Canonical double covers of circulants

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Abstract

The canonical double cover $B(X)$ of a graph X is the direct product of X and K_2. If $Aut(B(X)) \cong Aut(X) \times \mathbb{Z}_2$ then X is called stable; otherwise X is called unstable. An unstable graph is nontrivially unstable if it is connected, non-bipartite and distinct vertices have different neighborhoods. Circulant is a Cayley graph on a cyclic group. Qin et al. conjectured in [J. Combin. Theory Ser. B 136 (2019), 154-169] that there are no nontrivialy unstable circulants of odd order. In this paper we prove this conjecture.

Keywords: circulant, canonical double cover, unstable graph, automorphism group

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1 Introduction

All groups considered in this paper are finite and all graphs are finite, simple and undirected. For a graph X, we denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ the vertex set, the edge set and the automorphism group of X, respectively. *The canonical double cover* (also called *the bipartite double cover* or *the Kronecker cover*) of a graph X, denoted by $B(X)$, is the direct product $X \times K_2$ (where K_2 denotes the complete graph on two vertices). This means that $V(B(X)) = V(X) \times \mathbb{Z}_2$ and $E(B(X)) = \{\{(x,0),(y,1)\} \mid \{x,y\} \in E(X)\}$. Canonical double covers have been studied by several authors, see for example [3, 10, 16, 25, 27]. It is well-known that $B(X)$ is connected if and only if X is connected and non-bipartite, see [5]. It can also be easily seen that $\mathrm{Aut}(B(X))$ contains a subgroup isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$. However, determining the full automorphism group of $B(X)$ is not as trivial. Hammack and Imrich [4] investigated vertex-transitivity of direct product of graphs, and proved that for a non-bipartite graph X and a bipartite graph Y, direct product $X \times Y$ is vertex-transitive if and only if both $B(X)$ and Y are vertex-transitive. Hence, the problem of vertex-transitivity of direct product of graphs reduces to the problem of vertex-transitivity of canonical double covers. If $\mathrm{Aut}(B(X))$ is isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$ then the graph X is called *stable*, otherwise it is called *unstable*. This concept was first defined by Marušič et al. [14] and studied later most notably by Surowski [22, 23], Wilson [26], Lauri et al. [11] and Qin et al. [19]. A graph is said to be *irreducible* (or *vertex-determining*) if distinct vertices have different neighbours, and *reducible* otherwise. It is easy to see that the following graphs are all unstable: disconnected graphs, bipartite graphs with non-trivial automorphism group, and reducible graphs. An unstable graph is said to be *nontrivialy unstable* if it is non-bipartite, connected, and irreducible.

For a group G and an inverse closed subset $S \subseteq G \setminus \{1_G\}$, the *Cayley graph* $Cay(G,S)$ on G with connection set S is defined as the graph with vertex set G, with two vertices $x, y \in G$ being adjacent if and only if $x^{-1}y \in S$. A Cayley graph on a cyclic group is called *circulant*. Sabidussi [21] proved that a graph X is a Cayley graph on a group G if and only if $\mathrm{Aut}(X)$ has a regular subgroup isomorphic to G. It is easy to see that if X is a Cayley graph on a group G, then its canonical double cover $B(X)$ is a Cayley graph on $G \times \mathbb{Z}_2$. The converse is not true in general, that is $B(X)$ can be a Cayley graph even if the graph X is not Cayley. The problem of characterizing graphs with Cayley canonical double covers was studied by Marušič, Scapellato and Zagaglia Salvi [15], who introduced the class of generalized Cayley graphs (see also [7, 8]) and proved that if the canonical double cover of a generalized Cayley graph X is a stable graph, then X is a Cayley graph. The characterization of graphs whose canonical double covers are Cayley graphs was given by the second author in [6].

