# Reconfiguring colorings of graphs with bounded maximum average degree 

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#### Abstract

The reconfiguration graph $R_{k}(G)$ for the $k$-colorings of a graph $G$ has as vertex set the set of all possible $k$-colorings of $G$ and two colorings are adjacent if they differ in the color of exactly one vertex of $G$. Let $d, k \geq 1$ be integers such that $k \geq d+1$. We prove that for every $\epsilon>0$ and every graph $G$ with $n$ vertices and maximum average degree $d-\epsilon, R_{k}(G)$ has diameter $O\left(n(\log n)^{d-1}\right)$. This significantly strengthens several existing results.


## 1 Introduction

Let $k$ be a positive integer. A $k$-coloring of a graph $G$ is a function $f$ : $V(G) \rightarrow\{1, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E(G)$. The reconfiguration graph $R_{k}(G)$ for the $k$-colorings of a graph $G$ has as vertex set the set of all possible $k$-colorings of $G$ and two colorings are adjacent if they differ in the color of exactly one vertex of $G$.

Given a non-negative integer $d$, a graph $G$ is $d$-degenerate if every subgraph of $G$ contains a vertex of degree at most $d$. Expressed differently, $G$ is $d$-degenerate if there there exists an ordering $v_{1}, \ldots, v_{n}$ of the vertices in $G$, called a d-degenerate ordering, such that each $v_{i}$ has at most $d$ neighbors $v_{j}$ with $j<i$. The maximum average degree of a graph $G$ is defined as

$$
\max \left\{\frac{2|E(H)|}{|V(H)|}: H \subseteq G\right\}
$$

[^0]In particular, if $G$ has maximum average degree strictly less than some positive integer $d$, then $G$ is $(d-1)$-degenerate.

Consider the following conjecture of Cereceda 3].
Conjecture 1. For every integers $k$ and $\ell, \ell \geq k+2$, and every $k$-degenerate graph $G$ on $n$ vertices, $R_{\ell}(G)$ has diameter $O\left(n^{2}\right)$.

The conjecture appears difficult to prove or disprove, with the case $k=1$ only being known despite some efforts; for a recent exposition on the conjecture and the results surrounding it see [4, 1]. The most important breakthrough is Theorem 1 in [1] due to Bousquet and Heinrich, which addresses a number of cases for Conjecture 1, generalising several existing results. For instance, it is shown in [1] that there exists a constant $c>0$ independent of $k$ such that $R_{\ell}(G)$ has diameter at most $(c n)^{k+1}$ for every $\ell \geq k+2$.

The purpose of this note is to prove the following theorem.
Theorem 1. Let $d, k \geq 1$ be integers such that $k \geq d+1$. For every $\epsilon>0$ and every graph $G$ with $n$ vertices and maximum average degree $d-\epsilon, R_{k}(G)$ has diameter $O\left(n(\log n)^{d-1}\right)$.

Theorem 1 is a generalisation of [2, Theorem 2]. In particular, it has the following immediate consequences. By Euler's formula, planar graphs, triangle-free planar graphs and planar graphs of girth 5 have maximum average degrees strictly less than, respectively, 6, 4 and $7 / 2$. Hence Theorem 1 affirms (and is stronger than) Conjecture 1 for planar graphs of girth 5 but is one color short of confirming the conjecture for planar graphs and triangle-free planar graphs. It nevertheless generalises some best known existing results. More precisely, our theorem subsumes both [2, Corollary 5] and [1, Theorem 1] restricted to planar graphs, as well as [2, Corollary 7] and [6, Corollary 1].

## 2 The proof

In this section, we prove Theorem 1. Our approach is essentially a combination of the ones found in [1, 5]. We begin with some definitions.

Definition 1. Given a graph $G$, a coloring $\alpha$ of $G$ and a subgraph $H$ of $G$, let $\alpha^{H}$ denote the restriction of $\alpha$ to $H$.

Definition 2. Let $G$ be a graph, and let $k$ be a nonnegative integer. A subset $S \subseteq V(G)$ is a $k$-independent set of $G$ if $S$ is an independent set of $G$ and every vertex of $S$ has degree at most $k$ in $G$.

