### List 4-colouring of planar graphs

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#### Abstract

This paper proves the following result: If G is a planar graph and L is a 4list assignment of G such that  $|L(x) \cap L(y)| \leq 2$  for every edge xy, then G is L-colourable. This answers a question asked by Kratochvíl, Tuza and Voigt in [Journal of Graph Theory, 27(1):43–49, 1998].

Keywords: planar graph; lists with separation; list colouring.

### 1 Introduction

A list assignment of a graph G is a mapping L which assigns to each vertex v of G a set L(v) of permissible colours. An L-colouring of G is a proper colouring f of G such that for each vertex v of G,  $f(v) \in L(v)$ . We say G is L-colourable if G has an L-colouring. A k-list assignment of G is a list assignment L with  $|L(v)| \ge k$  for each vertex v. We say G is k-choosable if G is L-colourable for any k-list assignment L of G. The choice number ch(G) of G is the minimum integer k such that G is k-choosable.

It is known that there are planar graphs G and 4-list assignments L of G such that G is not L-colourable [15]. A natural direction of research is to put restrictions on the list assignments so that for any planar graph G and any 4-list assignment L of G satisfying the restrictions, G is L-colourable. Indeed, the Four Colour Theorem can be formulated as such a result: For any planar graph G, if L is a 4-list assignment of G with L(x) = L(y) for any edge xy of G, then G is L-colourable.

Are there other natural restrictions for which the corresponding "list 4-colouring theorem" is true?

By changing the equality to inequality in the above formulation of the Four Colour Theorem, one may ask the following question:

Is it true that for any planar graph G, any 4-list assignment L of G such that  $L(x) \neq L(y)$  (or equivalently,  $|L(x) \cap L(y)| \leq 3$ ), G is L-colourable?

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The answer is NO. Mirzakhani [13] constructed a planar graph G and a 4-list assignment L of G such that  $|L(x) \cap L(y)| \leq 3$ , and G is not L-colourable.

On the other hand, Kratochvíl, Tuza and Voigt [10] proved that for any planar graph G, for any 4-list assignment L of G such that for any edge xy,  $|L(x) \cap L(y)| \leq 1$ , G is L-colourable. Then they asked the following question:

**Question 1** [10] Is it true that for any planar graph G and any 4-list assignment L of G such that  $|L(x) \cap L(y)| \le 2$  for every edge xy, G is L-colourable?

This question received a lot of attention [1, 2, 4, 5, 6, 7, 9, 11, 16, 17, 18]. Most of the works deal with variations of this problem. There was not much progress on the question itself. In this paper, we answer this question in affirmative.

**Definition 1** Assume G is a graph and k, s are positive integers. A  $(\star, s)$ -list assignment of G is a list assignment L of G such that  $|L(x) \cap L(y)| \leq s$  for each edge xy. A (k, s)-list assignment of G is a  $(\star, s)$ -list assignment of G with  $|L(v)| \geq k$  for each vertex v. A graph G is called (k, s)-choosable if G is L-colourable for any (k, s)-list assignment L of G.

The following is the main result of this paper.

**Theorem 1** Every planar graph is (4, 2)-choosable.

## 2 The proof

It suffices to prove Theorem 1 for 2-connected near-triangulations of the plane: for each non-triangular face, we can add a new vertex adjacent to all vertices of the face, and assign new colours to the added vertex so that the resulting list assignment is still a (4,2)-list assignment. For a 2-connected plane graph G, we denote by B(G) the boundary cycle of G.

**Definition 2** A rooted plane graph is a pair  $(G, v_1v_2)$ , where G is a 2-connected neartriangulation of the plane, and  $v_1v_2$  is a boundary edge.

The vertices  $v_1, v_2$  are called the *root vertices* and  $v_1v_2$  is called the *root edge*.