The problem of characterization of vertex-transitive unstable graphs was posed in [2, Problem 5.7]. However, the problem is difficult even when restricted to the class of circulant graphs. The problem of classification of nontrivially unstable circulants was posed in [19, Problem 1.2]. Qin et al proved that every circulant of odd prime order is stable (see [19, Theorem 1.4], and posed the following conjecture.

Conjecture 1.1. *[19, Conjecture 1.3] There is no nontrivially unstable circulant of odd order.*

In this paper we prove Conjecture 1.1, that is we prove the following theorem.

Theorem 1.2. *Let X be a connected irreducible circulant of odd order and let $B(X)$ be its canonical double cover. Then* $\mathrm{Aut}(B(X)) \cong \mathrm{Aut}(X) \times \mathbb{Z}_2$.

Remark 1.3. *Using Theorem 1.2, the automorphism group of a canonical double cover of any circulant X of odd order can be determined in terms of the autormorphism group of X. Namely, if X is a connected reducible circulant, then there exist a positive integer $d > 1$ and a connected irreducible circulant Y, such that $X \cong Y \wr \overline{K_d}$. It is now easy to see that $B(X) \cong (B(Y) \wr \overline{K_d})$. Using the result of Sabidussi [20] on the automorphism group of wreath product of graphs, we obtain $Aut(B(X)) \cong \mathrm{Aut}(B(Y) \wr S_d \cong (Aut(Y) \times \mathbb{Z}_2) \wr S_d$.*

2 Preliminaries

Following [18], an automorphism γ of a bipartite graph Y is called a *strongly switching involution* if γ is an involution that swaps the two colour classes and fixes no edge. For a graph X and a partition $\mathcal{P}$ of $V(X)$, the *quotient graph of X with respect to $\mathcal{P}$* is the graph whose vertex set is $\mathcal{P}$ with $B_1, B_2 \in \mathcal{P}$ being adjacent if and only if there exist $x \in B_1$ and $y \in B_2$ such that $\{x, y\} \in E(X)$. The following result is proved in [18, Proposition 2.4].

Lemma 2.1. *A bipartite graph Y is isomorphic to the canonical double cover of a graph X if and only if* $\mathrm{Aut}(Y)$ *contains a strongly switching involution γ such that X is isomorphic to the quotient graph of Y with respect to the orbits of $\langle\gamma\rangle$.*

The following lemma gives a necessary and sufficient condition for the stability of a graph, and it has been proved in [6, Corollary 2.5]. However, we give the proof here for the sake of completeness.

Lemma 2.2. *Let X be a connected non-bipartite graph, and let $B(X)$ be its canonical double cover. Let τ be the automorphism of $B(X)$ defined by $\tau(x, i) = (x, i + 1)$. Then X is stable if and only if τ is central in* $\mathrm{Aut}(B(X))$.

Proof. Suppose first that X is stable. Then $\mathrm{Aut}(B(X)) \cong \mathrm{Aut}(X) \times \langle\tau\rangle$, hence τ is central in $\mathrm{Aut}(B(X))$. Suppose now that τ is central in $\mathrm{Aut}(B(X))$. Let $\alpha \in \mathrm{Aut}(B(X))$ be arbitrary.

Suppose first that α fixes the color classes of $B(X)$. Let g be the permutation of $V(X)$ such that $\alpha(x,0) = (g(x),0)$. Since τ commutes with α it follows that $\alpha(x,1) = \alpha(\tau(x,0)) = \tau(\alpha(x,0)) = (g(x),1)$. Let $\{x,y\} \in E(X)$. Then $\{(x,0),(y,1)\} \in E(B(X))$, and since $\alpha \in \mathrm{Aut}(B(X))$ it follows that $\{(g(x),0),(g(y),1)\} \in E(B(X))$. By definition of the canonical double cover, it follows that $\{g(x),g(y)\} \in E(X)$. It follows that $g \in \mathrm{Aut}(X)$. Hence, α is induced by the automorphism of X, that is $\alpha \in \mathrm{Aut}(X) \times \mathbb{Z}_2$. If α permutes the color classes of $B(X)$, then we apply the above arguments for $\alpha\tau$ which fixes the color classes, and conclude that $\alpha\tau \in \mathrm{Aut}(X) \times \mathbb{Z}_2$, hence $\alpha \in \mathrm{Aut}(X) \times \mathbb{Z}_2$. This finishes the proof. □

