Reconfiguring colorings of graphs with bounded maximum average degree

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Abstract

The reconfiguration graph $R_k(G)$ for the k-colorings of a graph Ghas as vertex set the set of all possible k-colorings of G and two colorings are adjacent if they differ in the color of exactly one vertex of G. Let $d, k \ge 1$ be integers such that $k \ge d + 1$. We prove that for every $\epsilon > 0$ and every graph G with n vertices and maximum average degree $d - \epsilon$, $R_k(G)$ has diameter $O(n(\log n)^{d-1})$. This significantly strengthens several existing results.

1 Introduction

Let k be a positive integer. A k-coloring of a graph G is a function $f : V(G) \to \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E(G)$. The reconfiguration graph $R_k(G)$ for the k-colorings of a graph G has as vertex set the set of all possible k-colorings of G and two colorings are adjacent if they differ in the color of exactly one vertex of G.

Given a non-negative integer d, a graph G is d-degenerate if every subgraph of G contains a vertex of degree at most d. Expressed differently, G is d-degenerate if there there exists an ordering v_1, \ldots, v_n of the vertices in G, called a d-degenerate ordering, such that each v_i has at most d neighbors v_j with j < i. The maximum average degree of a graph G is defined as

$$\max\left\{\frac{2|E(H)|}{|V(H)|}: H \subseteq G\right\}.$$

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In particular, if G has maximum average degree strictly less than some positive integer d, then G is (d-1)-degenerate.

Consider the following conjecture of Cereceda [3].

Conjecture 1. For every integers k and ℓ , $\ell \ge k+2$, and every k-degenerate graph G on n vertices, $R_{\ell}(G)$ has diameter $O(n^2)$.

The conjecture appears difficult to prove or disprove, with the case k = 1only being known despite some efforts; for a recent exposition on the conjecture and the results surrounding it see [4, 1]. The most important breakthrough is Theorem 1 in [1] due to Bousquet and Heinrich, which addresses a number of cases for Conjecture 1, generalising several existing results. For instance, it is shown in [1] that there exists a constant c > 0 independent of k such that $R_{\ell}(G)$ has diameter at most $(cn)^{k+1}$ for every $\ell \ge k+2$.

The purpose of this note is to prove the following theorem.

Theorem 1. Let $d, k \geq 1$ be integers such that $k \geq d+1$. For every $\epsilon > 0$ and every graph G with n vertices and maximum average degree $d - \epsilon$, $R_k(G)$ has diameter $O(n(\log n)^{d-1})$.

Theorem 1 is a generalisation of [2, Theorem 2]. In particular, it has the following immediate consequences. By Euler's formula, planar graphs, triangle-free planar graphs and planar graphs of girth 5 have maximum average degrees strictly less than, respectively, 6, 4 and 7/2. Hence Theorem 1 affirms (and is stronger than) Conjecture 1 for planar graphs of girth 5 but is one color short of confirming the conjecture for planar graphs and triangle-free planar graphs. It nevertheless generalises some best known existing results. More precisely, our theorem subsumes both [2, Corollary 5] and [1, Theorem 1] restricted to planar graphs, as well as [2, Corollary 7] and [6, Corollary 1].

2 The proof

In this section, we prove Theorem 1. Our approach is essentially a combination of the ones found in [1, 5]. We begin with some definitions.

Definition 1. Given a graph G, a coloring α of G and a subgraph H of G, let α^H denote the restriction of α to H.

Definition 2. Let G be a graph, and let k be a nonnegative integer. A subset $S \subseteq V(G)$ is a k-independent set of G if S is an independent set of G and every vertex of S has degree at most k in G.

Definition 3. For integers $s \ge 0$ and $t \ge 1$, a graph G is said to have *degree* depth (s,t) if there exists a partition $\{V_1, \ldots, V_t\}$ of V(G), called an *s*-degree partition, such that V_1 is an *s*-independent set of G and, for $i \in \{2, \ldots, t\}$, V_i is an *s*-independent set of $G \setminus \bigcup_{j=1}^{i-1} V_j$.

In what follows, let G be a graph of degree depth (s, t) and with s-degree partition $\{V_1, \ldots, V_t\}$.

Definition 4. An ordering v_n, \ldots, v_1 of V(G) is said to be *embedded in* $\{V_1, \ldots, V_t\}$ if, for every pair $(v_i, v_j) \in V(G) \times V(G)$ such that $v_i \in V_p$ and $v_j \in V_q$, i < j implies $p \leq q$.

