# Minimal asymmetric hypergraphs* 

Yiting Jiang ${ }^{\dagger}$ Jaroslav Nešetřil ${ }^{\ddagger}$

September 20, 2023


#### Abstract

In this paper, we prove that for any $k \geq 3$, there exist infinitely many minimal asymmetric $k$-uniform hypergraphs. This is in a striking contrast to $k=2$, where it has been proved recently that there are exactly 18 minimal asymmetric graphs.

We also determine, for every $k \geq 1$, the minimum size of an asymmetric $k$ uniform hypergraph.


Keywords: asymmetric hypergraphs, $k$-uniform hypergraphs, automorphism.

## 1 Introduction

In this paper we deal with (undirected) graphs, oriented graphs and more general hypergraphs and relational structures. Let us start with (undirected) graphs: An (undirected) graph $G$ is called asymmetric if it does not have a non-identity automorphism. Any non-asymmetric graph is also called symmetric graph. A graph $G$ is called minimal asymmetric if $G$ is asymmetric and every non-trivial induced subgraph of $G$ is symmetric (here $G^{\prime}$ is a non-trivial subgraph of $G$ if $G^{\prime}$ is a subgraph of $G$ and $1<\left|V\left(G^{\prime}\right)\right|<|V(G)|$ ). In this paper all graphs are finite.

It is a folklore result that most graphs are asymmetric. In fact, as shown by Erdős and Rényi [3] most graphs on large sets are asymmetric in a very strong sense. The paper

This paper is part of a project that has received funding * from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 810115 - Dynasnet).


This work is partially supported by the ANR project HOSIGRA (ANR-17-CE40-0022).
The authors were also supported partially by the project 21-10775S of the Czech Science Foundation (GACR).
${ }^{\dagger}$ Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China. E-mail: ytjiang@zjnu.edu.cn.
${ }^{\ddagger}$ Computer Science Institute of Charles University (IUUK and ITI), Malostranské nám.25, 11800 Praha 1, Czech Republic. E-mail: nesetril@iuuk.mff.cuni.cz.
[3] contains many extremal results (and problems), which motivated further research on extremal properties of asymmetric graphs, see e.g. [6], [11]. This has been also studied in the context of the reconstruction conjecture [7], [5].

The second author bravely conjectured a long time ago that there are only finitely many minimal asymmetric graphs, see e.g. [2]. Partial results were given in [8], 9], [12] and recently this conjecture has been confirmed by Pascal Schweitzer and Patrick Schweitzer [10] (the list of 18 minimal asymmetric graphs has been isolated already in [8]):

Theorem 1 [10] There are exactly 18 minimal asymmetric undirected graphs up to isomorphism.

In this paper, we consider analogous questions for $k$-graphs (or $k$-uniform hypergraphs), i.e. pairs ( $X, \mathscr{M}$ ) where $\mathscr{M} \subseteq\binom{X}{k}=\{A \subseteq X ;|A|=k\}$. Induced subhypergraphs, asymmetric hypergraphs and minimal asymmetric hypergraphs are defined analogously as for graphs.

We prove two results related to minimal asymmetric $k$-graphs.
Denote by $n(k)$ the minimum number of vertices of an asymmetric $k$-graph.
Theorem $2 n(2)=6, n(3)=6, n(k)=k+2$ for $k \geq 4$.
Our second result disproves analogous minimality conjecture (i.e. a result analogous to Theorem 1) for $k$-graphs.

Theorem 3 For every integer $k \geq 3$, there exist infinitely many $k$-graphs that are minimal asymmetric.

In fact we prove the following stronger statement.
Theorem 4 For every integer $k \geq 3$, there exist infinitely many $k$-graphs $(X, \mathscr{M})$ such that

1. $(X, \mathscr{M})$ is asymmetric.
2. If $\left(X^{\prime}, \mathscr{M}^{\prime}\right)$ is a non-trivial sub-k-graph of $(X, \mathscr{M})$ with at least two vertices, then $\left(X^{\prime}, \mathscr{M}^{\prime}\right)$ is symmetric.

We call $k$-graphs that satisfy the two above properties strongly minimal asymmetric. So strongly minimal asymmetric $k$-graphs do not contain any non-trivial (not necessarily induced) asymmetric sub- $k$-graph. Note that some of the minimal asymmetric graphs fail to be strongly minimal. For instance, as depicted in Figure 1, the graph $X_{2}$ is minimal asymmetric but not strongly minimal asymmetric, since $X_{1}$ is a minimal asymmetric subgraph of $X_{2}$.

An involution of a graph $G$ is any non-identity automorphism $\phi$ for which $\phi \circ \phi$ is an identity. It was proved in [10] that all minimal asymmetric graphs are in fact minimal involution-free graphs. However, it is not the case for $k$-graphs: there are $k$-graphs that are (strongly) minimal asymmetric but not minimal involution-free (see examples after the proof of Theorem 4 in Section 3.1). We prove the following form of Theorem 4 relating minimal asymmetric $k$-graphs for involutions.

Theorem 5 For every integer $k \geq 6$, there exist infinitely many $k$-graphs ( $X, \mathscr{M}$ ) such that

1. $(X, \mathscr{M})$ is asymmetric.
2. If $\left(X^{\prime}, \mathscr{M}^{\prime}\right)$ is a sub-k-graph of $(X, \mathscr{M})$ with at least two vertices, then $\left(X^{\prime}, \mathscr{M}^{\prime}\right)$ has an involution.

Theorem 4 and Theorem 5 are proved by constructing a sequence of strongly minimal asymmetric $k$-graphs. We have two different constructions of increasing strength. In Section 3.1 we give a construction with all vertex degrees bounded by 3. A stronger construction which yields minimal asymmetric $k$-graphs ( $k \geq 6$ ) with respect to involutions is given in the proof of Theorem 5 in Section 3.2. In Section 4 we consider minimal asymmetric relations and their multiplicities and conclude with several open problems.

## 2 The proof of Theorem 2

Lemma 6 For $k \geq 3$, we have $n(k) \geq k+2$.
Proof. Assume that there exists an asymmetric $k$-graph $(X, \mathscr{M})$ with $|X|=k+1$. If for each vertex $u \in X$, there is a hyperedge $M \in \mathscr{M}$ such that $u \notin M$, then $\mathscr{M}=\binom{X}{k}$, which is symmetric. Otherwise there exists $u, v \in X$ such that $\{u, v\} \subset M$ for every edge $M \in \mathscr{M}$, or there exist $u^{\prime}, v^{\prime} \in X$ and $M_{1}, M_{2} \in \mathscr{M}$ such that $u^{\prime} \notin M_{1}$ and $v^{\prime} \notin M_{2}$. In the former case, there is an automorphism $\phi$ of $(X, \mathscr{M})$ such that $\phi(u)=v$ and $\phi(v)=u$. In the latter case there is an automorphism $\phi$ of $(X, \mathscr{M})$ such that $\phi\left(u^{\prime}\right)=v^{\prime}$ and $\phi\left(v^{\prime}\right)=u^{\prime}$. In either case we have a contradiction.

