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Nonstandard Foundations of Equilibrium Search Models*

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Abstract. We develop an equilibrium sequential search model which includes most of the literature as special cases. In particular, the model can accommodate heterogeneity in buyers' search costs and demand functions and firms' cost functions, with general demand and cost functions. We identify conditions which ensure existence of equilibrium in pure strategies, utilizing recent progress in the theory of large games by Khan and Sun. These conditions elucidate the essential structure of equilibrium search models. Although we focus on sequential search, our methodology can be used for other classes of equilibrium search models as well.

Keywords: Search, equilibrium search models, nonstandard analysis.

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1. Introduction

Equilibrium search models have been used to study a wide variety of issues in labor economics, macroeconomics, and microeconomics. In microeconomics alone, a partial list would include: the existence of wage and price dispersion [2, 16, 17], advertising [18], market microstructure and the role of middlemen [20], and the effects of wage and price controls [15]. Despite numerous applications, the equilibrium search literature remains fragmented, containing many different variants of the basic model, each adapted to deal with some specific applied problem. It seems that little progress has been made in analyzing the general structure of this important class of models.

E.g., in Reinganum [16] buyers are identical and firms differ only with respect to their constant marginal costs. In that model, the existence and character of equilibrium is not an issue, since the equilibrium distribution (cdf) of prices is induced directly by the cdf of firms' costs. At the opposite extreme, in Rob [17] firms are identical but buyers may have different search costs. Although Rob's existence proof is ingenious, it does not seem to extend to more general equilibrium search models, particularly those where firms have different cost functions. The model in Carlson and McAfee [2] allows heterogeneity in buyers' search costs and firms' constant marginal costs, but under very specific assumptions in order to explicitly solve the model. The most general model seems to be Bénabou [1] which significantly extends [2], essentially combining the Reinganum and Rob models. Bénabou characterizes the equilibrium cdf of prices as a fixed point of a certain map, but is unable to prove existence for reasons discussed in the working paper version of [1].

In this paper, we develop an equilibrium search model which significantly extends [1] and prove existence of equilibrium in pure strategies. The proof is made possible by recent progress in the theory of large games by Khan and Sun [9]. We prove existence under two distinct sets of assumptions. We call the first set of assumptions *standard*, since they generalize the Bénabou model directly. We call the second set *general*, since they allow heterogeneity in buyers' search costs and demand functions as well as firms' cost functions, with very general cost and demand functions. Although the general assumptions seem much weaker than the standard ones from an economic point of view, they are

mathematically distinct. Furthermore, the standard assumptions are analytically tractable and have proved useful in applications.

In the next section, we present the model and prove existence of equilibrium in pure strategies. Section 3 concludes.

2. A General Model

Let \mathbf{R} and \mathbf{R}_+ denote the spaces of real numbers and nonnegative real numbers, respectively. We first characterize the demand side of the market. Let (I, \mathcal{I}, μ_I) be a finite measure space, where I is the set of buyers, \mathcal{I} is a σ -algebra of subsets of I , and μ_I is a finite measure on \mathcal{I} , with $\mu_I(I) = N > 0$. We assume there is a price $\bar{p} > 0$ above which buyers will not pay. Let $\tilde{p} < 0$ be the shut-down price chosen by firms which elect not to operate. The space of feasible prices is therefore $P = [0, \bar{p}] \cup \{\tilde{p}\}$, which is compact. Let \mathcal{X} be a set of nonincreasing functions $x : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ with the following properties: (i) $x(p) = 0$ for all $p > \bar{p}$, (ii) $Z(x) = \inf\{p \in \mathbf{R}_+ \mid x(p) = 0\} > 0$, and (iii) there exists a Lebesgue integrable function $g : [0, \bar{p}] \rightarrow \mathbf{R}_+$ such that for all $x \in \mathcal{X}$, $x(p) \leq g(p)$ for all $p \in [0, \bar{p}]$. We assume $g(0) > 0$; otherwise, the model is trivial. By theorem 3.7 in [6, p. 228], each $x \in \mathcal{X}$ is bounded and a.e. differentiable, so we may endow \mathcal{X} with the supremum metric. The demand side of the market is then characterized by a measurable function $B : I \rightarrow \mathbf{R}_+ \times \mathcal{X}$ which assigns to each buyer i a nonnegative search cost $s(i)$ and demand function $x(p \mid i)$ in \mathcal{X} .¹

