An improved lower bound of P(G, L) - P(G, k) for k-assignments L

Fengming Dong* and Meiqiao Zhang[†]
National Institute of Education, Nanyang Technological University, Singapore

Abstract

Let G=(V,E) be a simple graph with n vertices and m edges, P(G,k) be the chromatic polynomial of G, and P(G,L) be the number of L-colorings of G for any k-assignment L. In this article, we show that when $k \geq m-1 \geq 3$, P(G,L)-P(G,k) is bounded below by $\left((k-m+1)k^{n-3}+\frac{(k-m+3)c}{3}k^{n-5}\right)\sum\limits_{uv\in E}|L(u)\setminus L(v)|$, where $c\geq \frac{(m-1)(m-3)}{8}$, and in particular, if G is K_3 -free, then $c\geq \binom{m-2}{2}+2\sqrt{m}-3$. Consequently, $P(G,L)\geq P(G,k)$ whenever $k\geq m-1$.

Keywords: list-coloring, list-color function, chromatic polynomial, broken-cycle

Mathematics Subject Classification: 05C15, 05C30, 05C31

1 Introduction

For any graph G, let V(G) and E(G) be the vertex set and edge set of G respectively. Let \mathbb{N} be the set of positive integers, and any $k \in \mathbb{N}$, let $[k] := \{1, \ldots, k\}$. A proper k-coloring of G is a map $f: V(G) \to [k]$ such that $f(u) \neq f(v)$ for each pair of adjacent vertices u and v in G. Let P(G, k) denote the number of proper k-colorings of G. Introduced by Birkhoff [1] in 1912, P(G, k) is called the *chromatic polynomial* of G. More details on P(G, k) can be found in [1, 2, 3, 4, 9, 12, 13].

^{*}Corresponding Author. Email: fengming.dong@nie.edu.sg and donggraph@163.com.

[†]Email: nie21.zm@e.ntu.edu.sg and meiqiaozhang95@163.com.

The notion of list-coloring was introduced independently by Vizing [15] and by Erdős, Rubin and Taylor [7]. A map $L:V(G)\to 2^{\mathbb{N}}$ is called an assignment of G. For any $k\in\mathbb{N}$, a k-assignment of G is an assignment L of G with |L(v)|=k for all $v\in V(G)$. Given any assignment L of G, an L-coloring of G is a map $f:V(G)\to\mathbb{N}$ with the property that $f(v)\in L(v)$ for each $v\in V(G)$ and $f(u)\neq f(v)$ for each pair of adjacent vertices u and v in G. Let P(G,L) denote the number of L-colorings of G. For any $k\in\mathbb{N}$, let $P_l(G,k)$ be the minimum value of P(G,L) among all k-assignments L of G. Introduced by Kostochka and Sidorenko in 1990s, $P_l(G,k)$ is called the list-color function of G. More details on $P_l(G,k)$ can be found in [14].

It is known that P(G,k) is a polynomial in k of degree |V(G)| (see Theorem 3). However, due to Donner [6], $P_l(G,k)$ is in general not a polynomial in k. By the definitions of P(G,k) and $P_l(G,k)$, $P_l(G,k) \leq P(G,k)$ holds for every $k \in \mathbb{N}$. Clearly, $P_l(G,k) = P(G,k)$ does not hold for some graphs G and some numbers $k \in \mathbb{N}$. For example, $P(G,2) \geq 2$ holds for each bipartite graph G, but $P_l(G,2) = 0$ as long as G contains $K_{2,4}$ as a subgraph. On the other hand, it is not difficult to verify that $P(G,k) = P_l(G,k)$ holds for any chordal graph G and G and G (see [11]). From the big picture, for any simple graph G, Donner [6] showed that $P(G,k) = P_l(G,k)$ holds when G is sufficiently large, answering a problem proposed by Kostochka and Sidorenko [11], and Thomassen [14] proved that $P(G,k) = P_l(G,k)$ when G in 2017, Wang, Qian and Yan [16] significantly improved this result by showing that G in 2017, Wang, Qian and Yan [16] significantly improved this result by showing that G in G in G in G.