For a $k$-graph $G=(X, \mathscr{M})$, the set-complement of $G$ is defined as a $(|X|-k)$-graph $\bar{G}=(X, \overline{\mathscr{M}})=(X,\{X-M \mid M \in \mathscr{M}\})$. Denote by $\operatorname{Aut}(G)$ the set of all the automorphisms of $G$ and thus we have $\operatorname{Aut}(G)=\operatorname{Aut}(\bar{G})$. We define the degree of a vertex $v$ in a $k$-graph $G$ as $d_{G}(v)=|\{M \in \mathscr{M} ; v \in M\}|$.

Lemma 7 For $k \geq 4$, we have $n(k)=k+2$.
Proof. First, we construct an asymmetric 2-graph ( $X, \mathscr{M}$ ) with $|X|=k+2$ for each $k \geq 4$. Examples of such graphs $X_{1}$ and $T_{k+2}$ are depicted in Figure 1 .


Figure 1

For $k=4$, take the set-complement of $X_{1}$. For every $k \geq 5$, take the set-complement of $T_{k+2}$. It is easy to see that $X_{1}$ and $T_{k+2}(k \geq 5)$ are asymmetric. Thus set-complements $\bar{X}_{1}$ and $T_{k+2}^{-}(k \geq 5)$ are also asymmetric $k$-graphs.

Each of the set-complements $\bar{X}_{1}$ and $T_{k+2}^{-}$has $k+2$ vertices. Thus a non-trivial subgraph of each of them is symmetric, by Lemma 6 .

Lemma 8 For $k=3$, we have $n(3)=6$.


Figure 2: An asymmetric 3-graph with $|X|=6$
Proof. For $n(3) \leq 6$, consider the following 3-graph $G=(X, \mathscr{M})$ depicted on Figure 2 , $X=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\}, \mathscr{M}=\left\{\left\{v_{1}, v_{2}, v_{3}\right\},\left\{v_{2}, v_{4}, v_{5}\right\},\left\{v_{2}, v_{4}, v_{6}\right\},\left\{v_{3}, v_{4}, v_{5}\right\}\right\}$. Observe that $d_{G}\left(v_{1}\right)=d_{G}\left(v_{6}\right)=1, d_{G}\left(v_{2}\right)=d_{G}\left(v_{4}\right)=3, d_{G}\left(v_{3}\right)=d_{G}\left(v_{5}\right)=2$. It is not difficult to see $G$ is asymmetric.

Now we shall prove that $n(3) \geq 6$.
Assume that there exists an asymmetric 3 -graph $H=(X, \mathscr{M})$ with $|X|=5$. Let $X=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right\}$. Without loss of generality, let $M=\left\{v_{1}, v_{2}, v_{3}\right\} \in \mathscr{M}$. Then there exists an edge $M \in \mathscr{M}$ such that $v_{4} \in M$ and $v_{5} \notin M$, or $v_{4} \notin M$ and $v_{5} \in M$.
$k$-graph $H$ is asymmetric if and only if $\left(X,\binom{X}{3}-\mathscr{M}\right)$ is asymmetric. Thus we can sufficiently consider that $|\mathscr{M}| \leq \frac{\binom{5}{3}}{2}=5$. If $d_{H}\left(v_{4}\right)=d_{H}\left(v_{5}\right)$, which means both of $v_{4}$ and $v_{5}$ have degree 1 or 2 , then there exists an automorphism $\phi$ of $H$ such that $\phi\left(v_{4}\right)=v_{5}$. Assume that $d_{H}\left(v_{4}\right)>d_{H}\left(v_{5}\right)$.


Figure 3

Case 1. There is no edge $M \in \mathscr{M}$ such that $\left\{v_{4}, v_{5}\right\} \subseteq M$
It is sufficient to consider two subcases: $d_{H}\left(v_{4}\right)=2$ and $d_{H}\left(v_{5}\right)=1$, or $d_{H}\left(v_{4}\right)=3$ and $d_{H}\left(v_{5}\right)=1$. In the first subcase, up to isomorphism, we obtain two different graphs as Figure 3(a) and 3(b) shown. There exists an automorphism $\phi$ of $H$ such that $\phi\left(v_{2}\right)=v_{3}$ in (a) (resp. $\phi\left(v_{3}\right)=v_{4}$ in (b)). In the second subcase, there is only one possible graph as Figure 3(c) shown. Observe that there exists an automorphism $\phi$ of $H$ such that $\phi\left(v_{2}\right)=v_{3}$ or $\phi\left(v_{1}\right)=v_{4}$.

Case 2. There exists $M \in \mathscr{M}$ such that $\left\{v_{4}, v_{5}\right\} \subseteq M$
Let $\mathbb{M}=\left\{M \in \mathscr{M} ;\left\{v_{4}, v_{5}\right\} \subseteq M\right\}$, then by symmetric, $|\mathbb{M}| \neq 3$. Since $d_{H}\left(v_{4}\right)>d_{H}\left(v_{5}\right)$ and $|\mathscr{M}| \leq 5$, the graphs in this case we need to consider can be divide as follow:

1) $|\mathscr{M}-\mathbb{M}|=2$, as Figure $4(\mathrm{a}), 4(\mathrm{~b})$ and $4(\mathrm{c})$ shown.
2) $|\mathscr{M}-\mathbb{M}|=3$, as Figure $4(\mathrm{~d}), 4(\mathrm{e})$ and $4(\mathrm{f})$ shown.
3) $|\mathscr{M}-\mathbb{M}|=4$, as Figure $4(\mathrm{~g}), 4(\mathrm{~h}), 4(\mathrm{i}), 4(\mathrm{j}), 4(\mathrm{k})$ and $4(\mathrm{l})$ shown.

