# List 4-colouring of planar graphs 

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#### Abstract

This paper proves the following result: If $G$ is a planar graph and $L$ is a 4list assignment of $G$ such that $|L(x) \cap L(y)| \leq 2$ for every edge $x y$, then $G$ is $L$-colourable. This answers a question asked by Kratochvíl, Tuza and Voigt in [Journal of Graph Theory, 27(1):43-49, 1998].


Keywords: planar graph; lists with separation; list colouring.

## 1 Introduction

A list assignment of a graph $G$ is a mapping $L$ which assigns to each vertex $v$ of $G$ a set $L(v)$ of permissible colours. An $L$-colouring of $G$ is a proper colouring $f$ of $G$ such that for each vertex $v$ of $G, f(v) \in L(v)$. We say $G$ is $L$-colourable if $G$ has an $L$-colouring. A $k$-list assignment of $G$ is a list assignment $L$ with $|L(v)| \geq k$ for each vertex $v$. We say $G$ is $k$-choosable if $G$ is $L$-colourable for any $k$-list assignment $L$ of $G$. The choice number $\operatorname{ch}(G)$ of $G$ is the minimum integer $k$ such that $G$ is $k$-choosable.

It is known that there are planar graphs $G$ and 4-list assignments $L$ of $G$ such that $G$ is not $L$-colourable [15]. A natural direction of research is to put restrictions on the list assignments so that for any planar graph $G$ and any 4-list assignment $L$ of $G$ satisfying the restrictions, $G$ is $L$-colourable. Indeed, the Four Colour Theorem can be formulated as such a result: For any planar graph $G$, if $L$ is a 4-list assignment of $G$ with $L(x)=L(y)$ for any edge $x y$ of $G$, then $G$ is $L$-colourable.

Are there other natural restrictions for which the corresponding "list 4-colouring theorem" is true?

By changing the equality to inequality in the above formulation of the Four Colour Theorem, one may ask the following question:

Is it true that for any planar graph $G$, any 4-list assignment $L$ of $G$ such that $L(x) \neq$ $L(y)$ (or equivalently, $|L(x) \cap L(y)| \leq 3$ ), $G$ is L-colourable?

[^0]The answer is NO. Mirzakhani [13] constructed a planar graph $G$ and a 4-list assignment $L$ of $G$ such that $|L(x) \cap L(y)| \leq 3$, and $G$ is not $L$-colourable.

On the other hand, Kratochvíl, Tuza and Voigt [10] proved that for any planar graph $G$, for any 4 -list assignment $L$ of $G$ such that for any edge $x y,|L(x) \cap L(y)| \leq 1, G$ is $L$-colourable. Then they asked the following question:

Question 1 [10] Is it true that for any planar graph $G$ and any 4-list assignment $L$ of $G$ such that $|L(x) \cap L(y)| \leq 2$ for every edge $x y, G$ is $L$-colourable?

This question received a lot of attention [1, 2, 4, 5, 6, 6, 7, 2, 11, 16, 17, 18. Most of the works deal with variations of this problem. There was not much progress on the question itself. In this paper, we answer this question in affirmative.

Definition 1 Assume $G$ is a graph and $k, s$ are positive integers. $A(\star, s)$-list assignment of $G$ is a list assignment $L$ of $G$ such that $|L(x) \cap L(y)| \leq s$ for each edge $x y$. A $(k, s)$-list assignment of $G$ is a $(\star, s)$-list assignment of $G$ with $|L(v)| \geq k$ for each vertex $v$. A graph $G$ is called $(k, s)$-choosable if $G$ is L-colourable for any $(k, s)$-list assignment $L$ of $G$.

The following is the main result of this paper.
Theorem 1 Every planar graph is $(4,2)$-choosable.

## 2 The proof

It suffices to prove Theorem 1 for 2-connected near-triangulations of the plane: for each non-triangular face, we can add a new vertex adjacent to all vertices of the face, and assign new colours to the added vertex so that the resulting list assignment is still a $(4,2)$-list assignment. For a 2-connected plane graph $G$, we denote by $B(G)$ the boundary cycle of $G$.

Definition $2 A$ rooted plane graph is a pair $\left(G, v_{1} v_{2}\right)$, where $G$ is a 2-connected neartriangulation of the plane, and $v_{1} v_{2}$ is a boundary edge.

The vertices $v_{1}, v_{2}$ are called the root vertices and $v_{1} v_{2}$ is called the root edge.
Definition 3 Assume $\left(G, v_{1} v_{2}\right)$ is a rooted plane graph. Assume $v$ is a non-root boundary vertex, and $N_{G}(v) \cap B(G)=\left\{u_{1}, u_{2}, \ldots, u_{k}\right\}(k \geq 2)$, and $\left(v, u_{1}, u_{2}, \ldots, u_{k}\right)$ occur in $B(G)$ is this cyclic order. The vertices $u_{1}, \ldots, u_{k}$ are called the boundary neighbours of $v$. If the rooted edge is contained in the boundary path from $u_{i}$ to $u_{i+1}$, then $u_{i}$ and $u_{i+1}$ are called the primary boundary neighbours of $v$.

Note that each non-root boundary vertex of a rooted plane graph has exactly two primary boundary neighbours.

