Common graphs with arbitrary connectivity and chromatic number

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Abstract

A graph H is *common* if the number of monochromatic copies of H in a 2-edge-colouring of the complete graph K_n is asymptotically minimised by the random colouring. We prove that, given k, r > 0, there exists a k-connected common graph with chromatic number at least r. The result is built upon the recent breakthrough of Král, Volec, and Wei who obtained common graphs with arbitrarily large chromatic number and answers a question of theirs.

1 Introduction

A central concept in graph Ramsey theory is the Ramsey multiplicity of a graph H, which counts the minimum number of monochromatic copies of H in a 2-edge-colouring of the *n*-vertex complete graph K_n . There are some graphs H, the so-called common graphs, such that the number of monochromatic H-copies in a 2-edge-colouring of K_n is asymptotically minimised by the random colouring. For example, Goodman's formula [8] implies that a triangle K_3 is common, which is one of the earliest results in the area.

Partly inspired by Goodman's formula, Erdős [5] conjectured that every complete graph is common. This was subsequently generalised by Burr and Rosta [1], who conjectured that every graph is common. In the late 1980s, both conjectures were disproved by Thomason [18] and by Sidorenko [15], respectively. Since then, there have been numerous attempts to find new common (or uncommon) graphs, e.g., [6, 11, 17]. Although the complete classification seems to be still out of reach, new common graphs have been found during the last decade by using some advances on Sidorenko's conjecture [16] or the computer-assisted flag algebra method [14]. For more results along these lines, we refer the reader to one of the most recent results [4] on Sidorenko's conjecture and some applications of the flag algebra method [9, 10] with references therein.

Despite all these studies on common graphs, all the known common graphs only had chromatic numbers at most four. This motivated a natural question, appearing in [2, 10], to find a common graphs with arbitrarily large chromatic number. This question remained open until its very recent resolution by Král', Volec, and Wei [12]. Since their construction connects a graph with high chromatic number and girth to a copy of a complete bipartite graph by a long path, they asked [12, Problem 25] if highly connected common graphs with large chromatic number exist. We answer this question in the affirmative.

Theorem 1.1. Let k and r be positive integers. Then there exists a k-connected common graph with chromatic number at least r.

We remark that this short follow-up note to the recent result only partially presents various in-depth studies on common graphs and relevant questions. For a modern review of a variety of results in the area, we refer the reader to recent articles [7, 12].

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2 Proof of the main theorem

A useful setting to analyse commonality of graphs is to use the modern theory of dense graph limits [13]. A graphon is a two-variable symmetric measurable function $W : [0,1]^2 \rightarrow [0,1]$ and the homomorphism density of a graph H is defined by

$$t(H,W) := \int \prod_{uv \in E(H)} W(x_u, x_v) \ d\mu^{V(H)},$$

where μ denotes the Lebesgue measure on [0, 1]. In this language, a graph H is common if and only if $t(H, W) + t(H, 1 - W) \ge 2^{1-e(H)}$ for every graphon W, where e(H) denotes the number of edges in H.

The *q*-book H_I^q of H along an independent set $I \subseteq V(H)$ of H is the graph obtained by taking q vertex-disjoint copies of H and identifying the corresponding vertices in I. The following lemma is a straightforward consequence of Jensen's inequality.

Lemma 2.1. Let I be an independent set of a graph H. If H is common, then H_I^q is also common for every positive integer q.

Proof. By a standard application of Jensen's inequality, the inequality $t(H_I^q, W) \ge t(H, W)^q$ holds for every graphon W. Therefore,

$$\begin{split} t(H_I^q, W) + t(H_I^q, 1 - W) &\geq t(H, W)^q + t(H, 1 - W)^q \\ &\geq 2 \cdot \left(\frac{t(H, W) + t(H, 1 - W)}{2}\right)^q \\ &\geq 2^{1 - q \cdot e(H)} = 2^{1 - e(H_I^q)}, \end{split}$$

where the second inequality is again by convexity and the last inequality uses commonality of H. This proves commonality of H_I^q .

