Minimal asymmetric hypergraphs^{*}

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Abstract

In this paper, we prove that for any $k \ge 3$, there exist infinitely many minimal asymmetric k-uniform hypergraphs. This is in a striking contrast to k = 2, where it has been proved recently that there are exactly 18 minimal asymmetric graphs.

We also determine, for every $k \ge 1$, the minimum size of an asymmetric k-uniform hypergraph.

Keywords: asymmetric hypergraphs, k-uniform hypergraphs, automorphism.

1 Introduction

In this paper we deal with (undirected) graphs, oriented graphs and more general hypergraphs and relational structures. Let us start with (undirected) graphs: An (undirected) graph G is called *asymmetric* if it does not have a non-identity automorphism. Any non-asymmetric graph is also called *symmetric* graph. A graph G is called *minimal asymmetric* if G is asymmetric and every non-trivial induced subgraph of G is symmetric (here G' is a *non-trivial subgraph* of G if G' is a subgraph of G and 1 < |V(G')| < |V(G)|). In this paper all graphs are finite.

It is a folklore result that most graphs are asymmetric. In fact, as shown by Erdős and Rényi [3] most graphs on large sets are asymmetric in a very strong sense. The paper

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[3] contains many extremal results (and problems), which motivated further research on extremal properties of asymmetric graphs, see e.g. [6], [11]. This has been also studied in the context of the reconstruction conjecture [7], [5].

The second author bravely conjectured a long time ago that there are only finitely many minimal asymmetric graphs, see e.g. [2]. Partial results were given in [8], [9], [12] and recently this conjecture has been confirmed by Pascal Schweitzer and Patrick Schweitzer [10] (the list of 18 minimal asymmetric graphs has been isolated already in [8]):

Theorem 1 [10] There are exactly 18 minimal asymmetric undirected graphs up to isomorphism.

In this paper, we consider analogous questions for k-graphs (or k-uniform hypergraphs), i.e. pairs (X, \mathcal{M}) where $\mathcal{M} \subseteq {\binom{X}{k}} = \{A \subseteq X; |A| = k\}$. Induced subhypergraphs, asymmetric hypergraphs and minimal asymmetric hypergraphs are defined analogously as for graphs.

We prove two results related to minimal asymmetric k-graphs.

Denote by n(k) the minimum number of vertices of an asymmetric k-graph.

Theorem 2 n(2) = 6, n(3) = 6, n(k) = k + 2 for $k \ge 4$.

Our second result disproves analogous minimality conjecture (i.e. a result analogous to Theorem 1) for k-graphs.

Theorem 3 For every integer $k \ge 3$, there exist infinitely many k-graphs that are minimal asymmetric.

In fact we prove the following stronger statement.

Theorem 4 For every integer $k \ge 3$, there exist infinitely many k-graphs (X, \mathcal{M}) such that

- 1. (X, \mathcal{M}) is asymmetric.
- If (X', M') is a non-trivial sub-k-graph of (X, M) with at least two vertices, then (X', M') is symmetric.

We call k-graphs that satisfy the two above properties strongly minimal asymmetric. So strongly minimal asymmetric k-graphs do not contain any non-trivial (not necessarily induced) asymmetric sub-k-graph. Note that some of the minimal asymmetric graphs fail to be strongly minimal. For instance, as depicted in Figure 1, the graph X_2 is minimal asymmetric but not strongly minimal asymmetric, since X_1 is a minimal asymmetric subgraph of X_2 . An *involution* of a graph G is any non-identity automorphism ϕ for which $\phi \circ \phi$ is an identity. It was proved in [10] that all minimal asymmetric graphs are in fact minimal involution-free graphs. However, it is not the case for k-graphs: there are k-graphs that are (strongly) minimal asymmetric but not minimal involution-free (see examples after the proof of Theorem 4 in Section 3.1). We prove the following form of Theorem 4 relating minimal asymmetric k-graphs for involutions.

Theorem 5 For every integer $k \ge 6$, there exist infinitely many k-graphs (X, \mathcal{M}) such that

- 1. (X, \mathcal{M}) is asymmetric.
- 2. If (X', \mathcal{M}') is a sub-k-graph of (X, \mathcal{M}) with at least two vertices, then (X', \mathcal{M}') has an involution.

Theorem 4 and Theorem 5 are proved by constructing a sequence of strongly minimal asymmetric k-graphs. We have two different constructions of increasing strength. In Section 3.1 we give a construction with all vertex degrees bounded by 3. A stronger construction which yields minimal asymmetric k-graphs ($k \ge 6$) with respect to involutions is given in the proof of Theorem 5 in Section 3.2. In Section 4 we consider minimal asymmetric relations and their multiplicities and conclude with several open problems.

2 The proof of Theorem 2

Lemma 6 For $k \ge 3$, we have $n(k) \ge k+2$.