Definition $4 A$ list assignment of a rooted plane graph $\left(G, v_{1} v_{2}\right)$ is a mapping $L$ which assigns to each vertex $v \neq v_{1}, v_{2}$ a set $L(v)$ of colours, and assigns to the ordered pair $\left(v_{1}, v_{2}\right)$ a set $L\left(v_{1}, v_{2}\right)$ of ordered pairs of distinct colours. An L-colouring of $\left(G, v_{1} v_{2}\right)$ is a proper colouring $f$ of $G$ such that for each $v \neq v_{1}, v_{2}, f(v) \in L(v)$, and $\left(f\left(v_{1}\right), f\left(v_{2}\right)\right) \in$ $L\left(v_{1}, v_{2}\right)$.

Assume $L$ is a list assignment of $\left(G, v_{1} v_{2}\right)$. The list assignment $\tilde{L}$ of $G$ associated to $L$ is the list assignment of $G$ defined as $\tilde{L}(v)=L(v)$ for $v \neq v_{1}, v_{2}$ and $\tilde{L}\left(v_{1}\right)=\{c$ : $\left.\exists d,(c, d) \in L\left(v_{1}, v_{2}\right)\right\}$ and $\tilde{L}\left(v_{2}\right)=\left\{d: \exists c,(c, d) \in L\left(v_{1}, v_{2}\right)\right\}$. We say $L$ is a $(\star, 2)$-list assignment of $\left(G, v_{1} v_{2}\right)$ if $\tilde{L}$ is a $(\star, 2)$-list assignment of $G$.

Definition 5 Assume $L$ is a $(\star, 2)$-list assignment of $\left(G, v_{1} v_{2}\right)$, and $v \in B(G)$ is a nonroot vertex, and $u$ is a primary boundary neighbour of $v$. We say $u$ is a good neighbour of $v$, if one of the following holds:

- $|\tilde{L}(u) \cap \tilde{L}(v)| \leq 1$, or
- $|\tilde{L}(u)|=4$.

Definition $6 A(\star, 2)$-list assignment of a rooted plane graph $\left(G, v_{1} v_{2}\right)$ is valid $i f|L(v)|=$ 4 for each interior vertex $v$, and one of the following holds:
(A) $\left|L\left(v_{1}, v_{2}\right)\right| \geq 1$ and $|L(v)| \geq 3$ for each non-root boundary vertex $v$.
(B) $\left|L\left(v_{1}, v_{2}\right)\right| \geq 2$, and there exists a unique non-root boundary vertex $v^{*}$ such that $|L(v)| \geq 3$ for $v \in B(G)-\left\{v_{1}, v_{2}, v^{*}\right\},\left|L\left(v^{*}\right)\right|=2$ and $v^{*}$ has a good neighbour.

Assume $L$ is a valid list assignment of $\left(G, v_{1} v_{2}\right), v^{*}$ is a non-root boundary vertex with $\left|L\left(v^{*}\right)\right|=2$ and $u$ is a good neighbour of $v^{*}$. If $|L(u)|=4$, then we may delete one colour from $L(u) \cap L\left(v^{*}\right)$ so that $\left|L(u) \cap L\left(v^{*}\right)\right| \leq 1$. So if $L$ is a valid list assignment of $\left(G, v_{1} v_{2}\right)$, and $u$ is a good neighbour of $v^{*}$, then we assume that $|\tilde{L}(u) \cap L(v)| \leq 1$. However to prove that $u$ is a good neighbour of $v^{*}$, it suffices to prove that either $|L(u)|=4$ or $|\tilde{L}(u) \cap L(v)| \leq 1$.

Theorem 2 If $L$ is a valid list assignment of a rooted plane graph ( $G, v_{1} v_{2}$ ), then there exists an $L$-colouring of $\left(G, v_{1} v_{2}\right)$.

Proof. The proof is by induction on $|V(G)|$.
Assume first that $G$ is a triangle $\left(v_{1}, v_{2}, v_{3}\right)$.
If (A) holds, then $\left|L\left(v_{3}\right)\right|=3$. Assume $L\left(v_{1}, v_{2}\right)=\left\{\left(c_{1}, c_{2}\right)\right\}$. Let $c_{3} \in L\left(v_{3}\right)-\left\{c_{1}, c_{2}\right\}$. Then $f\left(v_{i}\right)=c_{i}$ for $i=1,2,3$ is an $L$-colouring of $\left(G, v_{1} v_{2}\right)$.

Assume (B) holds. Then $\left|L\left(v_{3}\right)\right|=2$ and $L\left(v_{1}, v_{2}\right)=\left\{\left(c_{1}, c_{2}\right),\left(c_{1}^{\prime}, c_{2}^{\prime}\right)\right\}$. We may assume that $v_{2}$ is a good neighbour of $v_{3}$. If $c_{1}=c_{1}^{\prime}$, then $c_{2} \neq c_{2}^{\prime}$. Let $c_{3} \in L\left(v_{3}\right)-\left\{c_{1}\right\}$. One of $c_{2}, c_{2}^{\prime}$ is distinct from $v_{3}$. Without loss of generality, we may assume that $c_{2} \neq c_{3}$. Then $f\left(v_{i}\right)=c_{i}$ for $i=1,2,3$ is an $L$-colouring of $\left(G, v_{1} v_{2}\right)$. The case $c_{2}=c_{2}^{\prime}$ is symmetric.