A common assumption in the equilibrium search literature is that all buyers have the same demand curve, which is perfectly inelastic at one unit up to some maximum price \bar{p} , after which quantity demanded is zero. The above framework allows buyers to have different maximum prices, bounded above by \bar{p} , and can also handle more standard linear demand curves, as well as different demand functions of a very general nature. Assuming the existence of a maximum price $\bar{p} > 0$ is a simple way of “compactifying” the model. For another way, see assumptions 1 in [12].

Let \mathcal{D} be the set of all cdfs with support contained in P with the Prohorov metric.

¹ Throughout the paper, metric space products are equipped with the product metric which metrizes the product topology.

Since P is compact, so is \mathcal{D} . As it turns out, the cdf $F \in \mathcal{D}$ such that $F(\tilde{p}) = 1$ (almost all firms shut down) causes certain technical difficulties, so for the time being we work with the subspace $\mathcal{D}_{<1} = \{F \in \mathcal{D} \mid F(\tilde{p}) < 1\}$.

Let $H : \mathbf{R} \times \mathcal{D}_{<1} \rightarrow \mathbf{R}_+$ be defined by

$$H(p \mid F) = \begin{cases} 0 & p < \tilde{p} \\ \frac{F(p) - F(\tilde{p})}{1 - F(\tilde{p})} & \tilde{p} \leq p \leq \bar{p} \\ 1 & p > \bar{p}. \end{cases} \quad (1)$$

$H(p \mid F)$ is the cdf of prices charged by operating firms. As is well-known (e.g., see [1]), under standard assumptions, a buyer with demand x and search cost s continues to search until she observes a price r satisfying

$$\gamma(r \mid x, F) \equiv \int_0^r x(p) H(p \mid F) dp \leq s, \quad (2)$$

where the left-hand side of (2) is the marginal benefit of another search, and the right-hand side is the marginal cost.² Let $\gamma : [0, \bar{p}] \times \mathcal{X} \times \mathcal{D}_{<1} \rightarrow \mathbf{R}_+$ be defined as in (2) and $z(F) = \sup\{p \in \mathbf{R} \mid H(p \mid F) = 0\} \geq 0$.

Proposition 1. *γ is continuous. For any $(x, F) \in \mathcal{X} \times \mathcal{D}_{<1}$, γ is a.e. differentiable on $[0, \bar{p}]$ (with respect to Lebesgue measure) with derivative $x(r)H(r \mid F)$. It equals zero on $[0, z(F)]$, is increasing on $[z(F), Z(x)]$ if $z(F) < Z(x)$, and constant on $[Z(x), \bar{p}]$.*

Figure 1 below depicts γ in bold.

Figure 1 Goes Here

If the search cost is s_1 , the buyer continues searching after observing prices $p > r_0$, and stops for all prices $0 \leq p \leq r_0$. Her reservation level is therefore r_0 . If the search cost is s_2 or s_3 , her reservation level is \bar{p} , so she stops at the first seller she visits. Generally, the reservation level of a buyer with (x, s) is given by $\sup\{r \in [0, \bar{p}] \mid \gamma(r \mid x, F) \leq s\}$.

Note that the reservation level is discontinuous at s_2 , since a small reduction in the search cost would cause a disproportionately large drop in the reservation level from \bar{p}

² As usual, we assume the first price quote is free. As in [19], the derivation of (2) assumes $H(0 \mid F) = 0$, which will be true in equilibrium. It is mathematically well-defined regardless.

to something less than $Z(x)$. Furthermore, there may be a positive mass of buyers with reservation level \bar{p} . E.g., suppose all firms charge \bar{p} . Since $z(F) = \bar{p}$, all buyers have reservation level \bar{p} . To take another example, common in the literature, suppose all buyers have the same γ as in Figure 1, because their demand functions are the same. Then all buyers with search costs $s \geq s_2$ would have reservation level \bar{p} .