Suppose that $X = Cay(\mathbb{Z}_k, S)$ is a circulant of odd order k. Then it is straightforward to verify that $B(X)$ is a Cayley graph on $\mathbb{Z}_k \times \mathbb{Z}_2$, with connection set $S \times \{1\}$. Since the map $\alpha : \mathbb{Z}_k \times \mathbb{Z}_2 \to \mathbb{Z}_{2k}$ defined by $\alpha(x,i) = 2x + ki$ is an isomorphism, it follows that $B(X) \cong \mathrm{Cay}(\mathbb{Z}_{2k}, k + 2S)$ (where elements of S are now considered as elements of $\mathbb{Z}_{2k}$), hence $B(X)$ is a circulant graph of order $2k$. Observe that the mapping $k_L : \mathbb{Z}_{2k} \to \mathbb{Z}_{2k}$ defined by $k_L(x) = x + k$ is the automorphism of $B(X)$ that corresponds to the map τ from Lemma 2.2. Hence, for circulants of odd order we have the following result.

Lemma 2.3. *Let $X = Cay(\mathbb{Z}_k, S)$ be a connected circulant of odd order k. Then X is stable if and only if the permutation k_L is central in the automorphism group of $Cay(\mathbb{Z}_{2k}, k+2S) = B(X)$.*

We will now give another characterization of stable vertex-transitive graphs.

Lemma 2.4. *Let X be a connected non-bipartite vertex-transitive graph. Let $B(X)$ be the canonical double cover of X, and let $A = Aut(B(X))$. Then X is stable if and only if $A_{(v,0)} = A_{(v,1)}$ for some $v \in V(X)$.*

Proof. Suppose first that X is stable. Then by Lemma 2.2 it follows that the map τ is central in A. If $\varphi \in A_{(v,0)}$ then we have $\varphi(v,1) = \varphi(\tau(v,0)) = \tau(\varphi(v,0)) = (v,1)$. This shows that $A_{(v,0)} \leq A_{(v,1)}$. Since $B(X)$ is vertex-transitive, all stabilizers are of the same order, hence $A_{(v,0)} = A_{(v,1)}$.

Suppose now that $A_{(v,0)} = A_{(v,1)}$ for some $v \in V(X)$. Let $w \in V(X)$ be arbitrary, let g be an automorphism of X that maps v into w, and let β be the automorphism of $B(X)$ defined by $\beta(x,i) = (g(x),i)$. Then $A_{(w,0)} = \beta A_{(v,0)}\beta^{-1} = \beta A_{(v,1)}\beta^{-1} = A_{(w,1)}$.

Let α be an automorphism of $B(X)$. We will prove that α and τ commute. Without loss of generality, we may assume that α fixes the color classes of $B(X)$. Suppose that $\alpha(w,0) = (u,0)$, for some $u \in V(X)$. Let h be an automorphism of X that maps u to w, and let γ be the automorphism of $B(X)$ defined by $\gamma(x,i) = (h(x),i)$. Observe that $\gamma\alpha \in A_{(w,0)} = A_{(w,1)}$, hence $\gamma(\alpha(w,1)) = (w,1)$. It follows that $\alpha(w,1) = \gamma^{-1}(w,1) = (u,1)$. It is now straightforward to conclude that $\alpha\tau(w,0) = \tau\alpha(w,0)$. Since $w \in V(X)$ was arbitrary, it follows that α and τ commute. We conclude that τ is central in $\mathrm{Aut}(B(X))$. By Lemma 2.2 it follows that X is stable. □

For circulants, the above lemma implies the following.