**Definition 3** Assume  $(G, v_1v_2)$  is a rooted plane graph. Assume v is a non-root boundary vertex, and  $N_G(v) \cap B(G) = \{u_1, u_2, \ldots, u_k\}$   $(k \ge 2)$ , and  $(v, u_1, u_2, \ldots, u_k)$  occur in B(G) is this cyclic order. The vertices  $u_1, \ldots, u_k$  are called the boundary neighbours of v. If the rooted edge is contained in the boundary path from  $u_i$  to  $u_{i+1}$ , then  $u_i$  and  $u_{i+1}$ are called the primary boundary neighbours of v.

Note that each non-root boundary vertex of a rooted plane graph has exactly two primary boundary neighbours. **Definition 4** A list assignment of a rooted plane graph  $(G, v_1v_2)$  is a mapping L which assigns to each vertex  $v \neq v_1, v_2$  a set L(v) of colours, and assigns to the ordered pair  $(v_1, v_2)$  a set  $L(v_1, v_2)$  of ordered pairs of distinct colours. An L-colouring of  $(G, v_1v_2)$  is a proper colouring f of G such that for each  $v \neq v_1, v_2, f(v) \in L(v)$ , and  $(f(v_1), f(v_2)) \in$  $L(v_1, v_2)$ .

Assume L is a list assignment of  $(G, v_1v_2)$ . The list assignment  $\tilde{L}$  of G associated to L is the list assignment of G defined as  $\tilde{L}(v) = L(v)$  for  $v \neq v_1, v_2$  and  $\tilde{L}(v_1) = \{c : \exists d, (c, d) \in L(v_1, v_2)\}$  and  $\tilde{L}(v_2) = \{d : \exists c, (c, d) \in L(v_1, v_2)\}$ . We say L is a  $(\star, 2)$ -list assignment of  $(G, v_1v_2)$  if  $\tilde{L}$  is a  $(\star, 2)$ -list assignment of G.

**Definition 5** Assume L is a  $(\star, 2)$ -list assignment of  $(G, v_1v_2)$ , and  $v \in B(G)$  is a nonroot vertex, and u is a primary boundary neighbour of v. We say u is a good neighbour of v, if one of the following holds:

- $|\tilde{L}(u) \cap \tilde{L}(v)| \leq 1$ , or
- $|\tilde{L}(u)| = 4.$

**Definition 6**  $A(\star,2)$ -list assignment of a rooted plane graph  $(G, v_1v_2)$  is valid if |L(v)| = 4 for each interior vertex v, and one of the following holds:

- (A)  $|L(v_1, v_2)| \ge 1$  and  $|L(v)| \ge 3$  for each non-root boundary vertex v.
- (B)  $|L(v_1, v_2)| \ge 2$ , and there exists a unique non-root boundary vertex  $v^*$  such that  $|L(v)| \ge 3$  for  $v \in B(G) \{v_1, v_2, v^*\}$ ,  $|L(v^*)| = 2$  and  $v^*$  has a good neighbour.

Assume L is a valid list assignment of  $(G, v_1v_2)$ ,  $v^*$  is a non-root boundary vertex with  $|L(v^*)| = 2$  and u is a good neighbour of  $v^*$ . If |L(u)| = 4, then we may delete one colour from  $L(u) \cap L(v^*)$  so that  $|L(u) \cap L(v^*)| \le 1$ . So if L is a valid list assignment of  $(G, v_1v_2)$ , and u is a good neighbour of  $v^*$ , then we assume that  $|\tilde{L}(u) \cap L(v)| \le 1$ . However to prove that u is a good neighbour of  $v^*$ , it suffices to prove that either |L(u)| = 4 or  $|\tilde{L}(u) \cap L(v)| \le 1$ .

**Theorem 2** If L is a valid list assignment of a rooted plane graph  $(G, v_1v_2)$ , then there exists an L-colouring of  $(G, v_1v_2)$ .

**Proof.** The proof is by induction on |V(G)|.

Assume first that G is a triangle  $(v_1, v_2, v_3)$ .

If (A) holds, then  $|L(v_3)| = 3$ . Assume  $L(v_1, v_2) = \{(c_1, c_2)\}$ . Let  $c_3 \in L(v_3) - \{c_1, c_2\}$ . Then  $f(v_i) = c_i$  for i = 1, 2, 3 is an L-colouring of  $(G, v_1 v_2)$ .

