# CUBIC GRAPHS THAT CANNOT BE COVERED WITH FOUR PERFECT MATCHINGS 

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#### Abstract

A conjecture of Berge suggests that every bridgeless cubic graph can have its edges covered with at most five perfect matchings. Since three perfect matchings suffice only when the graph in question is 3 -edge-colourable, the rest of cubic graphs falls into two classes: those that can be covered with four perfect matchings, and those that need at least five. Cubic graphs that require more than four perfect matchings to cover their edges are particularly interesting as potential counterexamples to several profound and long-standing conjectures including the celebrated cycle double cover conjecture. However, so far they have been extremely difficult to find.

In this paper we build a theory that describes coverings with four perfect matchings as flows whose flow values and outflow patterns form a configuration of six lines spanned by four points of the 3 -dimensional projective space $\mathbb{P}_{3}\left(\mathbb{F}_{2}\right)$ in general position. This theory provides powerful tools for investigation of graphs that do not admit such a cover and offers a great variety of methods for their construction. As an illustrative example we produce a rich family of snarks (nontrivial cubic graphs with no 3 -edge-colouring) that cannot be covered with four perfect matchings. The family contains all previously known graphs with this property.


Keywords: cubic graph, snark, perfect matching, covering, cycle double cover conjecture

AMS subject classifications: 05C21, 05C70, 05C15.

## 1 Introduction

Ever since Petersen proved his Perfect Matching Theorem [29], perfect matchings in cubic graphs have been regarded as their fundamental structures. An extension of Petersen's theorem due to Schönberger [32], later generalised by Plesník [30], implies that every bridgeless cubic graph $G$ contains a set of perfect matchings that together cover all the edges of $G$ (see also [21, p. 192] and [22, Corollary 3.4.3]). A natural question arises as
to what is the minimum number of perfect matchings needed to cover all edges of a given bridgeless cubic graph $G$; this number is called the perfect matching index of $G$ and is denoted by $\pi(G)$ (an alternative term excessive index also occurs, see [1, 7].)


Figure 1: Graphs of order 34 and 46 that cannot be covered with four perfect matchings
Clearly, $\pi(G) \geq 3$ for every bridgeless cubic graph $G$ with equality achieved if and only if the graph is 3 -edge-colourable. On the other hand, no constant upper bound for the perfect matching index of a cubic graphs is known. However, a conjecture attributed to Berge (see [27] or [33]) suggests that $\pi(G) \leq 5$ for every bridgeless cubic graph $G$. If this conjecture is true, then the perfect matching index of every snark, a 2 -connected cubic graph with no 3 -edge-colouring, is to be either 4 or 5 .

Cubic graphs with $\pi=4$ enjoy several important properties well known to hold for 3 -edge-colourable graphs. For example, every cubic graph with $\pi=4$ satisfies the 5 -cycle double conjecture (see [17, Theorem 3.1] and [34, Theorem 3.1 (2)]), the $7 / 5$-conjecture of Alon and Tarsi and Jaeger [34, Theorem 3.1 (1)] on shortest cycle covers, the Fan-Raspaud [9] conjecture on three perfect matchings with empty intersection, and others. Therefore cubic graphs with $\pi \geq 5$ are of particular interest, providing potential counterexamples to these and several other related conjectures such as the Fulkerson conjecture.

Very little is known about cubic graphs with $\pi \geq 5$. One major difficulty in their study comes from the fact that they appear to be extremely rare and therefore hard to find. The smallest such graph is the Petersen graph. Arbitrarily large examples with connectivity 2 can be easily derived from the Petersen graph, so the real problem is constructing nontrivial examples different from the Petersen graph. In 2009, Fouquet and Vanherpe [11, Problem 4.3] asked whether there exists a cyclically 4-edge-connected cubic graph with $\pi \geq 5$ different from the Petersen graph. The first such graph was reported by Brinkmann et al. [3] in 2013 as a result of an exhausting computer search. In the list comprising all 64326024 cyclically 4-edge-connected cubic snarks of girth at least 5 with $\pi \geq 4$ on up to 36 vertices there are only two that cannot be covered with four perfect matchings: the Petersen graph and the graph on 34 vertices displayed in Figure 1 (left). Another sporadic example, on 46 vertices, was discovered by Hägglund [14, Section 3]; it is depicted on Figure 1 (right). Esperet and Mazzuoccolo [7] generalised the former graph to an infinite family of windmill snarks with $\pi \geq 5$. A similar infinite family, which takes the other graph from Figure 1 as its basis, was provided by Chen [4]. Both graphs from Figure 1 are included in another infinite family of graphs with $\pi \geq 5$, the family of treelike snarks constructed by Abreu et al. [1], which we now discuss in a greater detail.

In some sense, treelike snarks are much richer in form than the windmill snarks and the snarks of Chen. Their construction makes use of cubic Halin graphs. A Halin graph $H$ consists of a plane tree $S$ with no 2 -valent vertices and a circuit $C$ passing through
all the leaves of $S$ in such a way that $H=C \cup S$ remains embedded in the plane. It follows that the circuit $C$, called the perimeter circuit, forms the boundary of the outer face of $H$, while the tree $S$, the inscribed tree of $H$, is contained in the interior of $C$. Halin [15] introduced these graphs in 1971 as a class of minimally 3 -connected graphs. However, the cubic Halin graphs were studied much earlier by Rademacher [31] in 1965 and by Kirkman [19] as early as in 1856 .

A treelike snark is formed from a cubic Halin graph $H$ by substituting each vertex of the perimeter circuit $C$ of $H$ with a 11-vertex subgraph $F_{P s}$ found in both graphs from Figure 1, called the Petersen fragment of [1]. This makes a significant difference in comparison with the windmill snarks and the family of Chen where more general building blocks can be used.

One of the notable aspects of the study of cubic graphs with $\pi \geq 5$ is the increasing difficulty of proving that four perfect matchings are not enough to cover their edges. This problem becomes evident if we compare the proof of Esperet and Mazzuoccolo [7] for the windmill snarks with that for the family constructed by Chen [4, and even more so if we take into account the proof for treelike snarks [1], where a substatial computer assistance is required. The reason for this lies in the lack of a simple characterisation of graphs with perfect matching index 5 or more. This fact was realised already by Hägglund [14] who posed the following problem.

Problem (Hägglund [14], Problem 3). Is it possible to give a simple characterisation of cubic graphs with perfect matching index equal to 5 ?

In response to Hägglund's question we provide a characterisation of graphs with perfect matching index not exceeding 4 . We describe them as the graphs that admit a proper edgecolouring by points of the configuration $T$ of ten points and six lines in the 3 -dimensional projective space $\mathbb{P}_{3}\left(\mathbb{F}_{2}\right)=P G(3,2)$, over the 2 -element field, spanned by four points in general position (see Figure 22). The defining property of the colouring requires any three edges incident with the same vertex to carry colours that form a line of $T$. Although this result seems to be nothing more than a simple observation, it provides a surprisingly powerful tool for the study of both cubic graphs that can, as well as those that cannot, be covered with four perfect matchings. Especially useful is the property that the such colourings are in fact nowhere-zero flows with values in the group $\mathbb{Z}_{2}^{4}$.


Figure 2: A tetrahedron in $P G(3,2)$ spanned by points $p_{1}, p_{2}, p_{3}$, and $p_{4}$
We demonstrate the power of our approach by presenting a far-reaching generalisation of treelike snarks. The new family of graphs, which we call Halin snarks, contains all
previously known nontrivial examples of graphs with $\pi \geq 5$. In particular, the treelike snarks, the windmill snarks, and the snarks constructed by Chen [4] are all Halin snarks.


Figure 3: A Halin snark
The construction of a Halin snark is similar to that of a treelike snark. It starts with a cubic Halin graph $H=C \cup S$ where $C$ is the perimeter circuit and $S$ is the inscribed tree of $H$. Each vertex of $C$ is substituted by a Halin fragment $F$, a generalisation of the Petersen fragment $F_{P s}$ mentioned above. It is formed from the union of two graphs $D$ and $B$ : the graph $D$ is obtained from an arbitrary bridgeless cubic graph $G$ with $\pi(G) \geq 5$ by removing two adjacent vertices $u$ and $v$ and retaining the four dangling edges; the graph $B$ is obtained from an arbitrary cubic bipartite graph $G^{\prime}$ by removing a path $u^{\prime} w^{\prime} v^{\prime}$ of length 2 and retaining the five dangling edges. The dangling edges of $D$ formerly incident in $G$ with $v$ are then welded with those of $B$ formerly incident in $G^{\prime}$ with $u^{\prime}$, thereby producing $F$.

Each Halin fragment $F$ has five dangling edges arranged in two pairs, which correspond to the vertices $u$ and $v^{\prime}$, and one singleton, which corresponds to the vertex $w^{\prime}$. The construction of a Halin snark is now finished by substituting each vertex $v_{i}$ of the perimeter circuit $C$ of $H$ with a Halin fragment $F_{i}$ in such a way that the lonely dangling edge of $F_{i}$ will replace the edge of $S$ incident with $V_{i}$, and each of pairs of dangling edges of $F_{i}$ will replace one edge of $C$ incident with $v_{i}$ following a cyclic orientation of $C$. Finally, the pairs of dangling edges from Halin fragments $F_{i}$ and $F_{j}$ corresponding to adjacent vertices $v_{i}$ and $v_{j}$ of $C$ are welded in such a way that a cubic graph is obtained (see Figure 3). We denote the resulting graph by $H^{\sharp}=C^{\sharp} \cup S$ where $S$ is the inherited inscribed tree of $H$ and $C^{\sharp}$ is the graph obtained from the perimeter circuit by performing all required substitutions.

If each Halin fragment $F_{i}$ used for the construction of $H^{\sharp}$ is created from a cubic graph $G_{i}$ with $\pi(G) \geq 5$ and from a bipartite cubic graph $G_{i}^{\prime}$, both cyclically 4-edge-connected and of girth at least 5 , then the same holds for the resulting graph $H^{\sharp}$.

The proof that every Halin snark has perfect matching index at least 5 relies on simple geometric considerations involving the projective space $P G(3,2)$ combined with nowherezero flow arguments. It is completely computer-free.

The methods developed along the way permit further extensions of the family of

Halin snarks, suggesting that the class of all cubic graphs with perfect matching index at least 5 may have a very complicated structure. In particular, the following theorem is a consequence of our results.

Theorem. For every even integer $n \geq 42$ there exists a cyclically 4-edge-conncected cubic graph $G$ of girth at least 5 on $n$ vertices such that $\pi(G) \geq 5$.

Taking into account the exhaustive computer search performed by Brinkmann et al. [3] mentioned earlier, Theorem 1 leaves the existence of a cyclically 4-edge-connected cubic graph $G$ of girth at least 5 on $n$ vertices with $\pi(G) \geq 5$ open only for $n=38$ and $n=40$. This is a significant progress as all previously known constructions combined are capable of producing only graphs of order $n=12 m-2$, where $m \geq 3$ (see [1, 4, 7]).

This paper is organised as follows. The next section contains a brief survey of terminology and notation used later in the paper. Section 3 presents a characterisation of cubic graphs that cannot be covered with four perfect matching in terms of flows with additional geometric structure within the projective space $P G(3,2)$. Geometric language is further developed in Section 4 with focus on objects consisting of two or three points. Transformation of geometric objects via the flows corresponding to coverings with four perfect matchings is studied in the next three sections. In Section 8, the accumulated knowledge is applied to proving that all Halin snarks have perfect matching index at least 5 and that they are nontrivial snarks whenever the building blocks are taken from cyclically 4 -edge-connected graphs with girth at least 5 . The next section deals with circular flow number of Halin snarks, which is shown to be at least 5 whenever the building have this property. A few final remarks conclude the paper.

The present paper serves as an introduction to the topic and its results will be extensively used in our subsequent papers.

## 2 Preliminaries

All graphs in this paper are finite and for the most part simple and cubic (3-valent). Multiple edges and loops may occur, but they will usually be excluded by imposing additional restrictions.

Graphs can be assembled from smaller building blocks called multipoles. Similarly to graphs, each multipole $M$ has its vertex set $V(M)$, its edge set $E(M)$, and an incidence relation between vertices and edges. Each edge of $M$ has two ends, and each end may, but need not be, incident with a vertex of $M$. An edge that is not incident with a vertex is called a dangling edge; its free end is called a semiedge. A multipole with $k$ semiedges is called a $k$-pole. Any two edges $s$ and $t$, each with a free end, can be coalesced into a new edge $s * t$, the junction of $s$ and $t$, by identifying a free end of $s$ with a free end of $t$; the incidences of $s * t$ are naturally inherited from $s$ and $t$.

Semiedges in multipoles are often grouped into pairwise disjoint sets, called connectors. A semiedge not occurring in any connector is said to be residual. In this paper, most multipoles will be dipoles. A dipole is a multipole with two connectors referred to as the input connector and the output connector. A dipole $D(I, O)$ with input connector $I$ of size $a$, output connector $O$ of size $b$, and with $c$ residual semiedges, will be called an $(a, b ; c)$-pole. If $c=0$, we speak of an $(a, b)$-pole. For an $(a, a)$-pole $D(I, O)$ we define its closure $[D]$ to be a graph obtained from $D$ by ordering the semiedges in both connectors and performing the junction of the $i$-th semiedge of the output connector to the $i$-th semiedge of the input connector.