In this article, we will establish a lower bound of P(G, L) - P(G, k) for an arbitrary k-assignment L with $k \ge m - 1$. Obviously, P(G, L) = P(G, k) holds whenever L(u) = L(v) for every edge uv in G. This article shows how large the gap between P(G, L) and P(G, k) can be when $L(u) \ne L(v)$ for some edge uv in G.

Theorem 1. Let G = (V, E) be a simple graph with n vertices and $m \ge 4$ edges. Then, for any k-assignment L of G with $k \ge m - 1$,

$$P(G, L) - P(G, k) \ge \left((k - m + 1)k^{n-3} + \frac{(k - m + 3)c}{3}k^{n-5} \right) \sum_{uv \in E} |L(u) \setminus L(v)|, \quad (1)$$

where $c \geq \frac{(m-1)(m-3)}{8}$, and particularly, when G is K_3 -free, $c \geq {m-2 \choose 2} + 2\sqrt{m} - 3$.

Note that any graph with less than 4 edges is a chordal graph. Thus, the following conclusion follows from Theorem 1 directly.

Corollary 2. For any simple graph G with m edges, $P_l(G, k) = P(G, k)$ holds for each $k \in \mathbb{N}$ with $k \ge m - 1$.

Let G = (V, E) be a simple graph with n vertices and m edges and η be a fixed bijection from E to [m]. A broken cycle of G (with respect to η) is a path $B = v_1v_2 \dots v_r$ of G, where $r \geq 3$, such that $v_1v_r \in E$ and $\eta(v_1v_r) < \eta(v_iv_{i+1})$ for each $i = 1, 2, \dots, r-1$. Let $\mathscr{B}(G)$ be the collection of edge sets E(B) over all broken cycles B of G, and let $\mathscr{NB}(G)$ be the set of subsets A of E that is broken-cycle free with respect to η (i.e., $E_0 \not\subseteq A$ for each $E_0 \in \mathscr{B}(G)$). Obviously, for each $A \in \mathscr{NB}(G)$, the spanning subgraph (V, A) has no cycles, implying that $0 \leq |A| \leq n-1$. For each i with $0 \leq i \leq n-1$, let $\mathscr{NB}_i(G)$ be the set of $A \in \mathscr{NB}(G)$ with |A| = i.

For any $e \in E$ and $1 \le i \le n-1$, let $\mathscr{NB}_i(G, e)$ be the set of $A \in \mathscr{NB}_i(G)$ with $e \in A$. Note that $|\mathscr{NB}_i(G, e)|$ depends on η although η is not included in the notation. For example, if G is K_3 , then $|\mathscr{NB}_2(G, e)|$ is either 1 or 2. Let $Q_{\eta}(G, e, x)$ denote the polynomial defined below:

$$Q_{\eta}(G, e, x) := \sum_{\substack{1 \le i \le n-1 \\ i \text{ odd}}} \frac{|\mathscr{N}\mathscr{B}_i(G, e)|}{i} x^{n-i} - \sum_{\substack{2 \le i \le n-1 \\ i \text{ even}}} |\mathscr{N}\mathscr{B}_i(G, e)| x^{n-i}.$$
 (2)

For any $e \in E$, let G/e denote the simple graph obtained from G by contracting e and deleting all but one of the multiple edges, if they arise. Thus, |E(G/e)| = m - 1 - t, where t is the number of 3-cycles in G containing e.