Notice that the ordering in Definition 4 is an s-degenerate ordering of G. If H is a subgraph of G such that $V(H) = \bigcup_{j=1}^{h} V_j$ for some index $h \in \{1, \ldots, t\}$, then H is called a *layered* subgraph of G, and h is its *boundary*. In the next definition, we shall slightly abuse Definition 3.

Definition 5. If *H* is a layered subgraph of *G* with boundary *h*, then we say that *H* has degree depth (s', t) if, for each index $j \in \{1, \ldots, h\}$, each $v \in V(H) \cap V_j$ has at most s' neighbors in $\bigcup_{i=j+1}^t V_i$.

We have the following crucial lemma.

Lemma 1. Let $s \ge 0$ and $t \ge 1$ be integers, let G be a graph with degree depth (s,t), and let F be a layered subgraph of G. Any (s+2)-coloring of G can be recolored, using only colors $1, \ldots, s+2$, to some coloring of G in which color s+2 is not used in F by $O((s+1)2^{s-1}t^s)$ recolorings per vertex of F and by not recoloring any vertex of $G \setminus F$.

Proof. Let $\{V_1, \ldots, V_t\}$ be an s-degree partition of G, and let $V(F) = V_1 \cup \ldots \cup V_b$, where $b \ge 1$ is the boundary of F. Let v_m, \ldots, v_1 be an ordering of V(F) that is embedded in $\{V_1, \ldots, V_b\}$. Let α be an (s+2)-coloring of G, and let $h \in \{1, \ldots, b\}$ be the smallest index such that V_h contains a vertex with color s + 2 under α . Let W denote the subset of vertices of V_h with color s + 2. For each color $a \in \{1, \ldots, s + 1\}$, define W_a to be the subset

of W whose vertices have no neighbor earlier in the ordering with color a. More formally,

$$W_a = \{v_i \in W : \alpha(v_i) \neq a \text{ for all neighbors } v_i \text{ of } v_i \text{ with } j > i\},\$$

and notice that

$$W = \bigcup_{i=1}^{s+1} W_i.$$

Claim 1. Let $U = \bigcup_{i=1}^{h-1} V_i$. For each $a \in \{1, \ldots, s+1\}$, there is a sequence of recolorings in $R_{s+2}(G)$ such that

- each vertex of U is recolored $O((2t)^{s-1})$ times,
- each vertex of W_a is recolored at most once,
- no vertex of $V(G) \setminus (U \cup W_a)$ is recolored, and
- at the end of the sequence, no vertex of $U \cup W_a$ has color s + 2.

Let us first show how to use the claim to prove the lemma. Applying the sequence described in Claim 1 for each $a \in \{1, \ldots, s+1\}$, we obtain a coloring in which color s + 2 is not used in $U \cup V_h$ by $O((s + 1)(2t)^{s-1})$ recolorings. The smallest index h' such that $V_{h'}$ contains a vertex with color s + 2 has now increased; hence at most $b \leq t$ such repetitions are needed to obtain a coloring in which color s + 2 is not used in F, so each vertex is recolored $O((s + 1)2^{s-1}t^s)$ times and the lemma follows. It remains to prove the claim.

Proof of Claim 1. Let $F^* = F[U \cup W_a]$ and note that F^* has degree depth (s',t) for some $s' \in \{0,\ldots,s\}$. We are going to apply induction on s'. The base case s' = 0 is trivial (simply immediately recolor the vertices of W_a) so we can assume that $s \geq s' > 0$ and that Claim 1 and hence, by the observation following the statement of Claim 1, also the lemma holds for each subgraph K of G and layered subgraph of K of degree depth (s'-1,t).

In the inductive step, we are in fact going to establish the claim for the pair ((s', t), s + 2), where the first term of the pair corresponds to the degree depth of F^* and the second term to the number of colors, assuming its validity for the pair ((s' - 1, t), s + 1). Let u_k, \ldots, u_1 be an ordering of the vertices of U that is embedded in $\{V_1, \ldots, V_{h-1}\}$. We first try to recolor immediately, whenever possible, each vertex of U to color s+2 starting with u_k and moving forward towards u_1 . Let γ denote the resulting coloring, let $S = \{\gamma(v) = s + 2 : v \in V(G)\}$ and let $H = G[U \setminus S]$.

Subclaim 1. *H* has degree depth (s' - 1, t).

