# Common graphs with arbitrary connectivity and chromatic number 

Sejin Ko* Joonkyung Lee ${ }^{\dagger}$


#### Abstract

A graph $H$ is common if the number of monochromatic copies of $H$ in a 2 -edge-colouring of the complete graph $K_{n}$ is asymptotically minimised by the random colouring. We prove that, given $k, r>0$, there exists a $k$-connected common graph with chromatic number at least $r$. The result is built upon the recent breakthrough of Král', Volec, and Wei who obtained common graphs with arbitrarily large chromatic number and answers a question of theirs.


## 1 Introduction

A central concept in graph Ramsey theory is the Ramsey multiplicity of a graph $H$, which counts the minimum number of monochromatic copies of $H$ in a 2-edge-colouring of the $n$-vertex complete graph $K_{n}$. There are some graphs $H$, the so-called common graphs, such that the number of monochromatic $H$-copies in a 2-edge-colouring of $K_{n}$ is asymptotically minimised by the random colouring. For example, Goodman's formula [8] implies that a triangle $K_{3}$ is common, which is one of the earliest results in the area.

Partly inspired by Goodman's formula, Erdős [5] conjectured that every complete graph is common. This was subsequently generalised by Burr and Rosta [1], who conjectured that every graph is common. In the late 1980s, both conjectures were disproved by Thomason [18] and by Sidorenko [15], respectively. Since then, there have been numerous attempts to find new common (or uncommon) graphs, e.g., $[6,11,17]$. Although the complete classification seems to be still out of reach, new common graphs have been found during the last decade by using some advances on Sidorenko's conjecture [16] or the computer-assisted flag algebra method [14]. For more results along these lines, we refer the reader to one of the most recent results [4] on Sidorenko's conjecture and some applications of the flag algebra method $[9,10]$ with references therein.

Despite all these studies on common graphs, all the known common graphs only had chromatic numbers at most four. This motivated a natural question, appearing in $[2,10]$, to find a common graphs with arbitrarily large chromatic number. This question remained open until its very recent resolution by Král', Volec, and Wei [12]. Since their construction connects a graph with high chromatic number and girth to a copy of a complete bipartite graph by a long path, they asked [12, Problem 25] if highly connected common graphs with large chromatic number exist. We answer this question in the affirmative.

Theorem 1.1. Let $k$ and $r$ be positive integers. Then there exists a $k$-connected common graph with chromatic number at least $r$.

We remark that this short follow-up note to the recent result only partially presents various in-depth studies on common graphs and relevant questions. For a modern review of a variety of results in the area, we refer the reader to recent articles [7, 12].

[^0]
## 2 Proof of the main theorem

A useful setting to analyse commonality of graphs is to use the modern theory of dense graph limits [13]. A graphon is a two-variable symmetric measurable function $W:[0,1]^{2} \rightarrow[0,1]$ and the homomorphism density of a graph $H$ is defined by

$$
t(H, W):=\int \prod_{u v \in E(H)} W\left(x_{u}, x_{v}\right) d \mu^{V(H)}
$$

where $\mu$ denotes the Lebesgue measure on $[0,1]$. In this language, a graph $H$ is common if and only if $t(H, W)+t(H, 1-W) \geq 2^{1-e(H)}$ for every graphon $W$, where $e(H)$ denotes the number of edges in $H$.

The $q$-book $H_{I}^{q}$ of $H$ along an independent set $I \subseteq V(H)$ of $H$ is the graph obtained by taking $q$ vertex-disjoint copies of $H$ and identifying the corresponding vertices in $I$. The following lemma is a straightforward consequence of Jensen's inequality.

Lemma 2.1. Let I be an independent set of a graph $H$. If $H$ is common, then $H_{I}^{q}$ is also common for every positive integer $q$.

Proof. By a standard application of Jensen's inequality, the inequality $t\left(H_{I}^{q}, W\right) \geq t(H, W)^{q}$ holds for every graphon $W$. Therefore,

$$
\begin{aligned}
t\left(H_{I}^{q}, W\right)+t\left(H_{I}^{q}, 1-W\right) & \geq t(H, W)^{q}+t(H, 1-W)^{q} \\
& \geq 2 \cdot\left(\frac{t(H, W)+t(H, 1-W)}{2}\right)^{q} \\
& \geq 2^{1-q \cdot e(H)}=2^{1-e\left(H_{I}^{q}\right)}
\end{aligned}
$$

where the second inequality is again by convexity and the last inequality uses commonality of $H$. This proves commonality of $H_{I}^{q}$.