In Section 2, we show that if $x \ge m-1$ and $n \ge 4$, then

$$Q_{\eta}(G, e, x) \ge (x - m + 1)x^{n-2} + \frac{(x - m + 3)|\mathscr{N}\mathscr{B}_{2}(G/e)|}{3}x^{n-4}.$$

Then, in Section 3, we find a lower bound of $|\mathscr{NB}_2(G/e)|$ in terms of m. In Section 4, we prove that P(G,L) - P(G,k) is bounded below by $\frac{1}{k} \sum_{uv \in E} (|L(u) \setminus L(v)|Q_{\eta}(G,uv,k))$ for any k-assignment L of G with $k \geq m-1$. Theorem 1 then follows immediately. Finally, in Section 5, we propose two conjectures studying the relation between $P_l(G,k)$ and P(G,k).

2 A lower bound of $Q_{\eta}(G, e, x)$

In this section, we always assume that G = (V, E) is a simple graph with n vertices and m edges and η is a fixed bijection from E to [m]. Due to Whitney [17], the coefficients of P(G, x) can be expressed in terms of the sizes of $\mathscr{NB}_i(G)$'s.

Theorem 3 ([17]).
$$P(G, x)$$
 can be expressed as $P(G, x) = \sum_{i=0}^{n-1} (-1)^i | \mathcal{NB}_i(G) | x^{n-i}$.

In this section, we shall find a lower bound of $Q_{\eta}(G, e, x)$ for any edge e under the condition

 $x \geq m-1$. By the definition of $\mathcal{NB}_i(G,e)$, we first have the following relation between $|\mathcal{NB}_i(G,e)|$ and $|\mathcal{NB}_{i+1}(G,e)|$.

Lemma 4. For any $e \in E$ and $i \in [n-2]$, $i|\mathcal{NB}_{i+1}(G,e)| \leq (m-i)|\mathcal{NB}_i(G,e)|$.

Proof. When $i \geq m$, the inequality is trivial, as both sides are 0. Now assume that $1 \leq i \leq m-1$. Lemma 4 then follows directly from the following facts:

- (i). for each $A \in \mathcal{NB}_{i+1}(G, e)$ and $e' \in A \setminus \{e\}, A \setminus \{e'\} \in \mathcal{NB}_i(G, e)$; and
- (ii). for each $A' \in \mathcal{NB}_i(G, e)$, there are at most m-i edges e' in $E \setminus A'$ such that $A' \cup \{e'\} \in \mathcal{NB}_{i+1}(G, e)$.

We can now apply Lemma 4 to find a lower bound of $Q_{\eta}(G, e, x)$.

Theorem 5. Assume that $n \geq 3$. For any edge e in G and $x \geq 0$,

$$Q_{\eta}(G, e, x) \ge \sum_{\substack{1 \le i \le n-1 \\ i \text{ odd}}} \frac{|\mathscr{N}\mathscr{B}_i(G, e)|}{i} (x - m + i) x^{n-i-1}.$$

$$(3)$$

In particular, if n is even, then,

$$Q_{\eta}(G, e, x) \ge \sum_{\substack{1 \le i \le n-3 \\ i \text{ odd}}} \frac{|\mathscr{NB}_{i}(G, e)|}{i} (x - m + i) x^{n-i-1} + \frac{|\mathscr{NB}_{n-1}(G, e)|}{n-1} x. \tag{4}$$

Proof. By Lemma 4, for any $i \in [n-2]$, as $x \ge 0$,

$$\frac{|\mathscr{N}\mathscr{B}_{i}(G,e)|}{i}x^{n-i} - |\mathscr{N}\mathscr{B}_{i+1}(G,e)|x^{n-i-1}| \ge \frac{|\mathscr{N}\mathscr{B}_{i}(G,e)|}{i}x^{n-i} - \frac{(m-i)|\mathscr{N}\mathscr{B}_{i}(G,e)|}{i}x^{n-i-1}$$

$$= \frac{|\mathscr{N}\mathscr{B}_{i}(G,e)|}{i}(x-m+i)x^{n-i-1}.$$
(5)

By the definition of $Q_{\eta}(G, e, x)$, the result follows from (5).