Proof of Subclaim. By our choice of h, each vertex $u \in U \cap V_p$ for some $p \in \{1, \ldots, h-1\}$ either satisfies $\gamma(u) = s + 2$ or has a neighbor $u' \in V_q$ for some $q \in \{p+1, \ldots, t\}$ such that $\gamma(u') = s+2$. This implies the subclaim. \square

By the above subclaim, we can apply the induction hypothesis to the pair ((s'-1,t), s+1) with H and $G \setminus S$ playing the roles of F and G, respectively. This gives a sequence of recolorings (that uses only colors $1, \ldots, s+1$) from γ^{H} to some coloring ζ^{H} of H such that

- color a is not used in ζ^H ,
- the number of recolorings per vertex of H is $O(2^{s-2}t^{s-1})$, and
- no vertex of $G \setminus (S \cup H)$ is recolored.

Clearly, this sequence of recolorings vacuously translates to a sequence of recolorings in G from α to coloring ζ satisfying $\zeta(v) = \zeta^H(v)$ if $v \in V(H)$ and $\zeta(v) = \gamma(v)$ if $v \in V(G) \setminus V(H)$. From ζ , we can now immediately recolor each vertex of W_a to color a. It remains to recolor each vertex of U to a color distinct from s+2. To do so, we simply repeat the above steps with the roles of a and s + 2 interchanged. This takes again $O(2^{s-2}t^{s-1})$ recolorings per vertex of H. Hence each vertex of H is recolored in total $O((2t)^{s-1})$ times. This proves the claim and hence completes the proof of the lemma.

We can prove our final lemma, from which Theorem 1 follows easily.

Lemma 2. Let $s \ge 0$ and $t \ge 1$ be integers, and let G be a graph with n vertices and degree depth (s,t). Then $R_{s+2}(G)$ has diameter $O(ns(2t)^s)$.

Proof. As before, we proceed by induction on the pair ((s,t), s+2), where the first term corresponds to the degree depth of G and the second term to the number of colors. The base case s = 0 is trivial, so we can assume that s > 0 and that the lemma holds for the pair ((s-1,t), s+1).

Let α and β be two (s + 2)-colorings of G, and let $\{V_1, \ldots, V_t\}$ be an s-degree partition of G. It suffices to show that we can recolor α to β by $O(s(2t)^s)$ recolorings per vertex. By Lemma 1 with F = G, we can recolor α to some (s + 1)-coloring α_1 of G and β to some (s + 1)-coloring β_1 of G by $O(s2^{s}t^s)$ recolorings per vertex.

Let v_n, \ldots, v_1 be an ordering of V(G) that is embedded in $\{V_1, \ldots, V_t\}$. We recolor α_1 and β_1 to new colorings α_2 and β_2 of G by trying to recolor, from α_1 and β_1 , immediately whenever possible each vertex of G to color s+2 starting with v_n and moving forward towards v_1 . Let $S = \{v \in V(G) :$ $\alpha_2(v) = s + 2(=\beta_2(v))\}$. As before, the graph H = G - S has degree depth (s-1,t). So we can apply our induction hypothesis to recolor α_2^H to β_2^H by $O((s-1)(2t)^{s-1})$ recolorings per vertex using only colors $1, \ldots, s+1$ (as this sequence of recolorings does not use color s+2, we need not worry about adjacencies between H and S). This completes the proof.

Proof of Theorem 1. Let H be any subgraph of G, and let h = |V(H)|. An independent set I of H is said to be *special* if I is a (d-1)-independent set of H and $|I| \ge \epsilon h/d^2$. It was shown in [5] that H contains a special independent set. This means that there is a partition $\{I_1, I_2, \ldots, I_\ell\}$ of V(G) such that I_1 is a special independent set of G and, for $i \in \{2, \ldots, \ell\}$, I_i is a special independent set of G has degree depth $(d-1, \ell)$. As $\ell = f(n)$ satisfies the recurrence

$$f(n) \le f\left(n - \frac{\epsilon n}{d^2}\right) + 1,$$

it follows that $\ell = O(\log n)$, by the master theorem. The theorem now follows by Lemma 2 with $t = \log n$ and s = d - 1.

Similarly, we can slightly improve on the constant c in the aforementioned main result from [1].

Corollary 1. Let $k, n \ge 1$ be integers, and let G be a k-degenerate graph with n vertices. Then $R_{k+2}(G)$ has diameter $O(2^k n^{k+1})$.

Proof. Noting that every k-degenerate graph with n vertices has degree depth (k, n), the corollary immediately follows from Lemma 2.

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