Octopuses in the Boolean cube: families with pairwise small intersections, part I

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Abstract

Let $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ be families of subsets of $\{1, \dots, n\}$. Suppose that for distinct k, k' and arbitrary $F_1 \in \mathcal{F}_k, F_2 \in \mathcal{F}_{k'}$ we have $|F_1 \cap F_2| \leqslant m$. What is the maximal value of $|\mathcal{F}_1| \dots |\mathcal{F}_\ell|$? In this work we find the asymptotic of this product as n tends to infinity for constant ℓ and m.

This question is related to a conjecture of Bohn et al. that arose in the 2-level polytope theory and asked for the largest product of the number of facets and vertices in a two-level polytope. This conjecture was recently resolved by Weltge and the first author.

The main result can be rephrased in terms of colorings. We give an asymptotic answer to the following question. Given an edge coloring of a complete m-uniform hypergraph into ℓ colors, what is the maximum of $\prod M_i$, where M_i is the number of monochromatic cliques in i-th color?

1 Introduction

A polytope $P \subset \mathbb{R}^d$ is called *2-level* if for each facet F there are two parallel hyperplanes H, H' such that $F \subset H$ and all vertices of P are contained in $H \cup H'$ Several standard polytope families are 2-level, e.g. hypercubes, cross-polytopes, simplices. The class of 2-level polytopes includes a number of important polytopal families like Hanner polytopes, Birkhoff polytopes, the Hansen polytopes and others [6]. These polytopes arise in such areas of mathematics as the semidefinite programming, communication complexity and polyhedral combinatorics.

A number of authors studied combinatorial structure of 2-level polytopes [12, 8, 6, 3]. Bohn et al. [6] suggested a beautiful conjecture on the tradeoff between the number of vertices $f_0(P)$ and the number of $d-1$-dimensional facets $f_{d-1}(P)$ of a 2-level polytope $P \subset \mathbb{R}^d$. Concretely, they asked if it is true that $f_0(P) f_{d-1}(P) \leqslant d2^{d+1}$ for all d. This bound is sharp for cubes and cross-polytopes. Recently, Kupavskii and Weltge answered this question in the positive [3]. Actually, they proved the following variation of the conjecture of Bohn et al., from which it is easy to deduce the original conjecture. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ let $\langle \mathbf{a}, \mathbf{b} \rangle$ stand for their scalar product.

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Theorem 1 ([3])**.** *Let $\mathcal{A}$, $\mathcal{B}$ be families of vectors in $\mathbb{R}^n$ that both linearly span $\mathbb{R}^n$. Suppose that $\langle \mathbf{a}, \mathbf{b}\rangle \in \{0,1\}$ holds for all $\mathbf{a} \in \mathcal{A}$, $\mathbf{b} \in \mathcal{B}$. Then we have $|\mathcal{A}| \cdot |\mathcal{B}| \leqslant (n+1)2^n$.*

The bound in the theorem is tight since one can take $\mathcal{A} = \{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\mathcal{B} = \{0,1\}^n$.

Some of the previous works dealt with the particular case of Theorem 1 when $\mathcal{A}, \mathcal{B} \subset \{0,1\}^n$. The problem is then much simpler. Actually, we will present a very short and elegant argument due to Peter Frankl that proves Theorem 1 in the $\{0,1\}$ case.

In this work, we provide the generalization of Theorem 1 on $\{0,1\}^n$ to several families. Compared to the two families case for $\{0,1\}^n$, this problem becomes much more challenging, and it seems almost hopeless to determine the exact extremal function. The proofs involve some interesting ingredients, such as correlation inequalities for several families. We will say more on these points after we introduce the necessary notation and formulate the main result. In what follows, we will work with families of sets instead of families of $\{0,1\}$-vectors.

1.1 Notation

Put $[n] = \{1, \dots, n\}$, and, more generally, $[a,b] = \{a, a+1, \dots, b\}$ for positive integers a, b, n. Given a set X, we denote by 2^X the set of all subsets of X. We denote by $\binom{X}{k}$ ($\binom{X}{\leqslant k}$) the family of all subsets of X of cardinality k (at most k). We also denote $\binom{n}{\leqslant m} := \left|\binom{[n]}{\leqslant m}\right| = \sum_{t=0}^{m} \binom{n}{t}$.

In this paper, we study families of sets with the "$\mathbf{m}$-*overlapping property*", which is defined below.

Definition 1. *Fix a positive integer ℓ. Let $\mathbf{m} = (\mathbf{m}_S)_{S \in \binom{[\ell]}{2}}$ be a vector of non-negative integers indexed by unordered pairs $\{k, k'\} \in \binom{[\ell]}{2}$. For simplicity we suppress brackets in $\mathbf{m}_{\{k,k'\}}$ and assume that $\mathbf{m}_{k,k'}$, $\mathbf{m}_{k',k}$, and $\mathbf{m}_{\{k,k'\}}$ identify the same entry. Families $\mathcal{F}_1, \dots, \mathcal{F}_\ell \subset 2^{[n]}$* satisfy an $\mathbf{m}$-overlapping property *if for any distinct $k_1, k_2 \in [\ell]$ and any sets $F_1 \in \mathcal{F}_{k_1}$, $F_2 \in \mathcal{F}_{k_2}$ we have*

$$|F_1 \cap F_2| \leqslant \mathbf{m}_{k_1,k_2}.$$

*If $\mathbf{m}_{k_1,k_2} = m$ for all pairs k_1, k_2 then the property is referred to as m-*overlapping*, and* overlappling *if additionally $m = 1$.*

1.2 Problem statement and results

In this work, we address the following problem.

Problem 1. *Let n, ℓ be positive integers, $\mathbf{m}$ be a vector of $\binom{\ell}{2}$ non-negative integers and $\mathcal{F}_1, \dots, \mathcal{F}_\ell \subset 2^{[n]}$ be families with the $\mathbf{m}$-overlapping property. What is the maximal value $s^*(n, \ell, \mathbf{m})$ of the product $|\mathcal{F}_1| \cdot \ldots \cdot |\mathcal{F}_\ell|$?*

If all coordinates of $\mathbf{m}$ are equal to m, we denote $s^*(n, \ell, m) := s^*(n, \ell, \mathbf{m})$.

It is easy to see that $s^*(n, \ell, 0) = 2^n$: indeed, supports of sets in distinct families are disjoint. Recently, Aprile, Cevallos, and Faenza [8] showed that $s^*(n, 2, 1) = (n+1)2^n$. In personal communication, Peter Frankl [14] gave a simple and elegant proof that $s^*(n, 2, t) = 2^n \sum_{i=0}^{t} \binom{n}{i}$ using Harris–Kleitman correlation inequality (we present his proof in Theorem 6). In [9], Ryser studied a similar question for one family. In particular, he showed that if for $n \notin \{9, 10\}$ n sets of size at least 3 intersect each other in at most 1 element, then they form either a finite projective plane or a symmetric group divisible design.

The main result of this paper is the following theorem:

Theorem 2. *Let ℓ, be positive integers and let $\mathbf{m}$ be a vector of integers as above. Then, as $n \to \infty$, we have the following.*

$$s^*(n, \ell, \mathbf{m}) = \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \cdot 2^n \cdot \prod_{S \in \binom{[\ell]}{2}} \left(\frac{1}{\mathbf{m}_S!}\left(\frac{\mathbf{m}_S \cdot n}{\sum_{S' \in \binom{[\ell]}{2}} \mathbf{m}_{S'}}\right)^{\mathbf{m}_S}\right). \tag{1}$$

Unlike in the case $\ell = 2$, it seems extremely challenging to determine the exact behaviour of $s^*(n, \ell, \mathbf{m})$ for general ℓ. In the follow-up paper [13], we improve the precision of the asymptotic from $O(n^{-1/2})$ to $O(n^{-1})$. More importantly, we will show that all extremal examples must be superfamilies of a certain tuple of families that deliver lower bound in Theorem 2. However, even coming up with the right extremal construction for general ℓ seems to be very difficult.

From the theorem above we immediately derive a cleaner formula for the asymptotic of $s^*(n, \ell, m)$.

