ or $(S - z) \cup \{r, r''\}$ is a 4-element cosegment contained in X ; a contradiction.

Now we may assume that neither $S \cup r$ nor $T \cup r$ is a cosegment. So r is in a triad whose intersection with S or T has size at most one. By 4.1.3, R intersects $S \cup T$; then by 4.1.5, R intersects both S and T . Now $|R \cap S| = |R \cap T| = 1$, so $|R \cap (S \cup T)| \neq 1$ by 4.1.4, and $|R \cap (S \cup T)| \neq 2$ by 4.1.6. This contradiction implies there are no two triads S and T that meet X , and intersect at a single element in X .

Next, we claim that either X is the disjoint union of two triads, or X is a 5-element subset of the disjoint union of two triads. Certainly X contains a triad S of $M \setminus d$, and there is a triad T that meets X and is disjoint from S . By Lemma 4.7, $|T \cap X| \geq 2$. If $X - (S \cup T) = \emptyset$, then the claim holds. So suppose that $X - (S \cup T)$ is non-empty. Then there is a triad R , distinct from S and T , that meets X . So $|R \cap X| \geq 2$. If R and T are disjoint, then M has an N -detachable pair by 4.1.2; whereas if R intersects T in one element not in X , then $R \cup T$ is not a cosegment, by Lemma 4.6, so M has an N -detachable pair by 4.1.3. Hence $|R \cap T| = 2$, and $R \cup T$ is a 4-element cosegment. By Lemmas 4.6 and 4.7, $|(R \cup T) - X| = 1$. Let $T' = (R \cup T) \cap X$. Now T' and S are disjoint triads contained in X . If $X - (S \cup T')$ is non-empty, then there is another triad R' that meets X , and neither $R' \cup S$ nor $R' \cup T'$ is a cosegment, by Lemma 4.6. So if R' meets S or T' , it does so at a single element in X ; a contradiction. On the other hand, if this triad is disjoint from S and T' , then this contradicts 4.1.2. We deduce that $S \cup T' = X$, as required.

So we may assume that X is contained in the disjoint union of two triads S and T , and $|X| \in \{5, 6\}$. By 4.1.1, $\cap(S, T) = 1$. Let $X' = S \cup T$, let $W' = E(M \setminus d) - X'$, and observe that $\lambda_{M \setminus d}(X') = 2$ and $r(X') = 5$.

4.1.7. For all 2-element subsets $S' \subseteq S$ and $T' \subseteq T$ such that $S' \neq S \cap X$ and $T' \neq T \cap X$,

$$\square(S', T) = \square(S', W') = \square(T', S) = \square(T', W') = 0.$$

Subproof. Let $S - S' = \{s\}$, and note that $s \in X$. Suppose $\square(S', T) = 1$, so $r(T \cup S') = 4$. As $r(W' \cup s) = r(W') + 1$, the set $W' \cup s$ is 3-separating in $M \setminus d$, implying that $s \in \text{cl}_{M \setminus d}^*(W')$. But then $\text{co}(M \setminus d \setminus s)$ is not 3-connected; a contradiction. So $\square(S', T) = 0$.

Similarly, suppose $\square(S', W') = 1$, so $r(W' \cup S') = r(W') + 1$. As $r(T \cup s) = 4$ and $r(X') = 5$, we have $\lambda_{M \setminus d}(T \cup s) = \lambda_{M \setminus d}(X') = 2$, implying that $s \in \text{cl}_{M \setminus d}^*(T)$; a contradiction. By symmetry, $\square(T', S) = \square(T', W') = 0$. \triangleleft

Now, if $|X| = 6$, then 4.1.7 implies that X is a corank-3 circuit in $M \setminus d$. Suppose that $|X| = 5$. Without loss of generality, let $T \subseteq X$ and $s \in S - X$. If X is not a circuit, it contains a 4-element circuit, since no element of X is in a triangle. It follows that $(S - s) \cup (T - t)$ is a circuit for some $t \in T$. But then $((S - s) \cup (T - t), \{t\}, E(M \setminus d) - X)$ is a cyclic 3-separation of $M \setminus d$, implying that $\text{co}(M \setminus d \setminus t)$ is not 3-connected; a contradiction. Moreover, note that $s \notin \text{cl}(X)$ in this case. So, if $|X| \in \{5, 6\}$, then X is the only circuit contained in X' .

Let $S = \{s_1, s_2, s_3\}$ and let $T = \{t_1, t_2, t_3\}$, where $T \subseteq X$ and $S - s_2 \subseteq X$. Since X is a circuit, T is not contained in a 4-element fan in $M \setminus d / s$, for each $s \in S$. By Tutte's Triangle Lemma, at least two of $M \setminus d / s / t_1$, $M \setminus d / s / t_2$, and $M \setminus d / s / t_3$ are 3-connected, for each $s \in S$. Up to relabelling the elements of T , we may assume that $M \setminus d / s_1 / t_1$, $M \setminus d / s_1 / t_2$, and $M \setminus d / s_2 / t_1$ are 3-connected. As each of these matroids also has an N -minor, by Lemma 4.9(i), we see that either M has an N -detachable pair, or there are 4-element circuits $\{d, s_1, t_1, \alpha\}$, $\{d, s_1, t_2, \beta\}$, and $\{d, s_2, t_1, \gamma\}$ in M .

By circuit elimination, $\{t_1, t_2, \alpha, \beta\}$ contains a circuit of M / s_1 , so $\square_{M / s_1}(\{t_1, t_2\}, \{\alpha, \beta\}) = 1$. If $\{\alpha, \beta\} \subseteq W'$, then $\square_M(\{t_1, t_2\}, \{\alpha, \beta\}) = 1$, and so $\square(\{t_1, t_2\}, W') \geq 1$, contradicting 4.1.7. On the other hand, if $\{\alpha, \beta\} = \{s_2, s_3\}$, then $\square(S, \{t_1, t_2\}) \geq 1$, which again contradicts 4.1.7. So $\{\alpha, \beta\}$ meets both $\{s_2, s_3\}$ and W' .

Now, by a similar argument, $\{s_1, s_2, \alpha, \gamma\}$ contains a circuit of M / t_1 , where $\{\alpha, \gamma\}$ meets both $\{t_2, t_3\}$ and W' , by 4.1.7 and since $S \cap X \neq \{s_1, s_2\}$. So $\alpha \in W'$, $\beta \in \{s_2, s_3\}$ and $\gamma \in \{t_2, t_3\}$. Again by circuit elimination, $\{s_1, s_2, t_1, t_2, \beta, \gamma\}$ contains a circuit, where $\beta \in \{s_2, s_3\}$ and $\gamma \in \{t_2, t_3\}$. Since the only circuit contained in X' is X , we deduce that $\beta = s_3$ and $\gamma = t_3$.

Now, either $M \setminus d / s_2 / t_2$ or $M \setminus d / s_2 / t_3$ is also 3-connected. If $M \setminus d / s_2 / t_3$ is 3-connected, then, as M has no N -detachable pairs, $\{d, s_2, t_3, \zeta\}$ is a circuit, and, by circuit elimination with $\{d, s_2, t_1, t_3\}$, the set $\{s_2, t_1, t_3, \zeta\}$ contains a circuit in M . By orthogonality, $\zeta \in \{s_1, s_3\}$, but this contradicts 4.1.7. So we may assume that $M \setminus d / s_2 / t_2$ is 3-connected.

As M has no N -detachable pairs, there is a circuit $\{d, s_2, t_2, \eta\}$ in M . By circuit elimination, and since $\gamma = t_3$, the set $\{t_1, t_2, t_3, \eta\}$ contains a circuit in M/s_2 . Since X is the only circuit contained in X' , we have $\eta \in W'$. By orthogonality with S , the set $\{t_1, t_2, t_3, \eta\}$ is a circuit of M . But $M \setminus d / t_1 / t_3$ has an N -minor, by Lemma 4.9(iii), so η is N -deletable in $M \setminus d$, contradicting Lemma 4.11. This final contradiction completes the proof of Theorem 4.1. \square

5. THE NON-TRIAD CASE

In this section, we prove Theorem 5.4. We first prove a lemma that guarantees either the existence of a detachable pair, or specific structured outcomes. We then consider these structured outcomes relative to an N -minor later in the section.

Preserving 3-connectivity.

Lemma 5.1. *Let M be a 3-connected matroid with an element d such that $M \setminus d$ is 3-connected. Let (X, W) be a 3-separation of $M \setminus d$ with $|X| \geq 4$, $r(W) \geq 3$, $r_{M \setminus d}^*(W) \geq 4$, and, for each $x \in X$,*

- (a) $\text{co}(M \setminus d \setminus x)$ is 3-connected,
- (b) $M \setminus d / x$ is 3-connected, and
- (c) x is not contained in a triangle or triad of M .

Suppose X is minimal subject to these conditions, and X does not contain a triad of $M \setminus d$. Then either:

- (i) $M \setminus d \setminus x$ is 3-connected for some $x \in X$;
- (ii) $M/s/t$ is 3-connected for distinct $s, t \in \text{cl}_{M \setminus d}^*(X)$ such that $s \in S^*$ and $t \in X \cap (T^* - S^*)$ for distinct triads S^* and T^* of $M \setminus d$ that meet X ;
- (iii) $\{x, x', c, w\}$ is a 4-element circuit of $M \setminus d$ where $\{x, x'\} \subseteq X$, $c \in \text{cl}_{M \setminus d}^*(X) - X$, $w \in W - c$, and x and x' are in distinct triads of $M \setminus d$ contained in $X \cup c$; or
- (iv) $X = \{x_1, x'_1, x_2, x'_2\}$ is a quad in $M \setminus d$, there exists an element $c \in W$ such that $\{x_1, x'_1, c\}$ and $\{x_2, x'_2, c\}$ are triads of $M \setminus d$, and for each $x \in X$ there is a 4-element circuit of M containing $\{x, c, d\}$.

Proof. We assume that (i) does not hold, and show that one of (ii)–(iv) holds. We consider two cases: 5.1.3 and 5.1.4. We first prove two claims that hold in either case.