Assume $c_{1} \neq c_{1}^{\prime}, c_{2} \neq c_{2}^{\prime}$. As $v_{2}$ is a good neighbour of $v_{3},\left|L\left(v_{3}\right) \cap\left\{c_{2}, c_{2}^{\prime}\right\}\right| \leq 1$. Assume $c_{2} \notin L\left(v_{3}\right)$. Let $c_{3} \in L\left(v_{3}\right)-\left\{c_{1}\right\}$. Then $f\left(v_{i}\right)=c_{i}$ for $i=1,2,3$ is an $L$-colouring of ( $G, v_{1} v_{2}$ ).

Assume $|V(G)|=n \geq 4$ and the theorem is true for any smaller rooted plane graphs.
For a cycle $C$ of $G, \operatorname{Int}[C]$ is the graph of all vertices and edges inside or on $C$, $\operatorname{Ext}[C]$ is the graph of all vertices and edges outside or on $C$. If $G$ has a separating triangle $C=\left(u_{1}, u_{2}, u_{3}\right)$, then let $G_{1}=\operatorname{Ext}[C]$. Then $\left(G_{1}, v_{1} v_{2}\right)$ has an $L$-colouring $f$. Let $G_{2}=\operatorname{Int}[C]-\left\{u_{3}\right\}$. Let $L^{\prime}$ be the list assignment of $\left(G_{2}, u_{1} u_{2}\right)$ defined as $L^{\prime}\left(u_{1}, u_{2}\right)=\left\{\left(f\left(u_{1}\right), f\left(u_{2}\right)\right)\right\}$, and for $v \in V\left(G_{2}\right)-\left\{u_{1}, u_{2}\right\}$,

$$
L^{\prime}(v)= \begin{cases}L(v)-\left\{f\left(u_{3}\right)\right\}, & \text { if } v \in N_{G}\left(u_{2}\right) \\ L(v), & \text { otherwise }\end{cases}
$$

Then $L^{\prime}$ is a valid list assignment of $\left(G_{2}, u_{1} u_{2}\right)$. By induction hypothesis, there is an $L^{\prime}$-colouring $g$ of $\left(G_{2}, u_{1} u_{2}\right)$. The union of $f$ and $g$ is an $L$-colouring of $\left(G, v_{1} v_{2}\right)$.

In the following, we assume that $G$ has no separating triangle.
Case $1 B(G)$ has a chord $x y$.
Let $G_{1}, G_{2}$ be the two subgraphs of $G$ separated by $x y$, (i.e., $G_{1}, G_{2}$ are connected induced subgraphs of $G$ with $V\left(G_{1}\right) \cap V\left(G_{2}\right)=\{x, y\}$ and $\left.V\left(G_{1}\right) \cup V\left(G_{2}\right)=V(G)\right)$, and assume $G_{1}$ contains the root edge $v_{1} v_{2}$.
Case 1(i) There is a chord $x y$ such that $|L(v)|=3$ for all $v \in B\left(G_{2}\right)-\{x, y\}$.
Let $L_{1}$ be the restriction of $L$ to $\left(G_{1}, v_{1} v_{2}\right)$. Then $L_{1}$ is a valid list assignment of $\left(G_{1}, v_{1} v_{2}\right)$. By induction hypothesis, there exists an $L_{1}$-colouring $f$ of $\left(G_{1}, v_{1} v_{2}\right)$.

Let $L_{2}$ be the list assignment of $\left(G_{2}, x y\right)$ defined as $L_{2}(x, y)=\{(f(x), f(y))\}$ and $L_{2}(v)=L(v)$ for $v \in V\left(G_{2}\right)-\{x, y\}$. Then $L_{2}$ is a valid list assignment of $\left(G_{2}, x y\right)$. By induction hypothesis, there exists an $L_{2}$-colouring $g$ of $\left(G_{2}, x y\right)$. The union of $f$ and $g$ is an $L$-colouring of $\left(G, v_{1} v_{2}\right)$.
Case 1(ii) There is a vertex $v^{*} \in B(G)-\left\{v_{1}, v_{2}\right\}$ with $\left|L\left(v^{*}\right)\right|=2$, and every chord $x y$ separates $v^{*}$ and the root edge, i.e., $v_{1} v_{2} \in E\left(G_{1}\right)$ and $v^{*} \in V\left(G_{2}\right)-\{x, y\}$.

We choose the chord $x y$ so that $G_{1}$ is minimum. Then $B\left(G_{1}\right)$ has no chord.
As there is a vertex $v^{*} \in B(G)-\left\{v_{1}, v_{2}\right\}$ with $\left|L\left(v^{*}\right)\right|=2$, we know that $\left(G, v_{1} v_{2}\right)$ satisfies (B). We may assume that $\left|L_{1}\left(v_{1}, v_{2}\right)\right|=2$.

Similarly, $L_{1}$ is a valid list assignment of $\left(G_{1}, v_{1} v_{2}\right)$ and hence there is an $L_{1}$-colouring $f$ of $\left(G_{1}, v_{1} v_{2}\right)$.