It turns out that a positive mass of buyers with reservation level \bar{p} causes certain technical difficulties, so this problem must be finessed. To do this, we define $\Gamma : \mathbf{R}_+ \times \mathcal{X} \times \mathcal{D}_{<1} \rightarrow \mathbf{R}_+$ by

$$\Gamma(r | x, F) = \begin{cases} \gamma(r | x, F) & 0 \leq r \leq \bar{p} \\ \gamma(\bar{p} | x, F) + r - \bar{p} & r \geq \bar{p}. \end{cases} \quad (3)$$

I.e., we continuously paste a line of slope 1 to γ at the point $(\bar{p}, \gamma(\bar{p} | x, F))$. To see the effect of this, suppose all buyers have the same γ as in Figure 1, but are heterogeneous with respect to their search costs. Before, all buyers with $s \geq s_2$ had reservation level \bar{p} . Now, Γ spreads those buyers out above \bar{p} . Since buyers with reservation levels greater than \bar{p} also stop at the first store they visit, the economic situation is unchanged. We may therefore define the reservation level of a buyer with (x, s) to be

$$r(s, x, F) = \sup\{r \geq 0 \mid \Gamma(r | x, F) \leq s\}. \quad (4)$$

Although r is not continuous, it is measurable.

Proposition 2. *Γ is continuous. The function $r : \mathbf{R}_+ \times \mathcal{X} \times \mathcal{D}_{<1} \rightarrow \mathbf{R}_+$ defined in (4) is measurable. Let $R : I \times \mathcal{D} \rightarrow \mathbf{R}_+$ be defined by $R(i, F) = r(B(i), F)$. Then R is also measurable.*

Most equilibrium search models assume (i) all buyers have the same differentiable demand function and (ii) the cdf Q of search costs defined by $Q(s) = \mu_I(\{i \in I \mid s(i) \leq s\})/N$ is absolutely continuous (AC), which is equivalent to representation by a pdf. These assumptions guarantee that firms face continuous demand. The first ensures that a small change in the firm's price would cause a small change in sales from existing customers, while the second implies that the cdf of reservation levels defined by $G(r | F) = \mu_I(\{i \in I \mid R(i, F) \leq r\})/N$ is AC (see the argument below), so G has no atoms. If G had an

atom at r_0 , then a small increase in price above r_0 would lead to a disproportionately large reduction in the firm's customer base. The following assumption generalizes the situation considerably.

Assumption 1. (i) For each $(r, F) \in \mathbf{R}_+ \times \mathcal{D}_{<1}$, $B^{-1}(T(r, F))$ is μ_I -null, where $T(r, F) = \{(x, s) \in \mathcal{X} \times \mathbf{R}_+ \mid \Gamma(r \mid x, F) = s\}$. (ii) For each $p \in [0, \bar{p})$, the set of all buyers whose demand function is discontinuous at p is μ_I -null.

Assumption 1(ii) ensures the nonexistence of a positive mass of buyers having demand functions which all jump down at some particular price, creating a discontinuity in firms' demand there. 1(i) is equivalent to assuming G is continuous, although not necessarily AC. However, we prefer the above statement since it refers directly to the primitives of the model. To get a sense of what 1(i) entails, consider the set of buyers with reservation level \bar{p} . For example, the buyer could have search cost s_2 in Figure 1, which requires a special relationship between the buyer's demand function $Z(x)$ and her search cost. 1(i) rules out too much coordination of this type in the map B which assigns buyers their characteristics. Note that 1(i) implies Q is continuous, although not necessarily AC. For example, suppose all firms charge \bar{p} . In that case, Γ equals zero on $[0, \bar{p}]$ and $r - \bar{p}$ on $[\bar{p}, \infty)$, so to ensure that the mass of buyers with any particular reservation level $r_0 \geq \bar{p}$ is null, it is necessary to assume that the mass with search cost $s(i) = r_0 - \bar{p}$ is null.