Assume (B) holds. Then  $|L(v_3)| = 2$  and  $L(v_1, v_2) = \{(c_1, c_2), (c'_1, c'_2)\}$ . We may assume that  $v_2$  is a good neighbour of  $v_3$ . If  $c_1 = c'_1$ , then  $c_2 \neq c'_2$ . Let  $c_3 \in L(v_3) - \{c_1\}$ . One of  $c_2, c'_2$  is distinct from  $v_3$ . Without loss of generality, we may assume that  $c_2 \neq c_3$ . Then  $f(v_i) = c_i$  for i = 1, 2, 3 is an L-colouring of  $(G, v_1 v_2)$ . The case  $c_2 = c'_2$  is symmetric.

Assume  $c_1 \neq c'_1, c_2 \neq c'_2$ . As  $v_2$  is a good neighbour of  $v_3, |L(v_3) \cap \{c_2, c'_2\}| \leq 1$ . Assume  $c_2 \notin L(v_3)$ . Let  $c_3 \in L(v_3) - \{c_1\}$ . Then  $f(v_i) = c_i$  for i = 1, 2, 3 is an L-colouring of  $(G, v_1v_2)$ .

Assume  $|V(G)| = n \ge 4$  and the theorem is true for any smaller rooted plane graphs.

For a cycle C of G,  $\operatorname{Int}[C]$  is the graph of all vertices and edges inside or on C,  $\operatorname{Ext}[C]$  is the graph of all vertices and edges outside or on C. If G has a separating triangle  $C = (u_1, u_2, u_3)$ , then let  $G_1 = \operatorname{Ext}[C]$ . Then  $(G_1, v_1v_2)$  has an L-colouring f. Let  $G_2 = \operatorname{Int}[C] - \{u_3\}$ . Let L' be the list assignment of  $(G_2, u_1u_2)$  defined as  $L'(u_1, u_2) = \{(f(u_1), f(u_2))\}$ , and for  $v \in V(G_2) - \{u_1, u_2\}$ ,

$$L'(v) = \begin{cases} L(v) - \{f(u_3)\}, & \text{if } v \in N_G(u_2), \\ L(v), & \text{otherwise.} \end{cases}$$

Then L' is a valid list assignment of  $(G_2, u_1u_2)$ . By induction hypothesis, there is an L'-colouring g of  $(G_2, u_1u_2)$ . The union of f and g is an L-colouring of  $(G, v_1v_2)$ .

In the following, we assume that G has no separating triangle.

Case 1 B(G) has a chord xy.

Let  $G_1, G_2$  be the two subgraphs of G separated by xy, (i.e.,  $G_1, G_2$  are connected induced subgraphs of G with  $V(G_1) \cap V(G_2) = \{x, y\}$  and  $V(G_1) \cup V(G_2) = V(G)$ ), and assume  $G_1$  contains the root edge  $v_1v_2$ .

**Case 1(i)** There is a chord xy such that |L(v)| = 3 for all  $v \in B(G_2) - \{x, y\}$ .

Let  $L_1$  be the restriction of L to  $(G_1, v_1v_2)$ . Then  $L_1$  is a valid list assignment of  $(G_1, v_1v_2)$ . By induction hypothesis, there exists an  $L_1$ -colouring f of  $(G_1, v_1v_2)$ .

Let  $L_2$  be the list assignment of  $(G_2, xy)$  defined as  $L_2(x, y) = \{(f(x), f(y))\}$  and  $L_2(v) = L(v)$  for  $v \in V(G_2) - \{x, y\}$ . Then  $L_2$  is a valid list assignment of  $(G_2, xy)$ . By induction hypothesis, there exists an  $L_2$ -colouring g of  $(G_2, xy)$ . The union of f and g is an L-colouring of  $(G, v_1v_2)$ .