To summarise, commonality is preserved under the $q$-book operation. Another advantage of the operation is that it preserves chromatic numbers. Indeed, a proper colouring of $H$ can be naturally extended to $H_{I}^{q}$ by assigning the same colour as a vertex of $H$ to its 'clones' in $H_{I}^{q}$. Our key idea is to repeatedly apply the $q$-book operation to a common graph $H$, which increases connectivity while maintaining the chromatic number and commonality of $H$.

First, enumerate the vertices in an $r$-vertex graph $H$ by $V(H)=\left\{v_{1}, v_{2}, \cdots, v_{r}\right\}$. Let $H_{0}:=H$ and let $H_{i}:=\left(H_{i-1}\right)_{U_{i}}^{q}$, the $q$-book of $H_{i-1}$ along $U_{i}$, where $U_{i}$ is the set of all copies of $v_{i}$ in $H_{i-1}$. The $q$-bookpile $H(q)$ of $H$ is then the graph $H(q):=H_{r}$ after the full $r$-step iteration. It is not hard to see that this graph $H(q)$ is independent of the initial enumeration and hence well-defined. For example, if $H=K_{r}$ and $q=2$, then $H(2)$ is the line graph of an $r$-dimensional hypercube graph. This graph in fact appeared in [3] in a different context, which partly inspired our approach. As each $H_{i}$ decomposes to $q$ edge-disjoint copies of $H_{i-1}$, the $q$-bookpile $H(q)$ decomposes to $q^{r}$ edge-disjoint copies of $H$. To distinguish these, we say that the $q^{r}$ edge-disjoint $H$-subgraphs of $H(q)$ as the standard copies of $H$ in $H(q)$.

As already sketched, the following theorem together with the construction of connected common graphs $H$ with arbitrarily large chromatic number in [12] implies Theorem 1.1:

Theorem 2.2. Let $H$ be a connected graph. For every positive integer $k$, there exists $q=q(k, H)$ such that the $q$-bookpile $H(q)$ of $H$ is $k$-connected.

To analyse connectivity of $H(q)$, we consider an auxiliary hypergraph on $V(H(q))$ whose edge set consists of the standard copies of $H$ in $H(q)$. We shall first describe what this hypergraph looks like.

Write $[q]:=\{1,2, \cdots, q\}$ and let $\alpha$ be a variable. Let $V(q, r, \alpha)$ be the set of $r$-tuples $v=$ $\left(n_{1}, n_{2}, \cdots, n_{r}\right)$, where all but exactly one entry are in $[q]$ and the one exceptional entry is $\alpha$. We call this unique entry the $\alpha$-bit of $v$. Let $\mathcal{H}_{q}^{r}$ be the $r$-uniform hypergraph on $V(q, r, \alpha)$ with the edge set $[q]^{r}$, where a vertex $v=\left(n_{1}, n_{2}, \cdots, n_{r}\right)$ with $n_{i}=\alpha$ is incident to an edge $e$ if substituting $\alpha$ by an integer value in $[q]$ gives the edge $e \in[q]^{r}$. Note that $\mathcal{H}_{q}^{r}$ is always a linear $r$-graph. Indeed, the codegree of a vertex pair is one if they share all the non- $\alpha$-bits and zero otherwise. In particular, if $q=2$, then this is the line hypergraph of the $r$-dimensional hypercube graph.

Proposition 2.3. Let $\mathcal{H}$ be the auxiliary r-graph on $V(H(q))$ whose edge set consists of the standard copies of $H$ in $H(q)$. Then $\mathcal{H}$ is isomorphic to $\mathcal{H}_{q}^{r}$.