We now turn to the supply side of the market. Let (J, \mathcal{J}, μ_J) be a probability space, where we have normalized the mass of sellers to be one. A firm j is characterized by its cost function $C(y \mid j)$, where y is output. In the proof of proposition 3, we show that quantity demanded is bounded above by some constant \bar{y} , so let \mathcal{C} be the set of all continuous functions $[0, \bar{y}] \rightarrow \mathbf{R}_+$ with the supremum metric. The supply side of the market is then characterized by a measurable function $S : J \rightarrow \mathcal{C}$, which assigns a cost function to each firm. Note that we do not assume constant returns to scale or even that cost functions are nondecreasing. A non-operating firm sets $p = \tilde{p}$ and makes zero profit. We assume $C(y \mid j) \equiv 0$ for a positive mass α of firms, and that indifferent firms choose to operate, so in equilibrium the mass α will choose to operate, and $F(\tilde{p}) \leq 1 - \alpha$.

Since the structure of firms' demand schedule is well-known, we present a heuristic

derivation. Suppose firm j charges the price p . Firm j 's potential buyers are those with reservation levels p and above. Any such buyer i will search until she finds a price $R(i, F)$ or below. The mass of such firms is $H(R(i, F) | F)$. Hence, j expects to sell $\frac{x(p | i)}{H(R(i, F) | F)}$ units to i . Summing over all such buyers, we obtain

$$D(p | F) = \int_{\{i \in I \mid R(i, F) \geq p\}} \frac{x(p | i)}{H(R(i, F) | F)} d\mu_I. \quad (5)$$

At this point, we have no guarantee that the integral in (5) is well-defined, since the denominator in the integrand could be zero, and because the integrand becomes arbitrarily large for small reservation levels. Assumption 2 below eliminates these problems.

Assumption 2. *There exists an $\bar{s} > 0$ such that $Q(\bar{s}) = 0$.*

From now on, we refer to assumptions 1 and 2 as the *general assumptions*.

The demand function in (5) is very general, and is not the usual one in the literature. To obtain the latter, we must make some strong assumptions.

Standard Assumptions. *All buyers have the same demand function $x \in \mathcal{X}$, continuous on $[0, \bar{p}]$. Furthermore, Q is AC with bounded continuous pdf q and $Q(0) = q(0) = 0$.*

We are free to work with either the reservation levels determined by γ or Γ , and in this case γ is more convenient. If all buyers have the same demand function,

$$D(p | F) = x(p) \int_{\{i \in I \mid R(i, F) \geq p\}} \frac{1}{H(R(i, F) | F)} d\mu_I. \quad (6)$$

Now i enters only via $R(i, F)$, which is measurable by proposition 2. We may therefore apply theorem 4.1.11 in [4, p. 92] to get

$$D(p | F) = x(p) N \int_p^{\bar{p}} \frac{1}{H(r | F)} dG(r | F). \quad (7)$$

It may appear we have a division-by-zero problem when $p < z(F)$, but since $[p, z(F)]$ is G-null, the integral is zero on that interval, and we may define the integrand as we like there. Since all buyers have the same γ , $G(r | F) = Q(\gamma(r | x, F))$, which is AC since γ is nondecreasing in r , and γ and Q are AC. To find its pdf g , we differentiate a.e. to obtain

$$g(r | F) = G'(r | F) = Q'(\gamma(r | x, F))\gamma'(r | x, F) = q(\gamma(r | x, F))x(r)H(r | F). \quad (8)$$

Substituting into (7), we get

$$D(p|F) = x(p) N \int_p^{\bar{p}} \frac{g(r|F)}{H(r|F)} dr = x(p) N \int_p^{\bar{p}} q(\gamma(r|x, F)) x(r) dr, \quad (9)$$

which is essentially the same as that in [1]. Note that $q(0) = 0$ is necessary to ensure the integral is zero on $[p, z(F)]$ when $p < z(F)$. Under the standard assumptions, the integrand in (9) is well-behaved, so all integrability and boundedness problems have disappeared. So the standard assumptions effectively mask these issues.