5.1.1. W is fully closed in $M \setminus d$.

Subproof. If W is not closed, then there exists some $x \in X$ such that $M \setminus d / x$ fails to be 3-connected; a contradiction. Suppose $x \in X \cap \text{cl}_{M \setminus d}^*(W)$. Since $|X| \geq 4$ and X does not contain a triad, X contains a circuit, but $x \notin \text{cl}_{M \setminus d}(X - x)$, so this circuit does not contain x . Thus $(X - x, \{x\}, W)$ is a cyclic 3-separation, implying that $\text{co}(M \setminus d \setminus x)$ is not 3-connected; a contradiction. \triangleleft

5.1.2. *In $M \setminus d$, every element of X is in a triad, and every triad that meets X contains exactly one element of W .*

Subproof. It is clear that every element of X is in a triad, as otherwise (i) holds. Let T^* be a triad that contains some $x \in X$. Then $T^* \not\subseteq X$. If $\{x\} = T^* \cap X$, then $x \in \text{cl}_{M \setminus d}^*(W)$, which contradicts 5.1.1. So $|T^* \cap X| = 2$ and $|T^* \cap W| = 1$, as required. \triangleleft

We now consider two cases. We begin by analysing the situation where there is some element $c \in W$ such that every element of X is in a triad contained in $X \cup c$.

5.1.3. *Suppose there exists some $c \in W$ such that each element in X is in a triad contained in $X \cup c$. Then one of (ii)–(iv) holds.*

Subproof. Since X does not contain a triad, every element of X is in a triad with c and exactly one other element of X . The intersection of any two such triads is $\{c\}$, otherwise $X \cup c$ contains a 4-element cosegment, implying that X contains a triad. It follows that $|X|$ is even, and there is a partition of X into pairs $\{x_i, x'_i\}$ such that $\{x_i, x'_i, c\}$ is a triad, for $i \in [|X|/2]$. The element d blocks each of these triads, by (c), and so this partition of X extends to a collection of cocircuits $\{x_i, x'_i, c, d\}$ of M . Moreover, by the cocircuit elimination axiom and (c), $\{x_i, x'_i, x_j, x'_j\}$ is a cocircuit in $M \setminus d$ for any distinct $i, j \in [|X|/2]$.

5.1.3.1. *Either (ii) or (iii) holds, or, for each $x \in X$, there is a 4-element circuit of M containing $\{x, c, d\}$.*

Subproof. Let $x \in X$. The matroid $M \setminus d/x$ is 3-connected. As $|W| \geq 3$ and $c \in \text{cl}_{M \setminus d/x}^*(X - x)$, it follows from Lemmas 3.4 and 3.11 that $\text{co}(M \setminus d/x/c)$ is not 3-connected. By Bixby's Lemma, $\text{si}(M \setminus d/x/c)$ is 3-connected. But if c is in a triangle T of $M \setminus d/x$, then T meets X and $W - c$. Let $T = \{x', c, w\}$ with $x' \in X$ and $w \in W - c$. As $M \setminus d$ has at least one triad contained in $(X - x) \cup c$ that also meets T , by orthogonality $\{x', c\}$ is contained in a triad contained in $(X - x) \cup c$. So (iii) holds if c is in a triangle of $M \setminus d/x$. Thus we may assume that $M \setminus d/x/c$ is 3-connected. Now, if $M/x/c$ is 3-connected, then (ii) holds. As x was chosen arbitrarily, d is in a triangle with every element of X in M/c . Since each element of X is not in a triangle, 5.1.3.1 follows. \blacksquare

Suppose $|X| = 4$. Then X is a 3-separating cocircuit in $M \setminus d$. It follows that X is also a circuit, so X is a quad of $M \setminus d$, and (iv) holds by 5.1.3.1. Thus, in what follows, we may assume that $|X| \geq 6$. We also assume that (ii) does not hold.

5.1.3.2. *If $\{a, b\}$ is contained in a 4-element circuit of $M \setminus d$ for $a \in \{x_i, x'_i\}$ and $b \in \{x_j, x'_j\}$ where $i \neq j$, then this circuit is $\{x_i, x'_i, x_j, x'_j\}$.*

Subproof. Suppose $a \in \{x_1, x'_1\}$, $b \in \{x_2, x'_2\}$, and $\{a, b\}$ is contained in a 4-element circuit C of $M \setminus d$ that is not $\{x_1, x'_1, x_2, x'_2\}$. It follows from

orthogonality that $c \in C$. By orthogonality, C meets $\{x_3, x'_3\}$. But now $c \in \text{cl}_{M \setminus d}(X) \cap \text{cl}_{M^* \setminus d}(X)$, contradicting Lemma 3.6. \blacksquare

5.1.3.3. *If $|X| = 6$, then $r_{M^* \setminus d}(X) = 4$.*

Subproof. Clearly $X = \{x_1, x'_1, x_2, x'_2, x_3, x'_3\}$ and $r_{M^* \setminus d}(X) \in \{3, 4\}$. Assume that $r_{M^* \setminus d}(X) = 3$. Then, as X is 3-separating in $M^* \setminus d$, and W is closed, W is a hyperplane and X is a rank-3 cocircuit in $M^* \setminus d$. Take any $a \in \{x_1, x'_1\}$ and $b \in \{x_2, x'_2\}$. By Lemma 3.16, the matroid $\text{co}(M^* \setminus d \setminus a \setminus b)$ is 3-connected. Since X is a cocircuit of $M^* \setminus d$, 5.1.3.2 implies that $M^* \setminus d \setminus a \setminus b$ has no series pairs, thus $M^* \setminus d \setminus a \setminus b$ is 3-connected. It follows that since (ii) does not hold, there exists a 4-element cocircuit C_{ab} of M^* containing $\{a, b, d\}$ for each $a \in \{x_1, x'_1\}$ and $b \in \{x_2, x'_2\}$.

Consider $C_{x_1 x_2}$. This cocircuit meets the circuit $\{c, d, x_3, x'_3\}$, and so, by orthogonality, $C_{x_1 x_2} \subseteq X \cup \{c, d\}$ with $C_{x_1 x_2} \cap \{x_1, x'_1, x_2, x'_2\} = \{x_1, x_2\}$. Similarly, $C_{x_1 x'_2}$ and $C_{x'_1 x_2}$ are cocircuits contained in $X \cup \{c, d\}$, and of these three cocircuits, only $C_{x_1 x'_2}$ contains x'_2 , and only $C_{x'_1 x_2}$ contains x'_1 . Now

$$E(M^*) - (C_{x_1 x_2} \cup C_{x_1 x'_2} \cup C_{x'_1 x_2})$$

is a flat in M^* of rank at most $r(M^*) - 3$, and so

$$r_{M^*}(E(M^*) - (X \cup \{c, d\})) \leq r(M^*) - 3.$$

But then $\lambda(X \cup \{c, d\}) \leq r_{M^*}(X \cup \{c, d\}) - 3 = 1$, contradicting the fact that M is 3-connected. We conclude that $r_{M^* \setminus d}(X) = 4$. \blacksquare

5.1.3.4. *$\{x_1, x'_1, x_2, x'_2\}$ is a circuit of $M \setminus d$.*

Subproof. Suppose $|X| = 6$ and that $\{x_1, x'_1, x_2, x'_2\}$ is independent. Now $r_{M \setminus d}(X) = 4$, by 5.1.3.3 and since X is 3-separating in $M \setminus d$. Then $x_3 \in \text{cl}_{M \setminus d}(\{x_1, x'_1, x_2, x'_2\})$. But this contradicts that $\{x_3, x'_3, c\}$ is a triad in $M \setminus d$.

So we may assume that $|X| \geq 8$. Again, suppose that $\{x_1, x'_1, x_2, x'_2\}$ is independent in $M \setminus d$. By 5.1.3.2, each element $b \in \{x_2, x'_2\}$ is not contained in a triangle of $M \setminus d / x_1$. Thus, the triad $\{x_2, x'_2, c\}$ of $M \setminus d / x_1$ is not contained in a 4-element fan. It follows, by Tutte's Triangle Lemma, that either $M \setminus d / x_1 / x_2$ or $M \setminus d / x_1 / x'_2$ is 3-connected. Assume without loss of generality that $M \setminus d / x_1 / x_2$ is 3-connected. Now M has a 4-element circuit $C_1 = \{x_1, x_2, d, \alpha\}$ for some α , since $M / x_1 / x_2$ is not 3-connected. As $\{c, d, x_3, x'_3\}$ and $\{c, d, x_4, x'_4\}$ are cocircuits of M , we deduce that $\alpha = c$, by orthogonality. By repeating this argument in $M \setminus d / x'_1$, we obtain a distinct circuit C_2 of M which is either $\{c, d, x'_1, x_2\}$ or $\{c, d, x'_1, x'_2\}$. By circuit elimination on C_1 and C_2 , there is a circuit contained in $\{x_1, x_2, x'_1, x'_2, c\}$. By orthogonality with $\{c, d, x_3, x'_3\}$, and since no element in X is contained in a triangle of M , the circuit is $\{x_1, x_2, x'_1, x'_2\}$; a contradiction. \blacksquare

It now follows from 5.1.3.4 that $\{x_1, x_2, x'_1, x'_2\}$ is 3-separating in $M \setminus d$. But $|X| \geq 6$, contradicting the minimality of X . This completes the proof of 5.1.3. \triangleleft

We now turn our attention to the case where, for every $c \in W$, some element of X is not in a triad of $M \setminus d$ that is contained in $X \cup c$. Recall that we are under the assumption that (i) does not hold.

5.1.4. *Suppose that for each $c \in W$, there is some element $x \in X$ such that x is not in a triad of $M \setminus d$ contained in $X \cup c$. Then either (ii) or (iii) holds.*

Subproof. We start by showing the following:

5.1.4.1. *Let c and c' be distinct elements in W such that there are two triads of $M \setminus d$ that meet X , one containing c , and the other containing c' . Then either (ii) holds, or there is a 4-element circuit of M containing $\{d, c, c'\}$.*

Subproof. Let T_c and $T_{c'}$ be the triads containing c and c' respectively. By 5.1.2, $T_c - c \subseteq X$, so $(X, \{c\}, W - c)$ is a cyclic 3-separation of $M \setminus d$. By Lemma 3.12, either $M \setminus d/c$ is 3-connected, or c is in a triangle that meets X . But each $x \in X$ is not in a triangle, so $M \setminus d/c$ is 3-connected. Now $T_{c'}$ is a triad of $M \setminus d/c$, so, similarly, $(X, \{c'\}, W - \{c, c'\})$ is a vertical 3-separation. By Lemma 3.12 again, if $M \setminus d/c/c'$ is not 3-connected, then $\{c, c'\}$ is contained in a 4-element circuit of $M \setminus d$ that meets X and $W - \{c, c'\}$. But this contradicts 5.1.1, which says that W is fully closed. So $M \setminus d/c/c'$ is 3-connected. Now, either (ii) holds, or there exists some α such that $\{d, c, c', \alpha\}$ is a 4-element circuit of M . ■

5.1.4.2. *Either (ii) holds, or, for each $c \in W$ in a triad T_c^* that meets X , and each $x \in X - T_c^*$, there is a 4-element circuit of M containing $\{x, c\}$.*

Subproof. Let c be an element in a triad T^* that meets X , and consider the 3-connected matroid $M \setminus d/x$ for any $x \in X - T^*$. Since T^* is a triad in $M \setminus d/x$, and $T^* - c \subseteq X$ by 5.1.2, $c \in \text{cl}_{M \setminus d/x}^*(X - x)$, and hence $(X - x, \{c\}, W - c)$ is a cyclic 3-separation of $M \setminus d/x$. Thus $\text{si}(M \setminus d/x/c)$ is 3-connected, by Bixby's Lemma. Suppose there is no 4-element circuit containing $\{x, c\}$. If c is in a triangle, then this triangle meets X ; a contradiction. Since neither x nor c is in a triangle, $M/x/c$ is 3-connected. Thus (ii) holds. ■

We now assume that (ii) does not hold. Let $W' = \text{cl}_{M \setminus d}^*(X) - X$. Observe that, for any $c \in W'$, the partition $(X \cup (W' - c), \{c\}, W - W')$ is a cyclic 3-separation, so $c \notin \text{cl}(X \cup (W' - c))$. We use this often in what follows.