Proof. Assume that there exists an asymmetric k-graph (X, \mathscr{M}) with |X| = k + 1. If for each vertex $u \in X$, there is a hyperedge $M \in \mathscr{M}$ such that $u \notin M$, then $\mathscr{M} = \binom{X}{k}$, which is symmetric. Otherwise there exists $u, v \in X$ such that $\{u, v\} \subset M$ for every edge $M \in \mathscr{M}$, or there exist $u', v' \in X$ and $M_1, M_2 \in \mathscr{M}$ such that $u' \notin M_1$ and $v' \notin M_2$. In the former case, there is an automorphism ϕ of (X, \mathscr{M}) such that $\phi(u) = v$ and $\phi(v) = u$. In the latter case there is an automorphism ϕ of (X, \mathscr{M}) such that $\phi(u') = v'$ and $\phi(v') = u'$. In either case we have a contradiction.

For a k-graph $G = (X, \mathcal{M})$, the set-complement of G is defined as a (|X| - k)-graph $\overline{G} = (X, \overline{\mathcal{M}}) = (X, \{X - M | M \in \mathcal{M}\})$. Denote by Aut(G) the set of all the automorphisms of G and thus we have $Aut(G) = Aut(\overline{G})$. We define the degree of a vertex v in a k-graph G as $d_G(v) = |\{M \in \mathcal{M}; v \in M\}|$.

Lemma 7 For $k \ge 4$, we have n(k) = k + 2.

Proof. First, we construct an asymmetric 2-graph (X, \mathcal{M}) with |X| = k + 2 for each $k \ge 4$. Examples of such graphs X_1 and T_{k+2} are depicted in Figure 1.



Figure 1

For k = 4, take the set-complement of X_1 . For every $k \ge 5$, take the set-complement of T_{k+2} . It is easy to see that X_1 and T_{k+2} ($k \ge 5$) are asymmetric. Thus set-complements $\overline{X_1}$ and $\overline{T_{k+2}}$ ($k \ge 5$) are also asymmetric k-graphs.

Each of the set-complements X_1 and T_{k+2} has k+2 vertices. Thus a non-trivial subgraph of each of them is symmetric, by Lemma 6.

Lemma 8 For k = 3, we have n(3) = 6.



Figure 2: An asymmetric 3-graph with |X| = 6

Proof. For $n(3) \leq 6$, consider the following 3-graph $G = (X, \mathscr{M})$ depicted on Figure 2, $X = \{v_1, v_2, v_3, v_4, v_5, v_6\}, \mathscr{M} = \{\{v_1, v_2, v_3\}, \{v_2, v_4, v_5\}, \{v_2, v_4, v_6\}, \{v_3, v_4, v_5\}\}$. Observe that $d_G(v_1) = d_G(v_6) = 1, d_G(v_2) = d_G(v_4) = 3, d_G(v_3) = d_G(v_5) = 2$. It is not difficult to see G is asymmetric.

Now we shall prove that $n(3) \ge 6$.

Assume that there exists an asymmetric 3-graph $H = (X, \mathscr{M})$ with |X| = 5. Let $X = \{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality, let $M = \{v_1, v_2, v_3\} \in \mathscr{M}$. Then there exists an edge $M \in \mathscr{M}$ such that $v_4 \in M$ and $v_5 \notin M$, or $v_4 \notin M$ and $v_5 \in M$.

k-graph *H* is asymmetric if and only if $(X, \binom{X}{3} - \mathscr{M})$ is asymmetric. Thus we can sufficiently consider that $|\mathscr{M}| \leq \frac{\binom{5}{3}}{2} = 5$. If $d_H(v_4) = d_H(v_5)$, which means both of v_4 and v_5 have degree 1 or 2, then there exists an automorphism ϕ of *H* such that $\phi(v_4) = v_5$. Assume that $d_H(v_4) > d_H(v_5)$.



Case 1. There is no edge $M \in \mathcal{M}$ such that $\{v_4, v_5\} \subseteq M$

It is sufficient to consider two subcases: $d_H(v_4) = 2$ and $d_H(v_5) = 1$, or $d_H(v_4) = 3$ and $d_H(v_5) = 1$. In the first subcase, up to isomorphism, we obtain two different graphs as Figure 3(a) and 3(b) shown. There exists an automorphism ϕ of H such that $\phi(v_2) = v_3$ in (a) (resp. $\phi(v_3) = v_4$ in (b)). In the second subcase, there is only one possible graph as Figure 3(c) shown. Observe that there exists an automorphism ϕ of H such that $\phi(v_2) = v_3$ or $\phi(v_1) = v_4$.

Case 2. There exists $M \in \mathcal{M}$ such that $\{v_4, v_5\} \subseteq M$

Let $\mathbb{M} = \{M \in \mathcal{M}; \{v_4, v_5\} \subseteq M\}$, then by symmetric, $|\mathbb{M}| \neq 3$. Since $d_H(v_4) > d_H(v_5)$ and $|\mathcal{M}| \leq 5$, the graphs in this case we need to consider can be divide as follow:

- 1) $|\mathcal{M} \mathbb{M}| = 2$, as Figure 4(a), 4(b) and 4(c) shown.
- 2) $|\mathcal{M} \mathbb{M}| = 3$, as Figure 4(d), 4(e) and 4(f) shown.
- 3) $|\mathcal{M} \mathbb{M}| = 4$, as Figure 4(g), 4(h), 4(i), 4(j), 4(k) and 4(l) shown.