It is easily to observe that there is an automorphism $\phi$ such that $\phi\left(v_{1}\right)=v_{4}$ and $\phi\left(v_{3}\right)=v_{5}$ in Figure $4(\mathrm{a}), \phi\left(v_{1}\right)=v_{5}$ and $\phi\left(v_{2}\right)=v_{4}$ in Figure $4(\mathrm{~b}), \phi\left(v_{1}\right)=v_{2}$ in Figure 4(c), $\phi\left(v_{1}\right)=v_{3}$ in Figure $4(\mathrm{~d}), \phi\left(v_{1}\right)=v_{4}$ in Figure $4(\mathrm{e}), \phi\left(v_{2}\right)=v_{4}$ and $\phi\left(v_{3}\right)=v_{5}$ in Figure $4(\mathrm{f}), \phi\left(v_{1}\right)=v_{3}$ or $\phi\left(v_{2}\right)=v_{4}$ in Figure $4(\mathrm{~g}), \phi\left(v_{1}\right)=v_{3}$ in Figure $4(\mathrm{~h}), \phi\left(v_{3}\right)=v_{4}$ in Figure $4(\mathrm{i}), \phi\left(v_{1}\right)=v_{4}$ or $\phi\left(v_{3}\right)=v_{5}$ in Figure $4(\mathrm{j}), \phi\left(v_{1}\right)=v_{2}$ and $\phi\left(v_{3}\right)=v_{5}$ in Figure $4(\mathrm{k}), \phi\left(v_{3}\right)=v_{4}$ in Figure 4(l). In each case, we obtain a contradiction.

## 3 Minimal asymmetric $k$-graphs

In this section, we give proofs of Theorem 4 and Theorem 5 .

### 3.1 Proof of Theorem 4

We define the following $k$-graphs for $k \geq 3, t \geq k-2$. (Note that for each positive integer $p$, we denote by $[p$ ] the set $\{0,1,2, \ldots, p-1\}$.)

$$
G_{k, t}=\left(X_{k, t}, \mathscr{E}_{k, t}\right),
$$


(a)

(d)

(g)

(j)

(b)

(e)

(h)

(k)

(c)

(f)

(i)

(1)

Figure 4

$$
\begin{aligned}
& \left.X_{k, t}=\left\{u_{i} ; i \in[t k]\right\} \cup\left\{v_{i}^{j} ; i \in[t k], j \in[k-2]\right\}\right\}, \\
& \mathscr{E}_{k, t}=\left\{E_{i} ; i \in[t k]\right\} \cup\left\{E_{i, j} ; j \in\{1,2, \ldots, k-3\}, i=j+s k-1, s \in[t]\right\},
\end{aligned}
$$

where $E_{i}=\left\{v_{i}^{0}, u_{i}, v_{i}^{1}, v_{i}^{2}, \ldots, v_{i}^{k-3}, v_{i+1}^{0}\right\}, E_{i, j}=\left\{v_{i}^{j}, v_{i+1}^{j}, \ldots, v_{i+k-1}^{j}\right\}$, and using addition modulo $t k$.
$G_{k, t}^{\circ}=\left\{X_{k, t} \cup\{x\}, \mathscr{E}_{k, t} \cup\left\{E^{0}\right\}\right\}$, where $E^{0}=\left\{v_{0}^{0}, u_{0}, v_{0}^{1}, v_{0}^{2}, \ldots, v_{0}^{k-3}, x\right\}$.
The graphs $G_{k, t}$ and $G_{k, t}^{\circ}$ is schematically depicted on Figure 5 .


Figure 5

The proof of Theorem 4 follows from the following two lemmas.
Lemma 9 1) The graph $G_{k, t}$ is symmetric and every non-identity automorphism $\phi$ of $G_{k, t}$ satisfies one of the following properties.

- There exists a positive integer $c \neq 0 \bmod t k$ such that for every $i \in[t k]$, $j=(i+c), \phi\left(E_{i}\right)=E_{j}$ (i.e. for each vertex $\left.v \in E_{i}, \phi(v) \in E_{j}\right)$;
- There exists an $i \in[t k]$ such that $\phi\left(E_{i}\right)=E_{i+1}$;
- There exists an $i \in[t k]$ such that $\phi\left(E_{i}\right)=E_{i+2}$.

2) The only automorphism of $G_{k, t}$ which leaves the set $E_{0} \backslash\left\{v_{1}\right\}$ invariant (i.e. for each vertex $\left.v \in E_{0} \backslash\left\{v_{1}\right\}, \phi(v) \in E_{0} \backslash\left\{v_{1}\right\}\right)$ is the identity.
3) Every non-trivial subgraph of $G_{k, t}$ containing the vertices in $E_{0}$ has a non-identity automorphism $\phi$ which leaves the set $E_{0}$ invariant.

Proof. The first property can be seen to hold by considering the degrees of the vertices in $G_{k, t}$. The second property follows easily from this.

To prove the third property, let $G$ be a non-trivial subgraph of $G_{k, t}$ containing the vertices in $E_{0}$ and let $s$ be the maximal index such that $G$ contains the edges $E_{0}, E_{1}$, $\ldots, E_{s}$. Suppose first that $s \neq t k-1$. Since $E_{s+1}$ is not in $G$, the vertices $u_{s}$ and $v_{s+1}$ are of degree one. The automorphism $\phi$ of $G$ which interchanges $u_{s}$ and $v_{s+1}$ and leaves all the other vertices fixed is a (non-identity) involution. If $s=k t-1$, there is an edge $E_{l}$, $l \in\{i, i+1, \ldots, i+k-1\}$, the vertices $u_{l}$ and $v_{l}^{j}$ have degree one. So here also there is a (non-identity) involution of $G$ that interchanges $u_{l}$ and $v_{l}^{j}$ and leaves all other vertices fixed, in particular leaving $E_{0}$ invariant.

Lemma 10 1) The graph $G_{k, t}^{\circ}$ is asymmetric.
2) Every non-trivial subgraph of $G_{k, t}^{\circ}$ has a non-identity automorphism.

Proof. To prove the first property, we first suppose that $\phi$ is an automorphism of $G_{k, t}^{\circ}$. We can see that the edges $E^{0}$ and $E_{0}$ are invariant under $\phi$ by considering the degrees of the vertices in $G_{k, t}^{\circ}$. Since $G_{k, t}$ is a subgraph of $G_{k, t}^{\circ}$, the automorphism $\phi^{\prime}$ induced by $\phi$ on $G_{k, t}$ leaves the $E_{0} \backslash\left\{v_{1}\right\}$ invariant. By Lemma $9, \phi^{\prime}$ is identity, thus $\phi$ is identity. Therefore, $G_{k, t}^{\circ}$ is asymmetric.