Proposition 3. (i) Under the standard assumptions, $D : [0, \bar{p}] \times \mathcal{D}_{<1} \rightarrow \mathbf{R}_+$ defined by (9) is continuous. (ii) Alternatively, under the general assumptions, D in (5) is continuous.

Although the general assumptions seem much weaker than the standard assumptions from an economic point of view, they are not more general mathematically, since the standard assumptions do not require assumption 2. Furthermore, the standard assumptions are computationally convenient and, until now, most applications of the equilibrium sequential search model have used some version of them.

We have not shown that D is continuous on the whole space \mathcal{D} , which seems difficult to do. We do know, however, that D is continuous on $[0, \bar{p}] \times \mathcal{D}_{\leq 1-\alpha}$, where $0 < \alpha < 1$ is the mass of firms which produce at zero cost, and $\mathcal{D}_{\leq 1-\alpha} = \{F \in \mathcal{D} \mid F(\tilde{p}) \leq 1 - \alpha\}$. Since $[0, \bar{p}] \times \mathcal{D}$ is compact Hausdorff and $\mathcal{D}_{\leq 1-\alpha}$ is closed,³ we may apply the Tietze-Urysohn extension theorem [4, p. 48] to obtain a continuous, nonnegative extension $[0, \bar{p}] \times \mathcal{D} \rightarrow \mathbf{R}_+$, which we also denote by D . Since the equilibrium cdf of prices will belong to $\mathcal{D}_{\leq 1-\alpha}$, the non-economic region $\mathcal{D}_{>1-\alpha}$ will play no further role in the analysis.

Let $\pi : P \times \mathcal{D} \times \mathcal{C} \rightarrow \mathbf{R}$ be defined by

$$\pi(p, F, C) = \begin{cases} pD(p|F) - C(D(p|F)) & p \in [0, \bar{p}] \\ 0 & p = \tilde{p}. \end{cases} \quad (10)$$

Let \mathcal{P} be the space of continuous functions $P \times \mathcal{D} \rightarrow \mathbf{R}$ with the supremum metric, and $\Pi : J \rightarrow \mathcal{P}$ be defined by $\Pi(j) = \pi(p, F, S(j))$.

³ Take any sequence $\{F_n\}$ in $\mathcal{D}_{\leq 1-\alpha}$ which converges to some $F \in \mathcal{D}$. In the proof of proposition 1, we show that $F_n(\tilde{p}) \rightarrow F(\tilde{p})$. Since $F_n(\tilde{p}) \leq 1 - \alpha$ for all n , $F(\tilde{p}) \leq 1 - \alpha$, so $F \in \mathcal{D}_{\leq 1-\alpha}$.

Proposition 4. *Under either the general or standard assumptions, π is continuous and Π is measurable.*

A *search market equilibrium* is a measurable price profile $f : J \rightarrow P$ such that $\Pi(j)(f(j), F) \geq \Pi(j)(p, F)$ for all $p \in P$ and $j \in J$, where $F \in \mathcal{D}$ is the cdf induced by f . Until now, we have only assumed (J, \mathcal{J}, μ_J) is a probability space. In the following theorem, the main result of the paper, it is assumed to be a *uniform Loeb probability space*, as discussed below.

Theorem. *If (J, \mathcal{J}, μ_J) is a uniform Loeb probability space, then a search market equilibrium exists under either the general or standard assumptions.*