Lemma 2.5. *Let X be a connected circulant of odd order k, let $B(X)$ be its canonical double cover and $A = Aut(B(X))$. Then X is stable if and only if $A_0 = A_k$.*

A Cayley graph $\Gamma = Cay(G,S)$ is said to be *normal* if G_L is a normal subgroup of $\mathrm{Aut}(\Gamma)$, or equivalently if $\mathrm{Aut}(\Gamma)_0 = \mathrm{Aut}(G,S)$, where $\mathrm{Aut}(G,S) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S\}$.

Lemma 2.6. *Let G be an abelian group of odd order, and let $X = Cay(G, S)$ be a connected Cayley graph on G. If $B(X) = Cay(G \times \mathbb{Z}_2, S')$ is a normal Cayley graph then X is stable.*

Proof. Since $B(X)$ is normal Cayley graph, it follows that each element of $\mathrm{Aut}(B(X))$ is a composition of some element of $(G\times\mathbb{Z}_2)_L$ with some element of $\mathrm{Aut}(G\times\mathbb{Z}_2)$. Let τ be the automorphism of $B(X)$ defined by $\tau(x, i) = (x, i+1)$. Let t be the unique element of order 2 in $G \times \mathbb{Z}_2$. Observe that $\tau = t_L$. Since G is abelian, it follows that each element of $(G \times \mathbb{Z}_2)_L$ commutes with τ. Let $\varphi \in \mathrm{Aut}(G \times \mathbb{Z}_2)$. Then $\varphi(t) = t$, since group automorphisms preserve the order of elements, and t is the unique element of order 2 in $G\times\mathbb{Z}_2$. It follows that $(\varphi t_L)(x) = \varphi(tx) = t\varphi(x) = (t_L\varphi)(x)$. This shows that $\tau = t_L$ commutes with every element of $\mathrm{Aut}(G)$. The result now follows by Lemma 2.3. □

The following lemma tells that a normal circulant of even order not divisible by four has a unique regular cyclic subgroup.

Lemma 2.7. *[24, Theorem 5.2.2] Let k be an odd positive integer, and let $X = Cay(\mathbb{Z}_{2k}, S)$. Let $A = Aut(X)$ admitting a normal cyclic regular subgroup H. Then H is the unique regular cyclic subgroup contained in A.*

The *wreath (lexicographic) product* $\Sigma \wr \Gamma$ of a graph Γ by a graph Σ is the graph with vertex set $V(\Sigma) \times V(\Gamma)$ such that $\{(u_1, u_2), (v_1, v_2)\}$ is an edge if and only if either $\{u_1, v_1\} \in E(\Sigma)$, or $u_1 = v_1$ and $\{u_2, v_2\} \in E(\Gamma)$. Observe that $\Sigma \wr \Gamma$ is the graph obtained by substituting a copy of Γ for each vertex of Γ.

The *deleted wreath (deleted lexicographic) product* of a graph Σ and $\overline{K_d}$, denoted by $\Sigma \wr_d \overline{K_d}$, is the graph with vertex set $V(\Sigma) \times \mathbb{Z}_d$, such that $\{(u_1, i), (v_1, j)\}$ is an edge if and only if $\{u_1, v_1\} \in E(\Sigma)$ and $i \neq j$. Observe that $\Sigma \wr_d \overline{K_d}$ can be obtained from $\Sigma \wr \overline{K_d}$ by removing d disjoint copies of Σ. Observe that the canonical double cover of a graph X is isomorphic to $X \wr_d \overline{K_2}$ (see [19, Example 2.1]). The following result gives a characterization of all arc-transitive circulants.