It is easily to observe that there is an automorphism ϕ such that $\phi(v_1) = v_4$ and $\phi(v_3) = v_5$ in Figure 4(a), $\phi(v_1) = v_5$ and $\phi(v_2) = v_4$ in Figure 4(b), $\phi(v_1) = v_2$ in Figure 4(c), $\phi(v_1) = v_3$ in Figure 4(d), $\phi(v_1) = v_4$ in Figure 4(e), $\phi(v_2) = v_4$ and $\phi(v_3) = v_5$ in Figure 4(f), $\phi(v_1) = v_3$ or $\phi(v_2) = v_4$ in Figure 4(g), $\phi(v_1) = v_3$ in Figure 4(h), $\phi(v_3) = v_4$ in Figure 4(i), $\phi(v_1) = v_4$ or $\phi(v_3) = v_5$ in Figure 4(j), $\phi(v_1) = v_2$ and $\phi(v_3) = v_5$ in Figure 4(k), $\phi(v_3) = v_4$ in Figure 4(l). In each case, we obtain a contradiction.

3 Minimal asymmetric k-graphs

In this section, we give proofs of Theorem 4 and Theorem 5.

3.1 Proof of Theorem 4

We define the following k-graphs for $k \ge 3, t \ge k-2$. (Note that for each positive integer p, we denote by [p] the set $\{0, 1, 2, \ldots, p-1\}$.)

$$G_{k,t} = (X_{k,t}, \mathscr{E}_{k,t}),$$



Figure 4

$$\begin{split} X_{k,t} &= \{u_i; i \in [tk]\} \cup \{v_i^j; i \in [tk], j \in [k-2]\}\}, \\ \mathscr{E}_{k,t} &= \{E_i; i \in [tk]\} \cup \{E_{i,j}; j \in \{1, 2, \dots, k-3\}, i = j + sk - 1, s \in [t]\}, \\ \text{where } E_i &= \{v_i^0, u_i, v_i^1, v_i^2, \dots, v_i^{k-3}, v_{i+1}^0\}, \ E_{i,j} &= \{v_i^j, v_{i+1}^j, \dots, v_{i+k-1}^j\}, \text{ and using addition modulo } tk. \end{split}$$

 $G_{k,t}^{\circ} = \{X_{k,t} \cup \{x\}, \mathscr{E}_{k,t} \cup \{E^0\}\}, \text{ where } E^0 = \{v_0^0, u_0, v_0^1, v_0^2, \dots, v_0^{k-3}, x\}.$

The graphs $G_{k,t}$ and $G_{k,t}^{\circ}$ is schematically depicted on Figure 5.



Figure 5

The proof of Theorem 4 follows from the following two lemmas.

Lemma 9 1) The graph $G_{k,t}$ is symmetric and every non-identity automorphism ϕ of $G_{k,t}$ satisfies one of the following properties.

- There exists a positive integer $c \neq 0 \mod tk$ such that for every $i \in [tk]$, $j = (i + c), \ \phi(E_i) = E_j \ (i.e. \ for \ each \ vertex \ v \in E_i, \ \phi(v) \in E_j);$
- There exists an $i \in [tk]$ such that $\phi(E_i) = E_{i+1}$;
- There exists an $i \in [tk]$ such that $\phi(E_i) = E_{i+2}$.
- 2) The only automorphism of $G_{k,t}$ which leaves the set $E_0 \setminus \{v_1\}$ invariant (i.e. for each vertex $v \in E_0 \setminus \{v_1\}$, $\phi(v) \in E_0 \setminus \{v_1\}$) is the identity.
- 3) Every non-trivial subgraph of $G_{k,t}$ containing the vertices in E_0 has a non-identity automorphism ϕ which leaves the set E_0 invariant.

Proof. The first property can be seen to hold by considering the degrees of the vertices in $G_{k,t}$. The second property follows easily from this.

To prove the third property, let G be a non-trivial subgraph of $G_{k,t}$ containing the vertices in E_0 and let s be the maximal index such that G contains the edges E_0 , E_1 , ..., E_s . Suppose first that $s \neq tk - 1$. Since E_{s+1} is not in G, the vertices u_s and v_{s+1} are of degree one. The automorphism ϕ of G which interchanges u_s and v_{s+1} and leaves all the other vertices fixed is a (non-identity) involution. If s = kt - 1, there is an edge E_l , $l \in \{i, i + 1, \ldots, i + k - 1\}$, the vertices u_l and v_l^j have degree one. So here also there is a (non-identity) involution of G that interchanges u_l and v_l^j and leaves all other vertices fixed, in particular leaving E_0 invariant.

Lemma 10 1) The graph $G_{k,t}^{\circ}$ is asymmetric.

2) Every non-trivial subgraph of $G_{k,t}^{\circ}$ has a non-identity automorphism.

Proof. To prove the first property, we first suppose that ϕ is an automorphism of $G_{k,t}^{\circ}$. We can see that the edges E^{0} and E_{0} are invariant under ϕ by considering the degrees of the vertices in $G_{k,t}^{\circ}$. Since $G_{k,t}$ is a subgraph of $G_{k,t}^{\circ}$, the automorphism ϕ' induced by ϕ on $G_{k,t}$ leaves the $E_{0} \setminus \{v_{1}\}$ invariant. By Lemma 9, ϕ' is identity, thus ϕ is identity. Therefore, $G_{k,t}^{\circ}$ is asymmetric.