Proof. At the $i$-th iteration of the blow-up procedure, each copy of $v_{j}, j \neq i$, is replaced by $q$ copies of it, each of which is in the edge-disjoint copies of $H_{i-1}$ glued along copies of $v_{i}$ 's. By enumerating the $q$ edge-disjoint copies of $H_{i-1}$ in $H_{i}$, we label each copy of $v_{j} \in V(H)$ by a vector $\left(n_{1}, n_{2}, \cdots, n_{r}\right) \in V(q, r, \alpha)$, where $n_{j}=\alpha$ and $n_{i}, i \neq j$, indicates that the vertex is in the $n_{i}$-th copy of $H_{i-1}$ in $H_{i}$. Let $\phi: V(\mathcal{H}) \rightarrow V(q, r, \alpha)$ be this labelling map.

We claim that this function $\phi$ is an isomorphism from $\mathcal{H}$ to $\mathcal{H}_{q}^{r}$. Indeed, two vertices labelled by $\left(n_{1}, n_{2}, \cdots, n_{r}\right)$ and $\left(m_{1}, m_{2}, \cdots, m_{r}\right)$, respectively, are in the same standard $H$-copy if and only if $m_{i}=n_{i}$ for all $i$ except their $\alpha$-bits. Hence, $r$ vertices in $V(\mathcal{H})$ form an edge if and only if their labels by $\phi$ in $V(q, r, \alpha)$ form an edge in $V\left(\mathcal{H}_{q}^{r}\right)$, which proves the claim.

From now on, we shall identify the $r$-graph $\mathcal{H}$ with $\mathcal{H}_{q}^{r}$. In a linear hypergraph, a path $P$ from a vertex $u$ to another vertex $v$ is an alternating sequence $v_{0} e_{1} v_{1} e_{2} \cdots v_{\ell-1} e_{\ell} v_{\ell}$ of vertices and edges, where $v_{0}=u, v_{\ell}=v,\left\{v_{i}, v_{i+1}\right\} \subseteq e_{i+1}$, and any non-consecutive edges are disjoint. Two paths $u e_{1} v_{1} e_{2} \cdots v_{\ell-1} e_{\ell} v$ and $u e_{1}^{\prime} v_{1}^{\prime} e_{2}^{\prime} \cdots v_{t-1}^{\prime} e_{t}^{\prime} v$ from $u$ to $v$ are internally vertex-disjoint if $e_{1} \cap e_{1}^{\prime}=\{u\}$, $e_{\ell} \cap e_{t}^{\prime}=\{v\}$, and all the other pairs $e_{i}$ and $e_{j}^{\prime}$ are disjoint.

We say that the two paths are vertex-disjoint if all edges of one path are disjoint from all edges of the other. Multiple paths $P_{1}, P_{2}, \cdots, P_{k}$ are (internally) vertex-disjoint if they are pairwise (internally) vertex-disjoint. An $r$-graph $\mathcal{G}$ is $k$-connected if there are at least $k$ internally vertexdisjoint paths from a vertex $u$ to another vertex $v$ for all pairs of distinct vertices $u$ and $v$. We show that $\mathcal{H}_{q}^{r}$ is highly connected in this sense for large enough $q$ in the following proposition, whose proof will be postponed for a while.

Proposition 2.4. For integers $k, r \geq 2$, there exists $q=q_{k, r}$ such that $\mathcal{H}_{q}^{r}$ is $k$-connected.
Let $W_{i}, i \in[q]$, and $U_{r}$ be subsets of $V(q, r, \alpha)$ defined by

$$
W_{i}:=\left\{v=\left(n_{1}, \cdots, n_{r}\right): n_{r}=i\right\} \text { and } U_{r}:=\left\{u=\left(m_{1}, \cdots, m_{r}\right): m_{r}=\alpha\right\} .
$$

Let $\mathcal{H}_{i}=\mathcal{H}_{q}^{r}\left[W_{i} \cup U_{r}\right]$ for brevity. The $r$-extension of an $(r-1)$-graph $\mathcal{G}$ is the $r$-graph obtained by adding $e(\mathcal{G})$ extra vertices, each of which is added to a unique $(r-1)$-uniform edge in $\mathcal{G}$. Then $\mathcal{H}_{i}$ is isomorphic to the $r$-extension of a copy of the $(r-1)$-graph $\mathcal{H}_{q}^{r-1}$ on $W_{i}$ by the isomorphism that maps each vertex $v=\left(n_{1}, \cdots, n_{r-1}, i\right)$ in $\mathcal{H}_{i}$ to $\left(n_{1}, \cdots, n_{r-1}\right) \in V(q, r-1, \alpha)$ and $\left(n_{1}, \cdots, n_{r-1}, \alpha\right)$ to the extra vertex added to extend the edge ( $n_{1}, \cdots, n_{r-1}$ ).