Lemma 2.8. [9, 12] *Let Γ be a connected arc-transitive circulant of order n. Then one of the following holds:*

(i) $\Gamma \cong K_n$;

(ii) $\Gamma = \Sigma \wr \overline{K_d}$, *where $n = md$, $m, d > 1$ and Σ is a connected arc-transitive circulant of order m;*

(iii) $\Gamma = \Sigma \wr_d \overline{K_d}$ *where $n = md$, $d > 3$, $gcd(d, m) = 1$ and Σ is a connected arc-transitive circulant of order m;*

(iv) Γ *is a normal circulant.*

The following result is a direct consequence of [2, Theorem 5.3].

Lemma 2.9. *Let $d \geq 3$ be an integer, let Σ be an irreducible vertex-transitive graph whose order is not divisible by d, and let $\Gamma = \Sigma \wr_d \overline{K_d}$. Then* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Sigma) \times S_d$.

Proof. Since Σ is vertex-transitive, and $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic to $\mathrm{Aut}(\Sigma) \times S_d$, it follows that Γ is vertex-transitive. Since Σ is reducible, by [2, Theorem 5.3] it follows that $\mathrm{Aut}(\Gamma) \ncong \mathrm{Aut}(\Sigma) \times S_d$ if and only if the conditions $(i) - (iii)$ of [2, Theorem 5.3] hold. However, since $|V(\Sigma)|$ is not divisible by d, it follows that condition (ii) of [2, Theorem 5.3] doesn't hold. We conclude that $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Sigma) \times S_d$. □

Lemma 2.10. *[19, Lemma 2.3] The canonical double cover $B(X)$ of a graph X is reducible if and only if X is reducible.*

The following result can be extracted from the proof of [17, Proposition 2.1]. However, as the statement of [17, Proposition 2.1] is a bit weaker, we include the proof here for the sake of completeness.

Lemma 2.11. *Let $X = Cay(G,S)$, $A = Aut(X)$ and let $K \subset S$ such that $\varphi(K) = K$ for every $\varphi \in A_1$. Then $\varphi(\langle K\rangle) = \langle K\rangle$ for every $\varphi \in A_1$. Moreover, if K is inverse closed, then φ induces an automorphism of $Cay(\langle K\rangle, K)$.*

Proof. Let $x \in G$ and $\omega \in A_x$ be arbitrary, and let M be any subset of G such that $\varphi(M) = M$ for every $\varphi \in A_1$. Observe that $(x_L)^{-1}\omega x_L \in A_1$. By the assumption on A_1 and M it follows that $(x_L)^{-1}\omega x_L(M) = M$. Therefore, $\omega(xM) = xM$ for every $x \in G$ and every $\omega \in A_x$.

We claim that $\varphi(K^t) = K^t$ for every positive integer t. The proof is by induction on t. If $t = 1$, the claim follows from the hypothesis. Suppose that $\varphi(K^t) = K^t$. Let $\varphi \in A_1$ and let $k_1 \in K$ be arbitrary. Observe that $(k_1k_2^{-1})_L\varphi \in A_{k_1}$. Applying the first part of the proof with $M = K^t$ and $x = k_1$, it follows that $(k_1k_2^{-1})_L\varphi(k_1K^t) = k_1K^t$. This implies that $\varphi(k_1K^t) = k_2K^t = \varphi(k_1)K^t \subseteq K^{t+1}$. Since this is true for every $k_1 \in K$, it follows that $\varphi(K^{t+1}) = K^{t+1}$.

Since $\langle K\rangle = K \cup K^2 \cup \ldots \cup K^t$ for some positive integer t, it follows that $\varphi(\langle K\rangle) = \langle K\rangle$. Moreover, let e be an arbitrary edge of $Cay(\langle K\rangle, K)$. Then $e = \{k_1, k_1k_2\}$ for some $k_1, k_2 \in K$. As we proved that $\varphi(k_1K^t) = \varphi(k_1)K^t$ for every positive integer t, it follows that $\varphi(k_1K) = \varphi(k_1)K$. Since $\varphi(k_1k_2) \in \varphi(k_1K) = \varphi(k_1)K$ we conclude that $\varphi(k_1k_2) = \varphi(k_1)k$ for some $k \in K$. This shows that $\varphi(e)$ is an edge of $Cay(\langle K\rangle, K)$, hence φ induces an automorphism of $Cay(\langle K\rangle, K)$. □

3 Main result

The following lemma gives a partial generalization of [19, Theorem 1.6], where it is proved that there is no arc-transitive nontrivally unstable circulant. It is easy to see that $B(X)$ is arc-transitive if X is arc-transitive.