To prove the second property, let $G$ be a non-trivial subgraph of $G_{k, t}^{\circ}$. If $G$ contains the edge $E_{0}$, then either $G=G_{k, t}$ or $G$ contains a non-trivial subgraph of $G_{k, t}$ containing the vertices in $E_{0}$. In both of the cases, according to Lemma 9, there is a non-identity automorphism of $G$. Suppose that $G$ does not contain the edge $E_{0}$. If $E^{0}$ is in $G$, then there is a non-identity involution of $G$ that interchanges $x$ and $u_{0}$ and leaving all other vertices fixed. If $E^{0}$ is not in $G$, then either $G$ does not contain any edge $E_{i}$ for all $i \in[t k]$ or there exist some $i \in[t k] \backslash\{0\}$ such that $E_{i}$ is an edge of $G$. In the former case, $G$ is consists of some pairwise disjoint edges, which is trivially symmetric. In the latter case, let $s$ be the minimal index such that $E_{s}$ is an edge of $G$. Since $E_{s-1}$ is not in $G$, there is a non-identity involution of $G$ that interchanges $v_{s}^{0}$ and $u_{s}$ and leaving all other vertices fixed.

It is easy to observe that the $k$-graphs $G_{k, t}^{\circ}$ have vertex degrees at most three. However note that in this construction, some of the strongly minimal asymmetric $k$-graphs $G_{k, t}^{\circ}$ are not minimal involution-free. In fact, when $k \geq 3, t \geq k-2$ is odd, the sub- $k$-graph $G_{k, t}^{\circ}-x$ of $G_{k, t}^{\circ}$ is involution-free. The most interesting form of Theorem 4 relates to minimal asymmetric graphs for involutions. It will be proved next.

### 3.2 Proof of Theorem 5

Let us recall Theorem 5 ,
Theorem 5 For every $k \geq 6$, there exist infinitely many $k$-graphs ( $X, \mathscr{M}$ ) such that

1. $(X, \mathscr{M})$ is asymmetric.
2. If $\left(X^{\prime}, \mathscr{M}^{\prime}\right)$ is a sub- $k$-graph of $(X, \mathscr{M})$ with at least two vertices, then $\left(X^{\prime}, \mathscr{M}^{\prime}\right)$ has an involution.
(So we claim infinitely many strongly minimal involution-free $k$-graphs for every $k \geq 6$.)
In the proof, we first construct the following $k$-graphs for $k \geq 4$ :

$$
\begin{aligned}
& G_{k}=\left(X_{k}, \mathscr{M}_{k}\right), X_{k}=\left\{v_{1}, v_{2}, \ldots, v_{2 k-1}\right\}, \mathscr{M}_{k}=\left\{M_{i}=\left\{v_{i}, v_{i+1}, \ldots, v_{i+k-1}\right\} ; i \in\{1,2, \ldots, k\}\right\} . \\
& G_{k}^{*}=\left(X_{k}^{*}, \mathscr{M}_{k}^{*}\right), X_{k}^{*}=X_{k} \cup\{x\}, \mathscr{M}_{k}^{*}=\mathscr{M}_{k} \cup\left\{M^{*}\right\}, \text { where } M^{*}=\left\{x, v_{1}, \ldots, v_{k-2}, v_{k+2}\right\} .
\end{aligned}
$$

These $k$-graphs are depicted on Figure 6 and 7 .


Figure 6: The graph $G_{k}$


Figure 7: The graph $G_{k}^{*}$
They will be used as building blocks of our construction.
Lemma 11 1) The $k$-graph $G_{k}$ is symmetric and the only non-identity automorphism $\phi$ of $G_{k}$ satisfies that $\phi\left(v_{i}\right)=v_{2 k-i}$ for every $i \in\{1,2, \ldots, 2 k-1\}$.
2) The only automorphism of $G_{k}$ which leaves the set $\left\{v_{2 k-2}, v_{2 k-1}\right\}$ invariant (i.e. $\left.\left\{\phi\left(v_{2 k-2}\right), \phi\left(v_{2 k-1}\right)\right\}=\left\{v_{2 k-2}, v_{2 k-1}\right\}\right)$ is the identity.
3) Every non-trivial sub-k-graph of $G_{k}$ containing vertices $v_{2 k-2}, v_{2 k-1}$ has an involution $\phi$ which leaves the set $\left\{v_{2 k-2}, v_{2 k-1}\right\}$ invariant.
4) Every non-trivial sub-k-graph $G$ of $G_{k}$ with at least two vertices has a non-identity automorphism $\phi$, which is an involution (i.e. $\phi \circ \phi=1_{V(G)}$ ).

Proof. The first property holds by considering the degree of each vertex in $G_{k}$. Then also the second property follows.

To prove the third one, we assume that $G$ is a non-trivial sub- $k$-graph of $G_{k}$ such that $G$ contains vertices $v_{2 k-2}, v_{2 k-1}$ and $j$ is the maximal index such that $G$ contains the edge $M_{j}=\left\{v_{j}, v_{j+1}, \ldots, v_{j+k-1}\right\}$. Let $i$ be the minimal index such that $G$ contains the edges
$M_{i}, M_{i+1}, \ldots, M_{j}$. Since $G$ is a nontrival sub- $k$-graph of $G_{k}$, we have $j<k$ and $M_{j+1}$ is not in $G$ or $i>1$ and $M_{i-1}$ is not in $G$. It implies that $v_{i+k-2}, v_{i+k-1}$ share the same edges $M_{i}, M_{i+1}, \ldots, M_{j}$. If $i \notin\{k-1, k\}$ then there is an involution $\phi$ of $G$ which leaves the set $\left\{v_{2 k-2}, v_{2 k-1}\right\}$ invariant, $\phi\left(v_{i+k-2}\right)=v_{i+k-1}$ and $\phi\left(v_{i+k-1}\right)=v_{i+k-2}$. If $i \in\{k-1, k\}$ and $G$ contains an edge $M_{l}(1 \leq l<i-1)$, then there is an involution $\phi$ of $G$ which leaves the set $\left\{v_{2 k-2}, v_{2 k-1}\right\}$ invariant, $\phi\left(v_{i-2}\right)=v_{i-1}$ and $\phi\left(v_{i-1}\right)=v_{i-2}$, as $M_{l}$ contains the vertex $v_{i-1}$ and $M_{i-1}$ is not in $G$. Now the remaining case is the edge set of $G$ is contained in $\left\{M_{k-1}, M_{k}\right\}$, which is easy to observe that there is an involution $\phi$ of $G$ which leaves the set $\left\{v_{2 k-2}, v_{2 k-1}\right\}$ invariant.

As the proof of the last property is similar to the previous one we omit it.