5.1.4.3. *There are distinct elements $c_1, c_2 \in W$ such that every element $x \in X$ is in a triad of $M \setminus d$ contained in $X \cup \{c_1, c_2\}$.*

Subproof. Suppose 5.1.4.3 does not hold. Let T_1^* and T_2^* be triads of $M \setminus d$ that meet X , with $c_1 \in T_1^*$ and $c_2 \in T_2^*$ for distinct $c_1, c_2 \in W$, and let $x \in X - (T_1^* \cup T_2^*)$ where x is in a triad T_3^* of $M \setminus d$ and there is an element $c_3 \in T_3^* \cap (W - \{c_1, c_2\})$. Note that $T_i^* - c_i \subseteq X$ for each $i \in \{1, 2, 3\}$, by 5.1.2. We may assume that for distinct $i, j \in \{1, 2, 3\}$, the set $T_i^* \cup T_j^*$ is not a cosegment, for otherwise we can let $c_i = c_j$; in particular, $|T_i^* \cap T_j^*| \leq 1$.

By 5.1.4.2, there are 4-element circuits of M containing $\{x, c_1\}$ and $\{x, c_2\}$. Suppose neither of these circuits contains d . If T_1^* and T_3^* are

disjoint, then, by orthogonality, the 4-element circuit containing $\{x, c_1\}$ is contained in $X \cup \{c_1, c_3\}$, so $c_1 \in \text{cl}(X \cup c_3)$; a contradiction. So T_3^* meets T_1^* and, similarly, T_3^* meets T_2^* . Since $|T_1^* \cap T_2^*| \leq 1$, observe that $T_3^* = \{x, x', c_3\}$ for some $x' \in X - x$ such that $T_1^* \cap T_2^* \cap T_3^* = \{x'\}$. Then, by orthogonality, the circuit containing $\{x, c_1\}$ is contained in $X \cup \{c_1, c_2, c_3\}$, so $c_1 \in \text{cl}(X \cup \{c_2, c_3\})$; a contradiction. So we may assume that M has a 4-element circuit containing $\{x, c_1, d\}$.

Now, for some choice of $\{c', c''\} = \{c_2, c_3\}$, the matroid M has 4-element circuits $\{x, c_1, d, \beta\}$ and $\{d, c_1, c', \alpha\}$ where $\beta \neq c'$, by 5.1.4.1. By circuit elimination, $\{x, c_1, c', \alpha, \beta\}$ contains a circuit. It follows that $\{\alpha, \beta\} \not\subseteq X$, otherwise $c' \in \text{cl}(X \cup c_1)$, $c_1 \in \text{cl}(X \cup c')$, or x is in a triangle of M . Moreover, $\{\alpha, \beta\} \not\subseteq W$, otherwise W is not closed, $c_1 \notin \text{cl}_{M \setminus d}^*(X)$, or $c' \notin \text{cl}_{M \setminus d}^*(X)$. So $\{\alpha, \beta\}$ meets X and W .

By orthogonality, $\alpha \in X \cup c''$. Suppose that $\alpha = c''$. Then $\beta \in X$, and $\{x, c_1, c', c'', \beta\}$ contains a circuit. Since this circuit meets $\{c_1, c', c''\}$, we obtain a contradiction. So $\alpha \in X$. Now $X \cup c''$ also contains a triad of $M \setminus d$, so there is a 4-element circuit $\{d, c_1, c'', \gamma\}$ of M , by 5.1.4.1, where $\gamma \in X \cup c'$, by orthogonality. By circuit elimination with $\{d, c_1, c', \alpha\}$, we again obtain a contradictory circuit contained in $X \cup \{c_1, c', c''\}$ and meeting $\{c_1, c', c''\}$. ■

Let $c, c' \in W$ be distinct elements such that every element of X is in a triad of $M \setminus d$ contained in $X \cup \{c, c'\}$.

5.1.4.4. *If $|X| > 4$, then (iii) holds.*

Subproof. Suppose that $|X| > 4$. Then there are at least three distinct triads contained in $X \cup \{c, c'\}$, and it follows that, up to labels, there are distinct triads T_1^* and T_2^* containing c . Let $T_1^* = \{x_1, x'_1, c\}$ and $T_2^* = \{x_2, x'_2, c\}$. Since X does not contain a triad of $M \setminus d$, the elements x_1, x'_1, x_2, x'_2 are distinct. There exists $x_3 \in X - \{x_1, x'_1, x_2, x'_2\}$ such that x_3 is not in a triad contained in $X \cup c$. It follows that $\{x_3, c'\}$ is contained in a triad T_3^* .

At least one of T_1^* and T_2^* does not meet T_3^* , so we may assume that $T_1^* \cap T_3^* = \emptyset$. By 5.1.4.2, there is a 4-element circuit containing $\{x_1, c'\}$. If this circuit does not contain d , then, by orthogonality, $c' \in \text{cl}(X \cup c)$; a contradiction. So the circuit is $\{x_1, c', d, p\}$, where $p \in T_2^*$, by orthogonality. Similarly, there is a circuit $\{x'_1, c', d, p'\}$, where $p' \in T_2^*$. By 5.1.4.1, M also has a 4-element circuit $\{d, c, c', \alpha\}$ for some $\alpha \in E(M) - \{d, c, c'\}$.

We consider two cases depending on whether or not $T_2^* \cap T_3^* = \emptyset$. First consider the case where $T_2^* \cap T_3^* = \emptyset$. Suppose $\{x_1, c, c', d\}$ is not a circuit. Then, by circuit elimination on the circuits $\{x_1, c', d, p\}$ and $\{d, c, c', \alpha\}$, there is a circuit contained in $\{x_1, p, c, c', \alpha\}$, where $p \in T_2^*$ and $\alpha \neq d$. If $\alpha \in X$, then either $c \in \text{cl}(X \cup c')$ or $c' \in \text{cl}(X \cup c)$; a contradiction. So $\alpha \in W$. Now, by orthogonality with T_3^* , the circuit does not contain c' . It follows that $\{x_1, p, c, \alpha\}$ is a circuit for $p \in \{x_2, x'_2\}$, so (iii) holds. So we may also assume that $\{x_1, c, c', d\}$ is a circuit. By the same argument

with x'_1 in the role of x_1 , we deduce that $\{x'_1, c, c', d\}$ is a circuit. But then $\{x_1, x'_1, c, c'\}$ contains a circuit; a contradiction.

Now we may assume that $T_2^* \cap T_3^* = \{x_2\}$, so $T_3^* = \{x_2, x_3, c'\}$. By 5.1.4.2, M has a 4-element circuit $\{x_3, c, \beta, z\}$, for some $\{\beta, z\} \subseteq E(M) - \{x_3, c\}$. By orthogonality, $\{\beta, z\}$ meets $\{x_1, x'_1, d\}$ and $\{x_2, x'_2, d\}$. Thus, if $d \notin \{\beta, z\}$, then $c \in \text{cl}(X)$; a contradiction. So let $z = d$. Now $\{d, c, \beta, x_3\}$ and $\{d, c, c', \alpha\}$ are circuits, so if $\{\beta, x_3\} \neq \{c', \alpha\}$, then, by circuit elimination, there is a circuit contained in $\{x_3, c, c', \alpha, \beta\}$, where $d \notin \{\alpha, \beta\}$. If this circuit contains c , then, by orthogonality, $\{\alpha, \beta\} \subseteq \{x_1, x'_1, x_2, x'_2\}$. But now $c \in \text{cl}(X \cup c')$; a contradiction. So either $\{d, c, c', x_3\}$ or $\{x_3, c', \alpha, \beta\}$ is a 4-element circuit of M .

If $\{d, c, c', x_3\}$ is a circuit, then, by circuit elimination with $\{x_1, c', d, p\}$, the set $\{x_1, x_3, p, c, c'\}$ contains a circuit. Since each element in X is not in a triangle, the circuit has at least four elements. Thus $c \in \text{cl}(X \cup c')$ or $c' \in \text{cl}(X \cup c)$; a contradiction. So $\alpha \neq x_3$ and $\{x_3, c', \alpha, \beta\}$ is a 4-element circuit of M , where $d \notin \{\alpha, \beta\}$.

Observe that $\{\alpha, \beta\}$ meets both X and W , by 5.1.1 and since $c' \notin \text{cl}(X)$. Let $\{\alpha, \beta\} \cap X = \{x_4\}$. By orthogonality between $\{x_3, c', \alpha, \beta\}$ and either T_1^* or T_2^* , we have $x_4 \notin \{x_1, x'_1, x_2, x'_2\}$. Moreover $x_4 \neq x_3$, since $x_3 \notin \{\alpha, \beta\}$.

Now $\{x_4, c\}$ contains a circuit by 5.1.4.2, and this circuit contains d , by orthogonality with T_1^* and T_2^* , and since $c \notin \text{cl}(X)$. By orthogonality with T_3^* , either $\{x_4, c, d, c'\}$ or $\{x_4, c, d, x_j\}$ is a 4-element circuit for $j \in \{2, 3\}$. Recall that $\{x_1, c', d, p\}$ is a circuit for some $p \in T_2^*$. By circuit elimination, either $\{x_1, p, x_4, c, c'\}$ or $\{x_1, p, x_j, x_4, c, c'\}$ contains a circuit, respectively. In the former case, $c \in \text{cl}(X \cup c')$ or $c' \in \text{cl}(X \cup c)$; a contradiction. In the latter case, the only other possibility is that $\{x_1, p, x_j, x_4\}$ is a 4-element circuit contained in X . But then this set intersects T_1^* in a single element, contradicting orthogonality. \blacksquare

5.1.4.5. $|X| \neq 4$.