**Case 1(ii)** There is a vertex  $v^* \in B(G) - \{v_1, v_2\}$  with  $|L(v^*)| = 2$ , and every chord xy separates  $v^*$  and the root edge, i.e.,  $v_1v_2 \in E(G_1)$  and  $v^* \in V(G_2) - \{x, y\}$ .

We choose the chord xy so that  $G_1$  is minimum. Then  $B(G_1)$  has no chord.

As there is a vertex  $v^* \in B(G) - \{v_1, v_2\}$  with  $|L(v^*)| = 2$ , we know that  $(G, v_1v_2)$  satisfies (B). We may assume that  $|L_1(v_1, v_2)| = 2$ .

Similarly,  $L_1$  is a valid list assignment of  $(G_1, v_1v_2)$  and hence there is an  $L_1$ -colouring f of  $(G_1, v_1v_2)$ .

Claim 1 There is another L-colouring f' of  $(G_1, v_1v_2)$  for which  $(f'(x), f'(y)) \neq (f(x), f(y))$ .

Assume Claim 1 is true. Let  $L_2$  be the list assignment of  $(G_2, xy)$  defined as  $L_2(x, y) = \{(f(x), f(y)), (f'(x), f'(y))\}$  and  $L_2(v) = L(v)$  for  $v \in V(G_2) - \{x, y\}$ . Note that  $\tilde{L}_2(v) \subseteq L(v)$  for  $v \in \{x, y\}$ , and the primary neighbours of  $v^*$  in  $(G_2, xy)$  are the same as its primary neighbours in  $(G, v_1v_2)$ . So  $v^*$  has a good neighbour in  $(G_2, xy)$ . Thus  $L_2$  is a valid list assignment of  $(G_2, xy)$ .

By induction hypothesis, there exists an  $L_2$ -colouring g of  $(G_2, xy)$ . Depending on (g(x), g(y)) = (f(x), f(y)) or (f'(x), f'(y)), the union of g and f or the union of g and f' is an L-colouring of  $(G, v_1v_2)$ . To finish the proof of Case 1, it remains to prove Claim 1.

#### Proof of Claim 1

Without loss of generality, we assume that  $y \notin \{v_1, v_2\}$ . Let  $L'_1 = L_1$ , except that  $L'_1(y) = L(y) - \{f(y)\}$ . If  $L'_1$  is a valid list assignment of  $(G_1, v_1v_2)$ , then by induction hypothesis, there is an  $L'_1$ -colouring f' of  $(G_1, v_1v_2)$ , and we are done.

Thus we may assume that  $L'_1$  is not a valid list assignment of  $(G_1, v_1v_2)$ . This happens only if  $|L'_1(y)| = 2$  and y has no good neighbour in  $(G_1, v_1v_2)$ . Assume  $L'_1(y) = \{c_1, c_2\}$ .

As  $B(G_1)$  has no chord, y has exactly two boundary neighbours, and one of them is x. Let y' be the other boundary neighbour of y, i.e.,  $N_G(y) \cap B(G_1) = \{x, y'\}$ . Then

$$\{c_1, c_2\} \subseteq \tilde{L}'_1(x) \cap \tilde{L}'_1(y')$$

If x is a root vertex, say  $x = v_1$ , then  $\tilde{L}_1(x) = \{c_1, c_2\}$  (as  $L'_1(y) \subseteq \tilde{L}_1(x)$  and  $\tilde{L}_1(x) | \leq 2$ ). Assume  $L(v_1, v_2) = \{(c_1, c'_1), (c_2, c'_2)\}$  for some colours  $c'_1, c'_2$  (possibly  $c'_1 = c'_2$ ). Assume  $c_1 \neq f(x)$ .

Let  $L''_1 = L_1$ , except that  $L''_1(v_1, v_2) = (c_1, c'_1)$ . As  $|L''_1(v)| \ge 3$  for all  $v \in B(G_1) - \{v_1, v_2\}$ ,  $L''_1$  is a valid list assignment of  $(G_1, v_1v_2)$ . Hence there is an  $L''_1$ -colouring f' of  $(G_1, v_1v_2)$ . As  $f'(x) \ne f(x)$ , Claim 1 is proved.