Lemma 3.1. *Let X be a nontrivialy unstable circulant of odd order m. Then $B(X)$ is not arc-transitive.*

Proof. Suppose that $B(X)$ is an arc-transitive circulant. Since $B(X)$ is also a connected circulant, by Lemma 2.8, it follows that $B(X)$ is a complete graph, normal circulant, wreath product or a deleted wreath product. If $B(X)$ is a complete graph, then it must be isomorphic to K_1 or K_2, since they are the only bipartite complete graphs. However, it is easy to see that none of them is a canonical double cover of some graph. Lemma 2.6 implies that $B(X)$ is not a normal circulant. If $B(X)$ is a wreath product, then $B(X)$ is reducible, hence by Lemma 2.10 it follows that X is also reducible, contrary to the assumption that X is nontrvialy unstable. Therefore, we may assume that $B(X)$ is the deleted wreath product. Suppose that $B(X) \cong \Sigma \wr_d \overline{K_d}$, where $2m = td$, $d > 3$, $gcd(d,t) = 1$ and Σ is a connected arc-transitive circulant of order t. Since $B(X)$ is bipartite and $d > 3$ it follows that Σ is also bipartite, hence order of Σ is even.

If Σ is reducible, then $\Sigma \cong \Sigma_1 \wr \overline{K_{d_1}}$, where Σ_1 is an irreducible circulant. Then by [2, Proposition 4.5], it follows that $B(X) \cong \Sigma \wr_d \overline{K_d} \cong (\Sigma_1 \wr_d \overline{K_d}) \wr \overline{K_{d_1}}$, implying that $B(X)$ is reducible. By Lemma 2.10 it follows that X is also reducible, a contradiction.

Suppose now that Σ is irreducible. Then by Lemma 2.9 it follows that $Aut(B(X)) \cong \mathrm{Aut}(\Sigma) \times S_d$. Recall that Σ is a connected arc-transitive bipartite circulant of even order. If Σ is a wreath product, then again $B(X)$ is a wreath product, hence $B(X)$ is reducible, and consequently also X is reducible, contrary to the assumption that X is nontrivialy unstable. We conclude that Σ is a normal circulant, or $\Sigma \cong \Sigma_1 \wr_d \overline{K_{d_1}}$, where Σ_1 is an arc-transitive circulant. Applying the same arguments for Σ_1, we conclude that Σ_1 is a normal circulant, or a deleted wreath product. As Γ is finite, this process has to terminate, so eventually we will get $B(X) = (\Sigma_t \wr_d \overline{K_{d_t}}) \wr_d \ldots \wr_d \overline{K_d}$, where Σ_t is a normal circulant of even order and $\mathrm{Aut}(B(X)) \cong \mathrm{Aut}(\Sigma_t) \times S_{d_t} \times \ldots \times S_d$.

Observe that $S_{d_t} \times \ldots \times S_d$ is a normal subgroup of $\mathrm{Aut}(B(X))$, hence its orbits form a system of imprimitivity for $\mathrm{Aut}(B(X))$. Let $\mathcal{P}$ denote the set of orbits of $S_{d_t} \times \ldots \times S_d$ on $V(B(X))$. Observe that the quotient graph of $B(X)$ with respect to $\mathcal{P}$ is isomorphic to Σ_t. The group $(\mathbb{Z}_{2m})_L$ projects into cyclic regular subgroup of $\mathrm{Aut}(\Sigma_t)$. For $g \in \mathrm{Aut}(B(X))$, let $g/_{\mathcal{P}}$ denote the permutation induced by the action of g on $\mathcal{P}$. Since Σ_t is a normal circulant, by Lemma 2.7 it follows that $(\mathbb{Z}_{2m})_L/_{\mathcal{P}}$ is a normal cyclic regular subgroup of $\mathrm{Aut}(\Sigma_t)$, hence $m_L/_{\mathcal{P}}$ is central in $\mathrm{Aut}(\Sigma_t)$. If follows that m_L is central in $\mathrm{Aut}(\Sigma_t) \times S_{d_t} \times \ldots \times S_d$. The result now follows by Lemma 2.3. ◻