Definition 3. For integers $s \geq 0$ and $t \geq 1$, a graph $G$ is said to have degree depth $(s, t)$ if there exists a partition $\left\{V_{1}, \ldots, V_{t}\right\}$ of $V(G)$, called an $s$-degree partition, such that $V_{1}$ is an $s$-independent set of $G$ and, for $i \in\{2, \ldots, t\}$, $V_{i}$ is an $s$-independent set of $G \backslash \bigcup_{j=1}^{i-1} V_{j}$.

In what follows, let $G$ be a graph of degree depth $(s, t)$ and with $s$-degree partition $\left\{V_{1}, \ldots V_{t}\right\}$.

Definition 4. An ordering $v_{n}, \ldots, v_{1}$ of $V(G)$ is said to be embedded in $\left\{V_{1}, \ldots, V_{t}\right\}$ if, for every pair $\left(v_{i}, v_{j}\right) \in V(G) \times V(G)$ such that $v_{i} \in V_{p}$ and $v_{j} \in V_{q}, i<j$ implies $p \leq q$.

Notice that the ordering in Definition 4 is an $s$-degenerate ordering of $G$.
If $H$ is a subgraph of $G$ such that $V(H)=\bigcup_{j=1}^{h} V_{j}$ for some index $h \in$ $\{1, \ldots, t\}$, then $H$ is called a layered subgraph of $G$, and $h$ is its boundary.

In the next definition, we shall slightly abuse Definition 3.
Definition 5. If $H$ is a layered subgraph of $G$ with boundary $h$, then we say that $H$ has degree depth $\left(s^{\prime}, t\right)$ if, for each index $j \in\{1, \ldots, h\}$, each $v \in V(H) \cap V_{j}$ has at most $s^{\prime}$ neighbors in $\bigcup_{i=j+1}^{t} V_{i}$.

We have the following crucial lemma.
Lemma 1. Let $s \geq 0$ and $t \geq 1$ be integers, let $G$ be a graph with degree depth $(s, t)$, and let $F$ be a layered subgraph of $G$. Any $(s+2)$-coloring of $G$ can be recolored, using only colors $1, \ldots, s+2$, to some coloring of $G$ in which color $s+2$ is not used in $F$ by $O\left((s+1) 2^{s-1} t^{s}\right)$ recolorings per vertex of $F$ and by not recoloring any vertex of $G \backslash F$.

Proof. Let $\left\{V_{1}, \ldots, V_{t}\right\}$ be an $s$-degree partition of $G$, and let $V(F)=V_{1} \cup$ $\ldots \cup V_{b}$, where $b \geq 1$ is the boundary of $F$. Let $v_{m}, \ldots, v_{1}$ be an ordering of $V(F)$ that is embedded in $\left\{V_{1}, \ldots, V_{b}\right\}$. Let $\alpha$ be an $(s+2)$-coloring of $G$, and let $h \in\{1, \ldots, b\}$ be the smallest index such that $V_{h}$ contains a vertex with color $s+2$ under $\alpha$. Let $W$ denote the subset of vertices of $V_{h}$ with color $s+2$. For each color $a \in\{1, \ldots, s+1\}$, define $W_{a}$ to be the subset
of $W$ whose vertices have no neighbor earlier in the ordering with color $a$. More formally,

$$
W_{a}=\left\{v_{i} \in W: \alpha\left(v_{j}\right) \neq a \text { for all neighbors } v_{j} \text { of } v_{i} \text { with } j>i\right\}
$$

and notice that

$$
W=\bigcup_{i=1}^{s+1} W_{i} .
$$

Claim 1. Let $U=\bigcup_{i=1}^{h-1} V_{i}$. For each $a \in\{1, \ldots, s+1\}$, there is a sequence of recolorings in $R_{s+2}(G)$ such that

- each vertex of $U$ is recolored $O\left((2 t)^{s-1}\right)$ times,
- each vertex of $W_{a}$ is recolored at most once,
- no vertex of $V(G) \backslash\left(U \cup W_{a}\right)$ is recolored, and
- at the end of the sequence, no vertex of $U \cup W_{a}$ has color $s+2$.