A common way of constructing multipoles is by removing vertices from a graph. The following multipoles obtained from a cubic graph $G$ will be repeatedly used throughout the paper: Let $v, u v$, and $u w v$ be paths in $G$ of length 0,1 , and 2 , respectively. Denote by $G_{v}, G_{u v}$, and $G_{u w v}$, respectively, the 3-pole, the (2,2)-pole, and the ( 2,$2 ; 1$ )-pole obtained from $G$ by removing the corresponding path and arranging the resulting semiedges in such a way that the edges formerly incident with the same vertex belong to the same connector. To be more precise, in both $G_{u v}$ and $G_{u w v}$ the semiedges formerly incident with $u$ will be assigned to the input connector, those formerly incident with $v$ will be assigned to the output connector, and the semiedge incident with $w$ will be residual. The semiedges of $G_{v}$ will all belong to its single connector.

An edge colouring of a graph or a multipole $X$ is an assignment of colours from a set $Z$ of colours to the edges of $X$ in such a way that the edges with adjacent edge ends receive distinct colours. It means that all edge colourings in this paper are proper. A $k$-edge-colouring is one where $|Z|=k$ colours. A 3-edge-colouring is also known as a Tait colouring. A 2-connected cubic graph with no 3 -edge-colouring is called a snark. A snark is nontrivial if it is cyclically 4 -edge-connected and has girth at least 5 .

A natural generalisation of a Tait colouring is the concept of a local Tait colouring of a cubic graph proposed by Archdeacon [2] and developed in [16, 20, 24], and elsewhere. It allows for an unlimited number of colours but requires that the colours of any two edges meeting at a vertex always determine the same third colour. Local Tait colourings can be conveniently described in terms of colourings by points of a partial Steiner triple system such that the colours meeting at any vertex form a triple of the system; for details see, for example, [20].

Local Tait colourings that occur in the present paper are at the same time nowherezero flows. The pertinent definitions are therefore in order. Given an abelian group $A$, an $A$-flow on a graph $G$ consists of an orientation of $G$ and a function $\phi: E(G) \rightarrow A$ such that, at each vertex, the sum of all incoming values equals the sum of all outgoing ones (Kirchhoff's law). A flow which only uses non-zero elements of the group is said to be nowhere-zero. For the existence of an $A$-flow on $G$ the choice of the edge directions is immaterial for one can reverse the orientation of any edge and replace the value on it by its negative without violating the Kirchhoff law. Furthermore, if $x=-x$ for every $x \in A$, then orientation can be ignored altogether. This is the case of local Tait colourings encountered in this paper.

## 3 Covering cubic graphs with four perfect matchings

The aim of this section is to translate the problem of covering a cubic graph with four perfect matchings into the language of flows whose values and flow patterns around vertices are restricted to points and lines of a tetrahedron in the 3-dimensional projective space over the 2-element field $\mathbb{F}_{2}$. This new language will be crucial for the remainder of the paper. We start with the necessary geometric background.

The $n$-dimensional projective space $\mathbb{P}_{n}\left(\mathbb{F}_{2}\right)=P G(n, 2)$ over the 2-element field $\mathbb{F}_{2}$ is an incidence geometry whose points can be identified with the nonzero vectors of the $(n+1)$-dimensional vector space $\mathbb{F}_{2}^{n+1}$ and lines are formed by the triples $\{x, y, z\}$ of points such that $x+y+z=0$. The 2-dimensional projective space $P G(2,2)$ is the well-known Fano plane which has 7 points and 7 lines. The 3-dimensional projective space $P G(3,2)$ has 15 points and 35 lines.

An automorphism of $\operatorname{PG}(n, 2)$ is called a collineation. It maps lines to lines and hence collinear points to collinear points. It is well known [6] that each collineation of $P G(n, 2)$
is induced by a bijective linear transformation $\mathbb{F}_{2}^{n+1} \rightarrow \mathbb{F}_{2}^{n+1}$.
A tetrahedron in $P G(3,2)$ is a configuration $T$ consisting of ten points and six lines spanned by a set of four points of $P G(3,2)$ in general position; it means that no three of the points lie on the same line. Two points of $T$ that lie on the same line of $T$ are said to be collinear in $T$, otherwise they are non-collinear in $T$.

Consider a fixed tetrahedron $T=T\left(p_{1}, p_{2}, p_{3}, p_{3}\right)$ spanned by points $p_{1}, p_{2}, p_{3}$, and $p_{4}$ in general position; these four points are the corner points of $T$. With every pair $\left\{c_{1}, c_{2}\right\}$ of distinct corner points $T$ contains the entire line $\left\{c_{1}, c_{1}+c_{2}, c_{2}\right\}$. The point $c_{1}+c_{2}$ is the midpoint of the line $\left\{c_{1}, c_{1}+c_{2}, c_{2}\right\}$. Clearly, there are six midpoints in $T$. Thus, in total, $T$ has four corner points and six midpoints arranged in six lines (see Figure 2). Observe that each line $\ell=\left\{c_{1}, c_{1}+c_{2}, c_{2}\right\}$ of $T$ is uniquely determined by either its two corner points or by its midpoint; accordingly, $\ell$ will be denoted by $\left\langle c_{1}, c_{2}\right\rangle$ or by $\left\langle c_{1}+c_{2}\right\rangle$.

There are exactly five points of $P G(3,2)$ that are not included in $T$. In order to describe them in geometric terms recall that four points $x_{1}, x_{2}, x_{3}$, and $x_{4}$ of $P G(3,2)$ are in general position if and only if they form a basis of the vector space $\mathbb{F}_{2}^{4}$. Every vector $y \in \mathbb{F}_{2}^{4}$ can therefore be uniquely expressed as a linear combination

$$
y=\alpha_{1} p_{1}+\alpha_{2} p_{2}+\alpha_{3} p_{3}+\alpha_{4} p_{4}
$$

where the coefficients $\alpha_{i}$ are from $\mathbb{F}_{2}=\{0,1\}$. The number of nonzero coefficients in this expression is the weight of $y$ with respect to $T$, and will be denoted by $|y|_{T}$. The subscript $T$ will be dropped whenever $T$ is clear from the context. We emphasise that the weight $|y|_{T}$ coincides with the Hamming weight only when $T$ is spanned by the unit vectors of $\mathbb{F}_{2}$.

According to our definition of weight, the zero vector has weight 0 , the corner points of $T$ have weight 1 , and midpoints have weight 2 . The remaining five points of $P G(3,2)$ have weight 3 and 4 and can be characterised as follows. In $T$, consider a triangle $t$, by which we mean a set consisting of six points arranged in three lines spanned by three distinct corner points $c_{1}, c_{2}$, and $c_{3}$; we denote the triangle $t$ by $\left\langle c_{1}, c_{2}, c_{3}\right\rangle$. The point $c_{1}+c_{2}+c_{3}$ of $P G(3,2)$, which obviously does not belong to $T$, can be regarded as the centre of $t$. There are four triangles in $T$; their centres provide four of the five points of $P G(3,2)$ missing in $T$. The last missing point is $p_{1}+p_{2}+p_{3}+p_{4}$, the barycentre of the entire $T$.

Let us consider an arbitrary covering $\mathcal{M}$ of a cubic graph $G$ with four perfect matchings $M_{1}, M_{2}, M_{3}$, and $M_{4}$, not necessarily distinct; thus $E(G)=M_{1} \cup M_{2} \cup M_{3} \cup M_{4}$. Every vertex $v$ of $G$ is incident with all the members of $\mathcal{M}$ and each edge incident with $v$ belongs to a member of $\mathcal{M}$. It follows that one of the edges at $v$ is covered by two members of $\mathcal{M}$ while the remaining two are covered by a single member of $\mathcal{M}$. If we label each edge $e$ with the binary vector $\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ where $x_{i}=0$ if and only if $e$ belongs to $M_{i}$ we obtain a mapping

$$
\begin{equation*}
\psi=\psi_{\mathcal{M}}: E(G) \rightarrow \mathbb{Z}_{2}^{4} \tag{1}
\end{equation*}
$$

This mapping is easily seen to be a proper edge-colouring of $G$, in fact, a local Tait colouring in the sense of [20]. Even more, it is a nowhere-zero $\mathbb{Z}_{2}^{4}$-flow because for each $i \in\{1,2,3,4\}$ the $i$-th coordinate mapping

$$
e \mapsto \psi(e)_{i}
$$

coincides with the characteristic function of a cycle, the 2-factor complementary to $M_{i}$, and hence is a flow.

The flow $\psi_{\mathcal{M}}$ has an additional geometric structure evinced by the fact that the set of values of $\phi$ forms the tetrahedron spanned by the points $(0,1,1,1),(1,0,1,1),(1,1,0,1)$,
and ( $1,1,1,0$ ) (representing the perfect matchings $M_{1}, M_{2}, M_{3}$, and $M_{4}$, respectively) and the flow values around each vertex form a line of the tetrahedron. Such a flow will be called tetrahedral. To be more precise, let $\phi: E(G) \rightarrow \mathbb{Z}_{2}^{4}$ an edge-valuation of a cubic graph $G$ and let $T=T\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ be a tetrahedron in $P G(3,2)$. We say that $\phi$ is a $T$-flow if, for each edge $e$ of $G$, the value $\phi(e)$ is a point of $T$, and for each vertex $v$ of $G$, the set $\left\{\phi\left(e_{1}\right), \phi\left(e_{2}\right), \phi\left(e_{3}\right)\right\}$, where $e_{1}, e_{2}$, and $e_{3}$ are the edges incident with $v$, is a line of $T$. If the particular tetrahedron $T$ is irrelevant, we just say that $\phi$ is a tetrahedral flow. Note that any tetrahedral flow is a nowhere-zero flow because any three points constituting a line of $T$ sum to zero and the value $\phi(e)=0$ doe not occur. Since $\phi$ is also a proper edge-colouring, it will sometimes be referred to as a tetrahedral colouring (or a $T$-colouring whenever a tetrahedron $T$ is specific) and the values $\phi(e)$ as colours. It may be worth mentioning that a tetrahedral flow is also an instance of a $B$-flow in the sense of Jaeger [18, p. 73], where $B$ is a subset of an abelian group $A$ not containing zero such that $B=-B$. However, it is endowed with an additional geometric structure.

With this preparation we can proceed to the main result of this section.
Theorem 3.1. A cubic graph can have its edges covered with four perfect matchings if and only if it admits a tetrahedral flow. Moreover, there exists a one-to-one correspondence between coverings of $G$ with four perfect matchings and $T$-flows, where $T$ is an arbitrary fixed tetrahedron in $P G(3,2)$.

Proof. It suffices to prove the second statement. We first do it for the tetrahedron $T_{1}$ spanned by the points $(0,1,1,1),(1,0,1,1),(1,1,0,1)$, and ( $1,1,1,0$ ), and then supply a general argument.

As mentioned in the preceding paragraphs, given a covering $\mathcal{M}=\left\{M_{1}, M_{2}, M_{3}, M_{4}\right\}$ of $G$ with four perfect matchings, the mapping $\psi_{\mathcal{M}}$ defined by $(1)$ is a $T_{1}$-flow. For the converse, let $\phi$ be an arbitrary $T_{1}$-flow on $G$. For each $i \in\{1,2,3,4\}$ define $N_{i}$ to be the set of all edges where the $i$-coordinate of the point $\psi(e)$ equals 0 . Taking into account the structure of lines in $T_{1}$ it is not difficult to see that each $N_{i}$ is a perfect matching of $G$ and that $\mathcal{N}=\left\{N_{1}, N_{2}, N_{3}, N_{4}\right\}$ covers all the edges of $G$. Thus every $T_{1}$-flow $\phi$ on $G$ determines a covering $\mathcal{N}=\mathcal{N}_{\phi}$ of $G$ with four perfect matchings. Furthermore, if we start from a $T_{1}$-flow $\phi$ on $G$, construct the corresponding covering $=\mathcal{N}_{\phi}$, and then derive the $T_{1}$-flow $\psi_{\mathcal{N}}$ from it, we can easily check that $\psi_{\mathcal{N}}=\phi$. Similarly, if we start with a covering $\mathcal{M}$, derive $\psi=\psi_{\mathcal{M}}$, and then $\mathcal{N}_{\psi}$, we can conclude that $\mathcal{N}_{\psi}=\mathcal{M}$. This means that we have established a one-to-one correspondence between coverings of $G$ with four perfect matchings and $T_{1}$-flows.