Subproof. Let $X = \{x_1, x'_1, x_2, x'_2\}$, where $\{x_1, x_2, c\}$ and $\{x'_1, x'_2, c'\}$ are triads of $M \setminus d$. Since $\lambda_{M \setminus d}(X) = 2$, we have $r(X) + r_{M \setminus d}^*(X) = 6$, so X is a quad in $M \setminus d$. By 5.1.4.2, there are 4-element circuits containing $\{x, c'\}$ for $x \in \{x_1, x_2\}$, and 4-element circuits containing $\{x', c\}$ for $x' \in \{x'_1, x'_2\}$. It follows, by orthogonality and since $c \notin \text{cl}(X \cup c')$ and $c' \notin \text{cl}(X \cup c)$, that any such circuit must contain d .

Suppose that X is a cocircuit of M . Then each of the 4-element circuits containing $\{x, c, d\}$ or $\{x, c', d\}$ for $x \in X$ is contained in $X \cup \{c, d\}$ or $X \cup \{c', d\}$, by orthogonality. So M has distinct circuits $\{x_1, \alpha, c', d\}$ and $\{x'_1, \alpha', c, d\}$ for some $\alpha, \alpha' \in X$. Now these two circuits, together with the circuit X , imply that $r(X \cup \{c, c', d\}) \leq 4$. But $r^*(X \cup \{c, c', d\}) \leq r^*(X) + 1 = 4$, so $\lambda_M(X \cup \{c, c', d\}) \leq 1$; a contradiction, since $|W| \geq 4$.

So we may assume that $d \in \text{cl}^*(X)$, and $X \cup d$ is a cocircuit of M . By 5.1.4.1, $\{x_0, c, c', d\}$ is a circuit for some $x_0 \in X$. Recall that each element in X is in a 4-element circuit containing d and either c or c' . Suppose one

of these circuits is contained in $X \cup \{c, c', d\}$. Then, by circuit elimination, there is a circuit of $M \setminus d$ contained in $X \cup \{c, c'\}$, and containing at most three elements of X ; a contradiction. So, for each $x \in X$, there is a 4-element circuit containing x, d , either c or c' , and an element in W .

Without loss of generality, suppose that $\{x_1, c, c', d\}$ is a circuit. Then, by orthogonality, $\{x_2, c', d, y\}$, $\{x'_1, c, d, y'_1\}$, and $\{x'_2, c, d, y'_2\}$ are circuits for some $y \in W$ and $y'_1, y'_2 \in W - c$. Note that if $c' \in \{y'_1, y'_2\}$, then, by circuit elimination on $\{x_1, c, c', d\}$ and either $\{x'_1, c, d, y'_1\}$ or $\{x'_2, c, d, y'_2\}$, there is a circuit contained in $\{x_1, x'_1, x'_2, c, c'\}$; a contradiction.

Since $X \cup d$ and $\{x'_1, x'_2, c', d\}$ are cocircuits of M , there is a cocircuit C^* contained in $X \cup c'$, by cocircuit elimination. Since $c' \notin \{y'_1, y'_2\}$, orthogonality with the circuit $\{x'_1, c, d, y'_1\}$ implies that $x'_1 \notin C^*$, and orthogonality with the circuit $\{x'_2, c, d, y'_2\}$ implies that $x'_2 \notin C^*$. But then $\{x_1, x_2, c'\}$ contains a cocircuit; a contradiction. ■

It now follows from 5.1.4.4 and 5.1.4.5 that 5.1.4 holds. ◁

With that, the proof of Lemma 5.1 is complete. □

Retaining an N -minor. In this section, we consider specific outcomes of Lemma 5.1, relative to a cyclic 3-separation $(Y, \{d'\}, Z)$ for which a 3-connected N -minor is known to lie primarily in Z , with the goal of finding an N -detachable pair.

For the entirety of the section we work under the following assumptions. Let M be a 3-connected matroid with an element d such that $M \setminus d$ is 3-connected. Let N be a 3-connected minor of M , where every triangle or triad of M is N -grounded, and $|E(N)| \geq 4$. Suppose that $M \setminus d$ has a cyclic 3-separation $(Y, \{d'\}, Z)$ with $|Y| \geq 4$, where $M \setminus d \setminus d'$ has an N -minor with $|Y \cap E(N)| \leq 1$. Let X be a 3-separating subset of Y with $|X| \geq 4$, where X does not contain a triad of $M \setminus d$, and, for each $x \in X$, both $\text{co}(M \setminus d \setminus x)$ and $M \setminus d / x$ are 3-connected, and x is doubly N -labelled in $M \setminus d$. Let $W = E(M \setminus d) - X$, and observe that $r_{M \setminus d}^*(W) \geq 3$.

Since every triangle or triad of M is N -grounded, each element in X is not in a triangle or triad of M , by Lemma 3.19. In particular, note that as $|X| \geq 4$ and X does not contain any triangles, $r(X) \geq 3$ and therefore $r(M \setminus d) \geq 4$.

We now consider the case where Lemma 5.1(iii) holds.

Lemma 5.2. *Suppose that there are elements $c \in \text{cl}_{M \setminus d}^*(X) \cap W$ and $w \in W - c$ such that*

- (a) $\{x_1, x_2, c, w\}$ is a 4-element circuit of $M \setminus d$ where $\{x_1, x_2\} \subseteq X$, and x_1 and x_2 are in distinct triads of $M \setminus d$ contained in $X \cup c$; and
- (b) $|W - \{c, w\}| \geq 2$ and W contains a circuit.

Then either

- (i) M has an N -detachable pair, or

- (ii) *there exists a set $Q \subseteq W \cup d$ with $\{c, d\} \subseteq Q$ such that $X \cup Q$ is a double-quad 3-separator of M with associated partition $\{X, Q\}$.*

Proof. Note that $(X, \{c\}, W - c)$ is a cyclic 3-separation of $M \setminus d$. Let $x'_1, x'_2 \in X$ be such that $\{x_1, x'_1, c\}$ and $\{x_2, x'_2, c\}$ are distinct triads of $M \setminus d$. Observe that the elements x_1, x'_1, x_2, x'_2 are distinct, since otherwise the union of the two triads is a cosegment, in which case X contains a contradictory triad. Moreover, by cocircuit elimination on the triads $\{x_1, x'_1, c\}$ and $\{x_2, x'_2, c\}$ of $M \setminus d$, and since X does not contain a triad, $\{x_1, x_2, x'_1, x'_2\}$ is a cocircuit of $M \setminus d$. By Lemma 4.8, $M \setminus d / x_1 / x_2$ has an N -minor. Since $\{c, w\}$ is a parallel pair in this matroid, $M \setminus d \setminus w / x_1 / x_2$ and $M \setminus d \setminus c / x_1 / x_2$ have N -minors.

We claim that $\text{co}(M \setminus d \setminus w)$ is 3-connected. Since $X \cup c$ is exactly 3-separating, Lemma 3.7 implies that $X \cup \{c, w\}$ is also exactly 3-separating, and $w \in \text{cl}(W - \{c, w\})$. If $r(W - \{c, w\}) \geq 3$, then $(X \cup c, \{w\}, W - \{c, w\})$ is a vertical 3-separation, and, by Bixby's Lemma, $\text{co}(M \setminus d \setminus w)$ is 3-connected, as required. On the other hand, if $r(W - \{c, w\}) = 2$, then $W - c$ is a segment. If $|W - c| \geq 4$, then $M \setminus d \setminus w$ is 3-connected by Lemma 3.9. So we may assume that $W - c$ is a triangle. But W contains a cocircuit of $M \setminus d$ that contains c , and c is not in a triad, as c is N -deletable. So W is a 4-element cocircuit of $M \setminus d$. Then $\text{co}(M \setminus d \setminus w)$ is 3-connected by Lemma 3.16, thus proving the claim. Since $M \setminus d \setminus w$ has an N -minor, either $\{d, w\}$ is an N -detachable pair and, in particular, (i) holds, or w is in a triad of $M \setminus d$.

So we may assume that w is in a triad T^* of $M \setminus d$. By orthogonality, T^* meets $\{x_1, x_2, c\}$. Recall that $w \in \text{cl}(W - \{c, w\})$, so that $w \notin \text{cl}_{M \setminus d}^*(X \cup c)$. If, for some $x \in X$, we have $x \notin \text{cl}(X - x)$, then $x \in \text{cl}_{M \setminus d}^*(X - x)$ by Lemma 3.4, in which case $(X - x, \{x\}, W)$ is a cyclic 3-separation of $M \setminus d$. But then $\text{co}(M \setminus d \setminus x)$ is not 3-connected; a contradiction. So each $x \in X$ is in a circuit contained in X . Thus, it follows from orthogonality that if T^* meets X , then $w \in \text{cl}_{M \setminus d}^*(X)$; a contradiction. So $(X \cup \{c, w\}) \cap T^* = \{c, w\}$.

Let $T^* = \{c, w, c'\}$, where $c' \in W - \{c, w\}$. Recall that $M \setminus d \setminus c / x_1 / x_2$ has an N -minor, and observe that $\{w, c'\}$ is a series pair in this matroid. Thus each of $M / c' / x_1$ and $M / c' / x_2$ has an N -minor, and, as c' (and w) are N -contractible, c' is not in a triangle (and neither is w). Thus $r(W - \{c, w\}) = r(W - c) \geq 3$. Observe that $X \cup \{c, w\}$ is exactly 3-separating and $|W - \{c, w\}| \geq 3$, so the dual of Lemma 3.7 implies that $X \cup \{c, w, c'\}$ is also exactly 3-separating and $c' \in \text{cl}_{M \setminus d}^*(W - \{c, w, c'\})$.