For any edge e in G, let $\eta|_{E(G/e)}$ be the restriction of η to the edge set of G/e, and let $\mathscr{NB}_j(G/e)$ be the set of $A \subseteq E(G/e)$ with |A| = j such that A is broken-cycle free with respect to $\eta|_{E(G/e)}$. In the following, we will show that $|\mathscr{NB}_i(G,e)|$ is bounded below by $|\mathscr{NB}_{i-1}(G/e)|$.

Lemma 6. For any $e \in E$ and $i \in [n-1]$, $|\mathscr{NB}_i(G,e)| \ge |\mathscr{NB}_{i-1}(G/e)|$.

Proof. It suffices to show that $A \cup \{e\} \in \mathscr{NB}_i(G, e)$ for each $A \in \mathscr{NB}_{i-1}(G/e)$.

Suppose that $A \cup \{e\} \notin \mathcal{NB}_i(G, e)$. Then, there exists $B \in \mathcal{B}(G)$ with $B \subseteq A \cup \{e\}$. As $A \in \mathcal{NB}_{i-1}(G/e)$, $B \not\subseteq A$, which implies that $e \in B$ and $B \setminus \{e\} \subseteq A$. However, $B \in \mathcal{B}(G)$ implies that $B \setminus \{e\} \in \mathcal{B}(G/e)$, a contradiction to the assumption that $A \in \mathcal{NB}_{i-1}(G/e)$.

Combining Theorem 5 and Lemma 6, we obtain a lower bound of $Q_{\eta}(G, e, x)$ in terms of $|\mathscr{NB}_{2}(G/e)|$ and x.

Corollary 7. For any $e \in E$ and real number x with $x \ge m-1$, if $n \ge 4$, then

$$Q_{\eta}(G, e, x) \ge (x - m + 1)x^{n-2} + \frac{(x - m + 3)|\mathscr{N}\mathscr{B}_{2}(G/e)|}{3}x^{n-4}.$$
 (6)

3 Lower bounds of $|\mathscr{NB}_2(G/e)|$

In this section, we still assume that G = (V, E) is a simple graph with |V| = n and |E| = m, and we shall find a lower bound of $|\mathcal{NB}_2(G/e)|$ in terms of m for an arbitrary edge e in G.

Given any simple graph H, by the definition of $|\mathcal{NB}_2(H)|$ or Corollary 2.3.1 in [4], $|\mathcal{NB}_2(H)|$ has the following expression:

$$|\mathscr{N}\mathscr{B}_2(H)| = \binom{|E(H)|}{2} - \triangle(H),\tag{7}$$

where $\triangle(H)$ is the number of 3-cycles in H.

First consider the special case that G is K_3 -free. Let $c_4(G)$ be the minimum integer r such that each edge e in G is contained in at most r 4-cycles of G. For any $u \in V$, let $N_G(u)$ denote the set of vertices in G adjacent to u, and let $d_G(u) = |N_G(u)|$.

Lemma 8. For any $e \in E$, if G is K_3 -free and $m \ge 3$, then

$$|\mathcal{NB}_2(G/e)| \ge {m-1 \choose 2} - c_4(G) \ge {m-2 \choose 2} + 2\sqrt{m} - 3.$$
 (8)

Proof. As G is K_3 -free, then G/e has exactly m-1 edges and at most $c_4(G)$ 3-cycles. Thus, applying (7) implies that $|\mathscr{NB}_2(G/e)| \geq {m-1 \choose 2} - c_4(G)$ for any edge $e \in E$.

Note that $\binom{m-1}{2} - \binom{m-2}{2} - 2\sqrt{m} + 3 = (\sqrt{m} - 1)^2$. Thus, it remains to show that $c_4(G) \le (\sqrt{m} - 1)^2$. It suffices to show that for each edge e' in G, the number of 4-cycles in G containing e', denoted by $c_4(e')$, is at most $(\sqrt{m} - 1)^2$. Let $e' = uv \in E$, $N'(u) := N_G(u) \setminus \{v\} = (\sqrt{m} - 1)^2$.