Claim 1 There is another L-colouring $f^{\prime}$ of $\left(G_{1}, v_{1} v_{2}\right)$ for which $\left(f^{\prime}(x), f^{\prime}(y)\right) \neq(f(x), f(y))$.
Assume Claim 1 is true. Let $L_{2}$ be the list assignment of $\left(G_{2}, x y\right)$ defined as $L_{2}(x, y)=$ $\left\{(f(x), f(y)),\left(f^{\prime}(x), f^{\prime}(y)\right)\right\}$ and $L_{2}(v)=L(v)$ for $v \in V\left(G_{2}\right)-\{x, y\}$. Note that $\tilde{L}_{2}(v) \subseteq$ $L(v)$ for $v \in\{x, y\}$, and the primary neighbours of $v^{*}$ in $\left(G_{2}, x y\right)$ are the same as its primary neighbours in $\left(G, v_{1} v_{2}\right)$. So $v^{*}$ has a good neighbour in $\left(G_{2}, x y\right)$. Thus $L_{2}$ is a valid list assignment of $\left(G_{2}, x y\right)$.

By induction hypothesis, there exists an $L_{2}$-colouring $g$ of $\left(G_{2}, x y\right)$. Depending on $(g(x), g(y))=(f(x), f(y))$ or $\left(f^{\prime}(x), f^{\prime}(y)\right)$, the union of $g$ and $f$ or the union of $g$ and $f^{\prime}$ is an $L$-colouring of $\left(G, v_{1} v_{2}\right)$. To finish the proof of Case 1 , it remains to prove Claim 1.

## Proof of Claim 1

Without loss of generality, we assume that $y \notin\left\{v_{1}, v_{2}\right\}$. Let $L_{1}^{\prime}=L_{1}$, except that $L_{1}^{\prime}(y)=L(y)-\{f(y)\}$. If $L_{1}^{\prime}$ is a valid list assignment of $\left(G_{1}, v_{1} v_{2}\right)$, then by induction hypothesis, there is an $L_{1}^{\prime}$-colouring $f^{\prime}$ of $\left(G_{1}, v_{1} v_{2}\right)$, and we are done.

Thus we may assume that $L_{1}^{\prime}$ is not a valid list assignment of $\left(G_{1}, v_{1} v_{2}\right)$. This happens only if $\left|L_{1}^{\prime}(y)\right|=2$ and $y$ has no good neighbour in $\left(G_{1}, v_{1} v_{2}\right)$. Assume $L_{1}^{\prime}(y)=\left\{c_{1}, c_{2}\right\}$.

As $B\left(G_{1}\right)$ has no chord, $y$ has exactly two boundary neighbours, and one of them is $x$. Let $y^{\prime}$ be the other boundary neighbour of $y$, i.e., $N_{G}(y) \cap B\left(G_{1}\right)=\left\{x, y^{\prime}\right\}$. Then

$$
\left\{c_{1}, c_{2}\right\} \subseteq \tilde{L}_{1}^{\prime}(x) \cap \tilde{L}_{1}^{\prime}\left(y^{\prime}\right)
$$

If $x$ is a root vertex, say $x=v_{1}$, then $\tilde{L}_{1}(x)=\left\{c_{1}, c_{2}\right\}$ (as $L_{1}^{\prime}(y) \subseteq \tilde{L}_{1}(x)$ and $\left.\tilde{L}_{1}(x) \mid \leq 2\right)$.
Assume $L\left(v_{1}, v_{2}\right)=\left\{\left(c_{1}, c_{1}^{\prime}\right),\left(c_{2}, c_{2}^{\prime}\right)\right\}$ for some colours $c_{1}^{\prime}, c_{2}^{\prime}$ (possibly $\left.c_{1}^{\prime}=c_{2}^{\prime}\right)$. Assume $c_{1} \neq f(x)$.

Let $L_{1}^{\prime \prime}=L_{1}$, except that $L_{1}^{\prime \prime}\left(v_{1}, v_{2}\right)=\left(c_{1}, c_{1}^{\prime}\right)$. As $\left|L_{1}^{\prime \prime}(v)\right| \geq 3$ for all $v \in B\left(G_{1}\right)-\left\{v_{1}, v_{2}\right\}$, $L_{1}^{\prime \prime}$ is a valid list assignment of $\left(G_{1}, v_{1} v_{2}\right)$. Hence there is an $L_{1}^{\prime \prime}$-colouring $f^{\prime}$ of $\left(G_{1}, v_{1} v_{2}\right)$. As $f^{\prime}(x) \neq f(x)$, Claim 1 is proved.

Thus we may assume that $x$ is not a root vertex.
Let $L_{1}^{\prime \prime}=L_{1}$ except that $L_{1}^{\prime \prime}(x)=L_{1}(x)-\{f(x)\}$. If $L_{1}^{\prime \prime}$ is a valid list assignment of $\left(G_{1}, v_{1} v_{2}\right)$, then again we obtain an $L$-colouring $f^{\prime}$ of $\left(G_{1}, v_{1} v_{2}\right)$ with $\left(f^{\prime}(x), f^{\prime}(y)\right) \neq$ $(f(x), f(y))$ and we are done.