To summarise, commonality is preserved under the q-book operation. Another advantage of the operation is that it preserves chromatic numbers. Indeed, a proper colouring of H can be naturally extended to H_I^q by assigning the same colour as a vertex of H to its 'clones' in H_I^q . Our key idea is to repeatedly apply the q-book operation to a common graph H, which increases connectivity while maintaining the chromatic number and commonality of H.

First, enumerate the vertices in an r-vertex graph H by $V(H) = \{v_1, v_2, \dots, v_r\}$. Let $H_0 := H$ and let $H_i := (H_{i-1})_{U_i}^q$, the q-book of H_{i-1} along U_i , where U_i is the set of all copies of v_i in H_{i-1} . The q-bookpile H(q) of H is then the graph $H(q) := H_r$ after the full r-step iteration. It is not hard to see that this graph H(q) is independent of the initial enumeration and hence well-defined. For example, if $H = K_r$ and q = 2, then H(2) is the line graph of an r-dimensional hypercube graph. This graph in fact appeared in [3] in a different context, which partly inspired our approach. As each H_i decomposes to q edge-disjoint copies of H_{i-1} , the q-bookpile H(q) decomposes to q^r edge-disjoint copies of H. To distinguish these, we say that the q^r edge-disjoint H-subgraphs of H(q) as the standard copies of H in H(q).

As already sketched, the following theorem together with the construction of connected common graphs H with arbitrarily large chromatic number in [12] implies Theorem 1.1:

Theorem 2.2. Let H be a connected graph. For every positive integer k, there exists q = q(k, H) such that the q-bookpile H(q) of H is k-connected.

To analyse connectivity of H(q), we consider an auxiliary hypergraph on V(H(q)) whose edge set consists of the standard copies of H in H(q). We shall first describe what this hypergraph looks like.

Write $[q] := \{1, 2, \dots, q\}$ and let α be a variable. Let $V(q, r, \alpha)$ be the set of r-tuples $v = (n_1, n_2, \dots, n_r)$, where all but exactly one entry are in [q] and the one exceptional entry is α . We call this unique entry the α -bit of v. Let \mathcal{H}_q^r be the r-uniform hypergraph on $V(q, r, \alpha)$ with the edge set $[q]^r$, where a vertex $v = (n_1, n_2, \dots, n_r)$ with $n_i = \alpha$ is incident to an edge e if substituting α by an integer value in [q] gives the edge $e \in [q]^r$. Note that \mathcal{H}_q^r is always a linear r-graph. Indeed, the codegree of a vertex pair is one if they share all the non- α -bits and zero otherwise. In particular, if q = 2, then this is the line hypergraph of the r-dimensional hypercube graph.

Proposition 2.3. Let \mathcal{H} be the auxiliary r-graph on V(H(q)) whose edge set consists of the standard copies of H in H(q). Then \mathcal{H} is isomorphic to \mathcal{H}_{q}^{r} .

Proof. At the *i*-th iteration of the blow-up procedure, each copy of v_j , $j \neq i$, is replaced by q copies of it, each of which is in the edge-disjoint copies of H_{i-1} glued along copies of v_i 's. By enumerating the q edge-disjoint copies of H_{i-1} in H_i , we label each copy of $v_j \in V(H)$ by a vector $(n_1, n_2, \dots, n_r) \in V(q, r, \alpha)$, where $n_j = \alpha$ and $n_i, i \neq j$, indicates that the vertex is in the n_i -th copy of H_{i-1} in H_i . Let $\phi: V(\mathcal{H}) \to V(q, r, \alpha)$ be this labelling map.

We claim that this function ϕ is an isomorphism from \mathcal{H} to \mathcal{H}_q^r . Indeed, two vertices labelled by (n_1, n_2, \dots, n_r) and (m_1, m_2, \dots, m_r) , respectively, are in the same standard *H*-copy if and only if $m_i = n_i$ for all *i* except their α -bits. Hence, *r* vertices in $V(\mathcal{H})$ form an edge if and only if their labels by ϕ in $V(q, r, \alpha)$ form an edge in $V(\mathcal{H}_q^r)$, which proves the claim.