 $\{u_1, u_2, \ldots, u_p\}$ and $N'(v) := N_G(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_q\}$. As G is K_3 -free, $N'(u) \cap N'(v) = \emptyset$. If p = 0 or q = 0, then $c_4(e') = 0 < (\sqrt{m} - 1)^2$. Now, assume $p \ge 1$ and $q \ge 1$. Clearly, $c_4(e')$ is equal to the size of the edge set $E_G(N'(u), N'(v)) := \{u_i v_j \in E : i \in [p], j \in [q]\}$. Thus,

$$c_4(e') = |E_G(N'(u), N'(v))| \le m - 1 - p - q, \tag{9}$$

implying that $p + q \leq m - 1 - c_4(e')$, and therefore

$$c_4(e') = |E_G(N'(u), N'(v))| \le pq \le \frac{1}{4}(p+q)^2 \le \frac{1}{4}(m-1-c_4(e'))^2.$$
(10)

Solving the inequality $c_4(e') \leq \frac{1}{4}(m-1-c_4(e'))^2$ with the condition $c_4(e') < m$ gives that $c_4(e') \leq (\sqrt{m}-1)^2$. Hence $c_4(G) = \max_{e' \in E} c_4(e') \leq (\sqrt{m}-1)^2$. The result holds.

Now we are going to find a lower bound of $|\mathcal{NB}_2(G/e)|$ in terms of m for any edge e in G. We shall apply the following theorem obtained by Fisher in [8].

Theorem 9 ([8]). For any simple graph H, $\triangle(H) \leq \frac{1}{6}|E(H)|(\sqrt{8|E(H)|+1}-3)$.

By applying Theorem 9, we can find an upper bound of $\triangle(H)$ in terms of |E(H)| and t, where t is any number not larger than the maximum degree of H.

Lemma 10. For any simple graph H, if the maximum degree of H is at least t, then

$$\triangle(H) \le \frac{|E(H)| - t}{6} \left(3 + \sqrt{8(|E(H)| - t) + 1} \right).$$
 (11)

Proof. Let w be a vertex in H with $d_H(w) = s \ge t$. Let H_0 be the subgraph of H induced by $N_H(w)$, and let H-w be the subgraph of H induced by $V(H)\setminus\{w\}$. Then $|E(H_0)| \le |E(H)|-s$ and |E(H-w)| = |E(H)|-s. Then,

$$\Delta(H) = |E(H_0)| + \Delta(H - w)
\leq |E(H)| - s + \frac{1}{6}(|E(H)| - s) \left(\sqrt{8(|E(H)| - s) + 1} - 3\right)
= \frac{|E(H)| - s}{6} \left(3 + \sqrt{8(|E(H)| - s) + 1}\right),$$
(12)

where the penultimate expression follows from Theorem 9. As $s \geq t$, the lemma holds. \Box

Lemma 11. If $m \geq 4$, then for any $e \in E$, $|\mathscr{NB}_2(G/e)| \geq \frac{(m-1)(m-3)}{8}$.

Proof. Let e be any edge in G and let t be the number of 3-cycles in G containing e. Then

 $m \ge 2t + 1$ and |E(G/e)| = m - t - 1. By (7) and Theorem 9, $|\mathscr{NB}_2(G/e)| \ge g(t, m)$, where

$$g(t,m): = {m-t-1 \choose 2} - \frac{(m-t-1)}{6} \left(\sqrt{8(m-t-1)+1} - 3\right)$$
$$= \frac{(m-t-1)^2}{2} - \frac{(m-t-1)}{6} \sqrt{8(m-t-1)+1}.$$
(13)

Note that $f(x) := \frac{1}{2}x^2 - \frac{x}{6}\sqrt{8x+1}$ is strictly increasing for $x \ge 1$, implying that $f(m-1) \ge f(m-2)$. Since g(t,m) = f(m-1-t), it is routine to verify that when $m \ge 4$,

$$g(0,m) > g(1,m) = \frac{(m-2)^2}{2} - \frac{m-2}{6}\sqrt{8m-15} > \frac{(m-1)(m-3)}{8}.$$
 (14)