Corollary 3. *Suppose ℓ and m are fixed integers. Then, as $n \to \infty$, we have the following.*

$$s^*(n, \ell, m) = \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \left[\frac{1}{m!}\left(\frac{n}{\binom{\ell}{2}}\right)^m\right]^{\binom{\ell}{2}} 2^n.$$

There is an equivalent formulation of Problem 1 for $\mathbf{m} = (m, m, \dots, m)$.

Problem 2. *Let n, ℓ, m be integers and H be a complete $(m+1)$-uniform hypergraph on n vertices. Take some coloring of edges of H into ℓ colours. Let k_i, $i = 1, \dots, \ell$ be the number of monochromatic cliques of colour i in H. What is the maximum value $\tilde{s}(n, \ell, m)$ of $k_1 \cdot \ldots \cdot k_\ell$ over all possible colorings? (We assume that each of the sets of size $\leqslant m-1$ forms a monochromatic clique in each color.)*

In particular, $\tilde{s}(n, 2, 1)$ is the maximum of the product of the number of cliques and the number of independent sets in a graph G on n vertices. In [8] it was shown that Problem 1 and Problem 2 for are equivalent for $m = 1$. Generally, the following holds.

Proposition 4. *Let n, ℓ be integers, then $s^*(n, \ell, m) = \tilde{s}(n, \ell, m)$.*

The rest of the paper is organised as follows. In Section 2 we list some tools that we use and give Peter Frankl's proof that determines $s^*(n, 2, m)$. We prove Proposition 4 and discuss related questions in Section 3. In Section 4 we give the sketch of the proof of Theorem 2. In Section 5 we prove the lower bound in (1). In Section 6 we prove the upper bound.

In what follows, the standard asymptotic notation such as $f = o(g)$, $f = \Omega(g)$ etc. for some functions f, g is always with respect to $n \to \infty$.

2 Tools

There is a trivial bijection between $2^{[n]}$ and the Boolean cube $\{0,1\}^n$. Given a set X, we consider its characteristic vector $\mathbf{c}^X$ whose i-th coordinate $\mathbf{c}_i^X$ equals 0 if $i \notin X$ and 1 otherwise. We will take these two equivalent points of view on sets, families, etc. interchangeably.

2.1 Correlation inequalities

Given a probability measure on the Boolean cube, we can consider a family of subsets as an event. For example, consider a uniform measure on the Boolean cube $\{0,1\}^n$ and some family of sets $\mathcal{F} \subset 2^{[n]}$. For a random set X sampled from the uniform measure, the probability of an event $\{X \in \mathcal{F}\}$ is $\frac{|\mathcal{F}|}{2^n}$.

This point of view provides us with a range of tools that we use throughout this work. In this subsection, we discuss correlation inequalities which are a powerful tool in Combinatorics and Extremal Set Theory. Alon and Spencer in their book [10] write that the first appearance of correlation inequalities can probably be attributed to Harris [2] and Kleitman [7].

We say that a family of subsets is *down-closed (a downset)* if $A \in \mathcal{F}$ and $B \subset A$ implies $B \in \mathcal{F}$. Harris–Kleitman correlation inequality is as follows.

Theorem 5 (Harris–Kleitman correlation inequality)**.** *Let $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ be down-closed. Then*

$$|\mathcal{A}||\mathcal{B}| \leqslant 2^n |\mathcal{A} \cap \mathcal{B}|.$$

If we reformulate the statement in the following way:

$$\frac{|\mathcal{A}|}{2^n} \cdot \frac{|\mathcal{B}|}{2^n} \leqslant \frac{|\mathcal{A} \cap \mathcal{B}|}{2^n},$$

we see that it states that the events $\{X \in \mathcal{A}\}$ and $\{X \in \mathcal{B}\}$ are positively correlated for X that is uniformly distributed over the Boolean cube.

Somewhat surprisingly, this inequality alone can be used to solve our problem in the case of two families. This solution was given by Peter Frankl in a private conversation. We provide it here.

Theorem 6 (Frankl [14])**.** *Let $\mathcal{A} \subset 2^{[n]}$ and $\mathcal{B} \subset 2^{[n]}$ be such families that for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$ it holds that $|A \cap B| \leqslant m$. Then*

$$|\mathcal{A}||\mathcal{B}| \leqslant 2^n \sum_{t=0}^{m} \binom{n}{t}.$$

It is not difficult to see that $\mathcal{A} = 2^{[n]}$ and $\mathcal{B} = \binom{[n]}{\leqslant m}$ satisfy the conditions of the theorem and attain equality in the inequality above.

Proof. Consider families $\mathcal{A}$ and $\mathcal{B}$ that maximize $|\mathcal{A}||\mathcal{B}|$. They are down-closed, otherwise consider their down-closures $\mathcal{A}^{\downarrow}, \mathcal{B}^{\downarrow}$, where

$$\mathcal{F}^{\downarrow} = \{F \subset [n] : \exists F' \in \mathcal{F}\ F \subset F'\}. \tag{2}$$

Clearly, $\mathcal{A}^{\downarrow}, \mathcal{B}^{\downarrow}$ satisfy the m-overlapping property as well, which by maximality implies $\mathcal{A} = \mathcal{A}^{\downarrow}, \mathcal{B} = \mathcal{B}^{\downarrow}$.

Since $\mathcal{A}$ and $\mathcal{B}$ is down-closed, we can apply Theorem 5. Then

$$|\mathcal{A}||\mathcal{B}| \leqslant 2^n |\mathcal{A} \cap \mathcal{B}|.$$

But $\mathcal{A} \cap \mathcal{B}$ can consist only of sets of cardinality at most m. Thus, $\mathcal{A} \cap \mathcal{B} \subset \binom{[n]}{\leqslant m}$ and

$$|\mathcal{A}||\mathcal{B}| \leqslant 2^n \sum_{t=0}^{m} \binom{n}{t}. \qquad \square$$

Note that if $m = 1$ then Theorem 6 implies the bound of Aprile et al [8] and is a special case of Kupavskii and Weltge's result [3].

Meanwhile, Theorem 5 is not sufficient to resolve Problem 1 already for $\ell = 3$. There are several correlation inequalities that generalize Harris–Kleitman correlation inequality. One is Daykin's inequality [1]. Before we present it, we introduce some extra notation. Given vectors $\mathbf{x}$ and $\mathbf{y}$ from $\mathbb{R}^n$ we define vectors $\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \vee \mathbf{y}$ such that $(\mathbf{x} \vee \mathbf{y})_j = \max(\mathbf{x}_j, \mathbf{y}_j)$ and $(\mathbf{x} \wedge \mathbf{y})_j = \min(\mathbf{x}_j, \mathbf{y}_j)$.

It is easy to see that $\vee$ and $\wedge$ restricted on the Boolean cube relate to union and intersection of sets respectively. For two families $\mathcal{F}_1$ and $\mathcal{F}_2$ we denote by $\mathcal{F}_1 \wedge \mathcal{F}_2$, $\mathcal{F}_1 \vee \mathcal{F}_2$ the family of pairwise intersections and pairwise unions of sets from $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively:

$$\mathcal{F}_1 \wedge \mathcal{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathcal{F}_1,\ F_2 \in \mathcal{F}_2\},$$

$$\mathcal{F}_1 \vee \mathcal{F}_2 = \{F_1 \cup F_2 : F_1 \in \mathcal{F}_1,\ F_2 \in \mathcal{F}_2\}.$$

Note that for down-closed families $\mathcal{A}$ and $\mathcal{B}$ we have $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \cap \mathcal{B}$.

Then Daykin's correlation inequality states the following.