## Lemma 12 1) The $k$-graph $G_{k}^{*}$ is asymmetric.

2) Every non-trivial sub-k-graph of $G_{k}^{*}$ has an involution.

Proof. First, we prove that $G_{k}^{*}$ is asymmetric. Assume that $\phi$ is a non-identity automorphism of $G_{k}^{*}$. By considering the degrees of the vertices in the edges $M^{*}$ and $M_{k}$ we conclude that $\phi(x)=x$ and $\phi\left(v_{2 k-1}\right)=v_{2 k-1}$ since $x$ and $v_{2 k-1}$ are the only two vertices in $G_{k}^{*}$ with degree one. As $G_{k}$ is a sub- $k$-graph of $G_{k}^{*}$, by Lemma 11, we know that $\phi\left(v_{i}\right)=v_{i}$ for every $i \in\{1,2, \ldots, k\}$. Thus $G_{k}^{*}$ is asymmetric. (Here one needs $k \geq 4$, which leads below to $k \geq 6$ ).

To prove the second property of $G_{k}^{*}$, we assume $G$ is a non-trivial sub- $k$-graph of $G_{k}^{*}$. Then either $G$ is a sub- $k$-graph of $G_{k}$ or $G$ is obtained by adding the vertex $x$ and the edge $M^{*}=\left\{x, v_{1}, \ldots, v_{k-2}, v_{k+2}\right\}$ to a non-trivial sub- $k$-graph of $G_{k}$. In the former case, $G$ has an involution by Lemma 11. In the latter case, since $G$ contains $M^{*}$, if there exists some $i \in\{1,2, \ldots, k-2\}$ such that $M_{i}$ is not an edge of $G$, then $G$ has an involution $\phi$ with $\phi(x)=v_{i}$ and $\phi\left(v_{i}\right)=x$. Thus $G$ contains all of the edges $M_{1}, M_{2}, \ldots, M_{k-2}$. Let $j$ be the maximal index that $G$ contains the edges $M_{1}, M_{2}, \ldots, M_{j}$. Since $G$ is a nontrival sub- $k$-graph of $G_{k}^{*}$, we have $j<k$ and $M_{j+1}$ is not in $G$, hence $j \in\{k-2, k-1\}$. If $M_{k}$ is not an edge of $G$, then either $j=k-2$ or $j=k-1$ there is an involution $\phi$ such that $\phi\left(v_{k-1}\right)=v_{k}$ and $\phi\left(v_{k}\right)=v_{k-1}$. Thus $G$ contains all the edges of $G_{k}^{*}$ but $M_{k-1}$. So there is a (non-identity) involution of $G$ that interchanges $v_{2 k-2}$ and $v_{2 k-1}$ and leaves all other vertices fixed.

For a hypergraph $G=(X, \mathscr{M})$, let $\widetilde{G}=(\widetilde{X}, \widetilde{M})$ be a hypergraph with $\widetilde{X}=X \cup$ $\bigcup_{i=1}^{|\mathcal{M}|}\left\{a_{i}, b_{i}\right\}$ (where $\left\{a_{i}, b_{i}\right\} \cap\left\{a_{j}, b_{j}\right\}=\varnothing$ and $\left\{a_{i}, b_{i}\right\} \cap X=\varnothing$ for any $i, j \in[|\mathscr{M}|]$ ) and $\stackrel{i=1}{\mathscr{M}}=\left\{M_{i} \cup\left\{a_{i}, b_{i}\right\} ; M_{i} \in \mathscr{M}\right\}$.

Observation 13 For every hypergraph $G=(X, \mathscr{M})$, every automorphism of $\widetilde{G}$ which maps $X$ to $X$ is also an automorphism of $G$ and every automorphism of $G$ extends to an automorphism of $\widetilde{G}$.

Lemma 14 Suppose $\phi$ is an automorphism of $\widetilde{G}_{k}=\left(\widetilde{X}_{k}, \widetilde{\mathscr{M}_{k}}\right)$ which leaves the set $\left\{v_{1}, v_{2 k-2}, v_{2 k-1}\right\}$ invariant. Then $\phi$ restricted to $X_{k}$ is identity.

Proof. Observe that the degree of each vertex in $X_{k} \backslash\left\{v_{1}, v_{2 k-2}, v_{2 k-1}\right\}$ in $\widetilde{G}_{k}$ is at least 2 while every vertex in $\widetilde{X}_{k} \backslash X_{k}$ has degree one. As $\phi$ is an automorphism of $\widetilde{G}_{k}$ which leaves the set $\left\{v_{1}, v_{2 k-2}, v_{2 k-1}\right\}$ invariant, $\phi$ maps $X_{k}$ to $X_{k}$. By Lemma 11 and Observation 13, $\phi$ restricted to $X_{k}$ is identity.

Lemma 15 Suppose $\phi$ is an automorphism of $\widetilde{G}_{k}^{*}=\left(\widetilde{X}_{k}^{*}, \widetilde{\mathscr{M}_{k}^{*}}\right)$ which leaves the vertices $x$ and $v_{2 k-1}$ invariant. Then $\phi$ restricted to $X_{k}^{*}$ is identity.

Proof. The proof of this lemma is very similar to the above proof of Lemma 14. Observe that the degree of each vertex in $X_{k}^{*} \backslash\left\{x, v_{2 k-1}\right\}$ is at least 2 while every vertex in $\widetilde{X}_{k}^{*} \backslash X_{k}$ has degree one. As $\phi$ is an automorphism of $\widetilde{G}_{k}^{*}$ which leaves the set $\left\{x, v_{2 k-1}\right\}$ invariant, $\phi$ maps $X_{k}^{*}$ to $X_{k}^{*}$. By Lemma 12 and Observation 13, $\phi$ restricted to $X_{k}^{*}$ is identity.

After all these preparations we shall, for each $k \geq 6$ and any non-negative integer $s$, construct a $k$-graph $G_{k, s}=(X, \mathscr{M})$ with desired properties. Let $n=(k-1)(k-2)^{s}$. First, we construct a hypergraph $H=(X, \hat{\mathscr{M}})$, depicted as Figure 8 , which is consist of $s+2$ layers as follow:

- On layer 1, disjoint union of $n$ copies of $G_{k}$.
- On layer 2, disjoint union of $\frac{n}{k-2}$ copies of $G_{k-2}$.
- On layer 3, disjoint union of $\frac{n}{(k-2)^{2}}$ copies of $G_{k-2}$.
- ...
- On layer $(s+1)$, disjoint union of $\frac{n}{(k-2)^{s}}=k-1$ copies of $G_{k-2}$.
- On layer $(s+2)$, one copy of $G_{k-2}^{*}$.