It remains to consider the case $t \ge 2$. Note that |E(G/e)| = m - t - 1 and the vertex in G/e produced after contracting e is of degree at least t. As $m \ge 2t + 1$, by (7) and Lemma 10,

$$|\mathcal{NB}_{2}(G/e)| \geq \binom{m-t-1}{2} - \frac{(m-2t-1)}{6} \left(3 + \sqrt{8(m-2t-1)+1}\right)$$

$$= \frac{(m-1)(m-3)}{8} + \frac{m-2t-1}{24} \left(9m-6t-27-4\sqrt{8(m-2t-1)+1}\right)$$

$$\geq \frac{(m-1)(m-3)}{8}. \tag{15}$$

Hence Lemma 11 holds. □

By Corollary 7 and Lemmas 8 and 11, the following conclusion holds.

Theorem 12. For any $e \in E$ and real number x with $x \ge m-1 \ge 3$, $Q_{\eta}(G, e, x)$ is bounded below by $(x-m+1)x^{n-2} + \frac{(x-m+3)c}{3}x^{n-4}$, where $c \ge \frac{(m-1)(m-3)}{8}$, and in particular, if G is K_3 -free, then $c \ge {m-2 \choose 2} + 2\sqrt{m} - 3$.

4 Proving Theorem 1

In this section, we always assume that G = (V, E) is a simple graph with n vertices and m edges, η is a fixed bijection from E to [m], and L is a k-assignment of G, where $k \geq 2$.

For any integer i with $0 \le i \le n-1$, let $\mathscr{NBF}_i(G)$ be the set of spanning forests F = (V, A) of G with $A \in \mathscr{NB}_i(G)$. Clearly, each $F \in \mathscr{NBF}_i(G)$ has exactly n-i components. We can represent F by the set $\{T_1, T_2, \ldots, T_{n-i}\}$, where $T_1, T_2, \ldots, T_{n-i}$ are the components of F.

For any subgraph H of G, define $\beta(H) = \big| \bigcap_{v \in V(H)} L(v) \big|$. By applying the inclusion-exclusion

principle, it can be proved that

$$P(G, L) = \sum_{i=0}^{n-1} (-1)^i \sum_{\{T_1, \dots, T_{n-i}\} \in \mathcal{MBF}_i(G)} \prod_{j=1}^{n-i} \beta(T_j).$$
 (16)

By Theorem 3 and (16), we have

$$P(G, L) - P(G, k) = \sum_{i=1}^{n-1} (-1)^i \sum_{\{T_1, \dots, T_{n-i}\} \in \mathcal{MBF}_i(G)} \left(\prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \right).$$
 (17)

For any edge e = uv in G, let $\alpha(e) = |L(u) \setminus L(v)|$. For any $F = \{T_1, \dots, T_{n-i}\} \in \mathscr{NBF}_i(G)$, a lower bound for $\prod_{i=1}^{n-i} \beta(T_i) - k^{n-i}$ was obtained in [16], as stated below.

Lemma 13 ([16]). For any $i \in [n-1]$ and $F = \{T_1, \ldots, T_{n-i}\} \in \mathcal{NBF}_i(G)$,

$$\prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \ge -k^{n-i-1} \sum_{e \in E(F)} \alpha(e).$$
 (18)

We are now going to establish an upper bound for $\prod_{j=1}^{n-i} \beta(T_j) - k^{n-i}$. We first introduce the following result.