Thus we may assume that x is not a root vertex.

Let  $L_1'' = L_1$  except that  $L_1''(x) = L_1(x) - \{f(x)\}$ . If  $L_1''$  is a valid list assignment of  $(G_1, v_1v_2)$ , then again we obtain an L-colouring f' of  $(G_1, v_1v_2)$  with  $(f'(x), f'(y)) \neq (f(x), f(y))$  and we are done.

Thus assume that  $L''_1$  is not a valid list assignment. This means that |L''(x)| = 2 and x has no good neighbour. Let x' be the other boundary neighbour of x. Then we have  $L''_1(x) = \{c_1, c_2\}$  (so  $f(x) \neq c_1, c_2$ ), and

$$\{c_1, c_2\} = L(x) \cap L(y) \cap \tilde{L}_1(x') \cap \tilde{L}_1(y').$$

As G is a near-triangulation of the plane and G has no separating triangle, x and y have a unique common neighbour z in  $G_1$ , which is an interior vertex of  $G_1$ .

Since  $B(G_1)$  has no chord, it is easy see that at least one of the following holds:

- $N_{G_1}(x') \cap N_{G_1}(y) \{z\} = \emptyset.$
- $N_{G_1}(y') \cap N_{G_1}(x) \{z\} = \emptyset.$

By symmetry, we may assume that  $N_{G_1}(x') \cap N_{G_1}(y) - \{z\} = \emptyset$ .

As  $|L(z) \cap L(y)| \le 2$ , there exists  $i \in \{1, 2\}$ , that  $|L(z) \cap \{f(y), c_i\}| \le 1$ . Without loss of generality, we assume

$$|L(z) \cap \{f(y), c_1\}| \le 1$$

Let

$$G_1' = G_1 - \{x, y\}.$$

Let  $L_1^*$  be the list assignment of  $(G'_1, v_1v_2)$  defined as follows:

$$L_1^*(v) = \begin{cases} L(v) - \{c_1, f(y)\}, & \text{if } v = z, \\ L(v) - \{f(y)\}, & \text{if } v \in N_{G_1}(y) - \{z\}, \\ L(v) - \{c_1\}, & \text{if } v \in N_{G_1}(x) - \{z\}, \\ L(v), & \text{otherwise.} \end{cases}$$

and

$$L_1^*(v_1, v_2) = \begin{cases} L(v_1, v_2), & \text{if } x' \text{ is not a root vertex, or } c_1 \notin \tilde{L}(x'), \\ L(v_1, v_2) - \{(c_1, c_1')\}, & \text{if } x' = v_1 \text{ and } (c_1, c_1') \in L(v_1, v_2). \end{cases}$$

Note that  $|L_1^*(z)| \ge 3$ , and  $L_1^*(y') = L(y')$  (as  $f(y) \notin L(y')$ ). If  $L_1^*$  is a valid list assignment of  $(G'_1, v_1v_2)$ , then there is an  $L_1^*$ -colouring f' of  $(G'_1, v_1v_2)$ . By letting  $(f'(x), f'(y)) = (c_1, f(y))$ , we obtain an L-colouring of  $(G_1, v_1v_2)$  with  $(f'(x), f'(y)) \neq (f(x), f(y))$ , and we are done.

Thus we assume that  $L_1^*$  is not a valid list assignment of  $(G'_1, v_1v_2)$ . This means that

• x' is not a root vertex, x' is the only boundary vertex of  $G'_1$  with  $|L_1^*(x')| = 2$ , and x' has no good neighbour.

Assume  $L(x') = \{c_1, c_2, c_3\}$  (and hence  $L_1^*(x') = \{c_2, c_3\}$ ).

Let z' be the unique common neighbour of x and x', which is an interior vertex of  $G_1$ . Let x'' be the other neighbour of x' in  $B(G_1)$ . Then x'' is a primary boundary neighbour of x' in  $G'_1$ . Since  $N_{G_1}(x') \cap N_{G_1}(y) - \{z\} = \emptyset$  and G has no separating triangle, z' is the other primary boundary neighbour of x'.