In the following lemma, we consider graphs that can be realised as a canonical double cover of some connected arc-transitive circulant of odd order, and derive certain important properties of their automorphism groups.

Lemma 3.2. *Let m be an odd positive integer, and let Γ be a connected bipartite arc-transitive circulant of order $2m$ and even valency, and let $A = Aut(\Gamma)$. Then one of the following holds:*

(i) $A_0 = A_m$, *or*

(ii) $\Gamma \cong \Gamma_1 \wr \overline{K_d}$ *where Γ_1 is an irreducible arc-transitive circulant of even order $2m_1$ and* $\mathrm{Aut}(\Gamma_1)_0 = \mathrm{Aut}(\Gamma_1)_{m_1}$.

Proof. Since Γ is a circulant of order $2m$ and has even valency, it follows that $\Gamma \cong Cay(\mathbb{Z}_{2m}, S)$ where S does not contain element m. This implies that the automorphism $m_L : x \mapsto m + x$ is a strongly switching involution of Γ. By Lemma 2.1, it follows that Γ is the canonical double cover of the graph X obtained as the quotient graph of Γ with respect to the orbits of m_L. Observe that X is a connected and non-bipartite circulant of order m, since Γ is a connected circulant of order $2m$.

If X is a stable graph, then by Lemma 2.5 it follows that Γ satisfies condition (i), and we are done. As Γ is arc-transitive, by Lemma 3.1 it follows that X is not nontrivialy unstable. Therefore, X is trivially unstable, and since it is connected and non-bipartite, it follows that X is reducible. We conclude that $X \cong \Sigma \wr \overline{K_d}$, where Σ is a connected irreducible circulant of order m_1, with m_1 being odd.

Observe that $\Gamma \cong X \times K_2 \cong (\Sigma \wr \overline{K_d}) \times K_2 \cong (\Sigma \times K_2) \wr \overline{K_d}$. Let $\Gamma_1 = \Sigma \times K_2$. Since Γ is an arc-transitive circulant, it follows that Γ_1 is an arc-transitive circulant (see [13, Remark 1.2]). Since Γ_1 is a canonical double cover of Σ, by Lemma 3.1 it follows that Σ is not non-trivially unstable. Recall that Σ is irreducible, connected and non-bipartite. We conclude that Σ is stable, hence by Lemma 2.5 it follows that $\mathrm{Aut}(\Gamma_1)_0 = \mathrm{Aut}(\Gamma_1)_{m_1}$. ◻

We are now ready to prove the main result of this paper. We will show that there is no non-trivially unstable circulant of odd order.

Theorem 1.2 (restated). *Let X be a connected irreducible circulant of odd order and let $B(X)$ be its canonical double cover. Then* $\mathrm{Aut}(B(X)) \cong \mathrm{Aut}(X) \times \mathbb{Z}_2$.