Thus assume that $L_{1}^{\prime \prime}$ is not a valid list assignment. This means that $\left|L^{\prime \prime}(x)\right|=2$ and $x$ has no good neighbour. Let $x^{\prime}$ be the other boundary neighbour of $x$. Then we have $L_{1}^{\prime \prime}(x)=\left\{c_{1}, c_{2}\right\}\left(\right.$ so $\left.f(x) \neq c_{1}, c_{2}\right)$, and

$$
\left\{c_{1}, c_{2}\right\}=L(x) \cap L(y) \cap \tilde{L}_{1}\left(x^{\prime}\right) \cap \tilde{L}_{1}\left(y^{\prime}\right)
$$

As $G$ is a near-triangulation of the plane and $G$ has no separating triangle, $x$ and $y$ have a unique common neighbour $z$ in $G_{1}$, which is an interior vertex of $G_{1}$.

Since $B\left(G_{1}\right)$ has no chord, it is easy see that at least one of the following holds:

- $N_{G_{1}}\left(x^{\prime}\right) \cap N_{G_{1}}(y)-\{z\}=\varnothing$.
- $N_{G_{1}}\left(y^{\prime}\right) \cap N_{G_{1}}(x)-\{z\}=\varnothing$.

By symmetry, we may assume that $N_{G_{1}}\left(x^{\prime}\right) \cap N_{G_{1}}(y)-\{z\}=\varnothing$.
As $|L(z) \cap L(y)| \leq 2$, there exists $i \in\{1,2\}$, that $\left|L(z) \cap\left\{f(y), c_{i}\right\}\right| \leq 1$. Without loss of generality, we assume

$$
\left|L(z) \cap\left\{f(y), c_{1}\right\}\right| \leq 1
$$

Let

$$
G_{1}^{\prime}=G_{1}-\{x, y\} .
$$

Let $L_{1}^{*}$ be the list assignment of $\left(G_{1}^{\prime}, v_{1} v_{2}\right)$ defined as follows:

$$
L_{1}^{*}(v)= \begin{cases}L(v)-\left\{c_{1}, f(y)\right\}, & \text { if } v=z, \\ L(v)-\{f(y)\}, & \text { if } v \in N_{G_{1}}(y)-\{z\}, \\ L(v)-\left\{c_{1}\right\}, & \text { if } v \in N_{G_{1}}(x)-\{z\}, \\ L(v), & \text { otherwise } .\end{cases}
$$

and

$$
L_{1}^{*}\left(v_{1}, v_{2}\right)= \begin{cases}L\left(v_{1}, v_{2}\right), & \text { if } x^{\prime} \text { is not a root vertex, or } c_{1} \notin \tilde{L}\left(x^{\prime}\right) \\ L\left(v_{1}, v_{2}\right)-\left\{\left(c_{1}, c_{1}^{\prime}\right)\right\}, & \text { if } x^{\prime}=v_{1} \text { and }\left(c_{1}, c_{1}^{\prime}\right) \in L\left(v_{1}, v_{2}\right)\end{cases}
$$

Note that $\left|L_{1}^{*}(z)\right| \geq 3$, and $L_{1}^{*}\left(y^{\prime}\right)=L\left(y^{\prime}\right)\left(\right.$ as $\left.f(y) \notin L\left(y^{\prime}\right)\right)$. If $L_{1}^{*}$ is a valid list assignment of $\left(G_{1}^{\prime}, v_{1} v_{2}\right)$, then there is an $L_{1}^{*}$-colouring $f^{\prime}$ of $\left(G_{1}^{\prime}, v_{1} v_{2}\right)$. By letting $\left(f^{\prime}(x), f^{\prime}(y)\right)=\left(c_{1}, f(y)\right)$, we obtain an $L$-colouring of $\left(G_{1}, v_{1} v_{2}\right)$ with $\left(f^{\prime}(x), f^{\prime}(y)\right) \neq$ $(f(x), f(y))$, and we are done.

Thus we assume that $L_{1}^{*}$ is not a valid list assignment of $\left(G_{1}^{\prime}, v_{1} v_{2}\right)$.
This means that

- $x^{\prime}$ is not a root vertex, $x^{\prime}$ is the only boundary vertex of $G_{1}^{\prime}$ with $\left|L_{1}^{*}\left(x^{\prime}\right)\right|=2$, and $x^{\prime}$ has no good neighbour.

Assume $L\left(x^{\prime}\right)=\left\{c_{1}, c_{2}, c_{3}\right\}$ (and hence $L_{1}^{\star}\left(x^{\prime}\right)=\left\{c_{2}, c_{3}\right\}$ ).
Let $z^{\prime}$ be the unique common neighbour of $x$ and $x^{\prime}$, which is an interior vertex of $G_{1}$.
Let $x^{\prime \prime}$ be the other neighbour of $x^{\prime}$ in $B\left(G_{1}\right)$. Then $x^{\prime \prime}$ is a primary boundary neighbour of $x^{\prime}$ in $G_{1}^{\prime}$. Since $N_{G_{1}}\left(x^{\prime}\right) \cap N_{G_{1}}(y)-\{z\}=\varnothing$ and $G$ has no separating triangle, $z^{\prime}$ is the other primary boundary neighbour of $x^{\prime}$.

Since $z^{\prime}$ is not a good neighbour of $x^{\prime}$, we conclude that

- $z^{\prime}=z ;$
- $c_{1} \notin L(z), c_{2}, c_{3}, f(y) \in L(z)$ and $c_{3} \neq f(y)$.