Intuitively, $G_{k, s}$ is obtained from $H$ by associating to each $(k-2)$-edge in each copy of $G_{k-2}$ on layer ( $i+1$ ) (or $G_{k-2}^{*}$ on the last layer $(s+2)$ ) a copy of $G_{k-2}$ on layer $i$, $i \in\{1,2, \ldots, s+1\}$ (or $G_{k}$ on layer 1) and changing each ( $k-2$ )-edge into a $k$-edge by adding the last two vertices of the corresponding copy of $G_{k-2}$ (or $G_{k}$ ) to it.

Formally, the $k$-graph $G_{k, s}=(X, \mathscr{M})$ can be constructed in two steps as follows. As above, set $n=(k-1)(k-2)^{s}$. Consider first $n$ copies of $G_{k}, \frac{n}{(k-2)}+\frac{n}{(k-2)^{2}}+\cdots+(k-1)$ copies of $G_{k-2}$ and one copy of $G_{k-2}^{*}$ arranged into $s+2$ layers (see schematic Figure 8). We then have hypergraph $G_{k-2}^{*}$ on layer $(s+2)$. Graphs on layer $(s+1)$ are $k-1$ copies of $G_{k-2}$, which will be listed as $G(1), G(2), \ldots, G(k-1)$. Graphs on layer $l$, $s+1 \geq l \geq 1$, will be $\frac{n}{(k-2)^{l-1}}$ copies of $G_{k-2}$ (or $G_{k}$ when $l=1$ ) and they will be listed as $G\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right), 1 \leq i_{j} \leq k-2, j=l, l+1, \ldots, s, 1 \leq i_{s+1} \leq k-1$. Then the vertices of $G_{k, s}$ are obtained from the vertices of the disjoint union of all hypergraphs $G\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right)$,


Layer 1


Layer 2


Layer (s + 1)


Layer (s + 2)

Figure 8: The hypergraph $H$
$1 \leq l \leq s+1$ and $G_{k-2}^{*}$. All this can be made more precise at the cost of more notation. We leave this to the interested reader.

Next, modify the ( $k-2$ )-edges to $k$-edges, which enlarge $G\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right), 2 \leq l \leq s+1$ and $G_{k-2}^{*}$ to $\widetilde{G}\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right), 2 \leq l \leq s+1$ and $\widetilde{G}_{k-2}^{*}$, as follows. We start with the last layer $s+2$.

Recall that $\mathscr{M}_{k-2}=\left\{M_{1}, M_{2}, \ldots, M_{k-2}\right\}$ and $\mathscr{M}_{k-2}^{*}=\left\{M_{1}^{*}, M_{2}^{*}, \ldots, M_{k-2}^{*}, M_{k-1}^{*}\right\}$. The edge $M_{i_{l}}^{*}$ of $G_{k-2}^{*}$ is enlarged by two last vertices of each hypergraph $G\left(i_{s+1}\right)$ (on layer $s+1), 1 \leq i_{s+1} \leq k-1$.

The previous layer $l, 2 \leq l \leq s+1$, are treated similarly: the edge $M$ corresponding to $M_{i_{i-1}}$ in $G\left(i_{l}, \ldots, i_{s+1}\right)$ on layer $l$ is enlarged by the last two vertices of $G\left(i_{l-1}, i_{l}, \ldots, i_{s+1}\right)$ on layer $l-1,1 \leq i_{l} \leq k-2$.

This finishes the construction of the $k$-graph $G_{k, s}$. And it is easy to observe that all vertices of $k$-graphs $G_{k, s}$ have degree bounded by $k$.

In the remaining of the proof, we use $G\left(i_{1}, i_{2}, \ldots, i_{s+1}\right), \widetilde{G}\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right), 2 \leq l \leq s+1$ and $\widetilde{G}_{k-2}^{*}$ as the corresponding sub- $k$-graphs of $G_{k, s}$. The corresponding vertex sets of $G_{k, s}$ are denoted by $V\left(G\left(i_{1}, i_{2}, \ldots, i_{s+1}\right)\right) 1 \leq l \leq s+1$ and $V\left(G_{k-2}^{*}\right)$.

Since $s$ can be any non-negative integer, it is sufficient to prove that for each of $k$-graphs $G_{k, s}$, the properties in Theorem 5 hold.

First, we prove asymmetry. To the contrary, we assume that $G_{k, s}$ has a non-identity automorphism $\phi$.

Claim $16 \phi(v)=v$ for every vertex $v$ on layer $(s+2)$,
Proof. Observing the degree sequence of each edge in $G_{k, s}$, the degree one belongs to four different types of degree sequences:

- the first edge in each copy $G\left(i_{1}, i_{2}, \ldots, i_{s+1}\right)$ of $G_{k}:(1,2, \ldots, k)$;
- the corresponding edge $M$ of edge $M_{i_{l}}$ in each copy $G\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right)$ of $G_{k-2}$ on layer $l(2 \leq l \leq s+1):(1,2,2,3,3 \ldots, k-2)$;
- the two corresponding edges $M^{\prime}$ and $M^{\prime \prime}$ of two different edges $M_{1}^{*}$ and $M_{k-1}^{*}$ of $G_{k-2}^{*}$ on layer $(s+2):(1,2,2,3,3 \ldots, k-3, k-3)$ and $(1,2,2,3,3, \ldots, k-4, k-3, k-$ $3, k-2$ ).

Thus the two vertices with degree one on layer $(s+2)$ are different from the others, which implies that $\phi$ maps the $V\left(G_{k-2}^{*}\right)$ to itself. By Observation 13 and by Lemma 15 , we obtain that $\phi$ restricted to layer $(s+2)$ is identity.

Claim 17 If the only automorphism $\phi$ of $G_{k, s}$ restricted to layer $(l+1), l \geq 1$, is identity, then $\phi$ restricted to layer $l$ is also identity.

Proof. Since the automorphism $\phi$ of $G_{k, s}$ restricted to layer $(l+1)$ is identity, the corresponding edges $M$ of each edge $M_{i_{l}}$ in $G\left(i_{l+1}, i_{l+2}, \ldots, i_{s+1}\right)$ on layer ( $l+1$ ) are pairwise different. It implies that the copies $G\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right)$ on layer $l$ are pairwise different and $\phi$ maps each $\widetilde{G}\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right)$ to itself (if $l=1, \phi$ maps each $G\left(i_{1}, i_{2}, \ldots, i_{s+1}\right)$ to itself) and leaves the head vertex and the tail two vertices of $G\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right)$ invariant. By Observation 13, and by Lemma 14, the automorphism $\phi$ of $G$ restricted to each vertex subset $V\left(G\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right)\right)$ on layer $l$ is identity.