For vertex subsets $U$ and $V$, a $U-V$ path is a path $v_{0} e_{1} v_{1} e_{2} \cdots v_{\ell-1} e_{\ell} v_{\ell}$ such that $v_{0} \in U$, $v_{\ell} \in V$, and $v_{i} \notin U \cup V$ for each $i$ distinct from 0 and $\ell$.

Lemma 2.5. For $r \geq 3$ and $1 \leq s \leq q$, let $U$ and $V$ be subsets of $U_{r}$ of size at least $s$. Then there exist vertex-disjoint $U-V$ paths $Q_{1}, Q_{2}, \cdots, Q_{s}$ such that each $Q_{i}$ is a path in $\mathcal{H}_{i}$.

Proof. Consider the auxiliary graph $G$ on $U_{r}$ such that $w w^{\prime} \in E(G)$ if and only if $w$ and $w^{\prime}$ share a neighbour in $\mathcal{H}_{i}$. That is, $w w^{\prime}$ is an edge in $G$ if $w=\left(n_{1}, \cdots, n_{r-1}, \alpha\right)$ and $w^{\prime}=\left(n_{1}^{\prime}, \cdots, n_{r-1}^{\prime}, \alpha\right)$ differ by exactly one entry, which is at the $\alpha$-bit of their common neighbour in $W_{i}$. Hence, this graph $G$ is isomorphic to the graph $K_{q}^{r-1}$ obtained by taking Cartesian product of $r-1$ copies of $K_{q}$ and moreover, the graph $G$ is independent of the choice of $i \in[q]$. In particular, $G$ is $(r-1)(q-1)$ connected, see, e.g., Theorem 1 in [19]. By Menger's theorem, there are at least $s$ vertex-disjoint $U-V$ paths in $G$, which we denote by $P_{1}, P_{2}, \cdots, P_{s}$, provided $(r-1)(q-1) \geq q \geq s$.

Our goal is to construct a $U-V$ path $Q_{i}$ in $\mathcal{H}_{i}$ by using $P_{i}$. We may assume that $U$ and $V$ are disjoint, as otherwise, one may assign a trivial path at each vertex in the intersection and consider $U^{\prime}:=U \backslash V$ and $V^{\prime}=V \backslash U$ instead of $U$ and $V$, respectively. It then suffices to find $s^{\prime}$ vertex-disjoint $U^{\prime}-V^{\prime}$ paths, $s^{\prime}<s$, where induction on $s$ applies.

We choose vertex-disjoint paths $P_{1}, P_{2}, \cdots, P_{s}$ that minimise the sum of the length of each path. Then each $P_{i}$ is an induced path, i.e., there are no $G$-edges on $V\left(P_{i}\right)$ other than $E\left(P_{i}\right)$. To see this, let $P_{i}=u_{0} u_{1} \cdots u_{\ell}$ with $u_{0}=u$ and $u_{\ell}=v$. If there is an edge $u_{i} u_{j}$ with $i+1<j$, then one can shorten the length of $P_{i}$ by replacing the path $u_{i} u_{i+1} \cdots u_{j}$ by $u_{i} u_{j}$. The internal vertices of the shorter path $P_{i}^{\prime}$ is still non-empty as $u v \notin E(G)$ and disjoint from the internal vertices of other $P_{j}$ 's, so we strictly reduce the sum of the $s$ vertex-disjoint paths.