5.2.1. *$M \setminus d / c' / x$ is 3-connected for all $x \in \{x_1, x_2, x'_1, x'_2\}$.*

Subproof. Let $x \in \{x_1, x_2, x'_1, x'_2\}$. Since $c' \in \text{cl}_{M \setminus d}^*(W - \{c, w, c'\})$, we have that $r_{M \setminus d}^*(W - \{c, w, c'\}) = r_{M \setminus d}^*(W - \{c, w\}) \geq 2$. First we show that 5.2.1 holds when we have equality. In this case, $W - \{c, w\}$ is a cosegment in $M \setminus d / x$. If $|W - \{c, w\}| \geq 4$ then $M \setminus d / c' / x$ is 3-connected by the dual of Lemma 3.9, as required. On the other hand, if $W - \{c, w\}$ is a triad, then it follows that $W - c$ is a corank-3 circuit, since w is not in a triangle, and $\text{si}(M \setminus d / c' / x)$ is 3-connected by applying the dual of Lemma 3.16 in

the matroid $M \setminus d/x$. Moreover, if $\{c', x\}$ is in a 4-element circuit in $M \setminus d$, then, by orthogonality, this circuit meets $W - \{c, w, c'\}$ and $\{c, w\}$, implying $x \in \text{cl}(W)$. But this contradicts that $\{x_1, x'_1, x_2, x'_2\}$ is a cocircuit of $M \setminus d$.

So we may assume that $r^*(W - \{c, w, c'\}) \geq 3$. Since $M \setminus d/x$ is 3-connected, it follows that $((X - x) \cup \{c, w\}, \{c'\}, W - \{c, w, c'\})$ is a cyclic 3-separation. Hence $\text{si}(M \setminus d/c'/x)$ is 3-connected by Bixby's Lemma. Recall that c' is N -contractible, so it is not in an N -grounded triangle in M . Thus, if c' is in a triangle T in $M \setminus d/x$, then it is in a 4-element circuit $T \cup x$ in $M \setminus d$. As x is in a triad of $M \setminus d$ contained in $X \cup c$, orthogonality implies that $T \cap ((X - x) \cup c)$ is non-empty. Since $c' \notin \text{cl}_{M \setminus d/x}((X - x) \cup \{c, w\})$, the triangle T also contains an element in $W - \{c, w, c'\}$. It then follows that $c \notin T$, as otherwise $x \in \text{cl}(W)$, contradicting that $\{x_1, x'_1, x_2, x'_2\}$ is a cocircuit of $M \setminus d$. So $\{x, x', c', w'\}$ is a circuit for some $x' \in X - x$ and $w' \in W - \{c, w, c'\}$. But $\{c, w, c'\}$ is also a triad of $M \setminus d$, so we obtain a contradiction to orthogonality. We deduce that $M \setminus d/c'/x$ is 3-connected. \triangleleft

5.2.2. *Either*

- (I) $M \setminus d/c/x'_1$ and $M \setminus d/c/x'_2$ are 3-connected, or
- (II) $\{x'_1, x'_2, c, w\}$ is a circuit.

Subproof. Let $i \in \{1, 2\}$. Since $M \setminus d/x'_i$ is 3-connected, and $c \in \text{cl}_{M \setminus d}^*(X) \cap \text{cl}_{M \setminus d}^*(W - c)$, the matroid $\text{si}(M \setminus d/c/x'_i)$ is 3-connected. If c is neither in a triangle in $M \setminus d/x'_1$, nor in $M \setminus d/x'_2$, then (I) holds. So, without loss of generality, suppose that c is in a triangle T in $M \setminus d/x'_1$. Then T meets X and $W - c$. Moreover, as $\{c, w, c'\}$ is a triad in $M \setminus d$, the triangle T contains one of w or c' . But if $c' \in T$, then $c' \in \text{cl}(X \cup c)$, contradicting that $c' \in \text{cl}_{M \setminus d}^*(W - \{c, w, c'\})$. So $T = \{x, c, w\}$ for some $x \in X$. Now $\{x'_1, x, c, w\}$ is a circuit in $M \setminus d$, since x is not in a triangle. As $\{x_1, x_2, c, w\}$ is also a circuit, the set $\{x'_1, x, x_1, x_2, c\}$ contains a circuit. But $c \notin \text{cl}(X)$, and X does not contain a triangle of M , so $\{x'_1, x, x_1, x_2\}$ is a circuit for some $x \in X - \{x_1, x_2, x'_1\}$. By orthogonality, $x = x'_2$. This completes the proof of 5.2.2. \triangleleft

5.2.3. *There is a cocircuit C^* of M such that $\{x_1, x'_1, x_2, x'_2\} \subseteq C^* \subseteq \{x_1, x'_1, x_2, x'_2, c\}$.*

Subproof. Observe that $\{x_1, x'_1, c, d\}$ and $\{x_2, x'_2, c, d\}$ are cocircuits of M so, by cocircuit elimination, $\{x_1, x'_1, x_2, x'_2, c\}$ contains a cocircuit C^* of M . Since no element of X is in a triad of M , we have $|C^*| \geq 4$. Suppose $x_1 \notin C^*$, say. Then $r_M^*(\{x'_1, x_2, x'_2, c\}) = 3$. Due to the cocircuits $\{x_2, x'_2, c, d\}$ and $\{x_1, x'_1, c, d\}$, it follows that $r_M^*(\{x_1, x'_1, x_2, x'_2, c, d\}) = 3$, so $r_{M \setminus d}^*(\{x_1, x'_1, x_2, x'_2\}) = 2$; a contradiction. 5.2.3 follows by symmetry. \triangleleft

5.2.4. *When 5.2.2(I) holds, M has an N -detachable pair.*

Subproof. We are now in the case where $M \setminus d/c/x'_1$ and $M \setminus d/c/x'_2$ are 3-connected. For each $i \in \{1, 2\}$, the matroid $M/c/x'_i$ has an N -minor, by

Lemma 4.8, so we may assume that there are elements $\alpha_1, \alpha_2 \in E(M) - \{c, d\}$ such that $\{x'_1, \alpha_1, c, d\}$ and $\{x'_2, \alpha_2, c, d\}$ are 4-element circuits of M , for otherwise M has an N -detachable pair.

Let $x \in \{x_1, x_2\}$. Recall that $M \setminus d / c' / x$ is 3-connected, by 5.2.1. Suppose that $M / c' / x$ is not 3-connected. Then there is a 4-element circuit of M containing $\{d, x, c'\}$. By orthogonality with the cocircuit C^* of 5.2.3, this circuit intersects X in two elements. So there exists $x'' \in X - x$ such that $\{x, x'', c', d\}$ is a circuit of M . In particular $c' \in \text{cl}(X \cup d)$.

We work towards showing that $c \in \text{cl}(X \cup d)$. Clearly this holds if $\alpha_1 \in X$ or $\alpha_2 \in X$, so we assume that $\alpha_1, \alpha_2 \in W - c$. By circuit elimination, there is a circuit of $M \setminus d$ contained in $\{x'_1, x'_2, \alpha_1, \alpha_2, c\}$. Suppose that $\{\alpha_1, \alpha_2\} \cap \{w, c'\} = \emptyset$. Then, by orthogonality with the triad $\{c, w, c'\}$ of $M \setminus d$, and since neither x'_1 nor x'_2 is in a triangle, we deduce that $\{x'_1, x'_2, \alpha_1, \alpha_2\}$ is a circuit. But this contradicts orthogonality with the cocircuit $\{x_1, x'_1, c, d\}$ of M . So $\{\alpha_1, \alpha_2\} \cap \{w, c'\} \neq \emptyset$.

Suppose that $\alpha_1 = c'$. Then, by circuit elimination on $\{x'_1, c', c, d\}$ and $\{x, x'', c', d\}$, there is a circuit contained in $\{x, x'_1, x'', c, c'\}$. If this circuit contains c' , then $c' \in \text{cl}(X \cup \{c, w\})$; a contradiction. Since no element of X is in a triangle, c is in a 4-element circuit contained in $X \cup c$. But then $c \in \text{cl}(X)$; a contradiction. By symmetry, we deduce that $c' \notin \{\alpha_1, \alpha_2\}$.

Without loss of generality we may now assume that $\alpha_1 = w$, so $\{x'_1, w, c, d\}$ and $\{x, x'', c', d\}$ are circuits of M . By strong circuit elimination, there is a circuit contained in $\{x, x'_1, x'', c, w, c'\}$ that contains c' . But then $c' \in \text{cl}(X \cup \{c, w\})$; a contradiction. We deduce that either α_1 or α_2 is in X , so $c \in \text{cl}(X \cup d)$.

Thus $r(X \cup \{c, w, c', d\}) = r(X \cup d)$. Due to the cocircuits $\{c, w, c', d\}$ and $\{x_1, x'_1, c, d\}$, we have $r(W - \{c, w, c'\}) = r(W - w) - 2$. Since $w \in \text{cl}(W - w)$,

$$\begin{aligned} \lambda(X \cup \{c, w, c', d\}) &= r(X \cup d) + (r(W - w) - 2) - r(M) \\ &= \lambda(X \cup d) - 2 \leq 1; \end{aligned}$$

a contradiction.

We conclude that $M / c' / x$ is 3-connected for each $x \in \{x_1, x_2\}$. Since each of these matroids has an N -minor, M has an N -detachable pair. \triangleleft

It remains to consider 5.2.2(II); that is, the case where $\{x'_1, x'_2, c, w\}$ is a circuit. Note that the elements x'_1 and x'_2 are in distinct triads of $M \setminus d$ contained in $X \cup c$. Repeating the earlier argument, since $M \setminus d / x'_1 / x'_2$ has an N -minor, it follows that $M \setminus d \setminus c / x'_1 / x'_2$ has an N -minor, and hence each of $M / c' / x'_1$ and $M / c' / x'_2$ has an N -minor.

Let $x \in \{x_1, x'_1, x_2, x'_2\}$. Now, it follows from 5.2.1 that either M has an N -detachable pair, or there is a 4-element circuit of M containing $\{x, c', d\}$. We now consider the fourth element of this circuit. By orthogonality with the cocircuit C^* of 5.2.3, for each $x \in \{x_1, x'_1, x_2, x'_2\}$ there exists some $x' \in \{x_1, x_2, x'_1, x'_2\} - x$ such that $\{x, x', c', d\}$ is a circuit of M .

Recall that $M \setminus d$ has circuits $\{x_1, x_2, c, w\}$ and $\{x'_1, x'_2, c, w\}$. By orthogonality, any triad contained in $X \cup c$ contains an element in $\{x_1, x_2\}$ and an element in $\{x'_1, x'_2\}$. Thus $|X| = 4$. By circuit elimination on the circuits $\{x_1, x_2, c, w\}$ and $\{x'_1, x'_2, c, w\}$, the sets $\{x_1, x_2, x'_1, x'_2, c\}$ and $\{x_1, x_2, x'_1, x'_2, w\}$ contain circuits. But $c, w \notin \text{cl}(X)$, and no element in $\{x_1, x_2, x'_1, x'_2\}$ is in a triangle, so $\{x_1, x_2, x'_1, x'_2\}$ is a 4-element circuit. Hence X is a quad in $M \setminus d$.