Now let $T=T\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ be an arbitrary tetrahedron in $P G(3,2)$ with corner points $p_{1}, p_{2}, p_{3}$, and $p_{4}$. Since the set $\left\{p_{1}, p_{2}, p_{3}, p_{4}\right\}$ forms a basis of $\mathbb{F}_{2}^{4}$, there exists a linear transformation $\Theta$ of $\mathbb{F}_{2}^{4}$ which takes the basis $\left\{p_{1}, p_{2}, p_{3}, p_{4}\right\}$ to the basis $\{(0,1,1,1),(1,0,1,1),(1,1,0,1),(1,1,1,0)\}$. This mapping induces a collineation of $P G(3,2)$, which means that $\Theta$ takes a line to a line and, consequently, transforms $T$ into $T_{1}$. In particular, if $\psi$ is a $T$-flow on $G$, then $\Theta \psi$ is a $T_{1}$-flow, and if $\phi$ is a $T_{1}$-flow, then $\Theta^{-1} \phi$ is a $T$-flow. This allows us to conclude that the assignment $\psi \mapsto \Theta \psi$ establishes a one-to-one correspondence between $T$-flows and $T_{1}$-flows on $G$. Combining this correspondence with the one-to-one correspondence between $T_{1}$-flows on $G$ and coverings of $G$ with four perfect matchings we obtain the desired result.

The significance of Theorem 3.1 resides in the fact that it enables us to move freely between the coverings of cubic graphs with four perfect matchings and the tetrahedral flows. With this correspondence in hand, we can replace reasoning about 1-factors and 2-factors in cubic graphs with algebraic calculus in the group $\mathbb{Z}_{2}^{4}$ combined with the
geometry of the projective space $P G(3,2)$ and then translate the results back to graph structure.

Remark 3.2. We discuss the relationship between $T_{1}$-flows and $T$-flows described in the second part of the previous proof for the special case of the tetrahedron $T_{0}$ spanned by the points $(1,0,0,0),(0,1,0,0),(0,0,1,0)$, and $(0,0,0,1)$. Observe that both $T_{0}$ and $T_{1}$ have the same set of midpoints, namely the points with exactly two coordinates equal to 1 . The linear transformation $\Lambda$ of $\mathbb{F}_{2}^{4}$ determined by the matrix

$$
A_{\Lambda}=\left(\begin{array}{llll}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{array}\right)
$$

swaps each unit vector of $\mathbb{F}_{2}^{4}$ with its antipode, the vector obtained by from it by interchanging zeros with ones, while leaving the midpoints fixed; in particular, $\Lambda^{2}=\mathrm{id}$. The corresponding collineation maps $T_{0}$ isomorphically to $T_{1}$ and vice versa. Now, if $\phi$ is an arbitrary $T_{i}$-flow on a cubic graph $G$ for $i \in\{0,1\}$, then the corresponding $T_{1-i}$-flow $\Lambda \phi$ can be obtained simply by replacing each flow value of weight 1 with respect to $T_{i}$ with its antipode (which is of weight 1 in the other tetrahedron) while leaving the flow values of weight 2 intact.

This correspondence has a useful consequence that if $\phi$ is an arbitrary $T_{0}$-flow on a cubic graph $G$, then for each $i \in\{1,2,3,4\}$ the set $M_{i}=\left\{e \in E(G) ; \phi(e)_{i}=1\right\}$ is a perfect matching of $G$ and $\mathcal{M}=\left\{M_{1}, M_{2}, M_{3}, M_{4}\right\}$ is a covering of $G$ with four perfect matchings. The correspondence also works in the reverse direction.

## 4 Small geometric objects in $P G(3,2)$

One way of applying Theorem 3.1 to proving that a cubic graph cannot be covered with four perfect matchings is by analysing conflicting behaviour of tetrahedral flows on the components resulting from the removal of an edge-cut from the graph. The sets of flow values on the edges of the cut form geometric objects in $P G(3,2)$ which in turn can be used to describing the conflicts. In this section we introduce several types of objects, mainly of size 2 , that will serve for this purpose.

We say that two sets $A$ and $B$ of points of a tetrahedron $T$ in $P G(3,2)$ have the same shape if there exists a collineation of $P G(3,2)$ that preserves $T$ and takes $A$ to $B$. A geometric shape in $T$, or simply a shape, is an equivalence class of all point sets having the same shape. The shape of a set of points of $T$ is a geometric shape it belongs to.

Let $\boldsymbol{\Pi}=\boldsymbol{\Pi}(T)$ be the set of all pairs $\{p, q\}$ where $p$ and $q$ are points of $T$, not necessarily distinct.

We distinguish the following seven types of objects in $\boldsymbol{\Pi}(T)$ :
(i) A line segment is a pair $\left\{c_{1}, c_{2}\right\}$ where $c_{1}$ and $c_{2}$ are any two distinct corner points of $T$. A line segment is a subset of a line consisting of its two corner points. The set of all line segments of $T$ will be denoted by ls. Clearly, there are six line segments in $T$.
(ii) A half-line is a pair $\left\{c_{1}, c_{1}+c_{2}\right\}$ where $c_{1}$ and $c_{2}$ are any two distinct corner points of $T$. A half-line is a subset of a line consisting of a corner point and a midpoint. It means that each line has two half-lines. The point $c_{1}$ it the origin of the half-line, and $c_{2}$ is its target. The set of all half-lines of $T$ will be denoted by hl. There are twelve half-lines in $T$.
(iii) An angle is a pair $\left\{c_{1}+c_{2}, c_{1}+c_{3}\right\}$ where $c_{1}, c_{2}$, and $c_{3}$ are any three distinct corner points of $T$. An angle consists of the midpoints of two intersecting lines of $T$. Their corner points form a triangle. Clearly, each triangle $\left\langle c_{1}, c_{2}, c_{3}\right\rangle$ in $T$ has three angles. On the other hand, each angle belongs to a unique triangle. The set of all angles of $T$ will be denoted by ang. There are twelve distinct angles in $T$.
(iv) An altitude is a pair $\left\{c_{1}, c_{2}+c_{3}\right\}$ where $c_{1}, c_{2}$, and $c_{3}$ are any three distinct corner points of $T$. An altitude consist of a corner point of a triangle $t=\left\langle c_{1}, c_{2}, c_{3}\right\rangle$ and the midpoint of the line of $t$ not incident with the chosen the corner point. Each triangle has three altitudes. The set of all altitudes of $T$ will be denoted by alt. There are twelve altitudes in $T$.
(v) An axis is a pair $\left\{c_{1}+c_{2}, c_{3}+c_{4}\right\}$ where $c_{1}, c_{2}, c_{3}$, and $c_{4}$ are all four corner points of $T$ in some order. An axis consists of the midpoints of two skew (non-intersecting) lines of $T$. The set of all axes of $T$ will be denoted by ax. Any tetrahedron $T$ in $P G(3,2)$ has three axes. The lines of $P G(3,2)$ spanned by the axes of $T$ all meet at the barycentre of $T$.
(vi) A double corner point is a degenerate pair $\{c, c\}$ where $c$ is an arbitrary corner point of $T$. The set of all such pairs will be denoted by dc.
(vii) A double midpoint is pair $\{m, m\}$ where $m$ is an arbitrary midpoint of $T$. The set of all such pairs will be denoted by dm.

The pairs under items (i)-(ii) are collinear, those under (iii)-(v) are non-collinear, and the pairs under items (vi)-(vii) are degenerate. Clearly, for a pair $\{p, q\} \in \Pi(T)$ the sum $p+q$ is a point of $T$ if and only if $\{p, q\}$ is collinear.

We now show that the items (i)-(vii) represent all geometric shapes of pairs of points of any tetrahedron in $P G(3,2)$.

Proposition 4.1. The set $\boldsymbol{\sigma}=\{1 \mathrm{~s}, \mathrm{hl}$, ang, alt, ax, $\mathrm{dc}, \mathrm{dm}\}$ provides a complete list of shapes of pairs of points of an arbitrary tetrahedron in $P G(3,2)$.

Proof. It is not difficult to see that each of the sets ls, hl, ang, alt, ax, dc, and dm contains pairs of the same shape. On the other hand, different members of $\boldsymbol{\sigma}$ consist of pairs of different shape. Since

$$
|\mathrm{l} \mathrm{~s}|+|\mathrm{h} l|+|\mathrm{ang}|+|\mathrm{alt}|+|\mathrm{ax}|+|\mathrm{dc}|+|\mathrm{dm}|=55=\binom{10}{2}+10=|\Pi(T)|
$$

each pair of points of $T$ belongs to precisely one element of $\boldsymbol{\sigma}$. This implies that the set $\boldsymbol{\sigma}$ is the complete set of shapes of pairs of points of $T$.

From among the shapes of 3 -element point sets only two will be important for us - lines (of course) and circles. We define a circle in a tetrahedron $T$ as any subset $\left\{m_{1}, m_{2}, m_{3}\right\}$ of $T$ which consists of three distinct midpoints sharing a triangle. It means that there exist three corner points $c_{1}, c_{2}$, and $c_{3}$ of $T$ such that $m_{1}=c_{1}+c_{2}, m_{2}=c_{2}+c_{3}$, and $m_{3}=c_{3}+c_{1}$. In particular, $m_{1}+m_{2}+m_{3}=0$, so $\left\{m_{1}, m_{2}, m_{3}\right\}$ is a line of $\operatorname{PG}(3,2)$. Clearly, all circles have the same shape, but the shape of a circle in $T$ is different from that of a line in $T$.

Lemma 4.2. For arbitrary points $x, y$, and $z$ of a tetrahedron $T$ in $P G(3,2)$, the equality $x+y+z=0$ holds if and only if $\{x, y, z\}$ is a line or a circle of $T$.

Proof. If $\{x, y, z\}$ is a line of $T$ or a circle of $T$, then indeed $x+y+z=0$. For the converse let $x, y$, and $z$ be arbitrary points of $T$ such that $x+y+z=0$. Since all of them are non-zero vectors, they must be pairwise distinct. Consider any two of them, say $x$ and $y$. If they are collinear, then $\{x, y, z\}$ is clearly a line. If they are non-collinear, then $\{x, y\}$ is either an angle, an axis, or an altitude. In the latter two cases, $z=x+y$ is not a point of $T$. If $\{x, y\}$ is an angle, then $z$ is the third midpoint of the triangle spanned by the corner points of the lines $\langle x\rangle$ and $\langle y\rangle$. In other words, $\{x, y, z\}$ is a circle of $T$.

## 5 Transitions through (2,2)-poles

The next two sections provide a background material for the study of conflicts of tetrahedral flows on edge-cuts of size 4 and 5 . For this purpose it is convenient to partition the semiedges of 4 -poles and 5 -poles arising from the removal of these cuts into two connectors of size 2 and, in the latter case, one additional residual semiedge. After choosing an input connector we can follow how input values of a tetrahedral flow transform into output values, and from this information we can derive a transition relation for a dipole.

Consider an arbitrary (2,2)-pole $X=X(I, O)$ with input connector $I=\left\{g_{1}, g_{2}\right\}$ and output connector $O=\left\{h_{1}, h_{2}\right\}$, and let $T$ be a fixed tetrahedron in $\operatorname{PG}(3,2)$. We say that $X$ has a transition

$$
\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}
$$

or that $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ is a transition through $X$, if there exists a $T$-flow $\phi$ on $X$ such that $\left\{\phi\left(g_{1}\right), \phi\left(g_{2}\right)\right\}=\{x, y\}$ and $\left\{\phi\left(h_{1}\right), \phi\left(h_{2}\right)\right\}=\left\{x^{\prime}, y^{\prime}\right\}$. If $X$ admits both transitions $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ and $\left\{x^{\prime}, y^{\prime}\right\} \rightarrow\{x, y\}$, we write

$$
\{x, y\} \longleftrightarrow\left\{x^{\prime}, y^{\prime}\right\}
$$

The set of all transitions through $X$ forms a binary relation $\mathbf{T}_{\boldsymbol{\Pi}}(X)$ on the set $\boldsymbol{\Pi}=\boldsymbol{\Pi}(T)$ of point pairs of $T$.

For convenience, we often refer to the symbol $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ as a transition even without a reference to a particular (2,2)-pole, as opposed to a transition through a $(2,2)$ pole defined above. There is no danger of confusion.

From the Kirchhoff law we deduce that for each transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ through a (2,2)-pole we have $x+y=x^{\prime}+y^{\prime}$. This value will be called the trace of the transition. A transition whose trace is 0 is said to be vanishing. A non-vanishing transition $\{x, y\} \rightarrow$ $\left\{x^{\prime}, y^{\prime}\right\}$ is collinear if both pairs $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ are collinear, that is, if there exist lines $\ell$ and $\ell^{\prime}$ of $T$ such that $\{x, y\} \subseteq \ell$ and $\left\{x^{\prime}, y^{\prime}\right\} \subseteq \ell^{\prime}$.