To prove the second property, let G be a non-trivial subgraph of $G_{k,t}^{\circ}$. If G contains the edge E_0 , then either $G = G_{k,t}$ or G contains a non-trivial subgraph of $G_{k,t}$ containing the vertices in E_0 . In both of the cases, according to Lemma 9, there is a non-identity automorphism of G. Suppose that G does not contain the edge E_0 . If E^0 is in G, then there is a non-identity involution of G that interchanges x and u_0 and leaving all other vertices fixed. If E^0 is not in G, then either G does not contain any edge E_i for all $i \in [tk]$ or there exist some $i \in [tk] \setminus \{0\}$ such that E_i is an edge of G. In the former case, G is consists of some pairwise disjoint edges, which is trivially symmetric. In the latter case, let s be the minimal index such that E_s is an edge of G. Since E_{s-1} is not in G, there is a non-identity involution of G that interchanges v_s^0 and u_s and leaving all other vertices fixed.

It is easy to observe that the k-graphs $G_{k,t}^{\circ}$ have vertex degrees at most three. However note that in this construction, some of the strongly minimal asymmetric k-graphs $G_{k,t}^{\circ}$ are not minimal involution-free. In fact, when $k \ge 3$, $t \ge k - 2$ is odd, the sub-k-graph $G_{k,t}^{\circ} - x$ of $G_{k,t}^{\circ}$ is involution-free. The most interesting form of Theorem 4 relates to minimal asymmetric graphs for involutions. It will be proved next.

3.2 Proof of Theorem 5

Let us recall Theorem 5.

Theorem 5 For every $k \ge 6$, there exist infinitely many k-graphs (X, \mathcal{M}) such that

- 1. (X, \mathcal{M}) is asymmetric.
- 2. If (X', \mathscr{M}') is a sub-k-graph of (X, \mathscr{M}) with at least two vertices, then (X', \mathscr{M}') has an involution.

(So we claim infinitely many strongly minimal involution-free k-graphs for every $k \ge 6$.)

In the proof, we first construct the following k-graphs for $k \ge 4$:

 $G_{k} = (X_{k}, \mathscr{M}_{k}), X_{k} = \{v_{1}, v_{2}, \dots, v_{2k-1}\}, \mathscr{M}_{k} = \{M_{i} = \{v_{i}, v_{i+1}, \dots, v_{i+k-1}\}; i \in \{1, 2, \dots, k\}\}.$ $G_{k}^{*} = (X_{k}^{*}, \mathscr{M}_{k}^{*}), X_{k}^{*} = X_{k} \cup \{x\}, \mathscr{M}_{k}^{*} = \mathscr{M}_{k} \cup \{M^{*}\}, \text{ where } M^{*} = \{x, v_{1}, \dots, v_{k-2}, v_{k+2}\}.$ These k-graphs are depicted on Figure 6 and 7.



Figure 6: The graph G_k



Figure 7: The graph G_k^*

They will be used as building blocks of our construction.

- **Lemma 11** 1) The k-graph G_k is symmetric and the only non-identity automorphism ϕ of G_k satisfies that $\phi(v_i) = v_{2k-i}$ for every $i \in \{1, 2, ..., 2k-1\}$.
 - 2) The only automorphism of G_k which leaves the set $\{v_{2k-2}, v_{2k-1}\}$ invariant (i.e. $\{\phi(v_{2k-2}), \phi(v_{2k-1})\} = \{v_{2k-2}, v_{2k-1}\}$) is the identity.
 - 3) Every non-trivial sub-k-graph of G_k containing vertices v_{2k-2} , v_{2k-1} has an involution ϕ which leaves the set $\{v_{2k-2}, v_{2k-1}\}$ invariant.
 - 4) Every non-trivial sub-k-graph G of G_k with at least two vertices has a non-identity automorphism ϕ , which is an involution (i.e. $\phi \circ \phi = 1_{V(G)}$).

Proof. The first property holds by considering the degree of each vertex in G_k . Then also the second property follows.

To prove the third one, we assume that G is a non-trivial sub-k-graph of G_k such that G contains vertices v_{2k-2} , v_{2k-1} and j is the maximal index such that G contains the edge $M_j = \{v_j, v_{j+1}, \ldots, v_{j+k-1}\}$. Let i be the minimal index such that G contains the edges

 $M_i, M_{i+1}, \ldots, M_j$. Since G is a nontrival sub-k-graph of G_k , we have j < k and M_{j+1} is not in G or i > 1 and M_{i-1} is not in G. It implies that v_{i+k-2}, v_{i+k-1} share the same edges $M_i, M_{i+1}, \ldots, M_j$. If $i \notin \{k-1, k\}$ then there is an involution ϕ of G which leaves the set $\{v_{2k-2}, v_{2k-1}\}$ invariant, $\phi(v_{i+k-2}) = v_{i+k-1}$ and $\phi(v_{i+k-1}) = v_{i+k-2}$. If $i \in \{k-1, k\}$ and G contains an edge M_l $(1 \le l < i-1)$, then there is an involution ϕ of G which leaves the set $\{v_{2k-2}, v_{2k-1}\}$ invariant, $\phi(v_{i-2}) = v_{i-1}$ and $\phi(v_{i-1}) = v_{i-2}$, as M_l contains the vertex v_{i-1} and M_{i-1} is not in G. Now the remaining case is the edge set of G is contained in $\{M_{k-1}, M_k\}$, which is easy to observe that there is an involution ϕ of G which leaves the set $\{v_{2k-2}, v_{2k-1}\}$ invariant.