Theorem 7 (Daykin correlation inequality)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two families of sets. Then*

$$|\mathcal{A}||\mathcal{B}| \leqslant |\mathcal{A} \vee \mathcal{B}||\mathcal{A} \wedge \mathcal{B}|.$$

Actually, Daykin's inequality works in a more general setting which we omit. Fortuin, Kasteleyn and Ginibre proposed another generalization of the Harris–Kleitman inequality in [5] to a wide class of log-supermodular measures. We will refer to such measures as the *FKG-measures.*

Definition 2 (FKG-measure)**.** *A σ-finite (nonnegative) measure μ on $\mathbb{R}^n$ is said to be an FKG measure if μ has a density function φ with respect to some product measure $d\sigma$ on $\mathbb{R}^k$, (that is, $d\sigma(\mathbf{x}) = \prod_{j=1}^{k} d\sigma(\mathbf{x}_j)$, and $d\mu(\mathbf{x}) = \varphi(\mathbf{x})d\sigma(\mathbf{x})$), where φ satisfies for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$*

$$\varphi(\mathbf{x})\varphi(\mathbf{y}) \leqslant \varphi(\mathbf{x} \wedge \mathbf{y})\varphi(\mathbf{x} \vee \mathbf{y}). \tag{3}$$

This definition is slightly different from the definition used by Fortuin et al in [5]. It is taken from [4], where the authors prove a correlation inequality for several families that we will also need in this work.

Theorem 8 (Rinott—Saks correlation inequality [4]). *Let ℓ, n be positive integers. Let $f_1, f_2, \dots, f_\ell$ and $g_1, \dots, g_\ell$ be nonnegative real-valued functions defined on $\mathbb{R}^n$ that satisfy following condition: for every sequence $\mathbf{x}^1, \dots, \mathbf{x}^\ell$ of elements from $\mathbb{R}^n$ we have*

$$\prod_{i=1}^{\ell} f_i(\mathbf{x}^i) \leqslant \prod_{i=1}^{\ell} g_i \left(\bigvee_{S \in \binom{[n]}{i}} \bigwedge_{j \in S} \mathbf{x}^j \right). \tag{4}$$

Then, for any FKG-measure μ on $\mathbb{R}^n$ we have

$$\prod_{i=1}^{m} \int_{\mathbb{R}^n} f_i(\mathbf{x}) d\mu(\mathbf{x}) \leqslant \prod_{i=1}^{m} \int_{\mathbb{R}^n} g_i(\mathbf{x}) d\mu(\mathbf{x}). \tag{5}$$

With appropriate measures and functions, this inequality implies all of the previous ones we listed. We will need the following $\{0,1\}$-corollary of Theorem 8:

Corollary 9 (Theorem 4.1 from [4]). *For any families of sets $\mathcal{A}_1, \dots, \mathcal{A}_\ell$,*

$$\prod_{k=1}^{\ell} |\mathcal{A}_k| \leqslant \prod_{k=1}^{\ell} \left| \bigvee_{S \in \binom{[\ell]}{k}} \left(\bigwedge_{s \in S} \mathcal{A}_s \right) \right|. \tag{6}$$

Corollary 9 can be derived by taking integral over the counting (i.e., uniform) measure on $\{0,1\}^n$. The corresponding $f_1, \dots, f_\ell$ and $g_1, \dots, g_\ell$ are simply the indicator functions that for a given set indicate if it belongs to the corresponding family: f_i for $\mathcal{A}_i$ and g_j for $\bigvee_{S \in \binom{[\ell]}{j}} \left(\bigwedge_{s \in S} \mathcal{A}_s \right)$.

According to Theorem 8, we can replace the uniform with any FKG-measure. An important example of an FKG-measure is the p-biased measure with $0 \leqslant p \leqslant 1$ for the sets $X \subset [n]$ and families $\mathcal{F} \subset 2^{[n]}$, defined as follows.

$$\begin{aligned} \mu_p(X) = & p^{|X|}(1-p)^{n-|X|} = \left(\frac{p}{1-p} \right)^{|X|} (1-p)^n, \\ \mu_p(\mathcal{F}) = & \sum_{X \in \mathcal{F}} \mu_p(X). \end{aligned}$$

We say that two measures μ, μ' are *proportional* if there is a non-zero constant C such that $\mu(x) = C\mu'(x)$ for any x. We get the following corollary which slightly more general than Corollary 9.

Corollary 10. *For any collection of families $\mathcal{A}_1, \dots, \mathcal{A}_\ell \subset 2^{[n]}$ and $0 \leqslant p \leqslant 1$ we have*

$$\prod_{k=1}^{\ell} \mu_p(\mathcal{A}_k) \leqslant \prod_{k=1}^{\ell} \mu_p \left(\bigvee_{S \in \binom{[\ell]}{k}} \left(\bigwedge_{s \in S} \mathcal{A}_s \right) \right). \tag{7}$$

Moreover, the same holds for any measure that is proportional to μ_p.

2.2 Entropy

Another probabilistic tool that we use in our work is entropy. One can find a detailed survey in [10]. Consider a random variable X with finite support. Then the entropy $\mathbb{H}[X]$ is defined as

$$\mathbb{H}[X] = -\sum_{x \in \operatorname{supp} X} \mathbb{P}[X = x] \log_2 \mathbb{P}[X = x]. \tag{8}$$

It is a non-negative function that satisfies the following properties:

Claim 11. *We have*

(i) If X and Y are arbitrary random variables distributed over a finite set, then $\mathbb{H}[X, Y] \leqslant \mathbb{H}[X] + \mathbb{H}[Y]$.

(ii) Consider a random variable X distributed over the sets of a family $\mathcal{F}$. Then $\mathbb{H}[X] \leqslant \log_2 |\mathcal{F}|$. The equality holds if and only if X is uniformly distributed over $\mathcal{F}$.

The proof of both statements can be found in [10].

Another similar function that characterizes the difference between two distributions is the cross-entropy. For the particular application that it has in our work, we define it as follows:

$$\mathbb{H}\left((p_i)_{i \in S}, (q_i)_{i \in S}\right) = -\sum_{i \in S} p_i \log_2 q_i,$$

where $(p_i)_{i \in S}$, $(q_i)_{i \in S}$ are two arbitrary discrete distributions with the same support S.

Proposition 12. *Given a distribution $(p_i)_{i \in S}$, the cross-entropy as a function of $(q_i)_{i \in S}$ achieves its minimum when distributions $(p_i)_{i \in S}$ and $(q_i)_{i \in S}$ coincide.*

For the proof see, for example, [11].

2.3 Coverings and matchings

Our problem has a natural hypergraph interpretation, and so we will need some simple tools from hypergraph theory. We call a subset T of V a *covering* for a hypergraph (V, E), if for any edge $e \in E$ we have $e \cap T \neq \varnothing$. A subset $\mathcal{M}$ of E is a *matching* if edges of $\mathcal{M}$ are pairwise disjoint. A matching $\mathcal{M}$ is *maximal* if it is impossible to enlarge it by adding another $e \in E$. A covering T is called *minimum* if there is no covering of smaller cardinality.

The following statement is folklore.

Proposition 13. *Let (V, E) be an arbitrary hypergraph with edges of size at most t. Then the size of a minimum covering T is at most size of any maximal matching $\mathcal{M}$ times t.*

Proof. Note that $\sqcup\mathcal{M}$ is a covering. Otherwise, if an edge is not covered by $\sqcup\mathcal{M}$ then we can add it to $\mathcal{M}$, contradicting the maximality of $\mathcal{M}$. The cardinality of $\sqcup\mathcal{M}$ is $t|\mathcal{M}|$. Since T is a minimum covering, $|T| \leqslant t|\mathcal{M}|$. □

3 Counting monochromatic sets

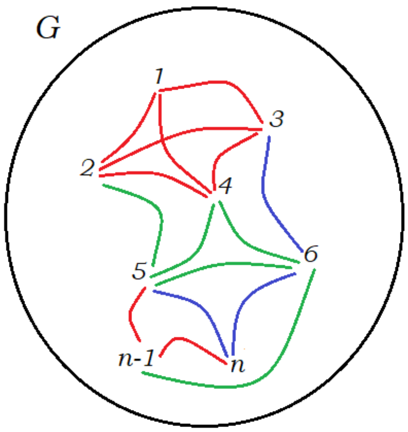

Figure 1: Graph coloring

First, consider the case $\mathbf{m}_S = 1$ for any $S \in \binom{\ell}{2}$. We claim that there is a correspondence between our families and cliques in a colored graph. This connection was previously discussed in [8]. To illustrate it, we need the following claim:

Claim 14. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_l$ satisfy the overlapping property and that are, moreover, maximal w.r.t. this property. Then for each $\mathcal{F}_k$ it holds that*

- *a subset K of $[n]$ is contained in $\mathcal{F}_k$ if and only if for each pair $i, j \in K$ $\{i, j\}$ is contained in $\mathcal{F}_k$,*
- *empty set and every singleton from $[n]$ belongs to $\mathcal{F}_k$,*

and, moreover, for each $\{i, j\} \subset [n]$ there is a unique k' such that $\{i, j\} \in \mathcal{F}_{k'}$.