Now $z^{\prime}=z$ implies that $N_{G_{1}}\left(y^{\prime}\right) \cap N_{G_{1}}(x)-\{z\}=\varnothing$. So we can repeat the same argument as above, but interchange the roles of $x, x^{\prime}$ and $y, y^{\prime}$. Then we conclude that the following hold:

- $y^{\prime}$ is not a root vertex,
- $z$ is adjacent to $y^{\prime}$,
- $L\left(y^{\prime}\right)=\left\{c_{1}, c_{2}, c_{3}^{\prime}\right\}$,
- $c_{2}, c_{3}^{\prime}, f(x) \in L(z)$.

As $\left\{c_{2}, c_{3}, c_{3}^{\prime}, f(x), f(y)\right\} \subseteq L\left(z_{1}\right)$, we have $c_{3}=c_{3}^{\prime}($ as $|L(z)|=4$ and the other colours are pairwise distinct). I.e.,

$$
L(z)=\left\{c_{2}, c_{3}, f(x), f(y)\right\} .
$$

As $L_{1}^{*}\left(x^{\prime}\right)=\left\{c_{2}, c_{3}\right\} \subseteq \tilde{L}\left(x^{\prime \prime}\right)$, we know that $c_{1} \notin \tilde{L}\left(x^{\prime \prime}\right)$.
Let

$$
G_{1}^{\prime \prime}=G_{1}-\left\{x^{\prime}, x, y\right\} .
$$

Let $L_{1}^{* *}$ be the list assignment of $\left(G_{1}^{\prime \prime}, v_{1} v_{2}\right)$ defined as follows:

$$
L_{1}^{* *}(v)= \begin{cases}L(v)-\left\{c_{1}, c_{2}, f(y)\right\}, & \text { if } v=z, \\ L(v)-\{f(y)\}, & \text { if } v \in N_{G_{1}}(y), \\ L(v)-\left\{c_{2}\right\}, & \text { if } v \in N_{G_{1}}(x), \\ L(v)-\left\{c_{1}\right\}, & \text { if } v \in N_{G_{1}}\left(x^{\prime}\right), \\ L(v), & \text { otherwise. }\end{cases}
$$

and

$$
L_{1}^{* *}\left(v_{1}, v_{2}\right)= \begin{cases}L\left(v_{1}, v_{2}\right), & \text { if } x^{\prime \prime} \text { is not a root vertex, or } c_{1} \notin \tilde{L}\left(x^{\prime \prime}\right), \\ L\left(v_{1}, v_{2}\right)-\left\{\left(c_{1}, c_{1}^{\prime}\right)\right\}, & \text { if } x^{\prime \prime}=v_{1} \text { and }\left(c_{1}, c_{1}^{\prime}\right) \in L\left(v_{1}, v_{2}\right) .\end{cases}
$$

If $L_{1}^{* *}$ is a valid list assignment of $\left(G_{1}^{\prime \prime}, v_{1} v_{2}\right)$, then there is an $L_{1}^{* *}$-colouring $f^{\prime}$ of $\left(G_{1}^{\prime \prime}, v_{1} v_{2}\right)$, which extends to an $L$-colouring of $\left(G_{1}, v_{1} v_{2}\right)$ by letting $f^{\prime}\left(x^{\prime}\right)=c_{1}, f^{\prime}(x)=c_{2}$ and $f^{\prime}(y)=f(y)$, and we are done.

Thus we may assume that $L_{1}^{* *}$ is not a valid list assignment of $\left(G_{1}^{\prime \prime}, v_{1} v_{2}\right)$. It is easy to check that $z$ is the only vertex of $B\left(G_{1}^{\prime \prime}\right)-\left\{v_{1}, v_{2}\right\}$ with $\left|L^{* *}(z)\right|<3$. Note that

$$
L^{* *}(z)=L(z)-\left\{c_{2}, f(y)\right\}=\left\{c_{3}, f(x)\right\} .
$$

The only reason that $L_{1}^{* *}$ is not a valid list assignment of $\left(G_{1}^{\prime \prime}, v_{1} v_{2}\right)$ is that $z$ has no good neighbour. Let $w_{1}, w_{2}$ be the two primary boundary neighbours of $z$ in $\left(G_{1}^{\prime \prime}, v_{1} v_{2}\right)$. We have

$$
\left\{c_{3}, f(x)\right\} \subseteq \tilde{L}\left(w_{1}\right), \tilde{L}\left(w_{2}\right) .
$$

This implies that $y^{\prime}$ is not a primary boundary neighbour of $z$ in $\left(G_{1}^{\prime \prime}, v_{1} v_{2}\right)$ (although $y^{\prime}$ is a boundary neighbour of $z$ in $\left.G_{1}^{\prime \prime}\right)$.

We repeat the above argument, but interchange the roles of $x, x^{\prime}$ and $y, y^{\prime}$. We conclude that for the two primary boundary neighbours $w_{1}^{\prime}, w_{2}^{\prime}$ of $z$ in $\left(G_{1}-\left\{x, y, y^{\prime}\right\}, v_{1} v_{2}\right)$,

$$
\left\{c_{3}, f(y)\right\} \subseteq \tilde{L}\left(w_{1}^{\prime}\right), \tilde{L}\left(w_{2}^{\prime}\right) .
$$

This means that $x^{\prime}$ is not a primary neighbour of $z$ in $\left(G_{1}-\left\{x, y, y^{\prime}\right\}, v_{1} v_{2}\right)$. But then the primary neighbours of $z$ in $\left(G_{1}-\left\{x, y, y^{\prime}\right\}, v_{1} v_{2}\right)$ and $\left(G_{1}-\left\{x^{\prime}, x, y\right\}, v_{1} v_{2}\right)$ are the same. I.e., $w_{1}^{\prime}=w_{1}$ and $w_{2}^{\prime}=w_{2}$. But then for $i=1,2$,

$$
\left\{c_{3}, f(x), f(y)\right\} \subseteq \tilde{L}\left(w_{i}\right) \cap L(z)
$$

contrary to the assumption that $L$ is $(\star, 2)$-list assignment of $\left(G, v_{1} v_{2}\right)$. This completes the proof of Claim 1, and hence the proof of Case 1.