As the proof of the last property is similar to the previous one we omit it. \blacksquare

Lemma 12 1) The k-graph G_k^* is asymmetric.

2) Every non-trivial sub-k-graph of G_k^* has an involution.

Proof. First, we prove that G_k^* is asymmetric. Assume that ϕ is a non-identity automorphism of G_k^* . By considering the degrees of the vertices in the edges M^* and M_k we conclude that $\phi(x) = x$ and $\phi(v_{2k-1}) = v_{2k-1}$ since x and v_{2k-1} are the only two vertices in G_k^* with degree one. As G_k is a sub-k-graph of G_k^* , by Lemma 11, we know that $\phi(v_i) = v_i$ for every $i \in \{1, 2, \ldots, k\}$. Thus G_k^* is asymmetric. (Here one needs $k \ge 4$, which leads below to $k \ge 6$).

To prove the second property of G_k^* , we assume G is a non-trivial sub-k-graph of G_k^* . Then either G is a sub-k-graph of G_k or G is obtained by adding the vertex x and the edge $M^* = \{x, v_1, \ldots, v_{k-2}, v_{k+2}\}$ to a non-trivial sub-k-graph of G_k . In the former case, G has an involution by Lemma 11. In the latter case, since G contains M^* , if there exists some $i \in \{1, 2, \ldots, k-2\}$ such that M_i is not an edge of G, then G has an involution ϕ with $\phi(x) = v_i$ and $\phi(v_i) = x$. Thus G contains all of the edges $M_1, M_2, \ldots, M_{k-2}$. Let j be the maximal index that G contains the edges M_1, M_2, \ldots, M_j . Since G is a nontrival sub-k-graph of G_k^* , we have j < k and M_{j+1} is not in G, hence $j \in \{k-2, k-1\}$. If M_k is not an edge of G, then either j = k-2 or j = k-1 there is an involution ϕ such that $\phi(v_{k-1}) = v_k$ and $\phi(v_k) = v_{k-1}$. Thus G contains all the edges of G_k^* but M_{k-1} . So there is a (non-identity) involution of G that interchanges v_{2k-2} and v_{2k-1} and leaves all other vertices fixed.

For a hypergraph $G = (X, \mathscr{M})$, let $\widetilde{G} = (\widetilde{X}, \widetilde{\mathscr{M}})$ be a hypergraph with $\widetilde{X} = X \cup \bigcup_{i=1}^{|\mathscr{M}|} \{a_i, b_i\}$ (where $\{a_i, b_i\} \cap \{a_j, b_j\} = \emptyset$ and $\{a_i, b_i\} \cap X = \emptyset$ for any $i, j \in [|\mathscr{M}|]$) and $\widetilde{\mathscr{M}} = \{M_i \cup \{a_i, b_i\}; M_i \in \mathscr{M}\}.$

Observation 13 For every hypergraph $G = (X, \mathcal{M})$, every automorphism of \tilde{G} which maps X to X is also an automorphism of G and every automorphism of G extends to an automorphism of \tilde{G} .

Lemma 14 Suppose ϕ is an automorphism of $\widetilde{G}_k = (\widetilde{X}_k, \widetilde{\mathcal{M}}_k)$ which leaves the set $\{v_1, v_{2k-2}, v_{2k-1}\}$ invariant. Then ϕ restricted to X_k is identity.

Proof. Observe that the degree of each vertex in $X_k \setminus \{v_1, v_{2k-2}, v_{2k-1}\}$ in \widetilde{G}_k is at least 2 while every vertex in $\widetilde{X}_k \setminus X_k$ has degree one. As ϕ is an automorphism of \widetilde{G}_k which leaves the set $\{v_1, v_{2k-2}, v_{2k-1}\}$ invariant, ϕ maps X_k to X_k . By Lemma 11 and Observation 13, ϕ restricted to X_k is identity.

Lemma 15 Suppose ϕ is an automorphism of $\widetilde{G}_k^* = (\widetilde{X}_k^*, \widetilde{\mathcal{M}}_k^*)$ which leaves the vertices x and v_{2k-1} invariant. Then ϕ restricted to X_k^* is identity.

Proof. The proof of this lemma is very similar to the above proof of Lemma 14. Observe that the degree of each vertex in $X_k^* \setminus \{x, v_{2k-1}\}$ is at least 2 while every vertex in $\widetilde{X}_k^* \setminus X_k$ has degree one. As ϕ is an automorphism of \widetilde{G}_k^* which leaves the set $\{x, v_{2k-1}\}$ invariant, ϕ maps X_k^* to X_k^* . By Lemma 12 and Observation 13, ϕ restricted to X_k^* is identity.