Proof. Let $X = Cay(\mathbb{Z}_m, S)$ be a connected irreducible circulant of odd order m. If X is stable, the result follows by the definition. It is clear that X is non-bipartite, as it is of odd order. Hence, we may assume that X is a nontrivially unstable. We have that $B(X) = Cay(\mathbb{Z}_{2m}, S')$ is a circulant of order $2m$, where $S' = m + 2S$. Let $A = \mathrm{Aut}(B(X))$. If A_0 is transitive on S', then $B(X)$ is arc-transitive, and the result follows by Lemma 3.1. Let $S_1, \ldots, S_k$ be the orbits of A_0 on S'. Observe that $S_i = -S_i$ (since mapping $i : x \mapsto -x$ is contained in A_0), and $m \notin S_i$, for every $i \in \{1, \ldots, k\}$. Let $\Gamma_i = Cay(\langle S_i \rangle, S_i)$. Observe that $\langle S_i \rangle$ is a subgroup of $\mathbb{Z}_{2m}$ of even order, hence it contains the element of order 2, that is $m \in \langle S_i \rangle$ for every $i \in \{1, \ldots, k\}$. By Lemma 2.11 every element of A_0 fixes Γ_i, hence it follows that every element of A_0 induces an automorphism of Γ_i. As A_0 acts transitively on S_i it follows that Γ_i is arc-transitive. Therefore, Γ_i is an arc-transitive circulant of even order (not divisible by 4) and even valency. Hence we can apply Lemma 3.2 to each of the graphs Γ_i. If for some i we have that Γ_i satisfies condition 1 of Lemma 3.2, it follows that every automorphism of $B(X)$ that fixes 0, must also fix m, hence $A_0 = A_m$ and by Lemma 2.5 it follows that X is stable, a contradiction.

We can now assume that $\Gamma_i \cong \Sigma_i \wr \overline{K_{d_i}}$, where Σ_i is an irreducible arc-transitive circulant of even order satisfying condition (i) from Lemma 3.2. Let R be the equivalence relation of "having the same neighbourhood" defined on $V(\Gamma_i)$. The equivalence classes of this relation are all of size d_i, and form a system of imprimitivity for $\mathrm{Aut}(\Gamma_i)$. Let H_i be the kernel of the action of $\langle S_i \rangle_L$ (which is a regular cyclic subgroup of $\mathrm{Aut}(\Gamma_i)$) on the partition induced by R. The permutation group $\langle S_i \rangle_L / H_i$ induced by the action of $\langle S_i \rangle_L$ on the classes of R is a cyclic regular group. Since H_i is semiregular on $V(\Gamma_i)$, it follows that the equivalence classes of R coincide with the orbits of H_i. It follows that S_i is a union of cosets of the subgroup H_i of $\langle S_i \rangle$. Observe that the element of order 2 in the quotient group $\langle S_i \rangle / H_i$ is $m + H_i$.

Recall that Σ_i has the property that the stabilizer of the identity and the element of order 2 in the cyclic regular subgroup of $\mathrm{Aut}(\Sigma_i)$ are equal, which implies that every automorphism of Γ_i that fixes 0, fixes setwise coset $m + H_i$. As every automorphism of Γ that fixes 0 induces automorphism of Γ_i, it follows that every automorphism of Γ that fixes 0 fixes setwise $m + H_i$. If $GCD(d_1, \ldots, d_k) = d > 1$, then $H = H_1 \cap \ldots \cap H_k$ has order d. It follows that S' is a union of cosets of H, hence $B(X)$ is a wreath product with $\overline{K_d}$. This shows that $B(X)$ is reducible, and by Lemma 2.10 it follows that X is also reducible, contrary to the assumption that X is nontrivially unstable circulant.

If $GCD(d_1, \ldots, d_k) = 1$, then it follows that $H_1 \cap \ldots \cap H_k = \{0\}$. As observed above, every automorphism of Γ fixes setwise each of the sets $m + H_i$, for $i \in \{1, \ldots, k\}$, hence it also fixes their intersection. Since $(m + H_1) \cap \ldots \cap (m + H_k) = \{m\}$, by Lemma 2.5 it follows that X is stable, a contradiction. This finishes the proof. $\square$

For further research, we propose the following problem.

Problem 3.3. *Does there exist a nontrivially unstable Cayley graph on an Abelian group of odd order.*

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