Let us first show how to use the claim to prove the lemma. Applying the sequence described in Claim 1 for each $a \in\{1, \ldots, s+1\}$, we obtain a coloring in which color $s+2$ is not used in $U \cup V_{h}$ by $O\left((s+1)(2 t)^{s-1}\right)$ recolorings. The smallest index $h^{\prime}$ such that $V_{h^{\prime}}$ contains a vertex with color $s+2$ has now increased; hence at most $b \leq t$ such repetitions are needed to obtain a coloring in which color $s+2$ is not used in $F$, so each vertex is recolored $O\left((s+1) 2^{s-1} t^{s}\right)$ times and the lemma follows. It remains to prove the claim.

Proof of Claim 1. Let $F^{*}=F\left[U \cup W_{a}\right]$ and note that $F^{*}$ has degree depth $\left(s^{\prime}, t\right)$ for some $s^{\prime} \in\{0, \ldots, s\}$. We are going to apply induction on $s^{\prime}$. The base case $s^{\prime}=0$ is trivial (simply immediately recolor the vertices of $W_{a}$ ) so we can assume that $s \geq s^{\prime}>0$ and that Claim 1 and hence, by the observation following the statement of Claim 1, also the lemma holds for each subgraph $K$ of $G$ and layered subgraph of $K$ of degree depth $\left(s^{\prime}-1, t\right)$.

In the inductive step, we are in fact going to establish the claim for the pair $\left(\left(s^{\prime}, t\right), s+2\right)$, where the first term of the pair corresponds to the degree depth of $F^{*}$ and the second term to the number of colors, assuming its validity for the pair $\left(\left(s^{\prime}-1, t\right), s+1\right)$. Let $u_{k}, \ldots, u_{1}$ be an ordering of the vertices of $U$ that is embedded in $\left\{V_{1}, \ldots, V_{h-1}\right\}$. We first try to recolor
immediately, whenever possible, each vertex of $U$ to color $s+2$ starting with $u_{k}$ and moving forward towards $u_{1}$. Let $\gamma$ denote the resulting coloring, let $S=\{\gamma(v)=s+2: v \in V(G)\}$ and let $H=G[U \backslash S]$.
Subclaim 1. $H$ has degree depth $\left(s^{\prime}-1, t\right)$.
Proof of Subclaim. By our choice of $h$, each vertex $u \in U \cap V_{p}$ for some $p \in\{1, \ldots, h-1\}$ either satisfies $\gamma(u)=s+2$ or has a neighbor $u^{\prime} \in V_{q}$ for some $q \in\{p+1, \ldots, t\}$ such that $\gamma\left(u^{\prime}\right)=s+2$. This implies the subclaim.

By the above subclaim, we can apply the induction hypothesis to the pair $\left(\left(s^{\prime}-1, t\right), s+1\right)$ with $H$ and $G \backslash S$ playing the roles of $F$ and $G$, respectively. This gives a sequence of recolorings (that uses only colors $1, \ldots, s+1$ ) from $\gamma^{H}$ to some coloring $\zeta^{H}$ of $H$ such that

- color $a$ is not used in $\zeta^{H}$,
- the number of recolorings per vertex of $H$ is $O\left(2^{s-2} t^{s-1}\right)$, and
- no vertex of $G \backslash(S \cup H)$ is recolored.

Clearly, this sequence of recolorings vacuously translates to a sequence of recolorings in $G$ from $\alpha$ to coloring $\zeta$ satisfying $\zeta(v)=\zeta^{H}(v)$ if $v \in V(H)$ and $\zeta(v)=\gamma(v)$ if $v \in V(G) \backslash V(H)$. From $\zeta$, we can now immediately recolor each vertex of $W_{a}$ to color $a$. It remains to recolor each vertex of $U$ to a color distinct from $s+2$. To do so, we simply repeat the above steps with the roles of $a$ and $s+2$ interchanged. This takes again $O\left(2^{s-2} t^{s-1}\right)$ recolorings per vertex of $H$. Hence each vertex of $H$ is recolored in total $O\left((2 t)^{s-1}\right)$ times. This proves the claim and hence completes the proof of the lemma.