After all these preparations we shall, for each $k \ge 6$ and any non-negative integer s, construct a k-graph $G_{k,s} = (X, \mathscr{M})$ with desired properties. Let $n = (k-1)(k-2)^s$. First, we construct a hypergraph $H = (X, \mathscr{M})$, depicted as Figure 8, which is consist of s + 2 layers as follow:

- On layer 1, disjoint union of n copies of G_k .
- On layer 2, disjoint union of $\frac{n}{k-2}$ copies of G_{k-2} .
- On layer 3, disjoint union of $\frac{n}{(k-2)^2}$ copies of G_{k-2} .
- ...
- On layer (s+1), disjoint union of $\frac{n}{(k-2)^s} = k-1$ copies of G_{k-2} .
- On layer (s+2), one copy of G_{k-2}^* .

Intuitively, $G_{k,s}$ is obtained from H by associating to each (k-2)-edge in each copy of G_{k-2} on layer (i+1) (or G_{k-2}^* on the last layer (s+2)) a copy of G_{k-2} on layer i, $i \in \{1, 2, \ldots, s+1\}$ (or G_k on layer 1) and changing each (k-2)-edge into a k-edge by adding the last two vertices of the corresponding copy of G_{k-2} (or G_k) to it.

Formally, the k-graph $G_{k,s} = (X, \mathscr{M})$ can be constructed in two steps as follows. As above, set $n = (k-1)(k-2)^s$. Consider first n copies of G_k , $\frac{n}{(k-2)} + \frac{n}{(k-2)^2} + \cdots + (k-1)$ copies of G_{k-2} and one copy of G_{k-2}^* arranged into s+2 layers (see schematic Figure 8). We then have hypergraph G_{k-2}^* on layer (s+2). Graphs on layer (s+1) are k-1copies of G_{k-2} , which will be listed as $G(1), G(2), \ldots, G(k-1)$. Graphs on layer l, $s+1 \ge l \ge 1$, will be $\frac{n}{(k-2)^{l-1}}$ copies of G_{k-2} (or G_k when l = 1) and they will be listed as $G(i_l, i_{l+1}, \ldots, i_{s+1}), 1 \le i_j \le k-2, j = l, l+1, \ldots, s, 1 \le i_{s+1} \le k-1$. Then the vertices of $G_{k,s}$ are obtained from the vertices of the disjoint union of all hypergraphs $G(i_l, i_{l+1}, \ldots, i_{s+1})$,



Figure 8: The hypergraph H

 $1 \le l \le s+1$ and G_{k-2}^* . All this can be made more precise at the cost of more notation. We leave this to the interested reader.

Next, modify the (k-2)-edges to k-edges, which enlarge $G(i_l, i_{l+1}, \ldots, i_{s+1}), 2 \le l \le s+1$ and G_{k-2}^* to $\widetilde{G}(i_l, i_{l+1}, \ldots, i_{s+1}), 2 \le l \le s+1$ and \widetilde{G}_{k-2}^* , as follows. We start with the last layer s+2.

Recall that $\mathscr{M}_{k-2} = \{M_1, M_2, \dots, M_{k-2}\}$ and $\mathscr{M}_{k-2}^* = \{M_1^*, M_2^*, \dots, M_{k-2}^*, M_{k-1}^*\}$. The edge $M_{i_l}^*$ of G_{k-2}^* is enlarged by two last vertices of each hypergraph $G(i_{s+1})$ (on layer s+1), $1 \le i_{s+1} \le k-1$.

The previous layer $l, 2 \le l \le s + 1$, are treated similarly: the edge M corresponding to $M_{i_{l-1}}$ in $G(i_l, \ldots, i_{s+1})$ on layer l is enlarged by the last two vertices of $G(i_{l-1}, i_l, \ldots, i_{s+1})$ on layer $l-1, 1 \le i_l \le k-2$.

This finishes the construction of the k-graph $G_{k,s}$. And it is easy to observe that all vertices of k-graphs $G_{k,s}$ have degree bounded by k.

In the remaining of the proof, we use $G(i_1, i_2, \ldots, i_{s+1})$, $\widetilde{G}(i_l, i_{l+1}, \ldots, i_{s+1})$, $2 \le l \le s+1$ and \widetilde{G}_{k-2}^* as the corresponding sub-k-graphs of $G_{k,s}$. The corresponding vertex sets of $G_{k,s}$ are denoted by $V(G(i_1, i_2, \ldots, i_{s+1}))$ $1 \le l \le s+1$ and $V(G_{k-2}^*)$.

Since s can be any non-negative integer, it is sufficient to prove that for each of k-graphs $G_{k,s}$, the properties in Theorem 5 hold.

First, we prove asymmetry. To the contrary, we assume that $G_{k,s}$ has a non-identity automorphism ϕ .

Claim 16 $\phi(v) = v$ for every vertex v on layer (s+2),

Proof. Observing the degree sequence of each edge in $G_{k,s}$, the degree one belongs to four different types of degree sequences:

- the first edge in each copy $G(i_1, i_2, \ldots, i_{s+1})$ of G_k : $(1, 2, \ldots, k)$;

- the corresponding edge M of edge M_{i_l} in each copy $G(i_l, i_{l+1}, \ldots, i_{s+1})$ of G_{k-2} on layer $l \ (2 \le l \le s+1)$: $(1, 2, 2, 3, 3 \ldots, k-2)$;
- the two corresponding edges M' and M'' of two different edges M_1^* and M_{k-1}^* of G_{k-2}^* on layer (s+2): (1, 2, 2, 3, 3, ..., k-3, k-3) and (1, 2, 2, 3, 3, ..., k-4, k-3, k-3, k-3).