Lemma 14. Let d_1, d_2, \ldots, d_r be any non-negative real numbers, and q_1, q_2, \ldots, q_r be any positive real numbers, where $r \geq 1$. If $x \geq \max_{1 \leq i \leq r} d_i$, then

$$(x-d_1)(x-d_2)\cdots(x-d_r) \le x^r - \frac{x^{r-1}}{q_1+\cdots+q_r} \sum_{i=1}^r q_i d_i.$$
 (19)

Proof. Assume that $d_1 \geq d_2 \geq \cdots \geq d_r$. It is trivial to verify that $d_1 \geq \frac{1}{q_1 + \cdots + q_r} \sum_{i=1}^r q_i d_i$. As $0 \leq (x - d_i) \leq x$ for all $2 \leq i \leq r$, the result follows immediately.

Lemma 15. For $i \in [n-1]$ and $F = \{T_1, \ldots, T_{n-i}\} \in \mathcal{NBF}_i(G)$,

$$\prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \le -\frac{k^{n-i-1}}{i} \sum_{e \in E(F)} \alpha(e).$$
 (20)

Proof. For each T_j with $E(T_j) \neq \emptyset$, we have

$$\beta(T_j) \le \min_{uv \in E(T_j)} |L(u) \cap L(v)| \le k - \max_{e \in E(T_j)} \alpha(e) \le k - \frac{1}{|E(T_j)|} \sum_{e \in E(T_j)} \alpha(e). \tag{21}$$

Assume that $E(T_j) \neq \emptyset$ for each j with $1 \leq j \leq s$ while $E(T_j) = \emptyset$ for each j with $s+1 \leq j \leq n-i$. As $|E(T_1)| + \cdots + |E(T_s)| = i$ and $k \geq \alpha(e)$ for each $e \in E(G)$, by (21) and Lemma 14,

$$\prod_{j=1}^{s} \beta(T_{j}) \leq \prod_{j=1}^{s} \left(k - \frac{1}{|E(T_{j})|} \sum_{e \in E(T_{j})} \alpha(e) \right)
\leq k^{s} - \frac{k^{s-1}}{|E(T_{1})| + \dots + |E(T_{s})|} \sum_{j=1}^{s} \sum_{e \in E(T_{j})} \alpha(e)
= k^{s} - \frac{k^{s-1}}{i} \sum_{e \in E(F)} \alpha(e).$$
(22)

As $\beta(T_j) = k$ for each j with $s + 1 \le j \le n - i$, (20) follows.

Recall that $Q_{\eta}(G, e, k)$ is the function defined in (2) and $\mathscr{NB}_{i}(G, e)$ is the set of $A \in \mathscr{NB}_{i}(G)$ with $e \in A$. We are now going to find a lower bound of P(G, L) - P(G, k) in terms of $\alpha(e)$ and $Q_{\eta}(G, e, k)$ for all edges e in G.

Lemma 16.
$$P(G, L) - P(G, k) \ge \frac{1}{k} \sum_{e \in E} (\alpha(e)Q_{\eta}(G, e, k))$$
.

Proof. By (17) and applying Lemma 13 for even i's and Lemma 15 for odd i's,

$$P(G, L) - P(G, k)$$

$$= \sum_{i=1}^{n-1} (-1)^{i} \sum_{\substack{\{T_{1}, \dots, T_{n-i}\} \in \mathcal{MF}_{i}(G)}} \left(\prod_{j=1}^{n-i} \beta(T_{j}) - k^{n-i} \right)$$

$$\geq \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{k^{n-i-1}}{i} \sum_{F \in \mathcal{MF}_{i}(G)} \sum_{e \in E(F)} \alpha(e) - \sum_{\substack{2 \leq i \leq n-1 \\ i \text{ even}}} k^{n-i-1} \sum_{F \in \mathcal{MF}_{i}(G)} \sum_{e \in E(F)} \alpha(e)$$

$$= \sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{k^{n-i-1}}{i} \sum_{e \in E} \alpha(e) |\mathcal{NB}_{i}(G, e)| - \sum_{\substack{2 \leq i \leq n-1 \\ i \text{ even}}} k^{n-i-1} \sum_{e \in E} \alpha(e) |\mathcal{NB}_{i}(G, e)|$$

$$= \frac{1}{k} \sum_{e \in E} \alpha(e) \left(\sum_{\substack{1 \leq i \leq n-1 \\ i \text{ odd}}} \frac{|\mathcal{NB}_{i}(G, e)|}{i} k^{n-i} - \sum_{\substack{2 \leq i \leq n-1 \\ i \text{ even}}} |\mathcal{NB}_{i}(G, e)| k^{n-i} \right). \tag{23}$$

By the definition of $Q_{\eta}(G, e, k)$, the result follows.