We can prove our final lemma, from which Theorem 1 follows easily.
Lemma 2. Let $s \geq 0$ and $t \geq 1$ be integers, and let $G$ be a graph with $n$ vertices and degree depth $(s, t)$. Then $R_{s+2}(G)$ has diameter $O\left(n s(2 t)^{s}\right)$.

Proof. As before, we proceed by induction on the pair $((s, t), s+2)$, where the first term corresponds to the degree depth of $G$ and the second term to the number of colors. The base case $s=0$ is trivial, so we can assume that $s>0$ and that the lemma holds for the pair $((s-1, t), s+1)$.

Let $\alpha$ and $\beta$ be two $(s+2)$-colorings of $G$, and let $\left\{V_{1}, \ldots, V_{t}\right\}$ be an $s$-degree partition of $G$. It suffices to show that we can recolor $\alpha$ to $\beta$ by $O\left(s(2 t)^{s}\right)$ recolorings per vertex. By Lemma 1 with $F=G$, we can recolor $\alpha$ to some $(s+1)$-coloring $\alpha_{1}$ of $G$ and $\beta$ to some $(s+1)$-coloring $\beta_{1}$ of $G$ by $O\left(s 2^{s} t^{s}\right)$ recolorings per vertex.

Let $v_{n}, \ldots, v_{1}$ be an ordering of $V(G)$ that is embedded in $\left\{V_{1}, \ldots, V_{t}\right\}$. We recolor $\alpha_{1}$ and $\beta_{1}$ to new colorings $\alpha_{2}$ and $\beta_{2}$ of $G$ by trying to recolor, from $\alpha_{1}$ and $\beta_{1}$, immediately whenever possible each vertex of $G$ to color $s+2$ starting with $v_{n}$ and moving forward towards $v_{1}$. Let $S=\{v \in V(G)$ : $\left.\alpha_{2}(v)=s+2\left(=\beta_{2}(v)\right)\right\}$. As before, the graph $H=G-S$ has degree depth $(s-1, t)$. So we can apply our induction hypothesis to recolor $\alpha_{2}^{H}$ to $\beta_{2}^{H}$ by $O\left((s-1)(2 t)^{s-1}\right)$ recolorings per vertex using only colors $1, \ldots, s+1$ (as this sequence of recolorings does not use color $s+2$, we need not worry about adjacencies between $H$ and $S$ ). This completes the proof.

Proof of Theorem 1. Let $H$ be any subgraph of $G$, and let $h=|V(H)|$. An independent set $I$ of $H$ is said to be special if $I$ is a $(d-1)$-independent set of $H$ and $|I| \geq \epsilon h / d^{2}$. It was shown in [5] that $H$ contains a special independent set. This means that there is a partition $\left\{I_{1}, I_{2}, \ldots, I_{\ell}\right\}$ of $V(G)$ such that $I_{1}$ is a special independent set of $G$ and, for $i \in\{2, \ldots, \ell\}, I_{i}$ is a special independent set of $G \backslash\left(\bigcup_{j=1}^{i-1} I_{j}\right)$. Thus $G$ has degree depth $(d-1, \ell)$. As $\ell=f(n)$ satisfies the recurrence

$$
f(n) \leq f\left(n-\frac{\epsilon n}{d^{2}}\right)+1
$$

it follows that $\ell=O(\log n)$, by the master theorem. The theorem now follows by Lemma 2 with $t=\log n$ and $s=d-1$.

Similarly, we can slightly improve on the constant $c$ in the aforementioned main result from [1].

Corollary 1. Let $k, n \geq 1$ be integers, and let $G$ be a $k$-degenerate graph with $n$ vertices. Then $R_{k+2}(G)$ has diameter $O\left(2^{k} n^{k+1}\right)$.

Proof. Noting that every $k$-degenerate graph with $n$ vertices has degree depth $(k, n)$, the corollary immediately follows from Lemma 2.

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