Since z' is not a good neighbour of x', we conclude that

- z' = z;
- $c_1 \notin L(z), c_2, c_3, f(y) \in L(z)$  and  $c_3 \neq f(y)$ .

Now z' = z implies that  $N_{G_1}(y') \cap N_{G_1}(x) - \{z\} = \emptyset$ . So we can repeat the same argument as above, but interchange the roles of x, x' and y, y'. Then we conclude that the following hold:

- y' is not a root vertex,
- z is adjacent to y',
- $L(y') = \{c_1, c_2, c'_3\},$

•  $c_2, c'_3, f(x) \in L(z).$ 

As  $\{c_2, c_3, c'_3, f(x), f(y)\} \subseteq L(z_1)$ , we have  $c_3 = c'_3$  (as |L(z)| = 4 and the other colours are pairwise distinct). I.e.,

$$L(z) = \{c_2, c_3, f(x), f(y)\}.$$

As  $L_1^*(x') = \{c_2, c_3\} \subseteq \tilde{L}(x'')$ , we know that  $c_1 \notin \tilde{L}(x'')$ . Let

$$G_1'' = G_1 - \{x', x, y\}.$$

Let  $L_1^{**}$  be the list assignment of  $(G_1'', v_1v_2)$  defined as follows:

$$L_{1}^{**}(v) = \begin{cases} L(v) - \{c_{1}, c_{2}, f(y)\}, & \text{if } v = z, \\ L(v) - \{f(y)\}, & \text{if } v \in N_{G_{1}}(y), \\ L(v) - \{c_{2}\}, & \text{if } v \in N_{G_{1}}(x), \\ L(v) - \{c_{1}\}, & \text{if } v \in N_{G_{1}}(x'), \\ L(v), & \text{otherwise.} \end{cases}$$

and

$$L_1^{**}(v_1, v_2) = \begin{cases} L(v_1, v_2), & \text{if } x'' \text{ is not a root vertex, or } c_1 \notin \tilde{L}(x''), \\ L(v_1, v_2) - \{(c_1, c_1')\}, & \text{if } x'' = v_1 \text{ and } (c_1, c_1') \in L(v_1, v_2). \end{cases}$$

If  $L_1^{**}$  is a valid list assignment of  $(G_1'', v_1v_2)$ , then there is an  $L_1^{**}$ -colouring f' of  $(G_1'', v_1v_2)$ , which extends to an *L*-colouring of  $(G_1, v_1v_2)$  by letting  $f'(x') = c_1, f'(x) = c_2$  and f'(y) = f(y), and we are done.

Thus we may assume that  $L_1^{**}$  is not a valid list assignment of  $(G_1'', v_1v_2)$ . It is easy to check that z is the only vertex of  $B(G_1'') - \{v_1, v_2\}$  with  $|L^{**}(z)| < 3$ . Note that

$$L^{**}(z) = L(z) - \{c_2, f(y)\} = \{c_3, f(x)\}.$$

The only reason that  $L_1^{**}$  is not a valid list assignment of  $(G_1'', v_1v_2)$  is that z has no good neighbour. Let  $w_1, w_2$  be the two primary boundary neighbours of z in  $(G_1'', v_1v_2)$ . We have

$$\{c_3, f(x)\} \subseteq L(w_1), L(w_2).$$

This implies that y' is not a primary boundary neighbour of z in  $(G''_1, v_1v_2)$  (although y' is a boundary neighbour of z in  $G''_1$ ).