We are now going to prove Theorem 1.

Proof of Theorem 1: As $m \ge 4$ and $k \ge m-1$, by Theorem 12 and Lemma 16,

$$P(G, L) - P(G, k) \ge \frac{1}{k} \sum_{e \in E} (\alpha(e) Q_{\eta}(G, e, k))$$

$$\ge \left((k - m + 1)k^{n-3} + \frac{(k - m + 3)c}{3} k^{n-5} \right) \sum_{e \in E} \alpha(e), \tag{24}$$

where $c \geq \frac{(m-1)(m-3)}{8}$, and if G is K_3 -free, then $c \geq {m-2 \choose 2} + 2\sqrt{m} - 3$.

5 Concluding remarks

Given any simple graph G, the list-chromatic number of G, denoted by $\chi_l(G)$, is the minimum integer r with $P_l(G,r) > 0$, and the list-color function threshold of G, denoted by $\tau(G)$, is the smallest integer $r \geq \chi(G)$ such that $P_l(G,k) = P(G,k)$ whenever $k \geq r$. Obviously, $\tau(G) \geq \chi_l(G) \geq \chi(G)$. By Corollary 2, $\tau(G) \leq |E(G)| - 1$ when $|E(G)| \geq 4$. The authors of this article also found some results on the upper bounds of $\tau(\mathcal{H})$ for r-uniform hypergraphs \mathcal{H} with m edges, where $r \geq 3$. In [5], they showed that $\tau(\mathcal{H})$ is bounded above by $\min\{m-1,0.6(m-1)+0.5\gamma(\mathcal{H})\}$, where $\gamma(\mathcal{H}) = \max_{e \in E(\mathcal{H})} |E_{r-1}(e)|$ and $E_{r-1}(e)$ is the set of edges e' in \mathcal{H} with $|e \cap e'| = r-1$, and in [18], they further showed that if $\rho(\mathcal{H}) := \min_{e,e' \in E(\mathcal{H})} |e \setminus e'| \geq 2$ and $m \geq \rho(\mathcal{H})^3/2 + 1$, then $\tau(\mathcal{H}) \leq \frac{2.4(m-1)}{\rho(\mathcal{H})\log(m-1)}$.

Thomassen [14] asked if there exists a universal constant α such that $\tau(G) - \chi_l(G) \leq \alpha$ holds for every simple graph G. Recently, Kaul et al [10] gave a negative answer to Thomassen's question by showing that $\tau(K_{2,s}) - \chi_l(K_{2,s}) \geq C\sqrt{s}$ for a constant C and all $s \in \mathbb{N}$ with $s \geq 16$.

We end this article with two conjectures.

Conjecture 1. There exists a constant c > 0 such that $\tau(G) \le c|V(G)|$ or $\tau(G) \le c\Delta(G)$ for every simple graph G, where $\Delta(G)$ is the maximum degree of G.

Conjecture 2. For any simple graph G, if L is a k-assignment of G with $k \geq \tau(G)$ and $L(u) \neq L(v)$ for some edge uv in G, then P(G, L) > P(G, k).

Clearly, Conjecture 2 holds for all chordal graphs. If this conjecture fails, then there exists a non-chordal graph G such that P(G, L) = P(G, k) for some k-assignment L, where $\tau(G) \le k \le |E(G)| - 2$ and $L(u) \ne L(v)$ for some edge uv in G.

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