Claim 16 states that the automorphism $\phi$ of $G_{k, s}$ induced on layer $(s+2)$ is identity. Then by Claim 17, $\phi$ of $G_{k, s}$ restricted to layer ( $s+1$ ) is identity. Continuing this way, we obtain that $\phi$ restricted to layer $i$ is identity, $i \in\{1,2, \ldots, s+2\}$. Thus $G_{k, s}$ is asymmetric.

The involution property of Theorem 5, follows from the following claim.
Claim 18 For every $k \geq 6$ and $s \geq 1$, any proper sub- $k$-graph of $G_{k, s}$ with at least 2 vertices has an involution.

Proof. For contradiction, assume that $G_{k, s}$ contains a non-trivial sub-k-graph $H$ such that $H$ has no involution. Without loss of generality, let us assume that $H$ is connected.

Let $l$ be the minimal layer such that there exists a copy $G=G\left(i_{l}, i_{l+1}, \ldots, i_{s+1}\right)$ of $G_{k-2}$ ( $G$ is a copy of $G_{k}$ if $l=1$ and $G=G_{k-2}^{*}$ if $l=s+2$ ) with $1<|V(H) \cap V(G)|<|V(G)|$. Let $G^{\prime}$ be the sub- $\left(k-2\right.$ )-graph (or sub- $k$-graph) of $G$ induced by $V\left(G^{\prime}\right)=V(H) \cap V(G)$, and let $\widetilde{G}^{\prime}$ be the corresponding sub- $k$-graph of $G^{\prime}$ in $H$. We distinguish two cases.
Case 1. Such an $l$ exists.
Let $x, y$ be the tail two vertices of $G$. If $G^{\prime}$ is an empty graph, then $V\left(G^{\prime}\right)=\{x, y\}$, hence $H$ has an involution interchanging $x$ and $y$. Assume that $G^{\prime}$ is a non-trivial sub-$(k-2)$-graph (or sub- $k$-graph) of $G$. If $G \neq G_{k-2}^{*}$, by Lemma 11, $G^{\prime}$ has an involution which leaves $x, y$ invariant if $x$ or $y$ belongs to $V\left(G^{\prime}\right)$. If $G=G_{k-2}^{*}$, by Lemma 12, $G^{\prime}$ has an involution. Then by Observation 13, $\widetilde{G}^{\prime}$ has an involution $\phi$ that maps $V\left(G^{\prime}\right)$ to $V\left(G^{\prime}\right)$, which can be easily extended to $\vec{H}$.

Case 2. Such an $l$ does not exist. This means for each copy $G=G\left(i_{m}, i_{m+1}, \ldots, i_{s+1}\right)$, $i \in\{1,2, \ldots, s+1\}$, or $G=G_{k-2}^{*}$, either $V(G) \cap V(H)=V(G)$ and $\widetilde{G}$ is a sub- $(k-2)$-graph (or sub- $k$-graph) of $H$ or $V(G) \cap V(H)=\varnothing$.
Assume $p$ is the maximal layer such that there is a copy $G^{\prime}=G\left(i_{p}, i_{p+1}, \ldots, i_{s+1}\right)$ of $G_{k-2}$ ( $G^{\prime}$ is a copy of $G_{k}$ if $p=1$ and $G^{\prime}=G_{k-2}^{*}$ if $p=s+2$ ) with $V\left(G^{\prime}\right) \cap V(H)=V\left(G^{\prime}\right)$. It is easy to check that the vertices of every $G\left(i_{q}, i_{q+1}, \ldots, i_{p-1}, i_{p}, \ldots, i_{s+1}\right)\left(i_{q} \in\{1,2, \ldots, k-2\}\right.$, $2 \leq q \leq p-1$ and $i_{q} \in\{1,2, \ldots, k-1\}$ if $q=1$ ) is contained in $H$, otherwise there exists a copy $G\left(i_{t}, i_{t+1}, \ldots, i_{p-1}, i_{p}, \ldots, i_{s+1}\right)$ for some $2 \leq t \leq p-1$, the vertices of which contained in $H$ are the tail two vertices, a contradiction. Since $H$ is a non-trivial sub- $k$-graph of $G_{k, s}, G^{\prime} \neq G_{k-2}^{*}$. By Lemma 11, $G^{\prime}$ has an involution. Then by Observation 13, $\widetilde{G^{\prime}}$ has an involution $\phi$ that maps $V\left(G^{\prime}\right)$ to $V\left(G^{\prime}\right)$, which can be extended to $H$.

This concludes the proof of Theorem 5.

## 4 Concluding remarks

1. Of course one can define the notion of asymmetric graph also for directed graphs.

One has then the following analogy of Theorem 1: there are exactly 19 minimal asymmetric binary relations. (These are symmetric orientations of 18 minimal asymmetric (undirected) graphs and the single arc graph $(\{0,1\},\{(0,1)\})$.)

Here is a companion problem about extremal asymmetric oriented graphs. This is one of the original motivation, see e.g. [2].

Let $G=(V, E)$ be an asymmetric graph with at least two vertices. We say that $G$ is critical asymmetric if for every $x \in V$ the graph $G-x=(V \backslash\{x\},\{e \in E ; x \notin e\})$ fails to be asymmetric or it is exactly a single vertex. Recall that an oriented graph is a relation not containing two opposite arcs.
Conjecture 1 There is no critical oriented asymmetric graph.
Explicitly: For every oriented asymmetric graph $G$ with at least two vertices, there exists $x \in V(G)$ such that $G-x$ is asymmetric.

Wójcik [12] proved that a critical oriented asymmetric graph has to contain a directed cycle. In general, Conjecture 1 is open.
2. More generally, we could consider $k$-ary relational structures $(X, R)$. We say the multiplicity $m(R)$ of a relation $R$ is at most $s$ if on every $k$-set there are at most $s$ tuples, $1 \leq s \leq k$ !. Thus oriented graphs are binary relations with multiplicity 1 . It is natural to ask for which multiplicities there are finitely many minimal asymmetric $k$-ary relational structures $(X, R)$.

There is exactly one minimal asymmetric $k$-ary relational structure with multiplicity 1 , which is a single $k$-set. And by Theorem 4 we know there are infinitely many minimal asymmetric $k$-ary relational structures $(X, R)$ with $m(R)=k!$.