Proof. First, if K belongs to $\mathcal{F}_k$ then each pair $\{i, j\}$ from K belongs to $\mathcal{F}_k$ because of down-closeness. Conversely, suppose that every $\{i, j\} \subset K$ is contained in $\mathcal{F}_k$. Then for each $k' \neq k$ and a set $F \in \mathcal{F}_{k'}$ we have $|F \cap K| \leqslant 1$. Therefore, by maximality of $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$, $K \in \mathcal{F}_k$.

Second, adding the empty set and singletons does not break the overlapping property, so by maximality they must belong to each family.

Third, if $\{i, j\}$ does not belong to any $\mathcal{F}_k$ then adding it to any of $\mathcal{F}_i$ does not break the overlapping property, and so by maximality each $\{i, j\}$ must belong to some $\mathcal{F}_k$. $\square$

Next, let us construct a coloring of the complete graph K_n on the vertex set $[n]$ into ℓ colors based on a maximal collection of families $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ with overlapping property. Color the edge $\{i, j\}$ with color k iff $\{i, j\} \in \mathcal{F}_k$ and put $E_k = \{\{i, j\} : i \neq j \text{ and } \{i, j\} \in \mathcal{F}_k\}$. Note that the cardinality of $\mathcal{F}_k$ is equal to the number of cliques in the graph $G_k := ([n], E_k)$, induced by the edges of k-th color. Conversely, any coloring of any graph G on vertex set $[n]$ is associated with some families of subsets of $[n]$ with overlapping property. For example, the coloring on Figure 3 produces the following families of sets:

$$\mathcal{F}_1 = \binom{[n]}{\leqslant 1} \cup 2^{\{1,2,3,4\}} \cup \{\{5, n-1\}, \{n-1, n\}\},$$
$$\mathcal{F}_2 = \binom{[n]}{\leqslant 1} \cup 2^{\{4,5,6\}} \cup \{\{6, n-1\}, \{2, 5\}\},$$
$$\mathcal{F}_3 = \binom{[n]}{\leqslant 1} \cup \{\{5, n\}, \{6, n\}, \{3, 6\}\}.$$

Let us briefly discuss how to generalize this correspondence to the $\mathbf{m}$-overlapping setting.

Previously, we could as well consider the complement of G_k and and count independent sets in this complement. It is natural to generalize this point of view. Consider a maximal collection

$\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ that are $\mathbf{m}$-overlapping. For each family $\mathcal{F}_k$ we construct the following hypergraph:

$$H_k = \left([n], \bigcup_{k' \in [\ell] \setminus \{k\}} \mathcal{F}_{k'}^{(\mathbf{m}_{k,k'}+1)} \right),$$

where $\mathcal{F}^{(t)}$ denotes $\mathcal{F} \cap \binom{[n]}{t}$.

We call $I \subset [n]$ *independent* in the hypergraph H_k if it does not contain any edge of H_k. We claim that the number of independent sets in H_k equals $|\mathcal{F}_k|$.

Claim 15. *If $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ are maximal and $\mathbf{m}$-overlapping then $\mathcal{F}_k$ consists of all independent sets in H_k, where H_k is defined above.*

Sketch of the proof. It follows from two implications:

1. If I is independent in H_k then it intersects any set of $\mathcal{F}_{k'}$, $k' \in [\ell] \setminus \{k\}$, in at most $\mathbf{m}_{k,k'}$ elements. Thus, it is contained in $\mathcal{F}_k$ due to maximality.

2. If $F \in \mathcal{F}_k$ then it intersects any set of $\mathcal{F}_{k'}$, $k' \in [\ell] \setminus \{k\}$, in at most $\mathbf{m}_{k,k'}$ elements. Thus, F is an independent set in the hypergraph H_k.

$\square$

In addition, if all entries of $\mathbf{m}$ are equal to m, then H_k becomes the complement to the subhypergraph of K_n^{m+1} consisting of all edges colored into color k. This proves Proposition 4. (Recall that we assume that any m or less vertices form a monochromatic clique in any color.)

4 Sketch of the proof of Theorem 2

Throughout this section, we assume that $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ are $\mathbf{m}$-overlapping and extremal (that is, maximize the product of cardinalities).

First, we find the lower bound that matches the asymptotic in (1). The construction of the example is based on the following guess: since $s^*(n, \ell, \mathbf{m})$ is proportional to 2^n, each $\mathcal{F}_k$, $k \in [\ell]$ should contain some "center" set C_k^* such that $\mathcal{F}_k$ contains all subsets of C^* and these sets together cover $[n]$ completely. That guarantees that $2^{C_k^*} \subset \mathcal{F}_k$ and that the "exponential part" of the product $\prod_{k \in [\ell]} |\mathcal{F}_k|$ is 2^n. The polynomial part of $|\mathcal{F}_k|$ arises from concatenations of subsets from C_k^* and elements from other centers. That makes a family $\mathcal{F}_k$ look like an "octopus" with "body" C_k^* and "tentacles" directed to the centers of others.

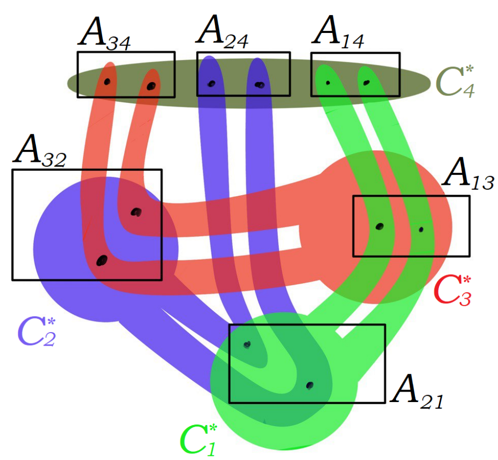

Figure 2: Octopuses described in Section 4.

For instance, in the case of $m = 1$, a family $\mathcal{F}_1$ presented on Figure 2 can be decomposed as follows:

$$\mathcal{F}_1 \simeq 2^{C_1^*} \vee \binom{A_{13}}{\leqslant 1} \vee \binom{A_{14}}{\leqslant 1}$$

with the body C_1^* and tentacles $\binom{A_{13}}{\leqslant 1} \vee \binom{A_{14}}{\leqslant 1}$. We use "$\simeq$" instead of "$=$" because in extremal examples we need to add all small sets, but they typically account for a negligibly small fraction of the family.

This summarizes the rough structure on which we based the example for the lower bound in Section 5. In what follows, we discuss the upper bound.

The proof of the upper bound is based on a bootstrapping idea: establishing the asymptotic helps to obtain understanding of the structure of extremal examples and vice versa. First, we prove that a set M_k of maximal cardinality from each family $\mathcal{F}_k$ can be considered as a proxy of the center C_k^*. More precisely, we prove that $\left|[n] \setminus \bigcup_{k\in[\ell]} M_k\right| = O(\log n)$ using Rinott—Saks inequality for p-biased measures (Corollary 10). The detailed argument is given in Lemma 17.

In what follows, we will use the definition of a (normalized) *degree* of a set with respect to some family. The degree d_k of a set F in a family $\mathcal{F}_k$ is

$$d_k(F) := \frac{|\{F' \in \mathcal{F}_k : F \subset F'\}|}{|\mathcal{F}_k|}.$$

The degree $d_k(x)$ of an element $x \in [n]$ is just $d_k(\{x\})$.