Thus the two vertices with degree one on layer (s + 2) are different from the others, which implies that ϕ maps the $V(G_{k-2}^*)$ to itself. By Observation 13 and by Lemma 15, we obtain that ϕ restricted to layer (s + 2) is identity.

Claim 17 If the only automorphism ϕ of $G_{k,s}$ restricted to layer (l+1), $l \ge 1$, is identity, then ϕ restricted to layer l is also identity.

Proof. Since the automorphism ϕ of $G_{k,s}$ restricted to layer (l+1) is identity, the corresponding edges M of each edge M_{i_l} in $G(i_{l+1}, i_{l+2}, \ldots, i_{s+1})$ on layer (l+1) are pairwise different. It implies that the copies $G(i_l, i_{l+1}, \ldots, i_{s+1})$ on layer l are pairwise different and ϕ maps each $\tilde{G}(i_l, i_{l+1}, \ldots, i_{s+1})$ to itself (if $l = 1, \phi$ maps each $G(i_l, i_{l+1}, \ldots, i_{s+1})$ to itself) and leaves the head vertex and the tail two vertices of $G(i_l, i_{l+1}, \ldots, i_{s+1})$ invariant. By Observation 13,and by Lemma 14, the automorphism ϕ of G restricted to each vertex subset $V(G(i_l, i_{l+1}, \ldots, i_{s+1}))$ on layer l is identity.

Claim 16 states that the automorphism ϕ of $G_{k,s}$ induced on layer (s+2) is identity. Then by Claim 17, ϕ of $G_{k,s}$ restricted to layer (s+1) is identity. Continuing this way, we obtain that ϕ restricted to layer i is identity, $i \in \{1, 2, \ldots, s+2\}$. Thus $G_{k,s}$ is asymmetric.

The involution property of Theorem 5, follows from the following claim.

Claim 18 For every $k \ge 6$ and $s \ge 1$, any proper sub-k-graph of $G_{k,s}$ with at least 2 vertices has an involution.

Proof. For contradiction, assume that $G_{k,s}$ contains a non-trivial sub-k-graph H such that H has no involution. Without loss of generality, let us assume that H is connected.

Let l be the minimal layer such that there exists a copy $G = G(i_l, i_{l+1}, \ldots, i_{s+1})$ of G_{k-2} (G is a copy of G_k if l = 1 and $G = G_{k-2}^*$ if l = s+2) with $1 < |V(H) \cap V(G)| < |V(G)|$. Let G' be the sub-(k-2)-graph (or sub-k-graph) of G induced by $V(G') = V(H) \cap V(G)$, and let \widetilde{G}' be the corresponding sub-k-graph of G' in H. We distinguish two cases. **Case 1.** Such an l exists.

Let x, y be the tail two vertices of G. If G' is an empty graph, then $V(G') = \{x, y\}$, hence H has an involution interchanging x and y. Assume that G' is a non-trivial sub-(k-2)-graph (or sub-k-graph) of G. If $G \neq G^*_{k-2}$, by Lemma 11, G' has an involution which leaves x, y invariant if x or y belongs to V(G'). If $G = G^*_{k-2}$, by Lemma 12, G'has an involution. Then by Observation 13, \tilde{G}' has an involution ϕ that maps V(G') to V(G'), which can be easily extended to H. **Case 2.** Such an *l* does not exist. This means for each copy $G = G(i_m, i_{m+1}, \ldots, i_{s+1})$, $i \in \{1, 2, \ldots, s+1\}$, or $G = G_{k-2}^*$, either $V(G) \cap V(H) = V(G)$ and \widetilde{G} is a sub-(k-2)-graph (or sub-k-graph) of *H* or $V(G) \cap V(H) = \emptyset$.

Assume p is the maximal layer such that there is a copy $G' = G(i_p, i_{p+1}, \ldots, i_{s+1})$ of G_{k-2} (G' is a copy of G_k if p = 1 and $G' = G^*_{k-2}$ if p = s+2) with $V(G') \cap V(H) = V(G')$. It is easy to check that the vertices of every $G(i_q, i_{q+1}, \ldots, i_{p-1}, i_p, \ldots, i_{s+1})$ ($i_q \in \{1, 2, \ldots, k-2\}$, $2 \le q \le p-1$ and $i_q \in \{1, 2, \ldots, k-1\}$ if q = 1) is contained in H, otherwise there exists a copy $G(i_t, i_{t+1}, \ldots, i_{p-1}, i_p, \ldots, i_{s+1})$ for some $2 \le t \le p-1$, the vertices of which contained in H are the tail two vertices, a contradiction. Since H is a non-trivial sub-k-graph of $G_{k,s}, G' \ne G^*_{k-2}$. By Lemma 11, G' has an involution. Then by Observation 13, $\tilde{G'}$ has an involution ϕ that maps V(G') to V(G'), which can be extended to H.

This concludes the proof of Theorem 5.