We repeat the above argument, but interchange the roles of x, x' and y, y'. We conclude that for the two primary boundary neighbours  $w'_1, w'_2$  of z in  $(G_1 - \{x, y, y'\}, v_1v_2)$ ,

$$\{c_3, f(y)\} \subseteq \tilde{L}(w_1'), \tilde{L}(w_2').$$

This means that x' is not a primary neighbour of z in  $(G_1 - \{x, y, y'\}, v_1v_2)$ . But then the primary neighbours of z in  $(G_1 - \{x, y, y'\}, v_1v_2)$  and  $(G_1 - \{x', x, y\}, v_1v_2)$  are the same. I.e.,  $w'_1 = w_1$  and  $w'_2 = w_2$ . But then for i = 1, 2,

$$\{c_3, f(x), f(y)\} \subseteq \tilde{L}(w_i) \cap L(z),$$

contrary to the assumption that L is  $(\star, 2)$ -list assignment of  $(G, v_1v_2)$ . This completes the proof of Claim 1, and hence the proof of Case 1.

Case 2 B(G) has no chord.

Case 2(i) (A) holds, and  $L(v_1, v_2) = \{(c_1, c_2)\}.$ 

Let u be the other boundary neighbour of  $v_2$  in G. Similarly, as G has no separating triangle and B(G) has no chord,  $v_1$  and  $v_2$  has a unique common neighbour w, and u and  $v_2$  have a unique common neighbour z, and w, z are interior vertices of G (and possibly w = z).

Let  $G' = G - v_2$  and let L' be the list assignment of  $(G', v_1w)$  defined as

$$L'(v) = \begin{cases} L(v) - \{c_2\}, & \text{if } v \in N_G(v_2) - \{v_1, w\}, \\ L(v), & \text{if } v \in V(G) - N_G(v_2), \end{cases}$$

and

$$L'(v_1, w) = \{(c_1, c_3), (c_1, c_4)\}, \text{ where } c_3, c_4 \in L(w) - \{c_1, c_2\}.$$

In the definition above, if  $|L(w) - \{c_1, c_2\}| \ge 3$ , then  $c_3, c_4$  are arbitrarily chosen from  $L(w) - \{c_1, c_2\}$ , with one exception:

If  $c_2 \notin L(w)$ , w = z and  $L(w) \cap L(u) \neq \emptyset$ , then let  $c' \in L(w) \cap L(u)$ , and we choose  $c_3, c_4 \in L(w) - \{c_1, c'\}$ .

We shall show that L' is valid list assignment of  $(G', v_1w)$ .

If  $c_2 \notin L(u)$ , then |L'(u)| = |L(u)| = 3, and (A) holds for L' and  $(G', v_1w)$ . So L' is a valid list assignment of  $(G', v_1w)$ .

Assume  $c_2 \in L(u)$  and hence |L'(u)| = 2. If  $z \neq w$ , then either  $c_2 \in L(z)$  and hence  $|L'(z) \cap L'(u)| \leq 1$  or |L'(z)| = 4. So z is a good neighbour of u in  $(G', v_1w)$ , and L' is a valid list assignment of  $(G', v_1w)$  ((B) holds for L' and  $(G', v_1w)$ ).

If z = w, then by our choice of  $c_3, c_4$ , we know that  $|L'(w) \cap L'(u)| \leq 1$ , and hence w is a good neighbour of u, and L' is a valid list assignment of  $(G', v_1w)$  ((B) holds for L' and  $(G', v_1w)$ ).

By induction hypothesis,  $(G', v_1w)$  has an L'-colouring f. By letting  $f(v_2) = c_2$ , we obtain an L-colouring of  $(G, v_1v_2)$ .

**Case 2(i)** (B) holds, and  $v^* \in B(G)$ ,  $|L(v^*)| = 2$ , and u is a good neighbour of  $v^*$ .

It may happen that  $v^*$  has two good neighbours. In this case, the good neighbour u is usually arbitrarily chosen, unless  $v^*$  is adjacent to a root vertex  $v_i$  for some  $i \in \{1, 2\}$  and  $|\tilde{L}(v_i)| = 1$ . In this case, we let  $u = v_i$ .