We next show that an inductive application of Daykin's inequality (Theorem 7) over $k \in \ell$ delivers the asympotic of $s^*(n, \ell, \mathbf{m})$ up to a constant factor. It allows to have a good control on the degrees: given a subset $K \subset [\ell]$ and sets $F_k \in \mathcal{F}_k$, $k \in K$, we have

$$\prod_{k\in K} d_k(F_k) = O\left(n^{-\sum_{\{k,k'\}\in\binom{K}{2}} |F_k \cap F_{k'}|}\right), \tag{9}$$

and, in particular,

$$d_{k_1}(x) d_{k_2}(x) d_{k_3}(x) = O(n^{-3}) \tag{10}$$

for any distinct k_1, k_2, k_3 and $x \in [n]$. Indeed, it is easy to see that families $\mathcal{F}_k(F_k)$, $\mathcal{F}_{k'}$ for $k \in K$ and $k' \in [n] \setminus K$ are $\mathbf{m}'$-overlapping for a suitable $\mathbf{m}' \leqslant \mathbf{m}$. Using the upper bound for $s^*(n, \ell, \mathbf{m}')$ that follows from iterative Daykin's inequality applications and the lower bound for $s^*(n, \ell, \mathbf{m})$ allows to obtain (9) and (10).

The entropy argument of Proposition 19 guarantees that most of the elements of $[n]$ have positive constant degrees in some $\mathcal{F}_k$. If we denote the k-th least normalized degree by $d_{(k)}(x)$ and the index of corresponding family by $(k)(x)$, we notice that for most of elements in $[n]$, $d_{(l)}(x)$ has constant lower bound, and, consequently, $d_{(l-2)}(x) = O(n^{-3/2})$ due to (10). In this way, removing suitable sets, we are able to prune families $\mathcal{F}_k$ such that their size changes by a factor $\left(1 - O(n^{-1/2})\right)$ and $d_k(x) = 0$ if $k \notin \{(\ell)(x), (\ell-1)(x)\}$. In other words, in the modified families each element has non-zero degree only in two families that correspond to two initial families in which $d_k(x)$ was the first and the second largest. While the set of maximal cardinality M_k is a proxy of the octopus's body of $\mathcal{F}_k$, the set $\{x \in [n] : k = (l-1)(x)\}$ is a proxy of its tentacles.

$$
\begin{array}{rc|l}
C^*_{\ell-1} & = & A_{\ell-1,\ell} \\
C^*_{\ell-2} & = & A_{\ell-2,\ell} \cup A_{\ell-2,\ell-1} \\
\dots & & \quad \dots \\
C^*_2 & = & \; A_{2,\ell} \;\cup\; A_{2,\ell-1} \cup \ldots \cup A_{2,3} \\
C^*_1 & = & \; A_{1,\ell} \;\cup\; A_{1,\ell-1} \cup \ldots \cup A_{1,3} \cup A_{1,2} \\
\hline
 & & \; \mathcal{F}_\ell \qquad \mathcal{F}_{\ell-1} \qquad \ldots \quad \mathcal{F}_3 \qquad \mathcal{F}_2
\end{array}
$$

Table 1: In our example for Theorem 16 each set in the family $\mathcal{F}_k$ consists of two parts: an arbitrary subset of the center $C^*_k = \bigcup_{k'>k} A_{k,k'}$ (note that $C^*_\ell = \varnothing$) and subsets ("tentacles") of size at most $\mathbf{m}_{k',k}$ in each of the domains $A_{k',k}$, $k' < k$. The rows of the table above are indexed by the corresponding domains of the families and the columns are indexed by the families, where the column indexed by $\mathcal{F}_i$ consists of the "target sets" of the tentacles of sets from $\mathcal{F}_i$. Note that $\mathcal{F}_1$ has no tentacles.

We denote these pruned families by $\mathcal{F}'_k$. By construction, for any $K \subset [\ell]$ of cardinality greater than 2, we have $\bigwedge_{k\in K} \mathcal{F}'_k = \{\varnothing\}$. That significantly simplifies the Rinott-Saks inequality for families of sets (Corollary 9), since among multipliers from the right-hand side only two factors remain. They can be bounded as follows:

$$
\left| \bigvee_{S\in\binom{[\ell]}{2}} \bigwedge_{s\in S} \mathcal{F}'_s \right| \leqslant \prod_{S\in\binom{[\ell]}{2}} \left| \binom{\operatorname{supp} \wedge_{s\in S} \mathcal{F}'_s}{\leqslant \mathbf{m}_S} \right|, \tag{11}
$$

$$
\left| \bigvee_{k\in[\ell]} \mathcal{F}'_k \right| \leqslant 2^n.
$$

Optimizing over sizes of disjoint sets $\operatorname{supp}(\wedge_{s\in S}\mathcal{F}'_s)$, we obtain tight upper bound of $s^*(n, \ell, \mathbf{m})$ up to a factor $1 + O(n^{-1/2})$. (Note that the error term $O(n^{-1/2})$ is an artefact of the pruning that we did.)

In the follow-up paper [13], we will improve the error term and determine a bulk of the structure of extremal examples. The tentacle analogy is very useful in understanding their structure.

5 Proof of the lower bound

In this section, we provide a construction that gives the lower bound in Theorem 2.

Theorem 16. *There are families $\mathcal{F}_1, \ldots, \mathcal{F}_\ell \in 2^{[n]}$ satisfying $\mathbf{m}$-overlapping property such that*

$$
\prod_{k=1}^{\ell} |\mathcal{F}_k| = \left(1 + O(n^{-1})\right) 2^n \cdot \prod_{S\in\binom{[\ell]}{2}} \left(\frac{1}{\mathbf{m}_S!} \left(\frac{\mathbf{m}_S \cdot n}{\sigma} \right)^{\mathbf{m}_S} \right) = \left(1 + O(n^{-1})\right) C n^\sigma 2^n,
$$

where $\sigma = \sum_{S\in\binom{[\ell]}{2}} \mathbf{m}_S$ and C is a constant depending on ℓ and $\mathbf{m}$ only.

Proof. Consider some vector $\mathbf{n}$ with coordinates indexed by S, $S \in \binom{[\ell]}{2}$, and a partition of the set $[n]$ into $\binom{l}{2}$ sets $A_{k,k'}$, $1 \leqslant k < k' \leqslant \ell$, such that $|A_{k,k'}| = \mathbf{n}_{k,k'}$. Then, define (cf. Table 1)

$$\mathcal{F}_k = \left(2^{\bigcup_{k'>k} A_{k,k'}}\right) \vee \bigvee_{k'<k} \binom{A_{k',k}}{\leqslant \mathbf{m}_{k,k'}}.$$

Obviously, if $F \in \mathcal{F}_{k_1}$, $G \in \mathcal{F}_{k_2}$ and $k_1 < k_2$, then the intersection of F and G is contained in A_{k_1,k_2}. At the same time, by definition, each set from $\mathcal{F}_{k_2}$ contains at most $\mathbf{m}_{k_1,k_2}$ elements in A_{k_1,k_2}. Thus, $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ satisfy the $\mathbf{m}$-overlapping property.