Each transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ between pairs of points induces the transition between their shapes. In this way we obtain a transition $s \rightarrow t$, where $s$ is the shape of $\{x, y\}, \mathrm{t}$ is the shape of $\left\{x^{\prime}, y^{\prime}\right\}$, and $\mathrm{s}, \mathrm{t} \in\{\mathrm{ls}, \mathrm{hl}$, ang, alt, $\mathrm{ax}, \mathrm{dc}, \mathrm{dm}\}=\boldsymbol{\sigma}$. Since our transitions are derived from flows, in most cases there is no need to distinguish between the shapes dc and dm as both of them represent the zero total flow through a connector. This permits us to merge the shapes dc and dm into a single shape $\mathrm{dc} \cup \mathrm{dm}$, which we denote by dpt and call the double point. Accordingly, we obtain the merged set of shapes

$$
\boldsymbol{\Sigma}=\{\mathrm{ls}, \mathrm{hl}, \text { ang }, \text { alt, ax, dpt }\}
$$

and denote the corresponding induced transition relation by $\mathbf{T}_{\boldsymbol{\Sigma}}(X)$ or simply by $\mathbf{T}(X)$. In what follows, this transition relation will become one of our main tools for the investigation of cubic graphs that cannot be covered with four perfect matchings.

Before stating our next theorem we need one more definition. We say that a transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ is stationary if $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ have the same shape in $\boldsymbol{\Sigma}$. If all transitions through a (2,2)-pole $X$ are stationary, then $X$ itself is called stationary.

Theorem 5.1. All transitions through an arbitrary $(2,2)$-pole $X$ are stationary except possibly those of the form $1 \mathrm{~s} \rightarrow$ ang or ang $\rightarrow$ ls. In particular, all transitions involving a half-line, an altitude, an axis, or a double point must have one of the following forms:

$$
\mathrm{hl} \rightarrow \mathrm{hl}, \quad \text { alt } \rightarrow \mathrm{alt}, \quad \mathrm{ax} \rightarrow \mathrm{ax}, \quad \text { and } \quad \mathrm{dpt} \rightarrow \mathrm{dpt} .
$$

Proof. Consider an arbitrary transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ through a (2,2)-pole $X$, and let $\phi$ be a $T$-flow on $X$ that induces it. Kirchhoff's law implies that $x+y=x^{\prime}+y^{\prime}$, which means that $x+y$ and $x^{\prime}+y^{\prime}$ must have the same weight. If $|x+y|=2$, then each of $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ is either an angle or a line segment. Thus if the transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ with $|x+y|=2$ is not stationary, then it has the form ang $\rightarrow$ ls or 1 s $\rightarrow$ ang.

It remains to verify that all the remaining transitions through $X$ are stationary. If $|x+y|=0$, then $x=y$ and $x^{\prime}=y^{\prime}$, and we obtain the transition dpt $\rightarrow \mathrm{dpt}$. If $|x+y|=1$, then $x+y=x^{\prime}+y^{\prime}$ is a corner point, and therefore both $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ must be halflines. Similarly, if $|x+y|=3$, then $x+y$ is the centre of a triangle, which means that both $\{x, y\}$ both $\left\{x^{\prime}, y^{\prime}\right\}$ are altitudes. Finally, if $|x+y|=4$, then $\{x, y\}$ both $\left\{x^{\prime}, y^{\prime}\right\}$ are necessarily axes. This yields the transitions $\mathrm{hl} \rightarrow \mathrm{hl}$, alt $\rightarrow \mathrm{alt}$, and ax $\rightarrow \mathrm{ax}$, respectively.

The previous theorem implies that the transition relation $\mathbf{T}(X)$ of every (2, 2, )-pole $X$ is contained in the set

$$
\begin{align*}
& \mathcal{A}=\{\mathrm{dpt} \rightarrow \mathrm{dpt}, \mathrm{hl} \rightarrow \mathrm{hl}, \mathrm{alt} \rightarrow \mathrm{alt}, \mathrm{ax} \rightarrow \mathrm{ax} \\
&\text { ang } \rightarrow \mathrm{ang}, \mathrm{ang} \rightarrow \mathrm{ls}, \mathrm{ls} \rightarrow \mathrm{ang}, \mathrm{ls} \rightarrow \mathrm{ls}\} . \tag{2}
\end{align*}
$$

The elements of $\mathcal{A}$ will be called admissible transitions.
Remark 5.2. It is obvious that Theorem 5.1 can be substantially strengthened as Kirchhoff's law provides additional restrictions to admissible transitions. The following statements hold for any transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ through an arbitrary $(2,2)$-pole $X$. Their proofs are easy and therefore are left to the reader.
(i) If $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ has the form $\mathrm{hl} \rightarrow \mathrm{hl}$, then $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ are half-lines with the same target, not necessarily identical.
(ii) If $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ has the form $1 \mathrm{~s} \rightarrow 1 \mathrm{~s}$, then $\{x, y\}=\left\{x^{\prime}, y^{\prime}\right\}$.
(iii) If $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ has the form ang $\rightarrow 1 \mathrm{~s}$, then $\left\{x^{\prime}, y^{\prime}\right\}$ is the segment of the line of the opposite to the angle $\{x, y\}$ in the triangle containing $\left\{x^{\prime}, y^{\prime}\right\}$. To be more precise, if $\{x, y\}=\left\{c_{1}+c_{2}, c_{1}+c_{3}\right\}$, then $\left\{x^{\prime}, y^{\prime}\right\}=\left\{c_{2}, c_{3}\right\}$ for suitable corner points $c_{1}, c_{2}$, and $c_{3}$.
(iv) If $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ has the form ang $\rightarrow$ ang, then either $\{x, y\}=\left\{x^{\prime}, y^{\prime}\right\}$, or $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ are opposite angles of the rectangle formed by the union of triangles determined by $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$, respectively. The value $x+y=x^{\prime}+y^{\prime}$ is the midpoint of the line forming a diagonal of the rectangle.
(v) If $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ has the form alt $\rightarrow$ alt, then $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ are altitudes of the same triangle.
(vi) There are no restrictions on transitions of the form $\mathrm{ax} \rightarrow \mathrm{ax}$ and dpt $\rightarrow \mathrm{dpt}$.

It is natural to ask whether all the admissible transitions can actually occur in some dipole. The answer is positive but we defer it until Example 5.6 which is preceded by the next two theorems.

Consider a cubic graph $G$ with perfect matching index greater than 4, pick two adjacent vertices $u$ and $v$, and form the (2,2,)-pole $D(I, O)=G_{u v}$. Our next theorem reveals the fundamental property of the dipole constructed in the just described way: for every nonvanishing transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ through $D(I, O)$ at most one of the pairs $\{x, y\}$ is collinear.

Theorem 5.3. Let $G$ be a cubic graph with $\pi(G) \geq 5$, let $u$ and $v$ be adjacent vertices of $G$, and let $D(I, O)=G_{u v}$. If $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ is an arbitrary transition through $D$, then at most one of the pairs $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ is collinear. In other words, $D$ has no collinear transition.

Proof. If $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ is a vanishing transition, then both pairs $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ are degenerate and hence not collinear. Assume that $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ is a non-vanishing transition through $D$, and let $\phi$ be a $T$-flow on $D$ which induces it. Suppose to the contrary that both pairs $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ are collinear. It follows that there exist lines $\ell$ and $\ell^{\prime}$ in $T$ such that $\{x, y\} \subseteq \ell$ and $\left\{x^{\prime}, y^{\prime}\right\} \subseteq \ell^{\prime}$. Since $x+y=x^{\prime}+y^{\prime}=t$ for some point $t$ of $T$, we see that $t \in \ell \cap \ell^{\prime}$. If we extend the flow $\phi$ on $D$ to the entire graph $G$ by assigning the value $t$ to the edge $u v$, the edges around $u$ will be properly coloured from the line $\ell$ and those around $v$ will be properly coloured from $\ell^{\prime}$. It follows that $G$ has a $T$-flow, contradicting Theorem 3.1. Hence $X$ has no collinear transition.

Theorem 5.3 implies that every $(2,2)$-pole $G_{u v}$ obtained from a cubic graph $G$ with $\pi(G) \geq 5$ by removing two adjacent vertices and arranging the dangling edges in the usual manner behaves like a collinearity destroying gadget: if the input pair of a transition is collinear, then the output pair must be non-collinear, and vice versa. More generally, every (2,2)-pole $X$ which has no collinear transitions will be called a collinearity destroying dipole, or briefly a decollineator.

The following result provides a characterisation of decollineators.
Theorem 5.4. The following statements are equivalent for an arbitrary (2,2)-pole $X$.
(i) $X$ is a decollineator, that is, $X$ admits no collinear transition.
(ii) $X$ has no transitions of the form $\mathrm{ls} \rightarrow \mathrm{ls}$ or $\mathrm{hl} \rightarrow \mathrm{hl}$.
(iii) The cubic graph $G$ created from $X$ by adding to $X$ two adjacent vertices and attaching each of them to a connector of $X$ has $\pi(G) \geq 5$.

Proof. (i) $\Rightarrow$ (ii): Let $X$ be a decollineator. Theorem 5.1 implies that the only possible transitions through $X$ are the admissible transitions constituting the set $\mathcal{A}$. Of them, only $1 \mathrm{~s} \rightarrow \mathrm{ls}$ and $\mathrm{hl} \rightarrow \mathrm{hl}$ are collinear. So $X$ has no transitions of the form $1 \mathrm{~s} \rightarrow 1 \mathrm{~s}$ or $\mathrm{hl} \rightarrow \mathrm{hl}$.
(ii) $\Rightarrow$ (iii): Let $X$ be a (2,2)-pole that admits no transitions of the form $1 \mathrm{~s} \rightarrow 1$ s or $\mathrm{hl} \rightarrow \mathrm{hl}$, and let $G$ be the cubic graph formed from $X$ by adding two adjacent vertices and attaching each of them to a connector of $X$. By Theorem 3.1 it is sufficient to show that $G$ has no $T$-flow. If $G$ had one, say $\phi$, then the three edges around $u$ and $v$ would receive values from lines $\ell_{u}$ and $\ell_{v}$ of $T$, respectively. Under the induced flow on $X$, the semiedges
of the input connector receive values $x$ and $y$ from $\ell_{u}$, and those of the output connector receive values $x^{\prime}$ and $y^{\prime}$ from $\ell_{v}$. It means that $X$ has a transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ which is collinear, contrary to the assumption.
(iii) $\Rightarrow$ (i): This implication follows directly from Theorem 5.3.

Theorems 5.1 5.4 combined readily imply the following.
Corollary 5.5. Every decollineator $D$ has its transition relation $\mathbf{T}(D)$ contained in the set

$$
\begin{equation*}
\mathcal{D}=\{\mathrm{dpt} \rightarrow \mathrm{dpt}, \mathrm{alt} \rightarrow \mathrm{alt}, \mathrm{ax} \rightarrow \mathrm{ax}, \text { ang } \rightarrow \text { ang, ang } \rightarrow \text { ls, } \mathrm{ls} \rightarrow \text { ang }\} . \tag{3}
\end{equation*}
$$

The next example shows that each admissible transition occurs in some (2,2)-pole.
Example 5.6. The decollineator $G_{u v}$ where $G$ is the Petersen graph has all admissible non-collinear transitions. The transition relation of the ( 2,2 )-pole consisting of two adjacent vertices, with input semiedges attached to one vertex and the output semiedges attached to the other vertex, is the set $\{1 \mathrm{~s} \rightarrow \mathrm{ls}, \mathrm{hl} \rightarrow \mathrm{hl}\}$ comprising the two collinear transitions.

We finish this section with two results that reveal interesting behaviour of bipartite cubic graphs.

Proposition 5.7. Let $G$ be a bipartite cubic graph and $v$ an arbitrary vertex of $G$. Then for every tetrahedral flow on the 3-pole $G_{v}$ the flow values of the three semiedges of $G_{v}$ form a line of the tetrahedron.

Proof. Let $x, y$, and $z$ be the values assigned to the semiedges of $G_{v}$. From Kirchhoff's law we know that $x+y+z=0$. By Lemma 4.2, the triple $\{x, y, z\}$ is either a line or a circle of $T$. If $\{x, y, z\}$ is a circle, then the edges carrying a value of weight 1 must form a 2-factor $F$ of $G-v$. Since $G-v$ has an odd number of vertices, $F$ contains an odd circuit, and hence $G$ is not bipartite, contrary to the assumption. Therefore $\{x, y, z\}$ is a line, as claimed.

Proposition 5.8. Let $G$ be an arbitrary bipartite cubic graph, and let $u$ and $v$ be adjacent vertices of $G$. Then $G_{u v}$ is a stationary (2,2)-pole.

Proof. Suppose to the contrary that $G_{u v}$ is not stationary. Theorem 5.1 then implies that the dipole $G_{u v}$ admits a transition ang $\rightarrow$ ls or its reverse. Without loss of generality we may assume that $G_{u v}$ admits ang $\rightarrow$ ls. Let $\phi$ be a $T$-flow of $G_{u v}$ that induces this transition. Then $\phi$ gives rise to a $T$-flow of $G-u$ under which the values on the semiedges of $G_{u}$ form a circle. This contradicts Proposition 5.7.