Let $Q = \{c, w, c', d\}$. Observe that since $\{c, w, c'\}$ is a triad of M and w is N -deletable in $M \setminus d$, this triad is blocked by d . So Q is a cocircuit of M . We will show that $X \cup Q$ is a double-quad 3-separator of M with associated partition $\{X, Q\}$, as illustrated in Figure 3.

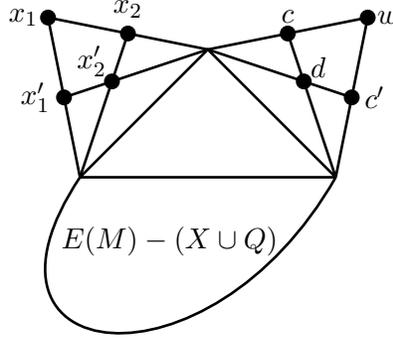


FIGURE 3. The double-quad 3-separator $X \cup Q$, when Lemma 5.2(ii) holds.

Since $d \in \text{cl}^*(Q - d)$, it follows that $d \notin \text{cl}^*(X)$, as otherwise $(X, \{d\}, W)$ is a cyclic 3-separation of M , contradicting that $M \setminus d$ is 3-connected. So X a cocircuit of M . As $\{x_1, x_2, c, w\}$ and $\{x_1, x_2, c', d\}$ are circuits of M , the circuit elimination axiom implies that $\{x_2, c, w, c', d\}$ contains a circuit. But since X is a cocircuit, $x_2 \notin \text{cl}(Q)$, and it follows that Q is a circuit, since c and c' are N -contractible. Now X and Q are quads of M , and $r(X \cup Q) \leq 5$ and $r^*(X \cup Q) \leq 5$; it follows that $\lambda(X \cup Q) = 2$ and $r(X \cup Q) = r^*(X \cup Q) = 5$.

Suppose $\{x_1, x'_1, c', d\}$ is a circuit. Then $\text{cl}((Q - w) \cup x'_1) = X \cup Q$, so $r(X \cup Q) \leq 4$; a contradiction. Similarly, $\{x_1, x'_2, c', d\}$ is not a circuit. We deduce that $\{x_1, x_2, c', d\}$ and $\{x'_1, x'_2, c', d\}$ are circuits of M .

It remains to show that $\{x_1, x'_1, c', w\}$ and $\{x_2, x'_2, c', w\}$ are cocircuits. Since X and Q are disjoint quads in M , and hence no element in X is in the coclosure of Q , it follows from Lemma 3.16 that $\text{co}(M \setminus w \setminus x)$ is 3-connected. Thus $\{w, x\}$ is contained in a 4-element cocircuit C_x^* for each $x \in X$. These cocircuits intersect X and Q in two elements each, by orthogonality. Suppose that for some $x \in X$, we have $c' \notin C_x^*$. Now

$$E(M) - (C_x^* \cup \{x_1, x'_1, c, d\} \cup X \cup Q)$$

is a flat of rank at most $r(M) - 4$. But then $\lambda(X \cup Q) \leq 1$; a contradiction. So $c' \in C_x^*$ for each $x \in X$. A similar argument shows that $x'_1 \in C_{x_1}$ and $x'_2 \in C_{x_2}$. Now $X \cup Q$ is now a double-quad 3-separator with associated partition $\{X, Q\}$, as required. \square

We now turn to the case where Lemma 5.1(iv) holds. In the analysis of this case, a 3-separator similar to a twisted cube-like 3-separator arises. Although the appearance of this 3-separator does not prevent us from guaranteeing the existence of an N -detachable pair, unlike when we encounter a twisted cube-like 3-separator, it does require special attention.

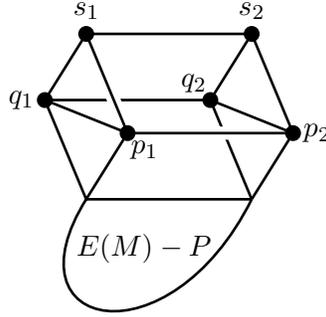


FIGURE 4. A Vámos-like 3-separator in M .

Let M be a matroid with a 6-element, rank-4, corank-4, exactly 3-separating set $P = \{p_1, p_2, q_1, q_2, s_1, s_2\}$ such that

- (a) $\{p_1, p_2, s_1, s_2\}$, $\{q_1, q_2, s_1, s_2\}$, and $\{p_1, p_2, q_1, q_2\}$ are the circuits of M contained in P ; and
- (b) $\{p_1, p_2, s_1, s_2\}$, $\{q_1, q_2, s_1, s_2\}$, $\{p_1, p_2, q_1, q_2, s_1\}$ and $\{p_1, p_2, q_1, q_2, s_2\}$ are the cocircuits of M contained in P .

Then we say P is a *Vámos-like 3-separator* of M . See Figure 4 for an illustration.

Lemma 5.3. *Suppose $X = \{x_1, x'_1, x_2, x'_2\}$ is a quad in $M \setminus d$, there exists an element $c \in W$ such that $\{x_1, x'_1, c\}$ and $\{x_2, x'_2, c\}$ are triads of $M \setminus d$, and for each $x \in X$ there is a 4-element circuit of M containing $\{x, c, d\}$. Then either*

- (i) M has an N -detachable pair,
- (ii) $X \cup \{c, d\}$ is an elongated-quad 3-separator of M ,
- (iii) $X \cup \{c, d\}$ is a spike-like 3-separator of M , or
- (iv) $X \cup \{c, d\}$ is a twisted cube-like 3-separator of M .

Proof. Note that $(X, \{c\}, W - c)$ is a cyclic 3-separation of $M \setminus d$. In what follows, we assume that (i) does not hold, and show that one of the other cases holds.

5.3.1. *There is a circuit $\{x, x', c, d\}$ for some distinct $x, x' \in X$.*

Subproof. For each $x \in X$, the set $\{x, c, d\}$ is contained in a 4-element circuit of M . Suppose that 5.3.1 does not hold. Then M has circuits $\{x_1, c, d, y_1\}$, $\{x'_1, c, d, y'_1\}$, $\{x_2, c, d, y_2\}$, and $\{x'_2, c, d, y'_2\}$, where $\{y_1, y'_1, y_2, y'_2\} \subseteq W - c$. If $y_1 = y_2$, say, then $\{x_1, x_2, c, d\}$ is a circuit, by circuit elimination and since neither x_1 nor x_2 is in a triangle. But then 5.3.1 holds. So we may assume that the elements y_1, y'_1, y_2 , and y'_2 are distinct. Moreover, by orthogonality, $X \cup d$ is a cocircuit of M .

Suppose that $\{y_1, y'_1, y_2, y'_2\}$ is an independent set; then, as $c \in \text{cl}^*(X \cup d)$ and $X \cup d$ is a cocircuit of M , the set $\{y_1, y'_1, y_2, y'_2, c, d\}$ is independent, so $r(X \cup \{c, d\}) = r(\text{cl}(X \cup \{c, d\})) \geq 6$; a contradiction. So we may assume, without loss of generality, that there is a circuit C of M with $\{y_1, y'_1\} \subseteq C \subseteq \{y_1, y'_1, y_2, y'_2\}$. Also, since $\{x_1, c, d, y_1\}$ and $\{x'_1, c, d, y'_1\}$ are circuits of M , $\{x_1, x'_1, c, y_1, y'_1\}$ contains a circuit, by circuit elimination. Due to the cocircuit $\{x_2, x'_2, c, d\}$, it follows that $\{x_1, x'_1, y_1, y'_1\}$ is a circuit.

Next, we show that $M \setminus x_2 \setminus y_1$ has an N -minor. By Lemma 4.8, $M \setminus d \setminus x_1 \setminus x'_2$ has an N -minor, and since $\{x'_1, x_2\}$ is a parallel pair in this matroid, $M \setminus d \setminus x_2 \setminus x_1$ has an N -minor too. Now $\{x'_2, c\}$ is a series pair in this matroid, so $M \setminus x_2 \setminus x_1 \setminus c$ has an N -minor. As $\{d, y_1\}$ is a parallel pair in this matroid, $M \setminus x_2 \setminus y_1$ has an N -minor, as claimed. By a similar argument, $M \setminus x'_2 \setminus y'_1$ also has an N -minor.

Now, if $M \setminus x_2 \setminus y_1$ is 3-connected, then (i) holds; so assume otherwise. Suppose $\{x_2, y_1\}$ is not contained in a 4-element cocircuit of M . Since y_1 is N -deletable, it is not in an N -grounded triad, so $M \setminus x_2 \setminus y_1$ has no series pairs or parallel pairs. Thus, if $M \setminus x_2 \setminus y_1$ is not 3-connected, then it has a 2-separation (P, Q) for which P is fully closed, by Lemma 3.17. Without loss of generality, we may assume that $\{x'_2, c, d\} \subseteq P$. If $x_1 \in P$, then $x'_1 \in P$, and $(P \cup \{x_2, y_1\}, Q)$ is a 2-separation of M ; a contradiction. So $x_1 \in Q$, and, similarly, $x'_1 \in Q$. If $y'_1 \in P$, then $x'_1 \in P$; a contradiction. So $y'_1 \in Q$. But now, since $\{x_1, x'_1, y_1, y'_1\}$ is a circuit, $(P, Q \cup y_1)$ is a 2-separation of $M \setminus x_2$; a contradiction. We deduce that $\{x_2, y_1\}$ is contained in a 4-element cocircuit of M . By a similar argument, $\{x'_2, y'_1\}$ is contained in a 4-element cocircuit of M .

Recall that $\{x_1, x'_1, x_2, x'_2\}$, $\{x_2, c, d, y_2\}$, $\{x_1, c, d, y_1\}$, and C are circuits of M . Thus, by orthogonality, the 4-element cocircuit containing $\{x_2, y_1\}$ is $\{x_1, x_2, y_1, y_2\}$. Similarly, the cocircuit containing $\{x'_2, y'_1\}$ is $\{x'_1, x'_2, y'_1, y'_2\}$. Since $\{c, d, x_1, x'_1\}$ and $\{c, d, x_2, x'_2\}$ are also cocircuits, we have $r^*(X \cup \{c, d, y_1, y_2, y'_1, y'_2\}) = 6$, so $\lambda(X \cup \{c, d, y_1, y_2, y'_1, y'_2\}) \leq 5 + 6 - 10 = 1$.