Case $2 B(G)$ has no chord.
Case 2(i) (A) holds, and $L\left(v_{1}, v_{2}\right)=\left\{\left(c_{1}, c_{2}\right)\right\}$.
Let $u$ be the other boundary neighbour of $v_{2}$ in $G$. Similarly, as $G$ has no separating triangle and $B(G)$ has no chord, $v_{1}$ and $v_{2}$ has a unique common neighbour $w$, and $u$ and $v_{2}$ have a unique common neighbour $z$, and $w, z$ are interior vertices of $G$ (and possibly $w=z$ ).

Let $G^{\prime}=G-v_{2}$ and let $L^{\prime}$ be the list assignment of $\left(G^{\prime}, v_{1} w\right)$ defined as

$$
L^{\prime}(v)= \begin{cases}L(v)-\left\{c_{2}\right\}, & \text { if } v \in N_{G}\left(v_{2}\right)-\left\{v_{1}, w\right\}, \\ L(v), & \text { if } v \in V(G)-N_{G}\left(v_{2}\right)\end{cases}
$$

and

$$
L^{\prime}\left(v_{1}, w\right)=\left\{\left(c_{1}, c_{3}\right),\left(c_{1}, c_{4}\right)\right\}, \text { where } c_{3}, c_{4} \in L(w)-\left\{c_{1}, c_{2}\right\}
$$

In the definition above, if $\left|L(w)-\left\{c_{1}, c_{2}\right\}\right| \geq 3$, then $c_{3}, c_{4}$ are arbitrarily chosen from $L(w)-\left\{c_{1}, c_{2}\right\}$, with one exception:

If $c_{2} \notin L(w), w=z$ and $L(w) \cap L(u) \neq \varnothing$, then let $c^{\prime} \in L(w) \cap L(u)$, and we choose $c_{3}, c_{4} \in L(w)-\left\{c_{1}, c^{\prime}\right\}$.

We shall show that $L^{\prime}$ is valid list assignment of $\left(G^{\prime}, v_{1} w\right)$.
If $c_{2} \notin L(u)$, then $\left|L^{\prime}(u)\right|=|L(u)|=3$, and (A) holds for $L^{\prime}$ and $\left(G^{\prime}, v_{1} w\right)$. So $L^{\prime}$ is a valid list assignment of $\left(G^{\prime}, v_{1} w\right)$.

Assume $c_{2} \in L(u)$ and hence $\left|L^{\prime}(u)\right|=2$. If $z \neq w$, then either $c_{2} \in L(z)$ and hence $\left|L^{\prime}(z) \cap L^{\prime}(u)\right| \leq 1$ or $\left|L^{\prime}(z)\right|=4$. So $z$ is a good neighbour of $u$ in $\left(G^{\prime}, v_{1} w\right)$, and $L^{\prime}$ is a valid list assignment of $\left(G^{\prime}, v_{1} w\right)\left((\mathrm{B})\right.$ holds for $L^{\prime}$ and $\left.\left(G^{\prime}, v_{1} w\right)\right)$.

If $z=w$, then by our choice of $c_{3}, c_{4}$, we know that $\left|\tilde{L}^{\prime}(w) \cap L^{\prime}(u)\right| \leq 1$, and hence $w$ is a good neighbour of $u$, and $L^{\prime}$ is a valid list assignment of $\left(G^{\prime}, v_{1} w\right)\left((\mathrm{B})\right.$ holds for $L^{\prime}$ and ( $\left.G^{\prime}, v_{1} w\right)$ ).

By induction hypothesis, $\left(G^{\prime}, v_{1} w\right)$ has an $L^{\prime}$-colouring $f$. By letting $f\left(v_{2}\right)=c_{2}$, we obtain an $L$-colouring of $\left(G, v_{1} v_{2}\right)$.

Case 2(i) (B) holds, and $v^{*} \in B(G),\left|L\left(v^{*}\right)\right|=2$, and $u$ is a good neighbour of $v^{*}$.
It may happen that $v^{*}$ has two good neighbours. In this case, the good neighbour $u$ is usually arbitrarily chosen, unless $v^{*}$ is adjacent to a root vertex $v_{i}$ for some $i \in\{1,2\}$ and $\left|\tilde{L}\left(v_{i}\right)\right|=1$. In this case, we let $u=v_{i}$.

Let $w$ be the other boundary neighbour of $v^{*}$, and let $z$ be the common neighbours of $v^{*}$ and $w$. Similarly, we know that the vertex $z$ is unique and is an interior vertex of $G$.