It is easy to see that

$$|\mathcal{F}_k| = \prod_{k'<k} \binom{|A_{k',k}|}{\leqslant \mathbf{m}_{k',k}} 2^{\sum_{k'>k} |A_{k,k'}|},$$

and, consequently,

$$\prod_{k=1}^{\ell} |\mathcal{F}_k| = \prod_{S \in \binom{[\ell]}{2}} \binom{\mathbf{n}_S}{\leqslant \mathbf{m}_S} 2^n.$$

Maximizing over $\mathbf{n}$ delivers the following optimization problem:

$$\begin{aligned} &\max_{\mathbf{n}} \prod_{S \in \binom{[\ell]}{2}} \left(\sum_{t=0}^{\mathbf{m}_S} \binom{\mathbf{n}_S}{t} \right), \\ &\text{s.t.} \sum_{S \in \binom{[\ell]}{2}} \mathbf{n}_S = n, \end{aligned} \tag{12}$$

where $\mathbf{n}$ is a vector of non-negative integers. To determine the asymptotic of the solution, note that

$$\sum_{t=0}^{\mathbf{m}_S} \binom{\mathbf{n}_S}{t} \sim \binom{\mathbf{n}_S}{\mathbf{m}_S} \sim \frac{\mathbf{n}_S^{\mathbf{m}_S}}{\mathbf{m}_S!}.$$

Maximizing the product of these expressions is equivalent to maximizing the sum of their logarithms. Thus, ignoring lower order terms, the target function of the optimization problem (12) can be changed to

$$\max_{\mathbf{n}} \sum_{S \in \binom{[\ell]}{2}} \frac{\mathbf{m}_S}{\sigma} \log \frac{\mathbf{n}_S}{n}.$$

The last expression is the minus cross-entropy between discrete distributions $(\mathbf{m}_S/\sigma)_{S \in \binom{[\ell]}{2}}$ and $(\mathbf{n}_S/n)_{S \in \binom{[\ell]}{2}}$. By Proposition 12, its maximum is achieved when distributions coincide, which proves the lower bound and, moreover, shows that the corresponding example is optimal in the class of examples that we considered. $\square$

6 Proof of the upper bound

We employ the following standard notation for a family $\mathcal{F}$ and sets $A \subset B$:

$$\mathcal{F}|_B := \{F \cap B : F \in \mathcal{F}\},$$

$$\mathcal{F}(A, B) := \{F \setminus B : F \in \mathcal{F} \text{ and } F \cap B = A\},$$
$$\mathcal{F}(A) := \mathcal{F}(A, A),$$
$$\mathcal{F}\left(\overline{A}\right) := \mathcal{F}(\varnothing, A).$$

When dealing with singletons, we suppress brackets for simplicity, i.e. $\mathcal{F}(x) = \mathcal{F}(\{x\})$ and $\mathcal{F}(\overline{x}) = \mathcal{F}\left(\overline{\{x\}}\right)$.

In what follows, we work with families $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ that are $\mathbf{m}$-overlapping and that are extremal, i.e., that maximize the product. Due to extremality, they possess certain useful properties, in particular, they must be down-closed. We call a collection of families $\mathcal{F}_k, k \in K \subset [\ell]$ *extremal* if they arise in some extremal example.

6.1 Maximal sets cover $[n]$ almost completely

Lemma 17. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be a collection of $\mathbf{m}$-overlapping extremal families and let $M_k \in \mathcal{F}_k$, $k \in [\ell]$ be the sets of maximal cardinality in the respective families. Define $R = [n] \setminus \bigcup_{k=1}^{l} M_k$. Then there is a constant C such that*

$$|R| \leqslant C \log_2 n.$$

Proof. We have the following decomposition for $\mathcal{F}_k$:

$$|\mathcal{F}_k| = \sum_{F \in \mathcal{F}_k|_R} |\mathcal{F}_k(F, R)|.$$

It follows from the definitions that

$$|\mathcal{F}_k(F, R)| \leqslant \left|\mathcal{F}_k(F, R)|_{M_k}\right| \left|\mathcal{F}_k|_{\bigcup_i M_i \setminus M_k}\right|.$$

To bound the size of $\mathcal{F}_k(F, R)|_{M_k}$, consider a hypergraph H:

$$H := \left(F \cup M_k, \bigcup_{k' \in [\ell] \setminus \{k\}} (\mathcal{F}_{k'}|_{F \cup M_k})^{(\mathbf{m}_{k,k'}+1)}\right)$$

and its induced hypergraph H':

$$H' := (M_k, E(H)|_{M_k}).$$

Any edge of H intersects both F and M_k since both F and M_k are contained in $\mathcal{F}_k$, and thus cannot contain a set of size $m_{k,k'} + 1$ from $\mathcal{F}_{k'}$. Consider a vertex cover T for H' and consider a set $(M_k \setminus T) \cup F$. This set does not contain any edge from H and hence it should belong to $\mathcal{F}_k$ due to maximality. At the same time, its size is equal to $|M_k| + |F| - |T|$, which is at most

$|M_k|$ due to maximality of M_k. Consequently, $|T| \geqslant |F|$, i.e. covering number of H' is at least $|F|$.

Slightly abusing notation, put $m = \max_{k,k'} \mathbf{m}_{k,k'}$. Any edge of H' has size at most m and the vertices of any maximal matching in H' form a vertex cover for H. Thus the size of the largest matching in H' is at least $|F|/m$ by Proposition 13. Denote one such matching by $\mathcal{M} \subset E(H')$. We bound from above $|\mathcal{F}_k(F,R)|_{M_k}|$ using that none of the sets in $\mathcal{F}_k(F,R)|_{M_k}$ can contain an edge from $\mathcal{M}$.

$$|\mathcal{F}_k(F,R)|_{M_k}| \leqslant 2^{|M_k|-|\bigsqcup \mathcal{M}|} \prod_{e \in \mathcal{M}} (2^{|e|}-1) = 2^{|M_k|} \prod_{e \in \mathcal{M}} (1-2^{-|e|}) \leqslant (1-2^{-m})^{\frac{|F|}{m}} 2^{|M_k|}.$$

At the same time, it is easy to see that $F \in \mathcal{F}_k$ satisfies $|F \cap M_i| \leqslant \mathbf{m}_{i,k}$ for every $i \neq k$, and thus $\mathcal{F}_k|_{\bigcup_i M_i \setminus M_k}$ has cardinality at most $n^{m(\ell-1)}$. Thus, the size of $\mathcal{F}_k$ can be bounded as follows:

$$|\mathcal{F}_k| \leqslant n^{m(\ell-1)} 2^{|M_k|} \sum_{F \in \mathcal{F}|_R} (1-2^{-m})^{\frac{|F|}{m}}.$$

Denote $(1-2^{-m})^{\frac{1}{m}}$ by ε_m and note that $\varepsilon_m < 1$ is some constant depending on m only.

Consequently, we can bound the product as follows

$$\prod_{k=1}^{\ell} |\mathcal{F}_k| \leqslant n^{m\ell(\ell-1)} 2^{\sum_{k=1}^{\ell} |M_k|} \prod_{k=1}^{\ell} \left(\sum_{F \in \mathcal{F}_k|_R} \varepsilon_m^{|F|} \right).$$

It is easy to see that $\sum_{k=1}^{\ell} |M_k| \leqslant |\bigcup_{k=1}^{\ell} M_k| + \frac{m\ell(\ell-1)}{2}$, and, thus,

$$\begin{aligned} \prod_{k=1}^{\ell} |\mathcal{F}_k| &\leqslant n^{m\ell(\ell-1)} 2^{|\bigcup_{k=1}^{\ell} M_k| + \frac{m\ell(\ell-1)}{2}} \prod_{k=1}^{\ell} \left(\sum_{F \in \mathcal{F}_k|_R} \varepsilon_m^{|F|} \right) \\ &\leqslant n^{m\ell(\ell-1)} 2^{n-|R|+\frac{m\ell(\ell-1)}{2}} \prod_{k=1}^{\ell} \left(\sum_{F \in \mathcal{F}_k|_R} \varepsilon_m^{|F|} \right). \end{aligned} \tag{13}$$

We neeed to bound the last product. To this end, note that the function that assigns to a set F the value $(\varepsilon_m)^{|F|}$ is proportional to the p-biased measure μ_p with some p, and thus we can apply Corollary 10 to it. Put

$$\mathcal{R}^{[k]} := \bigvee_{K \in \binom{[\ell]}{k}} \bigwedge_{k' \in K} (\mathcal{F}_{k'})\big|_R.$$

Note that

$$\bigwedge_{k' \in K} (\mathcal{F}_{k'})\big|_R = \bigcap_{k' \in K} (\mathcal{F}_{k'})\big|_R \subset \binom{R}{\leqslant m},$$

and, consequently,

$$\mathcal{R}^{[k]} \subset \binom{R}{\leqslant \binom{\ell}{k} m}.$$

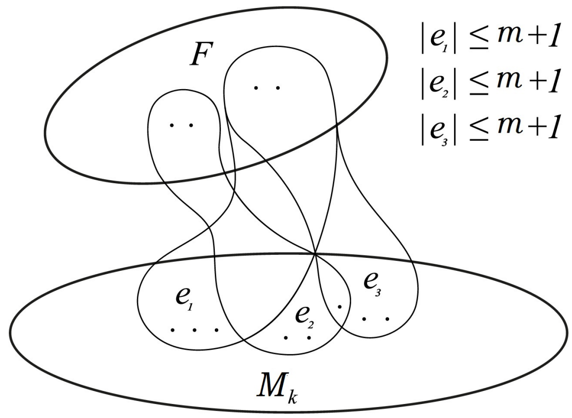

Figure 3: A hypergraph from the proof of Lemma 17.