Now $|E(M)| \leq 11$. If $r(X \cup \{c, d\}) = 4$, then 5.3.1 holds unless $(X - x) \cup \{c, d\}$ is a circuit for some $x \in X$. But no such circuit exists, by orthogonality with either $\{x_1, x_2, y_1, y_2\}$ or $\{x'_1, x'_2, y'_1, y'_2\}$. So we may assume $r(X \cup \{c, d\}) = 5$ and $|E(M)| = 11$. Let $\{w\} = E(M) - (X \cup \{c, d, y_1, y_2, y'_1, y'_2\})$. Since $\lambda(X \cup \{c, d\}) = 3$ and $\{y_1, y_2, y'_1, y'_2, w\}$ is coindependent, $r(\{y_1, y_2, y'_1, y'_2, w\}) = 3$. By orthogonality with the cocircuits $\{x_1, x_2, y_1, y_2\}$ and $\{x'_1, x'_2, y'_1, y'_2\}$, it follows that $\{y_1, y_2, w\}$ and $\{y'_1, y'_2, w\}$

are triangles. But since $M \setminus x_2 \setminus y_1$ has an N -minor, and $\{x_1, y_2\}$ is a series pair in this matroid, y_2 is N -contractible. Similarly, y_2' is N -contractible. So $\{y_1, y_2, w\}$ and $\{y_1', y_2', w\}$ are not N -grounded triangles; a contradiction. \triangleleft

Observe now that $r(X \cup \{c, d\}) \leq 4$. If $r(X \cup \{c, d\}) = 3$, then $\lambda(X \cup \{c, d\}) < 2$; a contradiction. So $r(X \cup \{c, d\}) = 4$ and $\lambda(X \cup \{c, d\}) = 2$. In particular, $d \notin \text{cl}(X)$.

5.3.2. Either

- (I) $\{x_1, x_2, c, d\}$ and $\{x_1', x_2', c, d\}$ are circuits of M , up to swapping the labels on x_2 and x_2' ; or
- (II) $\{x_1, x_1', c, d\}$ and $\{x_2, x_2', c, d\}$ are circuits of M .

Subproof. By 5.3.1, we may assume, up to labels, that either $\{x_1, x_2, c, d\}$ or $\{x_1, x_1', c, d\}$ is a circuit.

First, suppose that $\{x_1, x_2, c, d\}$ is a circuit. The elements x_1' and x_2' are also in 4-element circuits $\{x_1', c, d, y_1\}$ and $\{x_2', c, d, y_2\}$, respectively. If $y_1 \in X$ or $y_2 \in X$, then, by circuit elimination with $\{x_1, x_2, c, d\}$, and since $c \notin \text{cl}(X)$ and X does not contain a triangle, 5.3.2(I) holds. So let $y_1, y_2 \in W - c$. Note that $y_1 \neq y_2$, otherwise $\{x_1', x_2', c, d\}$ is a circuit, as required, by circuit elimination.

We claim that $M \setminus x_1' \setminus y_2$ has an N -minor. By Lemma 4.8, $M \setminus d \setminus x_1/x_2'$ has an N -minor, and since $\{x_2, x_1'\}$ is a parallel pair in this matroid, $M \setminus d \setminus x_1'/x_2'$ has an N -minor too. Now $\{x_1, c\}$ is a series pair in $M \setminus d \setminus x_1'/x_2'$, so the matroid $M \setminus x_1'/x_2'/c$ has an N -minor. As $\{d, y_2\}$ is a parallel pair in $M \setminus x_1'/x_2'/c$, the matroid $M \setminus x_1' \setminus y_2$ has an N -minor as required.

Suppose $\{x_1', y_2\}$ is contained in a 4-element cocircuit C^* of M . By orthogonality, C^* meets $\{c, d, y_1\}$ and $\{x_2', c, d\}$, so if neither c nor d is in C^* , then $C^* = \{x_1', x_2', y_1, y_2\}$. But then $r^*(X \cup \{c, d, y_1, y_2\}) = 5$, and, as $r(X \cup \{c, d, y_1, y_2\}) = 4$, we have $\lambda(X \cup \{c, d, y_1, y_2\}) = 1$. So $|E(M)| = 9$, in which case $E(M) - (X \cup \{c, d\})$ is a coindependent triangle containing $\{y_1, y_2\}$. But since $M \setminus x_1' \setminus y_2$ has an N -minor and $\{x_2', y_1\}$ is a series pair in this matroid, y_1 is N -contractible, so it is not in an N -grounded triangle; a contradiction. So C^* contains either c or d .

By orthogonality with $\{x_1, x_2, c, d\}$, the cocircuit C^* either contains $\{c, d\}$, or meets $\{x_1, x_2\}$. Now y_2 is in the closure and the coclosure of the 3-separating set $X \cup \{c, d\}$, so $|E(M)| = 8$. But then $r_{M \setminus d}^*(W) = 2$; a contradiction. We deduce that $\{x_1', y_2\}$ is not contained in a 4-element cocircuit of M . Since y_2 is N -deletable, it is not in an N -grounded triad, so $M \setminus x_1' \setminus y_2$ does not have any series pairs.

Now, if $M \setminus x_1' \setminus y_2$ is not 3-connected, then it has a 2-separation (P, Q) where we may assume $\{x_1, c, d\} \subseteq P$, and P is fully closed by Lemma 3.17. Then $x_2 \in P$, due to the circuit $\{x_1, x_2, c, d\}$, and $x_2' \in P$, due to the cocircuit $\{x_2, x_2', c, d\}$. But then $x_1', y_2 \in \text{cl}(P)$, so $(P \cup \{x_1', y_2\}, Q)$ is a 2-separation of M ; a contradiction. Thus $M \setminus x_1' \setminus y_2$ is 3-connected, so (i) holds.

Now we may assume that $\{x_1, x'_1, c, d\}$ is a circuit, and $\{x_2, c, d, y\}$ and $\{x'_2, c, d, y'\}$ are circuits for some distinct $y, y' \in W - c$. By circuit elimination, $\{x_2, x'_2, c, y, y'\}$ contains a circuit, and, by orthogonality with the triad $\{x_1, x'_1, c\}$ of $M \setminus d$, this circuit is $\{x_2, x'_2, y, y'\}$.

We will show that $M \setminus x'_1 \setminus y'$ is 3-connected and has an N -minor, using a similar approach as in the case where $\{x_1, x_2, c, d\}$ is a circuit. Firstly, observe that $M \setminus x'_1 \setminus y'$ has an N -minor, using a similar argument as in this other case.

Suppose that $\{x'_1, y'\}$ is contained in a 4-element cocircuit C^* of M . We claim that $C^* \subseteq X \cup \{c, d, y'\}$. By orthogonality, C^* meets $X - x'_1$ and $\{x_1, c, d\}$. Thus if $C^* \not\subseteq X \cup \{c, d, y'\}$, then $\{x_1, x'_1, y\} \subseteq C^*$. But C^* also meets $\{x'_2, c, d\}$, so $C^* \subseteq X \cup \{c, d, y'\}$ as claimed. In particular, $y' \in \text{cl}^*(X \cup \{c, d\})$. Observe that since $r(X \cup \{c, d, y, y'\}) = 4$ and $\lambda(X \cup \{c, d, y, y'\}) \geq 2$, we have $r^*(X \cup \{c, d, y, y'\}) \geq 6$. But $r^*(X \cup \{c, d\}) = 4$, so $y' \notin \text{cl}^*(X \cup \{c, d\})$; a contradiction. So $\{x'_1, y'\}$ is not contained in a 4-element cocircuit C^* of M .

Now, if $M \setminus x'_1 \setminus y'$ is not 3-connected, then it has a 2-separation (P, Q) where we may assume $\{x_1, c, d\} \subseteq P$, and P is fully closed. If $x'_2 \in P$, then $y' \in \text{cl}_{M \setminus x'_1}(P)$, so $(P \cup y', Q)$ is a 2-separation of $M \setminus x'_1$; a contradiction. So $x'_2 \in Q$, and it follows that $x_2 \in Q$, due to the cocircuit $\{x_2, x'_2, c, d\}$, and $y \in Q$, due to the circuit $\{x_2, c, d, y\}$. Now $\{x_2, x'_2, y\} \subseteq Q$, so $y' \in \text{cl}(Q)$ and $(P, Q \cup y')$ is a 2-separation of $M \setminus x'_1$; a contradiction. Hence $M \setminus x'_1 \setminus y'$ is 3-connected, so (i) holds. \triangleleft

We consider two cases, depending on whether or not d fully blocks (X, W) . Since $d \notin \text{cl}(X)$, these correspond to either $d \in \text{cl}^*(X)$, or $d \notin \text{cl}^*(X)$, respectively.

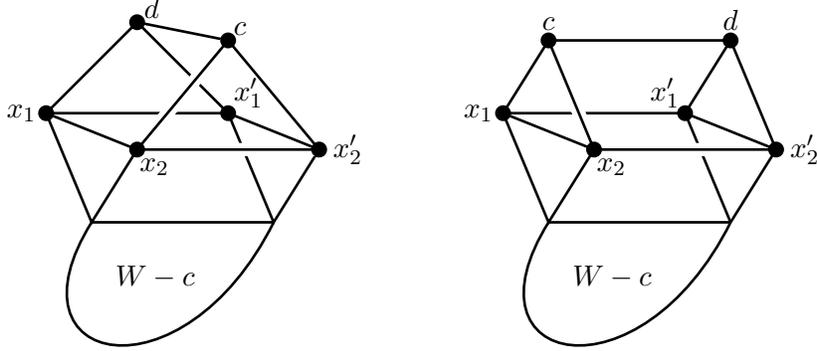
5.3.3. *If $d \notin \text{cl}^*(X)$, then (ii) or (iii) holds.*

Subproof. Suppose $d \notin \text{cl}^*(X)$. As X is a quad of $M \setminus d$, it follows that X is also a quad of M . If 5.3.2(I) holds, then $\{x_1, x_2, c, d\}$ and $\{x'_1, x'_2, c, d\}$ are circuits, up to swapping the labels on x_2 and x'_2 , in which case $X \cup \{c, d\}$ is an elongated-quad 3-separator, so (ii) holds. On the other hand, if 5.3.2(II) holds, then $X \cup \{c, d\}$ is a spike-like 3-separator of M , so (iii) holds. \triangleleft

We may now assume that $d \in \text{cl}^*(X)$, so $X \cup d$ is a cocircuit of M . Moreover, as $\{x_1, x'_1, c, d\}$ and $\{x_2, x'_2, c, d\}$ are cocircuits of M , cocircuit elimination, and the fact that X is not a cocircuit, implies that $X \cup c$ is a cocircuit. Hence, as illustrated in Figure 5, $X \cup \{c, d\}$ is a twisted cube-like 3-separator of M when 5.3.2(I) holds, or a Vámos-like 3-separator of M when 5.3.2(II) holds. In the former case, (iv) holds.