## 6 Weighted transitions through (2,2;1)-poles

Consider an arbitrary $(2,2 ; 1)$-pole $Y=Y(I, O ; r)$ with input connector $I=\left\{g_{1}, g_{2}\right\}$, output connector $O=\left\{h_{1}, h_{2}\right\}$, and residual semiedge $r$. Let $T$ be a tetrahedron in $P G(3,2)$, let $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$ be elements of $\Pi(T)$, and finally let $i \in\{1,2\}$. We say that $X$ has a transition

$$
\{x, y\} \xrightarrow{i}\left\{x^{\prime}, y^{\prime}\right\}
$$

if there exists a $T$-flow $\phi$ on $Y$ such that $\left\{\phi\left(g_{1}\right), \phi\left(g_{2}\right)\right\}=\{x, y\},\left\{\phi\left(h_{1}\right), \phi\left(h_{2}\right)\right\}=\left\{x^{\prime}, y^{\prime}\right\}$, and the residual value $\phi(r)$ has weight $i$. For emphasis, such a transition will be called
a weighted transition. However, whenever the context permits it, the adjective 'weighted' will be dropped.

Every weighted transition $\{x, y\} \xrightarrow{i}\left\{x^{\prime}, y^{\prime}\right\}$ between pairs of points induces a transition $\mathrm{s} \xrightarrow{i} \mathrm{t}$ where s is the shape of $\{x, y\}$ and t is the shape of $\left\{x^{\prime}, y^{\prime}\right\}$. Although the weight $i$ of a transition $\{x, y\} \xrightarrow{i}\left\{x^{\prime}, y^{\prime}\right\}$ is uniquely determined by $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$, the same might not hold for $\mathrm{s} \xrightarrow{i} \mathrm{t}$. The reason resides in the fact that the same shapes may correspond to different geometric positions of the corresponding pairs of points within the tetrahedron. Hence, the weight of a transition $s \xrightarrow{i} t$ carries an information that cannot be immediately recovered from $s$ and $t$.

Most of the terminology developed in Section 5 for unweighted transitions directly modifies to weighted transitions. In particular, a vanishing weighted transition is one where either the input pair or the output pair is a double point.

Our first aim is to study the weighted transitions through a $(2,2 ; 1)$-pole $G_{u w v}$ obtained from a bipartite cubic graph $G$ by removing a path uwv, where the input $I=\left\{g_{1}, g_{2}\right\}$ and the output $O=\left\{h_{1}, h_{2}\right\}$ are formed by the semiedges formerly incident with $u$ and $v$, respectively, and the residual semiedge is the one formerly incident with $w$. For brevity, such a $(2,2 ; 1)$-pole will be called bipartite. Analogously we say that a $(2,2)$-pole $G_{u v}$ arising from a bipartite cubic graph by removing two adjacent vertices is bipartite.

We need two lemmas.
Lemma 6.1. Let $G$ be a bipartite graph and let uwv be a path of length 2 in $G$. For an arbitrary tetrahedral flow on the $(2,2 ; 1)$-pole $G_{u w v}$, the total number of non-residual semiedges receiving a value of weight 2 does not exceed 2 .

Proof. Let $\{A, B\}$ be a bipartition of $G$. Since $G$ is regular, we have $|A|=|B|=m$ for some integer $m$. Without loss of generality we may assume that $u$ and $v$ belong to $A$ while $w$ belongs $B$. Under any given $T$-flow on $G-\{u, w, v\}$, each vertex of $G-\{u, w, v\}$ is incident with exactly one edge that carries a value of weight 2 . Let $\bar{p}$ denote the number of semiedges of $G_{u w v}$ formerly incident with the vertex $p \in\{u, w, v\}$ that receive a flow value of weight 2 . Counting the edges with a value of weight 2 leaving $A$ and those leaving $B$ yields

$$
\bar{u}+\bar{v}+(m-2)=\bar{w}+(m-1)
$$

which simplifies to

$$
\bar{u}+\bar{v}=\bar{w}+1
$$

However, $\bar{w} \leq 1$, so $\bar{u}+\bar{v} \leq 2$, which is equivalent to the statement of this lemma.
Lemma 6.2. In an arbitrary $k$-pole $Q$ endowed with a tetrahedral flow, the number of semiedges carrying a value of weight 2 has the same parity as $k$.

Proof. This is a direct consequence of the fact that in a multipole furnished with a tetrahedral flow every vertex is incident with exactly one edge that carries a value of weight 2 .

The combination of Lemmas 6.1 and 6.2 significantly restricts potential transitions through a $(2,2 ; 1)$-pole $G_{\text {uwv }}$ arising from a bipartite graph $G$. In the next theorem we specify a set $\mathcal{B}$ of weighted transitions that can occur for such a $(2,2 ; 1)$-pole. The subsequent Remark 6.4 confirms that $\mathcal{B}$ is a minimal transition set with this property because it actually occurs as the transition relation for a suitable bipartite $(2,2 ; 1)$-pole $G_{u w v}$.

The diagram of $\mathcal{B}$ is depicted in Figure 4. In this diagram, and in all other diagrams representing weighted transition relations, undirected connections between two shapes s
and t indicate the existence of both the transition $\mathrm{s} \xrightarrow{i} \mathrm{t}$ and $\mathrm{t} \xrightarrow{i} \mathrm{~s}$. Transitions of weight 2 are represented by bold lines.


Figure 4: The set of transitions $\mathcal{B}$ through bipartite ( 2,$2 ; 1$ )-poles

Theorem 6.3. Let $G$ be a bipartite cubic graph and let uwv be a path of length 2 in $G$. Then $\mathbf{T}\left(G_{u v w}\right) \subseteq \mathcal{B}$.

Proof. Let $\{x, y\} \xrightarrow{i}\left\{x^{\prime}, y^{\prime}\right\}$ be an arbitrary transition through $G_{u w v}$ and let $z$ be the corresponding residual value. Since $\mathcal{B}$ is a symmetric relation, from each pair of transitions $\{x, y\} \xrightarrow{i}\left\{x^{\prime}, y^{\prime}\right\}$ and $\left\{x^{\prime}, y^{\prime}\right\} \xrightarrow{i}\{x, y\}$ through $G_{u v w}$, whenever they exist, it is sufficient to check the one with $|x|+|y| \geq\left|x^{\prime}\right|+\left|y^{\prime}\right|$.

To prove the required inclusion we consider three main cases depending on the weight of $x$ and $y$. Throughout the proof we therefore distinguish between the shapes dc and dm . In our discussion we will say that a weighted transition $\mathrm{s} \xrightarrow{i} \mathrm{t}$ through $G_{u w v}$ is valid if it occurs in $\mathcal{B}$.

Case 1. $|x|=|y|=2$. From Lemma 6.1 we get that $\left|x^{\prime}\right|=\left|y^{\prime}\right|=1$.
Case 1.1. $x=y$. In this case $x^{\prime}+y^{\prime}=z \neq 0$, so $\left\{x^{\prime}, y^{\prime}\right\}$ is a line segment. Hence, we have a valid transition $\mathrm{dm} \xrightarrow{2} 1 \mathrm{~s}$.

Case 1.2. $x \neq y$. This time $\{x, y\}$ is either an axis or an angle. First assume that $\{x, y\}$ is an axis. Then $x^{\prime}+y^{\prime} \neq 0$ for otherwise we would have $z=x+y$, where $x+y$ is not a point of the tetrahedron. Therefore $\left\{x^{\prime}, y^{\prime}\right\}$ is a line segment. From Lemma 6.2 we infer that $|z|=2$, which yields a valid transition $\mathrm{ax}^{2}$ ls. Next assume that $\{x, y\}$ is an angle. Recall that $\left\{x^{\prime}, y^{\prime}\right\}$ is a pair of corner points. If $x^{\prime}=y^{\prime}$, then we obtain the valid transition ang $\xrightarrow{2}$ dc with residue $z=x+y$, which is the third point of the circle $\{x, y, x+y\}$. If $x^{\prime} \neq y^{\prime}$, then $\left\{x^{\prime}, y^{\prime}\right\}$ is a line segment, and by Lemma 6.2 the residual value is a midpoint. We have thus obtained the transition ang $\xrightarrow{2} 1 \mathrm{~s}$, which again is valid.
Case 2. $|x|=|y|=1$.
Case 2.1. $x=y$. If $z$ is a midpoint, then Lemma 6.2 implies that $\left|x^{\prime}\right|=\left|y^{\prime}\right|$. If $\left|x^{\prime}\right|=\left|y^{\prime}\right|=1$, then $x^{\prime}+y^{\prime}=z \neq 0$, so $\left\{x^{\prime}, y^{\prime}\right\}$ is a line segment. Thus we have obtained the transition $\mathrm{dc} \xrightarrow{2} 1 \mathrm{~s}$, which is again valid.

If $z$ is a corner point, then from Lemma 6.2 we infer that $\left|x^{\prime}\right| \neq\left|y^{\prime}\right|$, which in turn implies that $|x|+|y|<\left|x^{\prime}\right|+\left|y^{\prime}\right|$, a contradiction.

Case 2.2. $x \neq y$. In this case $\{x, y\}$ is a line segment. If $z$ is a corner point, we again get a cotradiction with $|x|+|y|<\left|x^{\prime}\right|+\left|y^{\prime}\right|$. Assume therefore that $z$ is a midpoint.

Then $\left|x^{\prime}\right|=\left|y^{\prime}\right|=1$. We claim that in this situation $x^{\prime}=y^{\prime}$. If not, then $\left\{x^{\prime}, y^{\prime}\right\}$ is a line segment, which gives rise to the transition 1s $\xrightarrow{2} 1$ s. However, any $T$-flow on $G_{u w v}$ corresponding to this transition would induce a $T$-flow on $G_{w}$ under which the values on the semiedges of $G_{w}$ form a circle $\left\{x+y, z, x^{\prime}+y^{\prime}\right\}$. This contradicts Proposition 5.7. Hence $x^{\prime}=y^{\prime}$, which again gives rise to a valid transition $1 \mathrm{~s} \xrightarrow{2} \mathrm{dc}$.
Case 3. $|x| \neq|y|$. This means that $\{x, y\}$ is either an altitude or a half-line. If $z$ is a midpoint, Lemma 6.2 yields that $\left|x^{\prime}\right| \neq\left|y^{\prime}\right|$, which implies that $\left\{x^{\prime}, y^{\prime}\right\}$ is either a half-line or an altitude as well. The resulting transitions are $\mathrm{hl} \xrightarrow{2} \mathrm{hl}$, alt $\xrightarrow{2}$ alt, and $\mathrm{hl} \xrightarrow{2}$ alt or its reverse, all of them valid.

If $z$ is a corner point, then $\left|x^{\prime}\right|=\left|y^{\prime}\right|$, by Lemma 6.2. However, Lemma 6.1 excludes the possibility that $\left|x^{\prime}\right|=\left|y^{\prime}\right|=2$, so $\left|x^{\prime}\right|=\left|y^{\prime}\right|=1$. If $x^{\prime}=y^{\prime}$, then $\{x, y\}$ cannot be an altitude, because otherwise $z=x+y$ whence $z$ would not be a point of $T$. Therefore $\{x, y\}$ is a half-line and we obtain the valid transition $\mathrm{hl} \xrightarrow{1}$ dc. Finally, if $x^{\prime} \neq y^{\prime}$, then $\left\{x^{\prime}, y^{\prime}\right\}$ is a line segment, which yields transitions $\mathrm{hl} \xrightarrow{1} 1$ s and alt $\xrightarrow{1} 1$ s, both of them valid.

To summarise, we have covered all the possibilities for the input pair $\{x, y\}$ and verified that all the transitions from $\{x, y\}$ through $G_{u v w}$, where $G$ is a bipartite graph, are contained in $\mathcal{B}$. Moreover, every transition contained in $\mathcal{B}$ occurs as a valid transition at least once. This completes the proof.

Remark 6.4. The inclusion $\mathbf{T}\left(G_{u v w}\right) \subseteq \mathcal{B}$ can actually be achieved with equality for suitable bipartite graphs. This situation takes place, for example, for both the complete bipartite graph $K_{3,3}$ and the Heawood graph, the unique cubic bipartite graph of girth 6 on 14 vertices [10]. In the former case, verification can be easily done directly, but for the Heawood graph a computer was necessary.

It may be interesting to mention that if the shapes dc and dm are distinguished, then the transition $\mathrm{dc} \xrightarrow{2}$ ls does not exist for $K_{3,3}$, while $\mathrm{dm} \xrightarrow{2}$ ls does. On the other hand, the Heawood graph admits both transitions $\mathrm{dc} \xrightarrow{2} 1$ s and $\mathrm{dm} \xrightarrow{2} 1$ s. In some cases, distinguishing between dc and dm does make sense and can be used for the construction of graphs with perfect matching index at least 5 .

## 7 Composing dipoles and their transitions

In order to be able to construct rich families of graphs with perfect marching index at least 5 we will employ several operations on dipoles as well as on their transition relations. The definitions come with no surprise, but we include them in order to avoid ambiguity.