Let w be the other boundary neighbour of  $v^*$ , and let z be the common neighbours of  $v^*$  and w. Similarly, we know that the vertex z is unique and is an interior vertex of G. By our choice of u, we know that either  $w \neq v_1, v_2$ , or  $w = v_i$  for some  $i \in \{1, 2\}$  and

 $|\tilde{L}(v_i)| = 2$  (for otherwise, we would have chosen w as the good neighbour of  $v^*$ ). Let

$$G' = G - \{v^*\}, \ c \in L(v^*) - L(u).$$

If w is not a root vertex, then let L' be the list assignment of  $(G', v_1v_2)$  defined as  $L'(v_1, v_2) = L(v_1, v_2)$ , and for  $v \in V(G') - \{v_1, v_2\}$ ,

$$L'(v) = \begin{cases} L(v) - \{c\}, & \text{if } v \in N_G(v^*), \\ L(v), & \text{if } v \in V(G) - N_G(v^*). \end{cases}$$

If  $|L'(w)| \ge 3$ , then  $|L'(v)| \ge 3$  for every  $v \in B(G') - \{v_1, v_2\}$  and hence L' is a valid list assignment of  $(G', v_1v_2)$ . Otherwise, w is the unique boundary vertex of G' with |L'(w)| = 2. Observe that either  $c \in L(z)$  and hence  $|L'(z) \cap L'(w)| \le 1$ , or  $|L'(z_1)| =$  $|L(z_1)| = 4$ . In any case, z is a good neighbour of w, and hence L' is a valid list assignment of  $(G, v_1v_2)$ . By induction hypothesis, there is an L'-colouring f of  $(G', v_1v_2)$ . By letting  $f(v^*) = c$ , we obtain an L-colouring of  $(G', v_1v_2)$ .

Assume w is a root vertex, say  $w = v_1$ . If  $c \notin \tilde{L}(v_1)$ , then the argument still works. Assume  $c \in \tilde{L}(v_1)$ . Without loss of generality, we may assume that  $|L(v_1, v_2)| = 2$ , say  $L(v_1, v_2) = \{(c, d), (c', d')\}$ . As observed above,  $|\tilde{L}(v_1)| = 2$ , i.e.,  $c \neq c'$  (and it is possible that d = d'). Let L' be the list assignment of  $(G', v_1v_2)$  defined as  $L'(v_1, v_2) = \{(c', d')\}$  and

$$L'(v) = \begin{cases} L(v) - \{c\}, & \text{if } v \in N_G(v^*), \\ L(v), & \text{if } v \in V(G) - N_G(v^*). \end{cases}$$

Then for all  $v \in B(G') - \{v_1, v_2\}, |L'(v)| \ge 3$ . Hence L' is a valid list assignment of  $(G', v_1v_2)$ . By induction hypothesis, there is an L'-colouring f of  $(G', v_1v_2)$ . By letting  $f(v^*) = c$ , we obtain an L-colouring of  $(G, v_1v_2)$ .

This completes the proof of Theorem 2.  $\blacksquare$ 

It is obvious that Theorem 1 follows from Theorem 2.

## **3** Some Remarks and Questions

For list colouring of planar graphs with list of separation, the following conjecture was propose in [16] and remains open:

#### **Conjecture 3** Every planar graph is (3,1)-choosable.

There are some other restrictions on list assignments are studied in the literature [3, 12, 19]. We say a list assignment L is *symmetric* if colours in the lists are integers

and for each v, for each integer  $i, i \in L(v)$  implies that  $-i \in L(v)$ . A graph G is called weakly k-choosable if G is L-colourable for any symmetric k-list assignment L of G. The following conjecture, which is a strengthening of the Four Colour Theorem, was proposed by Kündgen and Ramamurthi [12] and remains open.

Conjecture 4 Every planar graph is weakly 4-choosable.

A t-common k-list assignment of a graph G is a k-list assignment L of G such that  $|\bigcap_{v \in V(G)} L(v)| \ge t$ . It was asked by Choi and Kwon [3] whether every planar graph G is L-colourable for any 2-common 4-list assignment L. A positive answer would be a strengthening of the Four Colour Theorem. But Kemnitz and Voigt [8] proved that the answer to this question is negative.

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