Thus, using Corollary 10 in the first inequality below, we get

$$\prod_{k=1}^{\ell} \sum_{F \in \mathcal{F}_k|_R} \varepsilon_m^{|F|} \leqslant \prod_{k=1}^{\ell} \sum_{F \in \mathcal{R}^{[k]}} \varepsilon_m^{|F|} \leqslant \left(\sum_{i=0}^{|R|} \binom{|R|}{i} \varepsilon_m^i \right) \prod_{k=2}^{\ell} \binom{|R|}{\leqslant \binom{\ell}{k} m}$$
$$= (1+\varepsilon_m)^{|R|} \cdot \prod_{k=2}^{\ell} \binom{|R|}{\leqslant \binom{\ell}{k} m}.$$

Due to Theorem 16, $\prod_{k=1}^{\ell} |\mathcal{F}_k| = \Theta(n^\sigma 2^n)$, where $\sigma = \sum_{S \in \binom{[\ell]}{2}} \mathbf{m}_S$. Thus, combining the above with (13), we get

$$\Theta(n^\sigma 2^n) = |R|^{O(1)} (1+\varepsilon_m)^{|R|} 2^{n-|R|},$$
$$\left(\frac{2}{1+\varepsilon_m} \right)^{|R|} = n^{O(1)}.$$

Since $\varepsilon_m < 1$, the last inequality implies that

$$|R| = O(\log n).$$

□

6.2 Weak upper bound

In this section, we give a simple argument that allows to determine the value of $s^*(n, \ell, \mathbf{m})$ up to a constant. The argument is via an iterative application of Daykin's inequality and is due to Sergei Kiselev.

Proposition 18. *Put $\sigma = \sum_{S \in \binom{[\ell]}{2}} \mathbf{m}_S$. Then there is a constant C depending on ℓ and $\mathbf{m}$ such that*

$$s^*(n, \ell, \mathbf{m}) \leqslant C n^\sigma 2^n.$$

Proof. Consider a collection of families $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ satisfying the $\mathbf{m}$-overlapping property. We prove that

$$\prod_{k=1}^{\ell} |\mathcal{F}_k| \leqslant \prod_{k=1}^{\ell-1} \left| \left(\bigvee_{k'=1}^{k} \mathcal{F}_{k'} \right) \wedge \mathcal{F}_{k+1} \right| \cdot \left| \bigvee_{k=1}^{\ell} \mathcal{F}_k \right| \tag{14}$$

by induction on ℓ. The statement is true for $\ell = 2$ due to Theorem 7. Suppose that $\ell \geqslant 3$ and that the statement holds for $\ell - 1$. Then

$$\prod_{k=1}^{\ell} |\mathcal{F}_k| \leqslant \left[\prod_{k=1}^{\ell-2} \left| \left(\bigvee_{k'=1}^{k} \mathcal{F}_{k'} \right) \wedge \mathcal{F}_{k+1} \right| \cdot \left| \bigvee_{k=1}^{\ell-1} \mathcal{F}_k \right| \right] |\mathcal{F}_\ell|.$$

Due to Theorem 7

$$\left| \bigvee_{k=1}^{\ell-1} \mathcal{F}_k \right| \cdot |\mathcal{F}_\ell| \leqslant \left| \left(\bigvee_{k=1}^{\ell-1} \mathcal{F}_k \right) \wedge |\mathcal{F}_\ell| \right| \cdot \left| \bigvee_{k=1}^{\ell} \mathcal{F}_k \right|,$$

which proves (14). Next we bound its factors. Obviously,

$$|\mathcal{F}_1 \bigvee \mathcal{F}_2| \leqslant \left| \binom{[n]}{\leqslant \mathbf{m}_{1,2}} \right|$$

More generally, for any $k \in [2, \ell - 1]$ we have

$$\left| \left(\bigvee_{k'=1}^{k} \mathcal{F}_{k'} \right) \wedge \mathcal{F}_{k+1} \right| \leqslant \left| \binom{[n]}{\leqslant \sum_{k'=1}^{k} \mathbf{m}_{k',k+1}} \right|.$$

Finally,

$$\left| \bigvee_{k=1}^{\ell} \mathcal{F}_k \right| \leqslant 2^n.$$

Substituting these bounds in (14), we derive the statement of the proposition. □

6.3 Degrees

In what follows, we will be extensively working with the degrees of elements w.r.t. $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$. We use the notion of the *normalized degree* of an set F, defined as follows:

$$d(F, \mathcal{F}) = \frac{|\mathcal{F}(F)|}{|\mathcal{F}|},$$

For brevity, we write $d_k(F)$ instead of $d(F, \mathcal{F}_k)$ and $d_k(x)$ instead of $d_k(\{x\})$.

Proposition 19. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be a collection of $\mathbf{m}$-overlapping families that is extremal. Then there is a set I of size $O(\log n)$, such that for each $x \in [n] \setminus I$ there is $k \in [\ell]$ such that $d_k(x) \geqslant \frac{1}{3}$.*

Proof. Fix some $k \in [\ell]$ and take a uniformly random set $X \in \mathcal{F}_k$. Let $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be a random variable equal to the characteristic vector of X. For a subset S of $[n]$ we denote $(\mathbf{v}_i)_{i \in S}$ by $\mathbf{v}_S$. Thus, due to Claim 11 (i)

$$\mathbb{H}[\mathbf{v}] \leqslant \sum_{i \in M_k} \mathbb{H}\left[\mathbf{v}_i\right] + \mathbb{H}\left[\mathbf{v}_{[n]\setminus M_k}\right].$$

It easy to see that $\mathbf{v}_i = 1$ with probability $d_k(i)$. Moreover, by Claim 11 (ii) we have $\mathbb{H}[\mathbf{v}_{[n]\setminus M_k}] \leqslant \log_2 \left|\mathcal{F}_k|_{[n]\setminus M_k}\right|$. Obviously, $\left|\mathcal{F}_k|_{[n]\setminus M_k}\right| \leqslant 2^{|R|} \left|\mathcal{F}_k|_{\bigcup_i M_i \setminus M_k}\right|$ and, thus, $\log_2 \left|\mathcal{F}_k|_{[n]\setminus M_k}\right| \leqslant |R| + O(\log n) = O(\log n)$ due to Lemma 17. Because $\mathbb{H}[\mathbf{v}] = \log_2 |\mathcal{F}_k| \geqslant |M_k|$, we observe

$$|M_k| \leqslant \sum_{i \in M_k} h_2\left(d_k(x)\right) + O(\log n),$$

where $h_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$ is the binary entropy. Using the fact that $|\bigcup_{k \neq k'} M_k \cap M_{k'}| \leqslant \binom{\ell}{2} = O(1)$, we get

$$\left|\bigcup_{k=1}^{\ell} M_k\right| \leqslant \sum_{k=1}^{\ell} |M_k| \leqslant \sum_{i \in [n]\setminus R} \max_k h_2\left(d_k(i)\right) + O(\log n).$$

Using Lemma 17, we get

$$n \leqslant \sum_{i=1}^{n} \max_k h_2\left(d_k(i)\right) + O(\log n)$$

For each real-valued $\varepsilon \in [0,1]$ define I_ε as

$$I_\varepsilon := \left\{ i \in [n] : \max_k h_2\left(d_k(i)\right) < \varepsilon \right\}.$$

Since $h_2(p) \leqslant 1$ for any $0 \leqslant p \leqslant 1$, we obtain

$$n - O(\log n) \leqslant \varepsilon |I_\varepsilon| + (n - |I_\varepsilon|),$$
$$|I_\varepsilon| \leqslant \frac{O(\log n)}{1-\varepsilon}.$$

Note that $h_2(1/3) = \log_2 3 - 2/3$. Putting $\varepsilon = \log_2 3 - 2/3$ in the expression above, we get that for each $x \in [n] \setminus I_\varepsilon$ there is a family of sets $\mathcal{F}_k$ such that $d_k(x) \geqslant \frac{1}{3}$ and that $|I_\varepsilon| = O(\log n)$. $\square$

In addition, we observe the following property of degrees:

Proposition 20. *Let $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ be a collection of $\mathbf{m}$-overlapping families that is extremal. Then for any subset of indices of $K \subset [\ell]$ and sets $F_k \in \mathcal{F}_k$, $k \in K$*

$$\prod_{k \in K} d_k(F_k) \leqslant C_D n^{-\sum_{\{k,k'\} \in \binom{K}{2}} |F_k \cap F_{k'}|},$$

where C_D is some constant depending on $\mathbf{m}$, ℓ.