From now on, we shall identify the r-graph \mathcal{H} with \mathcal{H}_q^r . In a linear hypergraph, a path P from a vertex u to another vertex v is an alternating sequence $v_0e_1v_1e_2\cdots v_{\ell-1}e_\ell v_\ell$ of vertices and edges, where $v_0 = u$, $v_\ell = v$, $\{v_i, v_{i+1}\} \subseteq e_{i+1}$, and any non-consecutive edges are disjoint. Two paths $ue_1v_1e_2\cdots v_{\ell-1}e_\ell v$ and $ue'_1v'_1e'_2\cdots v'_{t-1}e'_t v$ from u to v are internally vertex-disjoint if $e_1 \cap e'_1 = \{u\}$, $e_\ell \cap e'_t = \{v\}$, and all the other pairs e_i and e'_i are disjoint.

We say that the two paths are *vertex-disjoint* if all edges of one path are disjoint from all edges of the other. Multiple paths P_1, P_2, \dots, P_k are *(internally) vertex-disjoint* if they are pairwise (internally) vertex-disjoint. An *r*-graph \mathcal{G} is *k*-connected if there are at least *k* internally vertexdisjoint paths from a vertex *u* to another vertex *v* for all pairs of distinct vertices *u* and *v*. We show that \mathcal{H}_q^r is highly connected in this sense for large enough *q* in the following proposition, whose proof will be postponed for a while.

Proposition 2.4. For integers $k, r \geq 2$, there exists $q = q_{k,r}$ such that \mathcal{H}_q^r is k-connected.

Let $W_i, i \in [q]$, and U_r be subsets of $V(q, r, \alpha)$ defined by

$$W_i := \{ v = (n_1, \cdots, n_r) : n_r = i \}$$
 and $U_r := \{ u = (m_1, \cdots, m_r) : m_r = \alpha \}.$

Let $\mathcal{H}_i = \mathcal{H}_q^r[W_i \cup U_r]$ for brevity. The *r*-extension of an (r-1)-graph \mathcal{G} is the *r*-graph obtained by adding $e(\mathcal{G})$ extra vertices, each of which is added to a unique (r-1)-uniform edge in \mathcal{G} . Then \mathcal{H}_i is isomorphic to the *r*-extension of a copy of the (r-1)-graph \mathcal{H}_q^{r-1} on W_i by the isomorphism that maps each vertex $v = (n_1, \cdots, n_{r-1}, i)$ in \mathcal{H}_i to $(n_1, \cdots, n_{r-1}) \in V(q, r-1, \alpha)$ and $(n_1, \cdots, n_{r-1}, \alpha)$ to the extra vertex added to extend the edge (n_1, \cdots, n_{r-1}) .

For vertex subsets U and V, a U–V path is a path $v_0e_1v_1e_2\cdots v_{\ell-1}e_\ell v_\ell$ such that $v_0 \in U$, $v_\ell \in V$, and $v_i \notin U \cup V$ for each *i* distinct from 0 and ℓ .

Lemma 2.5. For $r \ge 3$ and $1 \le s \le q$, let U and V be subsets of U_r of size at least s. Then there exist vertex-disjoint U-V paths Q_1, Q_2, \cdots, Q_s such that each Q_i is a path in \mathcal{H}_i .

Proof. Consider the auxiliary graph G on U_r such that $ww' \in E(G)$ if and only if w and w' share a neighbour in \mathcal{H}_i . That is, ww' is an edge in G if $w = (n_1, \dots, n_{r-1}, \alpha)$ and $w' = (n'_1, \dots, n'_{r-1}, \alpha)$ differ by exactly one entry, which is at the α -bit of their common neighbour in W_i . Hence, this graph G is isomorphic to the graph K_q^{r-1} obtained by taking Cartesian product of r-1 copies of K_q and moreover, the graph G is independent of the choice of $i \in [q]$. In particular, G is (r-1)(q-1)-connected, see, e.g., Theorem 1 in [19]. By Menger's theorem, there are at least s vertex-disjoint U-V paths in G, which we denote by P_1, P_2, \dots, P_s , provided $(r-1)(q-1) \geq q \geq s$.