Note that if a $k$-ary relation $R$ has multiplicity $m(R)=2$, then on every $k$-set the 2 tuples should be different exactly at two places (if on every $k$-set the 2 tuples are not in
this form, then $R$ restricted to these $k$-tuples are asymmetric, which means that the minimal asymmetric $k$-ary relational structure with such $R$ is a single $k$-set). For example, for every $k$-set $\left\{x_{1}, x_{2}, x_{3}, \ldots, x_{k}\right\},\left(x_{1}, x_{2}, x_{3}, \ldots, x_{k}\right) \in R$ implies $\left(x_{2}, x_{1}, x_{3}, \ldots, x_{k}\right) \in R$.

We use the construction $G_{3, t}^{\circ}$ in Section 3.1 to prove that there are infinitely many minimal asymmetric $k$-ary relational structures ( $k \geq 3$ ) with multiplicity 2 . Since in the proof it makes no diference if the two tuples of $R$ differ at different places, we assume that in $R$ for every $k$-set the two tuples differ at the first two places.

We first construct infinitely many minimal asymmetric ternary relational structures $\left(X_{3, t}, R_{3, t}^{\prime}\right)$ such that for every 3 -set $\left\{x_{1}, x_{2}, x_{3}\right\},\left(x_{1}, x_{2}, x_{3}\right) \in R_{3, t}^{\prime}$ implies $\left(x_{2}, x_{1}, x_{3}\right) \in$ $R_{3, t}^{\prime}$. We use the construction of $G_{3, t}^{\circ}$ as before in Section 3.1.
$G_{3, t}=\left(X_{3, t}, \mathscr{E}_{3, t}\right)$,
$G_{3, t}^{\circ}=\left(\left(X_{3, t}^{\prime}, \mathscr{E}_{3, t}^{\prime}\right)\right)=\left(X_{3, t} \cup\{x\}, \mathscr{E}_{3, t} \cup\left\{E_{3 t}\right\}\right)$, where $E_{3 t}=\left\{v_{0}, u_{0}, x\right\}$.
For every set $\{u, v, w\} \in \mathscr{E}_{3, t}^{\prime}$, we have $(u, v, w) \in R_{3, t}^{\prime}$ and $(v, u, w) \in R_{3, t}^{\prime}$.
The proof that $G_{3, t}^{\circ}$ is minimal asymmetric in Section 3 also works here. And then we obtain infinitely many minimal asymmetric $k$-ary relational structures $(X, R)$ with $m(R)=2$ by adding the extra $k-3$ (if $k>3$ ) vertices separately to each corresponding hyperedge as follows.
$H_{k, t}^{\circ}=\left(X_{k, t}^{\prime}, \mathscr{M}_{k, t}\right)$, where $X_{k, t}^{\prime}=X_{3, t} \cup\{x\} \cup \bigcup_{i=0}^{3 t}\left\{w_{i}^{1}, w_{i}^{2}, \ldots, w_{i}^{k-3}\right\}$ and $\mathscr{M}_{k, t}=\left\{E_{i}^{\prime}=\right.$ $\left.E_{i} \cup\left\{w_{i}^{1}, w_{i}^{2}, \ldots, w_{i}^{k-3}\right\} ; i \in[3 t+1]\right\}$.

Every vertex $w_{i}^{1}, w_{i}^{2}, \ldots, w_{i}^{k-3}$ in $E_{i}$ for every $i \in[3 t+1]$ maps to itself in any automorphism of $H_{k, t}^{\circ}$ according to the multiplicity of $R$, which complete the proof.

This also implies that for $k$-ary relation $R$ with $m(R)=k!-1$ there is only one minimal asymmetric relation while for $m(R)=k!-2$ we have infinitely many of them. Perhaps for every $m(R), 2 \leq m(R) \leq K!-2$ there are infinitely many minimal asymmetric relations.

Of interest are special cases such as cyclic relations. We call a relation $R$ cyclic if it has multiplicity $k$ and on every $k$-set $\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ it contains all the following tuples $\left(x_{1}, x_{2}, x_{3}, \ldots, x_{k}\right),\left(x_{2}, x_{3}, \ldots, x_{k}, x_{1}\right), \cdots,\left(x_{k}, x_{1}, x_{2}, \ldots, x_{k-1}\right)$.
Problem 2 Are there finitely many minimal asymmetric $k$-ary cyclic relational structures $(X, R)$ ?

It is not clear even for $k=3$.
Acknowledgement. The authors thank Dominik Bohnert and Christian Winter for finding a mistake in the original statement of Lemma 9.

## References

[1] L. Babai.Automorphism Groups, Isomorphism, Reconstruction, In: Handbook of Combinatorics. R. L. Graham, M. Grötschel, L. Lovǎsz (eds.) North Holland. Elsevier 1995, Vol. 2, pp.1447-1540.
[2] J. Bang-Jensen, B. Reed, M. Schacht, R. Šámal, B. Toft and U. Wagner. On Six Problems Posed by Jarik Nešetřil, In: Topics in Discrete Mathematics (eds.
M. Klazar, J. Kratochvíl, M. Loebl, J. Matoušek, R. Thomas, P. Valtr), Springer 2006, p.613-627.
[3] P. Erdős and A. Rényi. Asymmetric graphs, Acta Math. Acad, Sci. Hungar, 14 (1963), 295-315.
[4] Y. Jiang and J. Nešetřil. Minimal asymmetric hypergraphs, arXiv:2105.10031 [math.CO].
[5] V. Müller. Probabilistic reconstruction from subgraphs, Comment. Math. Univ. Carolin., 17 (1976), 709-719.
[6] J. Nešetřil. Graphs with small asymmetries, Comment. Math. Univ. Carolin., 11, 3 (1970), 403-419.
[7] J. Nešetřil. A congruence theorem for asymmetric trees, Facific Journal of Mathematics, Vol. 37, No. 3, 1971, 771-778.
[8] J. Nešetřil and G. Sabidussi. Minimal asymmetric graphs of length 4, Graphs and Combinatorics, 8 (1992), 343-359.
[9] G. Sabidussi. Clumps, minimal asymmetric graphs, and involutions, J. Comb. Theory, Ser. B, 53 (1991), $40-79$.
[10] P. Schweitzer and P. Schweitzer. Minimal asymmetric graphs, J. Comb. Theory, Ser. B, 127 (2017), $215-227$.
[11] S. Shelah. Graphs with prescribed asymmetry and minimal number of edges. In: Infinite and finite sets, Coll. Math. Soc. J. Bolyai, North Holland (1975) 12411256.
[12] P. Wójcik. On automorphisms of digraphs without symmetric cycles, Comment. Math. Univ. Carolin, 37 (3) (1996) 457-467.