Now each $U-V$ path $P_{i}=u_{0} u_{1} u_{2} \cdots u_{\ell}$ yields a $U-V$ path $Q_{i}$ in $\mathcal{H}_{i}$. Indeed, there exists a unique edge $e_{j} \in \mathcal{H}_{i}$ containing $u_{j}$ such that $e_{j}$ and $e_{j+1}$ share a vertex $w_{j}$ in $W_{i}$ by definition of $G$. Furthermore, two non-consecutive edges $e_{j}$ and $e_{j^{\prime}}, j+1<j^{\prime}$, are always disjoint, as otherwise $u_{j} u_{j^{\prime}} \in E(G)$. Therefore, $u_{0} e_{1} u_{1} e_{2} \cdots u_{\ell-1} e_{\ell} u_{\ell}$ is a path in $\mathcal{H}_{i}$. It then remains to check whether $Q_{1}, \cdots, Q_{k}$ in $\mathcal{H}_{q}^{r}$ are vertex-disjoint. The vertices in $Q_{i}$ and $Q_{j}$ are in $W_{i} \cup U_{r}$ and $W_{j} \cup U_{r}$, respectively. Indeed, the two sets $W_{i}$ and $W_{j}$ are disjoint and the vertices of $Q_{i}$ and $Q_{j}$ in $U_{r}$ are disjoint too, as they are exactly vertices of $P_{i}$ and $P_{j}$, respectively.

Proof of Proposition 2.4. If $r=2$ then $\mathcal{H}_{q}^{r}$ is a copy of $K_{q, q}$, which is $q$-connected. We may hence assume that $r \geq 3$. Take $q \geq \max \left\{q_{r-1, k}, 3(k+1)\right\}$ which is a multiple of 3 . Let $u, v$ be distinct vertices in $\mathcal{H}_{q}^{r}$. By induction on $r, \mathcal{H}_{q}^{r-1}$ is $k$-connected. As $\mathcal{H}_{i}$ is the $r$-extension of $\mathcal{H}_{q}^{r-1}$, there are at least $k$ internally vertex-disjoint paths in $\mathcal{H}_{i}$ from $u$ to $v$ if both vertices are in $W_{i}$.

Suppose that $u, v \in U_{r}$. For $1 \leq i \leq q / 3$, let $P_{i}$ be the path $u e_{i, 1} w_{i} e_{i, 2} u_{i}$ in $\mathcal{H}_{i}$, i.e., $w_{i} \in W_{i}$, $u_{i} \in U_{r}$, and $e_{i, 1}$ is the only edge in $\mathcal{H}_{i}$ containing $u$. We may further assume that all $u_{i}$ 's are distinct, as there are $q-1$ neighbours of $w_{j}$ in $U_{r}$ except $u$. Analogously, take paths $P_{j}^{\prime}=v e_{j, 1} w_{j} e_{j, 2} v_{j}$ for $q / 3<j \leq 2 q / 3$ in $\mathcal{H}_{j}$ where $v_{j}$ 's are all distinct. Applying Lemma 2.5 with $U=\left\{u_{i}: 1 \leq i \leq q / 3\right\}$, $V=\left\{v_{j}: q / 3<j \leq 2 q / 3\right\}$, and $s=q / 3$ gives $q / 3$ vertex-disjoint $U-V$ paths, each of which uses a unique $\mathcal{H}_{t}$ for some $t>2 q / 3$. Here we relabel $\mathcal{H}_{i}$ 's if necessary. Thus, concatenating these $U-V$ paths with $P_{i}$ and $P_{j}^{\prime}$ yields at least $k$ internally vertex-disjoint paths from $u$ to $v$.

Next, suppose that $u \in U_{r}$ and $v \in W_{j}$. For all $i \in[q / 3] \backslash\{j\}$, we analogously collect paths $P_{i}$ of length two from $u$ such that each $P_{i}$ is in $\mathcal{H}_{i}$ and ends at $u_{i} \in U_{r}$, where $u_{i}$ 's are all distinct. There are $q$ neighbours of $v$ in $U_{r}$, which we denote by $N\left(v ; U_{r}\right)$. Then again by Lemma 2.5, there are at least $q / 3-1$ vertex-disjoint $U-V$ paths from $U=\left\{u_{i}: i \in[q / 3] \backslash\{j\}\right\}$ to $V=N\left(v ; U_{r}\right)$, each of which uses distinct $\mathcal{H}_{t}$ such that $t \neq j$ and $t>q / 3$. Concatenating these $U-V$ paths with $P_{i}$ 's and the edges incident to $v$ gives $k$ internally vertex-disjoint paths from $u$ to $v$.