Our goal is to construct a U-V path Q_i in \mathcal{H}_i by using P_i . We may assume that U and V are disjoint, as otherwise, one may assign a trivial path at each vertex in the intersection and consider $U' := U \setminus V$ and $V' = V \setminus U$ instead of U and V, respectively. It then suffices to find s' vertex-disjoint U'-V' paths, s' < s, where induction on s applies.

We choose vertex-disjoint paths P_1, P_2, \dots, P_s that minimise the sum of the length of each path. Then each P_i is an induced path, i.e., there are no *G*-edges on $V(P_i)$ other than $E(P_i)$. To see this, let $P_i = u_0 u_1 \cdots u_\ell$ with $u_0 = u$ and $u_\ell = v$. If there is an edge $u_i u_j$ with i + 1 < j, then one can shorten the length of P_i by replacing the path $u_i u_{i+1} \cdots u_j$ by $u_i u_j$. The internal vertices of the shorter path P'_i is still non-empty as $uv \notin E(G)$ and disjoint from the internal vertices of other P_j 's, so we strictly reduce the sum of the *s* vertex-disjoint paths.

Now each U-V path $P_i = u_0 u_1 u_2 \cdots u_\ell$ yields a U-V path Q_i in \mathcal{H}_i . Indeed, there exists a unique edge $e_j \in \mathcal{H}_i$ containing u_j such that e_j and e_{j+1} share a vertex w_j in W_i by definition of G. Furthermore, two non-consecutive edges e_j and $e_{j'}$, j+1 < j', are always disjoint, as otherwise $u_j u_{j'} \in E(G)$. Therefore, $u_0 e_1 u_1 e_2 \cdots u_{\ell-1} e_\ell u_\ell$ is a path in \mathcal{H}_i . It then remains to check whether Q_1, \cdots, Q_k in \mathcal{H}_q^r are vertex-disjoint. The vertices in Q_i and Q_j are in $W_i \cup U_r$ and $W_j \cup U_r$, respectively. Indeed, the two sets W_i and W_j are disjoint and the vertices of Q_i and Q_j in U_r are disjoint too, as they are exactly vertices of P_i and P_j , respectively.

Proof of Proposition 2.4. If r = 2 then \mathcal{H}_q^r is a copy of $K_{q,q}$, which is q-connected. We may hence assume that $r \geq 3$. Take $q \geq \max\{q_{r-1,k}, 3(k+1)\}$ which is a multiple of 3. Let u, v be distinct vertices in \mathcal{H}_q^r . By induction on r, \mathcal{H}_q^{r-1} is k-connected. As \mathcal{H}_i is the r-extension of \mathcal{H}_q^{r-1} , there are at least k internally vertex-disjoint paths in \mathcal{H}_i from u to v if both vertices are in W_i .

Suppose that $u, v \in U_r$. For $1 \le i \le q/3$, let P_i be the path $ue_{i,1}w_ie_{i,2}u_i$ in \mathcal{H}_i , i.e., $w_i \in W_i$, $u_i \in U_r$, and $e_{i,1}$ is the only edge in \mathcal{H}_i containing u. We may further assume that all u_i 's are distinct, as there are q-1 neighbours of w_j in U_r except u. Analogously, take paths $P'_j = ve_{j,1}w_je_{j,2}v_j$ for $q/3 < j \le 2q/3$ in \mathcal{H}_j where v_j 's are all distinct. Applying Lemma 2.5 with $U = \{u_i : 1 \le i \le q/3\}$, $V = \{v_j : q/3 < j \le 2q/3\}$, and s = q/3 gives q/3 vertex-disjoint U-V paths, each of which uses a unique \mathcal{H}_t for some t > 2q/3. Here we relabel \mathcal{H}_i 's if necessary. Thus, concatenating these U-Vpaths with P_i and P'_i yields at least k internally vertex-disjoint paths from u to v.