So we may now assume that 5.3.2(II) holds, and $X \cup \{c, d\}$ is a Vámos-like 3-separator of M . We will show that M has an N -detachable pair.

5.3.4. *There exists some $p \in \{x_1, x'_1\}$ and $q \in \{x_2, x'_2\}$ such that $M \setminus p \setminus q$ has an N -minor.*



(a) Twisted cube-like 3-separator of M . (b) Vámos-like 3-separator of M .

FIGURE 5. The labellings of the twisted cube-like 3-separator or Vámos-like 3-separator when Lemma 5.3(iv) holds.

Subproof. Since $M \setminus d$ has an N -minor with $|X \cap E(N)| \leq 1$, we may assume, without loss of generality, that $X \cap E(N) = \{x_1\}$. Suppose that x'_1 is N -deletable in $M \setminus d$. If either x_2 or x'_2 is N -deletable in $M \setminus d \setminus x'_1$, then 5.3.4 holds, so we may assume that x_2 and x'_2 are N -contractible in $M \setminus d \setminus x'_1$. Since $\{x_1, c\}$ is a series pair in $M \setminus d \setminus x'_1$, the matroid $M \setminus d \setminus x'_1 / c / x'_2$ has an N -minor, as does $M \setminus x'_1 / c / x'_2$. As $\{d, x_2\}$ is a parallel pair in the latter matroid, $M \setminus x'_1 \setminus x_2$ has an N -minor, as required.

So we may assume that x'_1 is N -contractible in $M \setminus d$. If x_2 is N -contractible in $M \setminus d / x'_1$, then, as $\{x_1, x'_2\}$ is a parallel pair in $M \setminus d / x'_1 \setminus x_2$, the matroid $M \setminus d / x'_1 \setminus x_2$ has an N -minor. Similarly, if x'_2 is N -contractible in $M \setminus d / x'_1$, then $M \setminus d / x'_1 \setminus x_2$ has an N -minor. So there is some $q \in \{x_2, x'_2\}$ such that q is N -deletable in $M \setminus d / x'_1$. Let $\{x_2, x'_2\} - q = \{q'\}$. Since $M \setminus d \setminus q / x'_1$ has an N -minor and $\{c, q'\}$ is a series pair in this matroid, $M \setminus q / x'_1 / c$ has an N -minor. But $\{d, x_1\}$ is a parallel pair in this matroid, so $M \setminus q \setminus x_1$ has an N -minor, as required. \triangleleft

5.3.5. *If $p \in \{x_1, x'_1\}$ and $q \in \{x_2, x'_2\}$, then $M \setminus p \setminus q$ is 3-connected.*

Subproof. Pick p' and q' such that $\{p, p'\} = \{x_1, x'_1\}$ and $\{q, q'\} = \{x_2, x'_2\}$. First, observe that since $\{c, d, q, q'\}$ is a quad, and q is not in a triad, $M \setminus q$ is 3-connected by Lemma 3.16.

Suppose $\text{co}(M \setminus p \setminus q)$ is not 3-connected. Then $M \setminus p \setminus q$ has a 2-separation (P, Q) where we may assume either P or Q is fully closed, by Lemma 3.17. Thus, without loss of generality, we may assume that the triad $\{p', c, d\}$ is contained in P . Since $\{p, p', c, d\}$ is a circuit, $(P \cup p, Q)$ is a 2-separation of $M \setminus q$; a contradiction. So $\text{co}(M \setminus p \setminus q)$ is 3-connected.

Suppose that $M \setminus p \setminus q$ is not 3-connected. Then M has a 4-element cocircuit C^* containing $\{p, q\}$. Since $X \cup d$ and $X \cup c$ are cocircuits, C^* is not

contained in $X \cup d$ or $X \cup c$. So C^* meets $W' = E(M) - (X \cup \{c, d\})$. Suppose $C^* \cap (X \cup \{c, d\}) = \{p, q\}$. Then $p \in \text{cl}^*(W' \cup q)$, so $p \notin \text{cl}(\{p', q', c, d\})$. Since $r(X \cup \{c, d\}) = 4$, it follows that $\{p', q', c, d\}$ is a circuit of M ; a contradiction.

Now we may assume that $C^* \cap W' = \{w\}$, in which case $w \in \text{cl}^*(X \cup \{c, d\}) = \text{cl}^*(X)$. So $X \cup w$ contains a cocircuit. Since each $x \in X$ is not contained in an N -grounded triad, this cocircuit contains at least three elements of X . Then, by orthogonality, $X \cup w$ is a cocircuit. Now $(X, \{w\}, \{c\}, \{d\}, E(M) - (X \cup \{w, c, d\}))$ is a path of 3-separations where $w, c, d \in \text{cl}^*(X)$. But then $\{w, c, d\}$ is a triad in M ; a contradiction. \triangleleft

Now $\{p, q\}$ is an N -detachable pair by 5.3.4 and 5.3.5, thus completing the proof. \square

Putting the results of this section together we have:

Theorem 5.4. *Let M be a 3-connected matroid with an element d such that $M \setminus d$ is 3-connected. Let N be a 3-connected minor of M , where every triangle or triad of M is N -grounded, and $|E(N)| \geq 4$. Suppose that $M \setminus d$ has a cyclic 3-separation $(Y, \{d'\}, Z)$ with $|Y| \geq 4$, where $M \setminus d \setminus d'$ has an N -minor with $|Y \cap E(N)| \leq 1$. Let X be a minimal 3-separating subset of Y such that $|X| \geq 4$ and, for each $x \in X$,*

- (a) $\text{co}(M \setminus d \setminus x)$ is 3-connected,
- (b) $M \setminus d / x$ is 3-connected, and
- (c) x is doubly N -labelled in $M \setminus d$.

Suppose X does not contain a triad of $M \setminus d$. Then, either M has an N -detachable pair, or there exists some $c \in \text{cl}_{M \setminus d}^(X) - X$ such that one of the following holds:*

- (i) $X \cup \{c, d\}$ is a spike-like 3-separator;
- (ii) $X \cup \{c, d\}$ is an elongated-quad 3-separator;
- (iii) $X \cup \{c, d\}$ is a twisted cube-like 3-separator of M ; or
- (iv) there exists a set $Q \subseteq E(M) - X$ with $\{c, d\} \subseteq Q$ such that $X \cup Q$ is a double-quad 3-separator with associated partition $\{X, Q\}$.

Proof. If there is some element $y \in \text{cl}_{M \setminus d}^*(Y \cup d') \cap Z$, then $(Y \cup y, \{d'\}, Z - y)$ is a cyclic 3-separation with $|(Y \cup y) \cap E(N)| \leq 1$, since $|E(N)| \geq 4$, so, without loss of generality, we may assume that $Y \cup d'$ is coclosed in $M \setminus d$.

First, we remark that $r(Z \cup d') \geq 3$. Indeed, if not, then since $d' \notin \text{cl}(Z)$, we have $r(Z) \leq 1$; a contradiction. Note also that each $x \in X$ is not contained in a triangle or triad of M , since each $x \in X$ is N -contractible and N -deletable in $M \setminus d$. Let $W = E(M \setminus d) - X$. Now (X, W) is a 3-separation of $M \setminus d$ that satisfies the criteria of Lemma 5.1. So one of Lemma 5.1(i)–(iv) holds.

It is clear that if Lemma 5.1(i) holds, then M has an N -detachable pair by (c). If Lemma 5.1(ii) holds, then there exist elements s and t such that $M/s/t$ is 3-connected, and $M/s/t$ has an N -minor by Lemma 4.8; in

particular, if $s = d'$, then (ii) of the lemma applies since d' is in a triad meeting X that does not contain t . So, again, M has an N -detachable pair in this case.

Suppose Lemma 5.1(iii) holds. Since $c \in \text{cl}_{M \setminus d}^*(X)$ and $Y \cup d'$ is coclosed, $c \in Y \cup d'$. Thus, if $w \in Z$, then $w \notin \text{cl}^*(Y \cup d')$, so $w \in \text{cl}(Z - w)$. Hence $r(W - \{c, w\}) \geq r(Z - w) = r(Z) \geq 2$. Also, $r_{M \setminus d}^*(W - \{c, w\}) \geq r_{M \setminus d}^*(Z - w) \geq 2$. Now, by Lemma 5.2, either M has an N -detachable pair, or (iv) holds.

Finally, if Lemma 5.1(iv) holds, then, by Lemma 5.3, either M has an N -detachable pair, or (i), (ii), or (iii) holds. \square

6. CONCLUSION

Combining the two main results of this paper with the main result of [3], we have the following:

Theorem 6.1. *Let M be a 3-connected matroid and let N be a 3-connected minor of M where $|E(N)| \geq 4$, and every triangle or triad of M is N -grounded. Suppose, for some $d \in E(M)$, that $M \setminus d$ is 3-connected and has a cyclic 3-separation $(Y, \{d'\}, Z)$ with $|Y| \geq 4$, where $M \setminus d \setminus d'$ has an N -minor with $|Y \cap E(N)| \leq 1$. Then either*

- (i) M has an N -detachable pair; or
- (ii) there is a subset X of Y such that for some $c \in \text{cl}_{M \setminus d}^*(X) - X$, one of the following holds:
 - (a) $X \cup \{c, d\}$ is a skew-whiff 3-separator of M ,
 - (b) $X \cup \{c, d\}$ is a spike-like 3-separator of M ,
 - (c) $X \cup \{c, d\}$ is a twisted cube-like 3-separator of M or M^* ,
 - (d) $X \cup \{c, d\}$ is an elongated-quad 3-separator of M , or
 - (e) $X \cup \{a, b, c, d\}$ is a double-quad 3-separator of M with associated partition $\{X, \{a, b, c, d\}\}$ for some distinct $a, b \in E(M) - (X \cup \{c, d\})$.

Proof. By [3, Theorem 7.4], if neither (i) nor (ii) holds, then Y contains a 3-separating subset X such that $|X| \geq 4$ and for every $x \in X$, the matroids $\text{co}(M \setminus d \setminus x)$ and $M \setminus d \setminus x$ are 3-connected, and x is doubly N -labelled in $M \setminus d$. Let X be minimal subject to these properties. If X contains a triad, then (i) holds by Theorem 4.1. On the other hand, if X does not contain a triad, then, by Theorem 5.4, either (i) or (ii) holds. \square

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