Given an $\left(2,2 ; d_{1}\right)$-pole $M_{1}$ and an $\left(2,2 ; d_{2}\right)$-pole $M_{2}$, we can construct an $\left(2,2 ; d_{1}+d_{2}\right)$ pole $M_{1} \circ M_{2}$, the join of $M_{1}$ and $M_{2}$, by joining the output connector of $M_{1}$ with the input connector of $M_{2}$. The input and the output of $M_{1} \circ M_{2}$ are inherited from $M_{1}$ and $M_{2}$, respectively. The join is clearly associative, which means that $\left(M_{1} \circ M_{2}\right) \circ M_{3}=$ $M_{1} \circ\left(M_{2} \circ M_{3}\right)$. If both $M_{1}$ and $M_{2}$ are (2,2)-poles, we will also refer to $M_{1} \circ M_{2}$ as their composition.

Given two $(2,2 ; 1)$-poles $M_{1}$ and $M_{2}$, we can construct a new $(2,2 ; 1)$-pole $M_{1} \odot M_{2}$, the composition of $M_{1}$ and $M_{2}$, by attaching the two residual semiedges of $M_{1} \circ M_{2}$ to a new vertex, say $v$, and by adding a new dangling edge incident with $v$ which will become the residual edge of $M_{1} \odot M_{2}$.

Technically speaking, the dipoles $M_{1} \circ M_{2}$ and $M_{1} \odot M_{2}$ are not uniquely determined by $M_{1}$ and $M_{2}$ as they depend on the ordering of semiedges involved in the operation.

However, all our subsequent statements concerning $M_{1} \circ M_{2}$ or $M_{1} \odot M_{2}$ will equally hold for both possible outcomes of either of these operations.

Similarly to dipoles, we can also compose their transition relations. As expected, unweighted transitions $\mathrm{p} \rightarrow \mathrm{s}$ and $\mathrm{s} \rightarrow \mathrm{t}$ of dipoles $M_{1}$ and $M_{2}$, respectively, give rise to the transition $\mathrm{p} \rightarrow \mathrm{t}$ of $M_{1} \circ M_{2}$. Conversely, a transition $\mathrm{p} \rightarrow \mathrm{q}$ through $M_{1} \circ M_{2}$ occurs only when there exist transitions $\mathrm{p} \rightarrow \mathrm{s}$ through $M_{1}$ and $\mathrm{s} \rightarrow \mathrm{t}$ through $M_{2}$ for a suitable shape $s \in \boldsymbol{\Sigma}$. If exactly one of the transitions $\mathrm{p} \rightarrow \mathrm{s}$ and $\mathrm{s} \rightarrow \mathrm{t}$ is weighted, say $\mathrm{s} \xrightarrow{i} \mathrm{t}$, then its weight is inherited into the resulting transition $p \xrightarrow{i} t$. If both transitions are weighted, say $\mathrm{p} \xrightarrow{i} \mathrm{~s}$ and $\mathrm{s} \xrightarrow{j} \mathrm{t}$, then the composite transition $\mathrm{p} \xrightarrow{k} \mathrm{~s}$ through $M_{1} \odot M_{2}$ is defined if and only $i+j \leq 3$, in which case its weight is $k=3-i j$. The last rule is a consequence of the fact that each vertex of a $(2,2 ; 1)$-pole carrying a $T$-flow is incident with two edges with value of weight 1 and one edge with value of weight 2 . The same rules apply not just to transition relations of dipoles but also to arbitrary sets of transitions. So, if $\mathbf{T}\left(M_{1}\right) \subseteq \mathcal{R}_{1}$ and $\mathbf{T}\left(M_{2}\right) \subseteq \mathcal{R}_{2}$ we can conclude that $\mathbf{T}\left(M_{1} \circ M_{2}\right) \subseteq \mathcal{R}_{1} \circ \mathcal{R}_{2}$ or $\mathbf{T}\left(M_{1} \odot M_{2}\right) \subseteq \mathcal{R}_{1} \odot \mathcal{R}_{2}$, depending on which composition operation applies to $M_{1}$ and $M_{2}$.

The next two lemmas are easy but useful.
Lemma 7.1. The following statements hold:
(i) $\mathbf{T}\left(M_{1} \circ M_{2}\right)=\mathbf{T}\left(M_{1}\right) \circ \mathbf{T}\left(M_{2}\right)$ for any two (2,2)-poles $M_{1}$ and $M_{2}$.
(ii) $\mathbf{T}\left(M_{1} \odot M_{2}\right)=\mathbf{T}\left(M_{1}\right) \odot \mathbf{T}\left(M_{2}\right)$ for any two $(2,2 ; 1)$-poles $M_{1}$ and $M_{2}$.

Lemma 7.2. If $X$ is an arbitrary (2,2)-pole or $(2,2 ; 1)$-pole and $Y$ is a stationary $(2,2)$ pole, then $\mathbf{T}(X \circ Y) \subseteq \mathbf{T}(X)$ and $\mathbf{T}(Y \circ X) \subseteq \mathbf{T}(X)$.

Composition of dipoles can be conveniently utilised in constructing dipoles with various useful transition properties. Preventing collinear transitions, which is defining property of decollineators, is one of them. Another important type of a dipole is $(2,2)$-pole which does not permit any transition of the form ang $\rightarrow$ ang. We call it a deangulator. The smallest example of a deangulator is the (2,2)-pole of the form $G_{u v}$ obtained from $G=K_{4}$, the complete graph on four vertices, where $u v$ is any edge of $K_{4}$. Another example can be constructed from the Petersen graph by severing two edges at distance 2 and forming each connector from the half-edges of the same edge. Restricting the consideration to the weights of points it is easy to verify that neither of these dipoles has a transition $\{x, y\} \rightarrow\left\{x^{\prime}, y^{\prime}\right\}$ where $|x|=|y|=\left|x^{\prime}\right|=\left|y^{\prime}\right|=2$. Consequently, no transition of the form ang $\rightarrow$ ang through them is possible.

The next proposition can be applied to the construction of new decollineators and thereby, having in mind Theorem 5.4 (iii), to the construction of new cubic graphs with $\pi(G) \geq 5$.
Proposition 7.3. If $D_{1}$ and $D_{2}$ are decollineators and $U_{1}$ and $U_{2}$ are deangulators, then $D_{1} \circ U_{i} \circ D_{2}$ is a decollineator and $U_{1} \circ D_{i} \circ U_{2}$ is a deangulator for each $i \in\{1,2\}$.
Proof. By Theorem 5.4 it suffices to prove that the (2,2)-pole $D=D_{1} \circ U_{i} \circ D_{2}$ has no transitions of the form $\mathrm{hl} \rightarrow \mathrm{hl}$ and $\mathrm{ls} \rightarrow \mathrm{ls}$. The former transition is immediately excluded by Theorem 5.1. In order to exclude the latter, consider an arbitrary transition through $D$ starting with a line segment. Applying the assumptions and Theorem 5.1 to $D_{1}, U$, and $D_{2}$ we conclude that the only feasible sequence of transitions through them is $1 \mathrm{~s} \rightarrow$ ang $\rightarrow$ ls $\rightarrow$ ang, where the first transition is through $D_{1}$, the second is through $U_{i}$, and the third is through $D_{2}$. The resulting transition through $D$ is $1 \mathrm{~s} \rightarrow$ ang, which proves that $D$ has no transition $1 \mathrm{~s} \rightarrow 1 \mathrm{~s}$ as well. The proof for $U_{1} \circ D_{i} \circ U_{2}$ is similar.


Figure 5: The set of weighted transitions $\mathcal{D} \circ \mathcal{B}$

Example 7.4. We finish this section by exploring the transition relation of a Halin fragment. Recall that a Halin fragment is a $(2,2 ; 1)$-pole of the form $F=D \circ B$ where $D$ is a decollineator, which can be taken to be a $(2,2)$-pole $G_{u v}$ obtained from a cubic graph $G$ with $\pi(G) \geq 5$ by removing a path $u v$ of length 1 , and $B$ is a $(2,2 ; 1)$-pole $K_{u w v}$ arising from a cubic bipartite graph $K$ by removing a path uwv of length 2. Clearly, $\mathbf{T}(D \circ B) \subseteq \mathcal{D} \circ \mathcal{B}$. If $D$ is a decollineator constructed from the Petersen graph and the $(2,2 ; 1)$-pole $B$ is created from the complete bipartite graph $K_{3,3}$, then the resulting Halin fragment coincides with the Petersen fragment $F_{P s}$, the basic building block of treelike snarks. It can be shown that $\mathbf{T}\left(F_{P s}\right)=\mathcal{D} \circ \mathcal{B}$.

With this information we can easily generalise the family of windmill snarks introduced in [7]. Take three Halin fragments $F_{1}, F_{2}$, and $F_{3}$ and form a cubic graph $W=$ $W\left(F_{1}, F_{2}, F_{3}\right)$ by first creating their composition $F_{1} \circ F_{2} \circ F_{3}$, then attaching the three residual semiedges to a new vertex, and finally by taking a closure of the resulting (2, 2)pole. We claim that $\pi(W) \geq 5$. If $W$ had a covering with four perfect matchings, then the corresponding $T$-flow would induce a directed closed walk of length 3 in the diagram of the transition relation $\mathcal{D} \circ \mathcal{B}$ with the property that exactly one of the transitions has weight 2. A straightforward check with the help of Figure 5 reveals that such a closed walk does not exist. Therefore $\pi(W) \geq 5$, as claimed.

If the bipartite constituent $B_{i}$ of each of the three Halin fragments $F_{i}=D_{i} \circ B_{i}$ is constructed from $K_{3,3}$, then $W$ coincides with a windmill snark of Esperet and Mazzuoccolo [7]. Moreover, if we choose each $F_{i}$ to be the Petersen fragment, then $W$ becomes isomorphic to the snark shown in Figure 1 (left). It is therefore appropriate to call the family of graphs of the form $W\left(F_{1}, F_{2}, F_{3}\right)$, where $F_{1}, F_{2}$, and $F_{3}$ are arbitrary Halin fragments, the generalised windmill snarks.

## 8 Halin snarks

In this section we put to use the knowledge gathered in the previous sections and prove that Halin snarks cannot be covered with four perfect matchings. We also prove that they are nontrivial snarks whenever the building blocks employed for their construction originate from cyclically 4 -edge-connected graphs of girth at least 5 .

Recall that a Halin snark arises from a Halin graph $H=C \cup S$, where $C$ is the
perimeter circuit and $S$ is the inscribed tree, by substituting the vertices of $C$ with Halin fragments. As mentioned earlier, a Halin fragment is a $(2,2 ; 1)$-pole $F=D \circ B$ where $D$ is a decollineator and $B$ is a bipartite $(2,2 ; 1)$-pole. Let $v_{0}, v_{1}, \ldots, v_{k-1}$ be the vertices of $C$ arranged in a cyclic order, and let $F_{i}=D_{i} \circ B_{i}$, with $i \in\{0,1,2, \ldots, k-1\}$, be a Halin fragment that substitutes the vertex $v_{i}$ in the construction. We now form a Halin snark $H^{\sharp}$ by, first, performing a junction of the output connector of $F_{i}$ with the input connector of $F_{i+1}$ for each $i \in\{0,1, \ldots, k-1\}$ taken modulo $k$; then, by adding in the tree $S^{\prime}=S-\left\{v_{0}, \ldots, v_{k-1}\right\}$; and, finally, by attaching the residual semiedge of each $F_{i}$ to the unique vertex $v_{i}^{\prime}$ of $S^{\prime}$ adjacent in $S$ to $v_{i}$. For convenience, we identify the vertex of $F_{i}$ adjacent to $v_{i}^{\prime}$ with $v_{i}$. As a result, $S$ is inherited from $H$ to $H^{\sharp}$ in its entirety, and so we can write $H^{\sharp}=C^{\sharp} \cup S$ where $C^{\sharp}$ is a subgraph obtained from $H^{\sharp}$ by the removal of all 3 -valent vertices of $S$.

It is not difficult to show that every Halin snark $H^{\sharp}$ is indeed a snark, meaning that it admits no proper 3 -edge-colouring. One can argue, for example, by using the fact that in each Halin fragment $D_{i} \circ B_{i}$ the (2,2)-pole $D_{i}$ is isochromatic [5, p. 13]. Therefore, for every 3 -edge-colouring of $H^{\sharp}$ - which can be taken a nowhere-zero $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$-flow on $H^{\sharp}$ - the Kirchhoff law fails at each vertex $v_{i}$, forcing the zero value on the edge $v_{i} v_{i}^{\prime}$ that connects $B_{i}$ to $S$. We omit the details because in Theorem 8.5 we prove a stronger result, namely that $\pi\left(H^{\sharp}\right) \geq 5$.

According to our definition, Halin snarks need not be nontrivial in general. The next proposition shows that their nontriviality can easily be achieved if the building blocks are by properly choosen.

Proposition 8.1. If $H^{\sharp}$ is a Halin snark such that each Halin fragment $F_{i}=D_{i} \circ B_{i}$ has both the decollineator $D_{i}$ and the bipartite $(2,2 ; 1)$-pole $B_{i}$ constructed from a cyclically 4 -edge-connected cubic graph of girth at least 5 , then $H^{\sharp}$ is cyclically 4-edge-connected, with girth at least 5.