Next, suppose that $u \in U_r$ and $v \in W_j$. For all $i \in [q/3] \setminus \{j\}$, we analogously collect paths P_i of length two from u such that each P_i is in \mathcal{H}_i and ends at $u_i \in U_r$, where u_i 's are all distinct. There are q neighbours of v in U_r , which we denote by $N(v; U_r)$. Then again by Lemma 2.5, there are at least q/3 - 1 vertex-disjoint U-V paths from $U = \{u_i : i \in [q/3] \setminus \{j\}\}$ to $V = N(v; U_r)$, each of which uses distinct \mathcal{H}_t such that $t \neq j$ and t > q/3. Concatenating these U-V paths with P_i 's and the edges incident to v gives k internally vertex-disjoint paths from u to v.

Lastly, suppose that $u \in W_i$ and $v \in W_j$ for $i \neq j$. Let $N(u; U_r)$ and $N(v; U_r)$ be neighbours of u and v in U_r , respectively. Then Lemma 2.5 gives q vertex-disjoint $N(u; U_r) - N(v; U_r)$ paths. After deleting those paths in \mathcal{H}_i or \mathcal{H}_j , there are still at least k paths left, which allow us to make k internally vertex-disjoint path from u to v.

Theorem 2.2 follows from the fact that internally disjoint paths in \mathcal{H}_q^r translate to internally disjoint paths in H(q).

Lemma 2.6. Let H be a connected graph and let P_1, P_2, \dots, P_k be k internally vertex-disjoint paths from u to v in \mathcal{H}_q^r . Then there exist internally vertex-disjoint paths Q_1, Q_2, \dots, Q_k from u to v in H(q) such that each Q_i , $i \in [k]$, only uses those edges and vertices in the standard H-copies that correspond to edges in P_i .

Proof. Let $P_i = v_{i,0}e_{i,1}v_{i,1}e_{i,2}\cdots v_{i,\ell_{i-1}}e_{i,\ell_i}v_{i,\ell_i}$. As H is connected, there exists a path $Q_{i,j+1}$ from $v_{i,j}$ to $v_{i,j+1}$ in the standard copy of H that corresponds to the edge $e_{i,j+1}$. For paths P from x to y and P' from y to z, we write xPyP'z for the concatenation of the two paths from x to z. For each $i \in [k]$, let $Q_i = v_{i,0}Q_{i,1}v_{i,1}Q_{i,2}\cdots v_{i,\ell_{i-1}}Q_{i,\ell_i}v_{i,\ell_i}$. We claim that these paths Q_1, Q_2, \cdots, Q_k are internally vertex-disjoint. Indeed, $Q_{i,1}$ and $Q_{i',1}$ are in the H-copies that correspond to $e_{i,1}$ and $e_{i',1}$, respectively, who share the vertex $u = v_{i,0} = v_{i',0}$ only; by the same reason, Q_{i,ℓ_i} and $Q_{i',\ell_{i'}}$ are also disjoint except the vertex $v = v_{i,\ell_i} = v_{i',\ell_{i'}}$; the other $Q_{i,j}$ and $Q_{i',j'}$ are vertex-disjoint, since $e_{i,j}$ and $e_{i',j'}$ are disjoint edges in \mathcal{H}_q^r .

3 Concluding remarks

After Theorem 1.1, it would be natural to ask for examples of common graphs that are even more challenging to find. We suggest to find a common graph with arbitrarily large girth, chromatic number, and connectivity.

Question 3.1. Let $r, k, g \ge 3$ be integers. Does there exist an r-chromatic k-connected common graph with girth at least g?

We believe that such a common graph exists; however, our blown-up graph H(q) in Theorem 1.1 may decrease the girth of H, as the construction produces 4-cycles whenever $q \ge 2$ and E(H) is nonempty. This suggests that solving Question 3.1 might require new ideas.

Acknowledgements. This work is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government MSIT NRF-2022R1C1C1010300 and Samsung STF Grant SSTF-BA2201-02. The second author was also supported by IBS-R029-C4. We would like to thank Dan Král', Jan Volec, and Fan Wei for helpful discussions. We are also grateful to anonymous referees for their careful reviews.

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