Proof. If for some k, k' we have F_k, $F_{k'}$ such that $|F_k \cap F_{k'}| > \mathbf{m}_{k,k'}$, then either $d_k(F_k) = 0$ or $d_{k'}(F_{k'}) = 0$ and the inequality is trivial. Thus, assume $|F_k \cap F_{k'}| \leqslant \mathbf{m}_{k,k'}$ for each k, k'. Consider families $\mathcal{F}_k$, $k \in [\ell] \setminus K$, and $\mathcal{F}_k(F_k)$, $k \in K$ as families in $2^{[n]}$. They satisfy the $\mathbf{m}'$-overlapping property, where

$$\mathbf{m}'_S = \begin{cases} \mathbf{m}_S, & S \not\subset K \\ \mathbf{m}_S - \left|\bigcap_{s \in S} F_s\right|, & S \subset K. \end{cases}$$

According to Theorem 18,

$$\prod_{k \in [\ell] \setminus K} |\mathcal{F}_k| \prod_{k \in K} |\mathcal{F}_k(F_k)| = O\left(n^{\sigma - \sum_{\{k,k'\} \in \binom{K}{2}} |F_k \cap F_{k'}|} 2^n \right).$$

Meanwhile, due to Theorem 16

$$\prod_{k \in [\ell]} |\mathcal{F}_k| = \Theta(n^\sigma 2^n),$$

and, consequently,

$$\prod_{k \in K} d_k(F_k) = O(1) \cdot n^{-\sum_{\{k,k'\} \in \binom{K}{2}} |F_k \cap F_{k'}|}.$$

□

6.4 Proof

Proposition 20 implies the following Lemma which is crucial for the understanding the asymptotic of $s^*(n, \ell, \mathbf{m})$:

Lemma 21. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be families from the extremal example. Then there are subfamilies $\mathcal{F}'_k \subset \mathcal{F}_k$, $k \in [\ell]$ such that*

1. *Every element $x \in [n]$ is contained in at most two subfamilies $\mathcal{F}'_k$'s.*
2. *For every $k \in [\ell]$ it holds that $|\mathcal{F}'_k| \geqslant (1 - \delta_n)|\mathcal{F}_k|$, where $\delta_n = O(n^{-1/2})$.*

Proof. Given $x \in [n]$, consider two cases: $x \in I$ and $x \notin I$, where I is defined as in Proposition 19. If $x \in [n] \setminus I$, then there is $k^* \in [\ell]$ such that $d_{k^*}(x) \geqslant \frac{1}{3}$. Consider any subset K of $[\ell] \setminus \{k^*\}$ of cardinality 2. From Proposition 20 we get

$$d_{k^*}(x) \prod_{k \in K} d_k(x) = O\big(n^{-3}\big).$$

Thus, there is $k \in K$ such that

$$d_k(x) = O\big(n^{-3/2}\big).$$

It implies that there are $\ell - 2$ families in which x has normalized degree $O\big(n^{-3/2}\big)$. We put

$$T_x = \left\{k : d_k(x) = O\big(n^{-3/2}\big)\right\}.$$

If $x \in I$, there may be no $k^* \in K$ such that $d_{k^*}(x) \geqslant 1/3$. Thus, consider an arbitrary set $K \subset [\ell]$ of cardinality 3. Then

$$\prod_{k \in K} d_k(x) = O\big(n^{-3}\big),$$

and consequently, there is $k \in K$ such that

$$d_k(x) = O\big(n^{-1}\big).$$

Since K is arbitrary, there are at least $\ell - 2$ families for which this inequality holds. We put

$$S_x = \left\{k : d_k(x) = O\big(n^{-1}\big)\right\}.$$

Put $W_k = \{x : k \in T_x \cup S_x\}$. Define $\mathcal{F}'_k = \mathcal{F}_k(\overline{W_k})$. Then

$$\begin{aligned}
\frac{|\mathcal{F}'_k|}{|\mathcal{F}_k|} &\geqslant 1 - \big(|\{x : k \in S_x\}|\, O\big(n^{-1}\big) + |\{x : k \in T_x\}|\, O\big(n^{-3/2}\big)\big) \\
&\geqslant 1 - O\left(\frac{|I|}{n} + n^{-1/2}\right) =: 1 - \delta_n.
\end{aligned}$$

Since $|I| = O(\log n)$ due to Proposition 19, we obtain $\delta = O(n^{-1/2})$. $\square$

We are ready to prove the upper bound.

Proof of the upper bound in Theorem 2. Let $\mathcal{F}'_1, \ldots, \mathcal{F}'_\ell$ be the families from Lemma 21. Then

$$\prod_{k=1}^{l} |\mathcal{F}_k| \leqslant (1 - \delta_n)^{-\ell} \prod_{k=1}^{l} |\mathcal{F}'_k| \tag{15}$$

and each $\{x\}$, $x \in [n]$, is contained in the sets from at most two families $\mathcal{F}'_k$. Thus, for each $S \subset [l]$ with $|S| > 2$ we have

$$\bigwedge_{k \in S} \mathcal{F}'_k = \varnothing. \tag{16}$$

Consequently, the sets $\operatorname{supp}(\mathcal{F}_k \wedge \mathcal{F}_{k'})$ are disjoint for different pairs $\{k, k'\}$, where $\operatorname{supp} \mathcal{F} = \{x \in [n] : \{x\} \in \mathcal{F}\}$. Hence, we can use Corollary 9 and obtain

$$\prod_{k=1}^{l} |\mathcal{F}'_k| \leqslant \left| \bigvee_{1 \leqslant k < k' \leqslant l} (\mathcal{F}'_k \wedge \mathcal{F}'_{k'}) \right| \left| \bigvee_{k=1}^{l} \mathcal{F}'_k \right| \leqslant \prod_{1 \leqslant k < k' \leqslant \ell} \binom{|\operatorname{supp}(\mathcal{F}'_k \wedge \mathcal{F}'_{k'})|}{\leqslant \mathbf{m}_{k,k'}} 2^n$$

where the last inequality is due to the following obvious fact:

$$\mathcal{F}'_k \wedge \mathcal{F}'_{k'} \subset \binom{\operatorname{supp}(\mathcal{F}'_k \wedge \mathcal{F}'_{k'})}{\leqslant \mathbf{m}_{k,k'}}.$$

Optimizing over the choices for cardinalities of supports of $\mathcal{F}'_k \wedge \mathcal{F}_k$ leads us to the optimization problem (12). As a reminder, it is formulated as follows:

$$\zeta^* = \max_{\mathbf{n}} \prod_{S \in \binom{[\ell]}{2}} \binom{\mathbf{n}_S}{\leqslant t}$$
$$\text{s.t.} \sum_{S \in \binom{[\ell]}{2}} \mathbf{n}_S = n.$$

The asymptotics of the solution was obtained in the proof of Theorem 16. Substituting the obtained bound on ζ^* into (15) concludes the proof. □

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