Proof. Let $H^{\sharp}$ be a Halin snark satisfying the assumptions. It is easy to see that $H^{\sharp}$ has girth at least 5 . We claim that $H^{\sharp}$ is cyclically 4 -edge-connected. Obviously, none of the multipoles used in the construction has a $k$-edge-cut with $k<4$ separating a subgraph containing a cycle from the rest of $H^{\sharp}$. Moreover, all Halin graphs are 3-edgeconnected [15]. Thus, every cycle-separating edge-cut is either entirely contained in one of the multipoles - and then it is an $l$-cut for $l \geq 4$ - or its mapping to the original Halin graph cuts the perimeter circuit twice, and thus is a cycle separating $m$-edge cut with $m \geq 5$.

We now proceed to the proof that every Halin snark $H^{\sharp}$ has $\pi\left(H^{\sharp}\right) \geq 5$. The idea is to modify the argument of Example 7.4 by engaging induction. For this purpose we extend the weighted transition relation $\mathcal{D} \circ \mathcal{B}$ to the set

$$
\begin{equation*}
\mathcal{M}=\mathcal{D} \circ \mathcal{B} \cup\left\{\mathrm{ax}^{1} \mathrm{hl}, \mathrm{ls} \xrightarrow{1} \mathrm{hl}\right\} \tag{4}
\end{equation*}
$$

whose diagram of $\mathcal{M}$ is displayed in Figure 6(left).
The following property of $\mathcal{M}$ will be crucial for the inductive proof.
Proposition 8.2. If $M_{1}$ and $M_{2}$ are arbitrary $(2,2 ; 1)$-poles such that $\mathbf{T}\left(M_{1}\right) \subseteq \mathcal{M}$ and $\mathbf{T}\left(M_{2}\right) \subseteq \mathcal{M}$, then $\mathbf{T}\left(M_{1} \odot M_{2}\right) \subseteq \mathcal{M} \odot \mathcal{M}-\{\mathrm{dpt} \stackrel{1}{\longleftrightarrow} \mathrm{alt}\}$.

Proof. If $\mathbf{T}\left(M_{1}\right) \subseteq \mathcal{M}$ and $\mathbf{T}\left(M_{2}\right) \subseteq \mathcal{M}$, then clearly $\mathbf{T}\left(M_{1} \odot M_{2}\right) \subseteq \mathcal{M} \odot \mathcal{M}$; the diagram of $\mathcal{M} \odot \mathcal{M}$ is displayed in Figure 6 (right). We show that the transitions dpt ${ }^{1}$ alt and


Figure 6: The transition relations $\mathcal{M}$ (left) and $\mathcal{M} \odot \mathcal{M}$ and $\mathcal{M}^{\prime}$ (right); transitions added to $\mathcal{D} \circ \mathcal{B}$ are dotted, those of $\mathcal{M} \odot \mathcal{M}$ that do not occur in $\mathcal{M}^{\prime}$ are dashed.
alt $\xrightarrow{1} \mathrm{dpt}$, which occur in $\mathcal{M} \odot \mathcal{M}$ and in the diagram are represented by a dashed line, do not belong to $\mathbf{T}\left(M_{1} \odot M_{2}\right)$.

Suppose to the contrary that $M_{1} \odot M_{2}$ admits the transition dpt $\xrightarrow{1}$ alt, and let $\phi$ be the corresponding $T$-flow. Since the diagram of $\mathcal{M}$ has a single directed path from dpt to alt, namely dpt $\xrightarrow{2}$ ang $\xrightarrow{1}$ alt, the induced transitions through $M_{1}$ and $M_{2}$ must be $\mathrm{dpt} \xrightarrow{2}$ ang and ang $\xrightarrow{1}$ alt, respectively. It follows that the two edges joining $M_{1}$ to $M_{2}$ receive from $\phi$ the values forming an angle $\left\{c_{1}+c_{2}, c_{1}+c_{3}\right\}$, where $c_{1}, c_{2}$, and $c_{3}$ are corner points of $T$. Since the flow through the input connector of $M_{1}$ is 0 , the Kirchhoff law tells us that the residual edge of $M_{1}$ receives the value $\left(c_{1}+c_{2}\right)+\left(c_{1}+c_{3}\right)=c_{2}+c_{3}$. The value on the residual edge of $M_{2}$ must therefore be either $c_{2}$ or $c_{3}$, which in turn implies that the flow through the output connector of $M_{1} \odot M_{2}$ will be either $\left(c_{2}+c_{3}\right)+c_{2}=c_{3}$ or $\left(c_{2}+c_{3}\right)+c_{3}=c_{2}$. In both cases, the outflow from $M_{1} \odot M_{2}$ has weight 1 , while an altitude has trace of weight 3 . This contradiction excludes the transition dpt $\xrightarrow{1}$ alt. The reverse transition alt $\xrightarrow{1}$ dpt can be handled similarly.

For brevity we set

$$
\begin{equation*}
\mathcal{M}^{\prime}=\mathcal{M} \odot \mathcal{M}-\{\mathrm{dpt} \stackrel{1}{\longleftrightarrow} \mathrm{alt}\} . \tag{5}
\end{equation*}
$$

It is easy to check that $\mathcal{M}^{\prime} \subseteq \mathcal{M}$, which immediately implies the following result.
Corollary 8.3. If $M_{1}$ and $M_{2}$ are $(2,2 ; 1)$-poles such that $\mathbf{T}\left(M_{1}\right) \subseteq \mathcal{M}$ and $\mathbf{T}\left(M_{2}\right) \subseteq \mathcal{M}$, then $\mathbf{T}\left(M_{1} \odot M_{2}\right) \subseteq \mathcal{M}^{\prime} \subseteq \mathcal{M}$.

We divide the proof of the fact that $\pi\left(H^{\sharp}\right) \geq 5$ for every Halin snark into three smaller steps, Theorems 8.5, 8.6, and 8.8, each dealing with a particular dipole contained in it. All three statements have an independent value as they can be applied in the construction of graphs with perfect matching index at least 5 .

Definition 8.4. Let $H^{\sharp}=C^{\sharp} \cup S$ be a Halin snark, with $H=C \cup S$ being the underlying Halin graph, $C$ being the perimeter circuit, and $S$ the inscribed tree. A Halin $(2,2 ; 1)$-pole $X$ is a $(2,2,1)$-pole created from $H^{\sharp}$ by removing one Halin fragment $F=B \circ D$ from $C^{\sharp}$ and keeping the dangling edges. The connectors of $X$ are formed in a natural manner:
each consists of the semiedges joined in $H^{\sharp}$ to a connector of $F$. A Halin $(2,2)$-pole $Y$ is a $(2,2)$-pole created from $H^{\sharp}$ in a similar manner with the exception that one does not remove the entire Halin fragment $F=D \circ B$, where $D$ is a decollineator and $B$ is a bipartite $(2,2 ; 1)$-pole, but removes only $D$. Finally, an extended Halin (2,2)-pole $Z$ is a $(2,2)$-pole created from $H^{\sharp}$ by severing a pair of edges that connect two consecutive Halin fragments in $C^{\sharp}$. In all three cases the output connector coincides with the output connector of the Halin fragment preceding $F$ with respect to the cyclic ordering around $C^{\sharp}$.

Theorem 8.5. Every Halin $(2,2 ; 1)$-pole $X$ satisfies $\mathbf{T}(X) \subseteq \mathcal{M}^{\prime}$.
Proof. Consider a Halin $(2,2 ; 1)$-pole $X$ constructed from a Halin snark $H^{\sharp}=C^{\sharp} \cup S$ by removing one Halin fragment. Let $H=C \cup S$ be the underlying Halin graph. Throughout the proof we refer to the notation introduced in the beginning of this section.

We prove the result by induction on the number of trivalent vertices of the inscribed tree $S$ of $H$. If $S$ has only one trivalent vertex, then, up to a cyclic shift of indices, $X=F_{0} \odot F_{1}$ where $F_{0}$ and $F_{1}$ are Halin fragments. Since $\mathbf{T}\left(F_{0}\right) \subseteq \mathcal{M}$ and $\mathbf{T}\left(F_{1}\right) \subseteq \mathcal{M}$, Proposition 8.2 implies that $\mathbf{T}(X)=\mathbf{T}\left(F_{0} \odot F_{1}\right) \subseteq \mathcal{M}^{\prime}$. This establishes the basis of induction.

For the induction step consider an arbitrary Halin ( 2,$2 ; 1$ )-pole $X$ where the inscribed tree has $k \geq 2$ trivalent vertices, and assume that the result is true for all Halin ( 2,$2 ; 1$ )poles whose inscribed trees have fewer than $k$ trivalent vertices. The perimeter circuit of $H$ has $k+2$ vertices $v_{0}, v_{1}, \ldots, v_{k+1}$, which means that $H^{\sharp}$ comprises $k+2$ Halin fragments $F_{0}, F_{1}, \ldots, F_{k}, F_{k+1}$. Without loss of generality we may assume that $X$ is constructed by removing $F_{k+1}$. Let $w$ be the vertex of $X$ incident with its residual semiedge, and let $w_{1}$ and $w_{2}$ be the neighbours of $w$ in $X$. For $i \in\{1,2\}$ let $S_{i}$ denote the component of $S-w w_{i}$ containing the vertex $w_{i}$, and let $S_{i}^{\prime}$ be the tree obtained from $S_{i}$ by adding to it the vertex $w$ and the edge $w w_{i}$. Recall that $H$ is planar with $S$ being a plane tree. Therefore there exists an index $r$ with $0 \leq r<k$ such that the vertices $v_{0}, \ldots, v_{r}$ are the leaves of $S_{1}$ and the vertices $v_{r+1}, \ldots, v_{k}$ are the leaves of $S_{2}$. Now we can form two smaller Halin graphs $H_{1}=C_{1} \cup S_{1}^{\prime}$ and $H_{2}=C_{2} \cup S_{2}^{\prime}$, where $C_{1}$ is the circuit that runs through the vertices $v_{0}, \ldots, v_{r}$, and $w$, while $C_{2}$ is the circuit that runs through the vertices $v_{r+1}, \ldots, v_{k}$, and $w$. From these Halin graphs we can build Halin snarks $H_{1}^{\sharp}$ and $H_{2}^{\sharp}$ where $w$ is substituted (say) by $F_{k+1}$ and the remaining perimeter vertices are substituted by the same Halin fragments as in $H^{\sharp}$. By removing $F_{k+1}$ from both $H_{1}^{\sharp}$ and $H_{2}^{\sharp}$ we obtain Halin ( 2,$2 ; 1$ )poles $X_{1}$ and $X_{2}$, respectively. It is easy to see that $X_{1} \odot X_{2}$ is isomorphic to $X$. The induction hypothesis implies that $\mathbf{T}\left(X_{1}\right) \subseteq \mathcal{M}^{\prime}$ and $\mathbf{T}\left(X_{2}\right) \subseteq \mathcal{M}^{\prime}$. Since $\mathcal{M}^{\prime} \subseteq \mathcal{M}$, from Proposition 8.2 eventually get $\mathbf{T}(X)=\mathbf{T}\left(X_{1} \circ X_{2}\right) \subseteq \mathcal{M}^{\prime}$. This concludes the induction step as well as the entire proof.

Theorem 8.6. Every transition through a Halin (2,2)-pole has the form

$$
\mathrm{ls} \rightarrow \mathrm{ls} \quad \text { or } \quad \mathrm{hl} \rightarrow \mathrm{hl} .
$$

Proof. Let $Y$ be an arbitrary Halin $(2,2)$-pole. Definition 8.4 implies that $Y$ can be obtained from a bipartite $(2,2 ; 1)$-pole $B$ and a Halin $(2,2 ; 1)$-pole $X$ by taking the join $B \circ X$ and then performing the junction of the residual semiedges of $B$ and $X$. It follows that a transition $s \rightarrow t$ through $Y$ can only exist provided that there exists a transition $\mathrm{s} \xrightarrow{i} \mathrm{q}$ through $B$ and a transition $\mathrm{q} \xrightarrow{j} \mathrm{t}$ through $X$ such that $i=j$. Furthermore, by Theorem 5.1, the resulting transition $\mathrm{s} \rightarrow \mathrm{t}$ must be admissible, which means that it belongs to the set $\mathcal{A}$ displayed in Equation (2). Theorem 6.3 and Theorem 8.5 imply that $\mathbf{T}(B) \subseteq \mathcal{B}$ and $\mathbf{T}(X) \subseteq \mathcal{M}^{\prime}$. By inspecting the relations $\mathcal{B}$ and $\mathcal{M}^{\prime}$ and excluding
all results that are not admissible we conclude that only the following five combinations $(\mathrm{s} \xrightarrow{i} \mathrm{q}) \in \mathcal{B}$ and $(\mathrm{q} \xrightarrow{i} \mathrm{t}) \in \mathcal{M}^{\prime}$ are possible:

$$
\begin{array}{ll}
\mathrm{hl} \xrightarrow{1} \text { ls } \xrightarrow{1} \mathrm{hl} & \text { ls } \xrightarrow{2} \text { ang } \xrightarrow{2} \text { ls } \\
\mathrm{hl} \xrightarrow{2} \text { alt } \xrightarrow{2} \mathrm{hl} & \text { ls } \xrightarrow{1} \text { alt } \xrightarrow{1} \text { ls } \\
\mathrm{hl} \xrightarrow{1} \text { dpt } \xrightarrow{1} \mathrm{hl} &
\end{array}
$$

In all the cases we obtain either $1 \mathrm{~s} \rightarrow \mathrm{ls}$ or $\mathrm{hl} \rightarrow \mathrm{hl}$. The theorem is proved.
We have just proved that all transitions through a Halin (2, 2)-pole are collinear. A Halin (2,2)-pole is thus an example of a collineator, a (2,2)-pole whose transition relation is contained in the set $\mathcal{C}=\{1 \mathrm{~s} \rightarrow \mathrm{ls}, \mathrm{hl} \rightarrow \mathrm{hl}\}$ of all collinear transitions.