By our choice of $u$, we know that either $w \neq v_{1}, v_{2}$, or $w=v_{i}$ for some $i \in\{1,2\}$ and $\left|\tilde{L}\left(v_{i}\right)\right|=2$ (for otherwise, we would have chosen $w$ as the good neighbour of $v^{*}$ ).

Let

$$
G^{\prime}=G-\left\{v^{*}\right\}, c \in L\left(v^{*}\right)-L(u)
$$

If $w$ is not a root vertex, then let $L^{\prime}$ be the list assignment of $\left(G^{\prime}, v_{1} v_{2}\right)$ defined as $L^{\prime}\left(v_{1}, v_{2}\right)=L\left(v_{1}, v_{2}\right)$, and for $v \in V\left(G^{\prime}\right)-\left\{v_{1}, v_{2}\right\}$,

$$
L^{\prime}(v)= \begin{cases}L(v)-\{c\}, & \text { if } v \in N_{G}\left(v^{*}\right) \\ L(v), & \text { if } v \in V(G)-N_{G}\left(v^{*}\right)\end{cases}
$$

If $\left|L^{\prime}(w)\right| \geq 3$, then $\left|L^{\prime}(v)\right| \geq 3$ for every $v \in B\left(G^{\prime}\right)-\left\{v_{1}, v_{2}\right\}$ and hence $L^{\prime}$ is a valid list assignment of $\left(G^{\prime}, v_{1} v_{2}\right)$. Otherwise, $w$ is the unique boundary vertex of $G^{\prime}$ with $\left|L^{\prime}(w)\right|=2$. Observe that either $c \in L(z)$ and hence $\left|L^{\prime}(z) \cap L^{\prime}(w)\right| \leq 1$, or $\left|L^{\prime}\left(z_{1}\right)\right|=$ $\left|L\left(z_{1}\right)\right|=4$. In any case, $z$ is a good neighbour of $w$, and hence $L^{\prime}$ is a valid list assignment of $\left(G, v_{1} v_{2}\right)$. By induction hypothesis, there is an $L^{\prime}$-colouring $f$ of $\left(G^{\prime}, v_{1} v_{2}\right)$. By letting $f\left(v^{*}\right)=c$, we obtain an $L$-colouring of $\left(G^{\prime}, v_{1} v_{2}\right)$.

Assume $w$ is a root vertex, say $w=v_{1}$. If $c \notin \tilde{L}\left(v_{1}\right)$, then the argument still works. Assume $c \in \tilde{L}\left(v_{1}\right)$. Without loss of generality, we may assume that $\left|L\left(v_{1}, v_{2}\right)\right|=2$, say $L\left(v_{1}, v_{2}\right)=\left\{(c, d),\left(c^{\prime}, d^{\prime}\right)\right\}$. As observed above, $\left|\tilde{L}\left(v_{1}\right)\right|=2$, i.e., $c \neq c^{\prime}$ (and it is possible that $\left.d=d^{\prime}\right)$. Let $L^{\prime}$ be the list assignment of $\left(G^{\prime}, v_{1} v_{2}\right)$ defined as $L^{\prime}\left(v_{1}, v_{2}\right)=\left\{\left(c^{\prime}, d^{\prime}\right)\right\}$ and

$$
L^{\prime}(v)= \begin{cases}L(v)-\{c\}, & \text { if } v \in N_{G}\left(v^{*}\right) \\ L(v), & \text { if } v \in V(G)-N_{G}\left(v^{*}\right)\end{cases}
$$

Then for all $v \in B\left(G^{\prime}\right)-\left\{v_{1}, v_{2}\right\},\left|L^{\prime}(v)\right| \geq 3$. Hence $L^{\prime}$ is a valid list assignment of $\left(G^{\prime}, v_{1} v_{2}\right)$. By induction hypothesis, there is an $L^{\prime}$-colouring $f$ of $\left(G^{\prime}, v_{1} v_{2}\right)$. By letting $f\left(v^{*}\right)=c$, we obtain an $L$-colouring of $\left(G, v_{1} v_{2}\right)$.

This completes the proof of Theorem 2,
It is obvious that Theorem 1 follows from Theorem 2.

## 3 Some Remarks and Questions

For list colouring of planar graphs with list of separation, the following conjecture was propose in [16] and remains open:

Conjecture 3 Every planar graph is (3,1)-choosable.
There are some other restrictions on list assignments are studied in the literature [3, 12, 19]. We say a list assignment $L$ is symmetric if colours in the lists are integers
and for each $v$, for each integer $i, i \in L(v)$ implies that $-i \in L(v)$. A graph $G$ is called weakly $k$-choosable if $G$ is $L$-colourable for any symmetric $k$-list assignment $L$ of $G$. The following conjecture, which is a strengthening of the Four Colour Theorem, was proposed by Kündgen and Ramamurthi [12] and remains open.

Conjecture 4 Every planar graph is weakly 4-choosable.
A $t$-common $k$-list assignment of a graph $G$ is a $k$-list assignment $L$ of $G$ such that $\left|\bigcap_{v \in V(G)} L(v)\right| \geq t$. It was asked by Choi and Kwon [3] whether every planar graph $G$ is $L$-colourable for any 2 -common 4 -list assignment $L$. A positive answer would be a strengthening of the Four Colour Theorem. But Kemnitz and Voigt [8] proved that the answer to this question is negative.

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