Lastly, suppose that $u \in W_{i}$ and $v \in W_{j}$ for $i \neq j$. Let $N\left(u ; U_{r}\right)$ and $N\left(v ; U_{r}\right)$ be neighbours of $u$ and $v$ in $U_{r}$, respectively. Then Lemma 2.5 gives $q$ vertex-disjoint $N\left(u ; U_{r}\right)-N\left(v ; U_{r}\right)$ paths.

After deleting those paths in $\mathcal{H}_{i}$ or $\mathcal{H}_{j}$, there are still at least $k$ paths left, which allow us to make $k$ internally vertex-disjoint path from $u$ to $v$.

Theorem 2.2 follows from the fact that internally disjoint paths in $\mathcal{H}_{q}^{r}$ translate to internally disjoint paths in $H(q)$.

Lemma 2.6. Let $H$ be a connected graph and let $P_{1}, P_{2}, \cdots, P_{k}$ be $k$ internally vertex-disjoint paths from $u$ to $v$ in $\mathcal{H}_{q}^{r}$. Then there exist internally vertex-disjoint paths $Q_{1}, Q_{2}, \cdots Q_{k}$ from $u$ to $v$ in $H(q)$ such that each $Q_{i}, i \in[k]$, only uses those edges and vertices in the standard $H$-copies that correspond to edges in $P_{i}$.

Proof. Let $P_{i}=v_{i, 0} e_{i, 1} v_{i, 1} e_{i, 2} \cdots v_{i, \ell_{i-1}} e_{i, \ell_{i}} v_{i, \ell_{i}}$. As $H$ is connected, there exists a path $Q_{i, j+1}$ from $v_{i, j}$ to $v_{i, j+1}$ in the standard copy of $H$ that corresponds to the edge $e_{i, j+1}$. For paths $P$ from $x$ to $y$ and $P^{\prime}$ from $y$ to $z$, we write $x P y P^{\prime} z$ for the concatenation of the two paths from $x$ to $z$. For each $i \in[k]$, let $Q_{i}=v_{i, 0} Q_{i, 1} v_{i, 1} Q_{i, 2} \cdots v_{i, \ell_{i-1}} Q_{i, \ell_{i}} v_{i, \ell_{i}}$. We claim that these paths $Q_{1}, Q_{2}, \cdots, Q_{k}$ are internally vertex-disjoint. Indeed, $Q_{i, 1}$ and $Q_{i^{\prime}, 1}$ are in the $H$-copies that correspond to $e_{i, 1}$ and $e_{i^{\prime}, 1}$, respectively, who share the vertex $u=v_{i, 0}=v_{i^{\prime}, 0}$ only; by the same reason, $Q_{i, \ell_{i}}$ and $Q_{i^{\prime}, \ell_{i^{\prime}}}$ are also disjoint except the vertex $v=v_{i, \ell_{i}}=v_{i^{\prime}, \ell_{i^{\prime}}}$; the other $Q_{i, j}$ and $Q_{i^{\prime}, j^{\prime}}$ are vertex-disjoint, since $e_{i, j}$ and $e_{i^{\prime}, j^{\prime}}$ are disjoint edges in $\mathcal{H}_{q}^{r}$.

## 3 Concluding remarks

After Theorem 1.1, it would be natural to ask for examples of common graphs that are even more challenging to find. We suggest to find a common graph with arbitrarily large girth, chromatic number, and connectivity.

Question 3.1. Let $r, k, g \geq 3$ be integers. Does there exist an $r$-chromatic $k$-connected common graph with girth at least $g$ ?

We believe that such a common graph exists; however, our blown-up graph $H(q)$ in Theorem 1.1 may decrease the girth of $H$, as the construction produces 4 -cycles whenever $q \geq 2$ and $E(H)$ is nonempty. This suggests that solving Question 3.1 might require new ideas.