4 Concluding remarks

1. Of course one can define the notion of asymmetric graph also for directed graphs.

One has then the following analogy of Theorem 1: there are exactly 19 minimal asymmetric binary relations. (These are symmetric orientations of 18 minimal asymmetric (undirected) graphs and the single arc graph $(\{0,1\},\{(0,1)\})$.)

Here is a companion problem about extremal asymmetric oriented graphs. This is one of the original motivation, see e.g. [2].

Let G = (V, E) be an asymmetric graph with at least two vertices. We say that G is *critical asymmetric* if for every $x \in V$ the graph $G - x = (V \setminus \{x\}, \{e \in E; x \notin e\})$ fails to be asymmetric or it is exactly a single vertex. Recall that an oriented graph is a relation not containing two opposite arcs.

Conjecture 1 There is no critical oriented asymmetric graph.

Explicitly: For every oriented asymmetric graph G with at least two vertices, there exists $x \in V(G)$ such that G - x is asymmetric.

Wójcik [12] proved that a critical oriented asymmetric graph has to contain a directed cycle. In general, Conjecture 1 is open.

2. More generally, we could consider k-ary relational structures (X, R). We say the multiplicity m(R) of a relation R is at most s if on every k-set there are at most s tuples, $1 \le s \le k!$. Thus oriented graphs are binary relations with multiplicity 1. It is natural to ask for which multiplicities there are finitely many minimal asymmetric k-ary relational structures (X, R).

There is exactly one minimal asymmetric k-ary relational structure with multiplicity 1, which is a single k-set. And by Theorem 4 we know there are infinitely many minimal asymmetric k-ary relational structures (X, R) with m(R) = k!.

Note that if a k-ary relation R has multiplicity m(R) = 2, then on every k-set the 2 tuples should be different exactly at two places (if on every k-set the 2 tuples are not in

this form, then R restricted to these k-tuples are asymmetric, which means that the minimal asymmetric k-ary relational structure with such R is a single k-set). For example, for every k-set $\{x_1, x_2, x_3, \ldots, x_k\}, (x_1, x_2, x_3, \ldots, x_k) \in R$ implies $(x_2, x_1, x_3, \ldots, x_k) \in R$.

We use the construction $G_{3,t}^{\circ}$ in Section 3.1 to prove that there are infinitely many minimal asymmetric k-ary relational structures $(k \ge 3)$ with multiplicity 2. Since in the proof it makes no difference if the two tuples of R differ at different places, we assume that in R for every k-set the two tuples differ at the first two places.

We first construct infinitely many minimal asymmetric ternary relational structures $(X_{3,t}, R'_{3,t})$ such that for every 3-set $\{x_1, x_2, x_3\}$, $(x_1, x_2, x_3) \in R'_{3,t}$ implies $(x_2, x_1, x_3) \in R'_{3,t}$. We use the construction of $G^{\circ}_{3,t}$ as before in Section 3.1.

 $G_{3,t} = (X_{3,t}, \mathscr{E}_{3,t}),$ $G_{3,t}^{\circ} = ((X'_{3,t}, \mathscr{E}'_{3,t})) = (X_{3,t} \cup \{x\}, \mathscr{E}_{3,t} \cup \{E_{3t}\}), \text{ where } E_{3t} = \{v_0, u_0, x\}.$ For every set $\{u, v, w\} \in \mathscr{E}'_{3,t}$, we have $(u, v, w) \in R'_{3,t}$ and $(v, u, w) \in R'_{3,t}$.

The proof that $G_{3,t}^{\circ}$ is minimal asymmetric in Section 3 also works here. And then we obtain infinitely many minimal asymmetric k-ary relational structures (X, R) with m(R) = 2 by adding the extra k - 3 (if k > 3) vertices separately to each corresponding hyperedge as follows.

 $H_{k,t}^{\circ} = (X_{k,t}', \mathscr{M}_{k,t}), \text{ where } X_{k,t}' = X_{3,t} \cup \{x\} \cup \bigcup_{i=0}^{3t} \{w_i^1, w_i^2, \dots, w_i^{k-3}\} \text{ and } \mathscr{M}_{k,t} = \{E_i' = E_i \cup \{w_i^1, w_i^2, \dots, w_i^{k-3}\}; i \in [3t+1]\}.$

Every vertex $w_i^1, w_i^2, \ldots, w_i^{k-3}$ in E_i for every $i \in [3t+1]$ maps to itself in any automorphism of $H_{k,t}^{\circ}$ according to the multiplicity of R, which complete the proof.

This also implies that for k-ary relation R with m(R) = k! - 1 there is only one minimal asymmetric relation while for m(R) = k! - 2 we have infinitely many of them. Perhaps for every m(R), $2 \le m(R) \le K! - 2$ there are infinitely many minimal asymmetric relations.

Of interest are special cases such as cyclic relations. We call a relation R cyclic if it has multiplicity k and on every k-set $\{x_1, x_2, \ldots, x_k\}$ it contains all the following tuples $(x_1, x_2, x_3, \ldots, x_k), (x_2, x_3, \ldots, x_k, x_1), \cdots, (x_k, x_1, x_2, \ldots, x_{k-1}).$

Problem 2 Are there finitely many minimal asymmetric k-ary cyclic relational structures (X, R)?

It is not clear even for k = 3.

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