Remark 8.7. Both transitions from Proposition 8.6 can actually be achieved by the Halin (2,2)-pole on 26 vertices obtained from the windmill snark of order 34 depicted in Figure 1 (left) by deleting one decollineator on 8 vertices.

Theorem 8.8. Every transition through an extended Halin (2,2)-pole has the form

$$
\text { ang } \rightarrow \text { ls. }
$$

Proof. Let $Z$ be an arbitrary extended Halin (2,2)-pole. Definition 8.4 implies that $Z=$ $D \circ Y$ where $D$ is a decollineator and $Y$ is a Halin (2,2)-pole. By using Corollary 5.5 and Theorem 8.6 we can conclude that a transition $\mathrm{s} \rightarrow \mathrm{t}$ through $Z$ can only exist when there exists a transition $\mathrm{s} \rightarrow \mathrm{q}$ in $\mathcal{D}$, and $\mathrm{q} \rightarrow \mathrm{t}$ is one of $\mathrm{ls} \rightarrow \mathrm{ls}$ and $\mathrm{hl} \rightarrow \mathrm{hl}$. However, $\mathcal{D}$ has no transition involving hl , so $\mathrm{q}=1 \mathrm{~s}$ and hence $\mathrm{s}=$ ang and $\mathrm{t}=1 \mathrm{~s}$, as required.

Theorem 8.9. Every Halin snark has perfect matching index at least 5.
Proof. Every Halin snark $G$ can be expressed as $G=[Z]$ where $Z$ is an extended Halin $(2,2)$-pole. Now, if we had $\pi(G) \leq 4$, then $G$ would have a tetrahedral flow, and hence the (2,2)-pole $Z$ would admit a stationary transition. However, Theorem 8.8 shows that this is not the case. Therefore $\pi(G) \geq 5$.

The next theorem makes use of the fact that the family of Halin snarks is quite rich.
Theorem 8.10. For every even integer $n \geq 42$ there exists a nontrivial snark of order $n$ with perfect matching index at least 5 .

Proof. We prove that there exists a nontrivial Halin snark of every even order $n \geq 42$. We know that there exists a treelike snark order $n=12 k-2$ for every integer $k \geq 3$. These snarks are clearly nontrivial and are Halin. It is therefore sufficient to establish the existence of nontrivial Halin snarks of order $n$ for the remaining five even residue classes modulo 12. To this end, we prove the following two claims.
Claim 1. If there exists a nontrivial Halin snark on $n$ vertices, then there is also one on $n+12$ vertices.

Proof of Claim 1. Let $H^{\sharp}=C^{\sharp} \cup T$ be a nontrivial Halin snark of order $n$. In $H^{\sharp}$, choose an arbitrary Halin fragment $F_{h}=D_{h} \circ B_{h} \subseteq C^{\sharp}$, where $D_{h}$ is a decollineator and $B_{h}$ is a bipartite $(2,2 ; 1)$-pole. Take the Heawood graph, denoted by $H w$, which is a bipartite vertex- and edge-transitive cubic graph of girth 6 on 14 vertices, and create the ( 2,2 )-pole $G_{u v}$ for any fixed edge $u v$ of $G=H w$; denote it by $M_{H w}$. Now, construct a new graph $K$ by replacing in $H^{\sharp}$ the Halin fragment $F_{h}=D_{h} \circ B_{h}$ with the $(2,2 ; 1)$-pole $D_{h} \circ B_{h} \circ M_{H w}$.

It is easy to see that $B_{h} \circ M_{H w}$ is again a bipartite $(2,2 ; 1)$-pole, so $D_{h} \circ\left(B_{h} \circ M_{H w}\right)$ is a Halin fragment and $K$ is a Halin snark. Its order is $n+12$, and since $H w$ is cyclically 6 -edge-connected (see [28, Theorem 17]), $K$ is also nontrivial. This completes the proof of Claim 1.

To finish the proof it suffices to prove the following.
Claim 2. There exist nontrivial Halin snarks of order 42, 44, 48, 50, and 52.
Proof of Claim 2. In order to construct a nontrivial Halin snark of order 42 we start with the windmill snark $W_{34}$ of order 34, shown in Figure 1 (left). It is created from the unique Halin graph of order 4 , the complete graph $K_{4}$, by substituting each vertex of the perimeter circuit with the Petersen fragment. Recall that the Petersen fragment can be expressed as $F_{P s}=D_{P s} \circ B$ where $D_{P s}$ is the decollineator constructed from the Petersen graph and $B$ is the bipartite $(2,2 ; 1)$-pole $G_{u w v}$ created from the complete bipartite graph $K_{3,3}$. Let us change $K_{3,3}$ to the Heawood graph to obtain a bipartite $(2,2 ; 1)$-pole $B_{H w}$ of order 11. Replacing in $W_{34}$ the Petersen fragment with the Halin fragment $D_{P s} \circ B_{H w}$ gives rise to a Halin snark $G_{42}$ of order 42. Again, $G_{42}$ is nontrivial because $H w$ is cyclically 6 -edge-connected.

Nontrivial Halin snarks of order 44, 48, 52, can be constructed in a manner similar to $G_{42}$. Instead of the Heawood graph one has to use, respectively, the generalised Petersen graph $G P(8,3)$ (known as the Möbius-Kantor graph), the generalised Petersen graph $G P(10,3)$ (known as the Desargues graph), and the generalised Petersen graph $G P(12,5)$ (nicknamed the Nauru graph), all of them vertex- and edge-transitive cubic bipartite graphs of girth 6 (see [10]). They are all cyclically 6 -edge-connected due to [28, Theorem 17], implying that the resulting Halin snarks are nontrivial.

Finally, we construct a nontrivial Halin snark of order 50. For this purpose we use the graph $G_{42}$ constructed above, choose one Petersen fragment $F_{P s}=D_{P s} \circ B$ left intact by the construction of $G_{42}$ from $W_{34}$, and replace it with $D_{P s} \circ B_{H w}$. The number of vertices increases by 8 to 50 , and the result is obviously a nontrivial Halin snark.

The proof is complete.

## 9 Circular flows on Halin snarks

In this section we show that Halin snarks provide a rich source of graphs with circular flow number at least 5 . Such graphs are particularly interesting for the outstanding 5flow conjecture of Tutte and therefore have been extensively studied in a number of recent papers [1, 8, 13, 23].

Given a real number $r \geq 2$, we define a nowhere-zero real-valued $r$-flow as an $\mathbb{R}$-flow $\phi$ such that $1 \leq|\phi(e)| \leq r-1$ for each edge $e$ of $G$. A nowhere-zero modular $r$-flow is an $\mathbb{R} / r \mathbb{Z}$-flow $\phi$ such that $1 \leq \phi(e)(\bmod r) \leq r-1$ for each edge $e$. Here $x(\bmod r)$ denotes the unique real number $x^{\prime} \in[0, r) \subseteq \mathbb{R}$ such that $x-x^{\prime}$ is a multiple of $r$. It is a well-known folklore fact that a graph admits a nowhere-zero real-valued $r$-flow if and only if it admits a nowhere-zero modular $r$-flow.

The circular flow number of a graph $G$, denoted by $\Phi_{c}(G)$, is the infimum of the set of all real numbers $r$ such that $G$ has a nowhere-zero $r$-flow. This parameter was introduced by Goddyn et al. in [12] as fractional flow number and was shown to be a minimum and a rational number for every (finite) bridgeless graph.

Theorem 9.1. Let $G$ be a Halin snark in which all decollineators have been obtained from snarks with circular flow number at least 5 . Then $\Phi_{c}(G) \geq 5$.

Proof. Let $H$ be a Halin graph with perimeter circuit $C=\left(v_{0} v_{1} \ldots v_{k-1}\right)$ and inscribed tree $S$, and let $H^{\sharp}$ be a Halin snark created from $H$ using decollineators obtained from graphs with circular flow number at least 5. Recall that every Halin graph contains a triangle; without loss of generality we may assume that $v_{0} v_{1} u$ is a triangle in $H$, with $u$ being a vertex of $S$. Set $e_{0}=v_{0} u$ and $e_{1}=v_{1} u$.

Suppose to the contrary that $\Phi_{c}\left(H^{\sharp}\right)=r<5$, and let $\phi$ be a nowhere-zero modular $r$-flow on $H^{\sharp}$. Define the flow through a dipole $X$ as the sum of flow-values (in $\mathbb{R} / r \mathbb{Z}$ ) on the dangling edges of the input connector directed towards the dipole; of course, this value coincides with the sum of flow-values on the dangling edges in the output connector of $X$ directed away from $X$.

For $i \in\{0,1, \ldots, k-1\}$ let $a_{i}$ denote the flow through the decollineator $D_{i}$ of $H^{\sharp}$. By our assumption, each $D_{i}$ arises from a snark with circular flow number at least 5 , so each $a_{i}$ lies in the interval $(-1,1)$. Without loss of generality we may assume that $a_{1} \in[0,1)$. As $\phi$ is an $r$-flow, both $\phi\left(e_{0}\right)$ and $\phi\left(e_{1}\right)$ are contained in the interval [1,r-1]. It follows that both $a_{0}$ and $a_{2}$ are contained in ( $\left.-1,0\right]$. This in turn implies that $\phi\left(e_{0}\right) \in[1,2)$ and $\phi\left(e_{1}\right) \in(-2,-1]$, provided that $e_{0}$ and $e_{1}$ are in directed from $u$. However, the third edge incident with $u$ now receives a value from the interval $(-1,1)$, which contradicts the definition of a nowhere-zero $r$-flow.

The just proved theorem significantly generalises the result of Abreu et al. [1, Theorem 9] that the circular flow number of every treelike snark is at least 5. Indeed, every treelike snark can serve as an ingredient for a Halin fragment that satisfies the assumptions of Theorem 9.1 and thus gives rises to new snarks with circular flow number at least 5. Clearly, this process can be iterated indefinitely.

## 10 Concluding remarks

The methods introduced in this paper can be used to produce a great variety of families of snarks with perfect matching index at least 5. For example, Halin dipoles treated in Theorems 8.6 and 8.8 can be variously combined with bipartite $(2 ; 2 ; 1)$-poles and dipoles constructed on the basis of Propositions 5.8 and 7.3 . The family of Halin snarks itself can be easily extended to a family of snarks of the form $H^{\sharp} \cup S$, where $S$ is a planar forest with any number of components rather than being just a tree. One possibility to construct such a snark is to start with a Halin snark expressed as $G=[D \circ Y]$ where $D$ is a decollineator and $Y$ is a Halin (2,2)-pole. Since $Y$ is a collineator and the composition of two collineators is again a collineator, we can create a graph $G_{1}^{+}=\left[D \circ\left(Y_{1} \circ \ldots \circ Y_{k}\right)\right]$ where $Y_{1}, \ldots, Y_{k}$ are arbitrary Halin (2,2)-poles. Corollary 5.5 and Theorem 8.6 immediately imply that $G_{1}^{+}$has no tetrahedral flow, whence $\pi\left(G_{1}^{+}\right) \geq 5$. Another possibility is to take a graph $G_{2}^{+}=\left[Z_{1} \circ \ldots \circ Z_{k}\right]$ where $Z_{1}, \ldots, Z_{k}$ are extended Halin (2,2)-poles. In this case, the fact that $\pi\left(G_{2}^{+}\right) \geq 5$ follows from Theorem 8.8 . We can also develop special constructions such as superposition [26] or a construction of nontrivial snarks with $\pi \geq 5$ and circular flow number strictly smaller than 5 , see [25].

Our methods unfortunately fail for nontrivial snarks with perfect matching index at least 5 of small order. We therefore leave the following open problem.

Problem 10.1. Do there exist nontrivial snarks of orders 38 and 40 with perfect matching index at least 5?

If 4 -cycles are permitted, then the answer to this question is positive. Cyclically 4-edge-connected snarks with $\pi \geq 5$ containing a quadrilateral can be easily created from
the snark shown in Figure 1 (left) by replacing one or two copies of the bipartite ( 2,$2 ; 1$ )pole on three vertices obtained from the complete bipartite graph $K_{3,3}$ with a similar $(2,2 ; 1)$-pole created from the graph of the 3 -dimensional cube.

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