Acknowledgements. This work is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government MSIT NRF-2022R1C1C1010300 and Samsung STF Grant SSTF-BA2201-02. The second author was also supported by IBS-R029-C4. We would like to thank Dan Král', Jan Volec, and Fan Wei for helpful discussions. We are also grateful to anonymous referees for their careful reviews.

## References

[1] Stefan A Burr and Vera Rosta. On the Ramsey multiplicities of graphs-problems and recent results. J. Graph Theory, 4(4):347-361, 1980. doi:10.1002/jgt.3190040403.
[2] David Conlon, Jacob Fox, and Benny Sudakov. Recent developments in graph ramsey theory. Surveys in combinatorics, 424(2015):49-118, 2015.
[3] David Conlon, Hiêp Hàn, Yury Person, and Mathias Schacht. Weak quasi-randomness for uniform hypergraphs. Random Structures Algorithms, 40:1-38, 2012. doi:10.1002/rsa. 20389.
[4] David Conlon and Joonkyung Lee. Sidorenko's conjecture for blow-ups. Discrete Anal., 2:1-14, 2021. doi:10.19086/da.
[5] Paul Erdős. On the number of complete subgraphs contained in certain graphs. Magyar Tud. Akad. Mat. Kutató Int. Közl, 7:459-464, 1962.
[6] Jacob Fox. There exist graphs with super-exponential Ramsey multiplicity constant. J. Graph Theory, 57(2):89-98, 2008. doi:10.1002/jgt. 20256.
[7] Jacob Fox and Yuval Wigderson. Ramsey multiplicity and the Turán coloring. arXiv:2207.07775.
[8] Adolph. W. Goodman. On sets of acquaintances and strangers at any party. Amer. Math. Monthly, 66:778-783, 1959. doi:10.2307/2310464.
[9] Andrzej Grzesik, Joonkyung Lee, Bernard Lidický, and Jan Volec. On tripartite common graphs. to appear in Combin. Probab. Comput. doi:10.1017/S0963548322000074.
[10] Hamed Hatami, Jan Hladký, Serguei Norine, Alexander Razborov, and Dan Král'. Non-three-colourable common graphs exist. Combin. Probab. Comput., 21(5):734-742, 2012. doi:10.1017/S0963548312000107.
[11] Chris Jagger, Pavel Štovíček, and Andrew Thomason. Multiplicities of subgraphs. Combinatorica, 16(1):123-141, 1996. doi:10.1007/BF01300130.
[12] Dan Král', Jan Volec, and Fan Wei. Common graphs with arbitrary chromatic number. arXiv:2206.05800.
[13] László. Lovász. Large Networks and Graph Limits. Amer. Math. Soc. Colloq. Publ. American Mathematical Society, 2012. URL: https://books.google.co.uk/books?id=FsFqHLid8sAC.
[14] Alexander A. Razborov. Flag algebras. J. Symbolic Logic, 72(4):1239-1282, 2007. doi:10.2178/jsl/1203350785.
[15] Alexander Sidorenko. Cycles in graphs and functional inequalities. Math. Notes, 46(5):877-882, 1989. doi:10.1007/BF01139620.
[16] Alexander Sidorenko. A correlation inequality for bipartite graphs. Graphs Combin., 9(2-4):201-204, 1993. URL: http://dx.doi.org/10.1007/BF02988307, doi:10.1007/BF02988307.
[17] Alexander Sidorenko. Randomness friendly graphs. Random Structures Algorithms, 8(3):229241, 1996. doi:10.1002/(SICI) 1098-2418(199605)8:3<229::AID-RSA6>3.3.C0;2-F.
[18] Andrew Thomason. A disproof of a conjecture of Erdős in Ramsey theory. J. London Math. Soc., 2(2):246-255, 1989. doi:10.1112/jlms/s2-39.2.246.
[19] Simon Špacapan. Connectivity of Cartesian products of graphs. Appl. Math. Lett., 21:682-685, 2008. doi:10.1016/j.aml.2007.06.010.


[^0]:    *Department of Mathematics, Hanyang University, Seoul. Email: tpwls960104@hanyang.ac.kr.
    ${ }^{\dagger}$ Department of Mathematics, Yonsei University, Seoul. Email: joonkyunglee@yonsei.ac.kr.