Note that the uniform Loeb probability space hypothesis applies to both the standard and general case. It is beyond the scope of this paper to give a precise definition of the uniform Loeb probability space; see [3, 5, 8]. Our discussion here is completely informal. One first constructs the space of *hyperreal* numbers (the *ultrapower construction*), which extends \mathbf{R} to include, for example, infinite or *unlimited* natural numbers, which are greater than any standard natural number. Let J be a *hyperfinite* set with cardinality N , where N is an unlimited natural number; e.g., $J = \{1, 2, \dots, N\}$. Let $\nu_J(\{j\}) = 1/N$ for all $j \in J$. This is an extension of the usual uniform distribution on a finite set, in which each player has *infinitesimal* weight. It is not a standard probability space because, for one thing, $1/N$ is not a real number. We fix this by defining $\mu_J = \text{sh } \nu_J$, where sh is the *standard part* or *shadow* map which assigns the nearest real number to each (limited) hyperreal. Now, each player has measure zero. The uniform Loeb probability space then follows from the standard outer measure construction applied to (J, μ_J) . It satisfies the usual definition of an atomless probability space, and is the nonstandard analogue of the uniform distribution.

The above theorem is now a direct application of the following result, which is a simplified version of theorem 1 in [9], stated in a form suitable for further applications to equilibrium search models.

Theorem. *(Khan and Sun) Let J be a uniform Loeb probability space, $P \subseteq \mathbf{R}$ be compact and nonempty, \mathcal{D} be the set of cdfs with support contained in P with the Prohorov metric,*

and \mathcal{P} the set of continuous functions $P \times \mathcal{D} \rightarrow \mathbf{R}$ with the supremum metric. If $\Pi : J \rightarrow \mathcal{P}$ is Loeb measurable, there exists a Loeb measurable function $f : J \rightarrow P$ such that $\Pi(j)(f(j), F) \geq \Pi(j)(p, F)$ for all $p \in P$ and $j \in J$, where $F \in \mathcal{D}$ is the cdf induced by f .

Note that the Khan-Sun theorem is false for the unit interval with Lebesgue measure: see the counterexample in [11] and further elaborated in [7]. Although the proof of the Khan-Sun theorem involves essentially classical reasoning, certain pieces of the argument related to the distribution of the best response correspondence break down for Lebesgue measure spaces. It remains an open question as to whether the theorem in this paper is valid for Lebesgue measure spaces. If so, it seems the proof will have to involve an original line of attack.

In the context of equilibrium search models, most of the focus is on the cdf of prices and comparative statics, not the space of players. Any mathematical model for J with appropriate economic content is therefore suitable. In particular, the only feature of the unit interval with Lebesgue measure with any economic content is that each player has measure zero, which is also a feature of the uniform Loeb probability space. The bottom line is that researchers can simply assume the latter, and then focus on those objects of primary interest such as the cdf of prices, which is completely standard. Furthermore, most equilibrium search models have the same general structure covered by the Khan-Sun theorem, so one can use the present paper as a road map for proving existence in other equilibrium search models, with different search technologies and equilibrium concepts.

Another approach is to assume buyers' search rules only depend on finitely many moments of the cdf of prices, as in [12]. In that case, the theorem in [14] guarantees existence, even for Lebesgue measure spaces. In this sense, the latter are game-theoretically incomplete because an equilibrium exists for any finite number of moments, but not in the limit.

3. Conclusion

In this paper, we developed an equilibrium sequential search model which includes most of the literature as special cases. In particular, the model can accommodate heterogeneity in buyers' search costs and demand functions and firms' cost functions, with very general

cost and demand functions. We identified conditions which ensure existence of equilibrium in pure strategies, and elucidate the essential structure of this important class of models. The impetus for our work was provided by recent progress in the theory of large games by Khan and Sun [9]. Although we focused on sequential search, our methodology can be used to prove existence for other classes of equilibrium search models as well.

Appendix

Proof of Proposition 1

The integrand in (2) is a.e. differentiable and bounded above by $g(0)$, hence integrable. We now prove γ is continuous. We can re-write it as $\gamma(r | x, F) = \int_{[0, \bar{p}]} x(p) H(p | F) 1_{[0, r]}(p) dp$, where $1_{[0, r]}(p)$ is the indicator function which equals 1 when $0 \leq p \leq r$ and 0 otherwise. Let $(r_n, x_n, F_n) \rightarrow (r, x, F)$. Consider the sequence $\{x_n(p) H(p | F_n) 1_{[0, r_n]}(p)\}$ of integrable functions. Since $x_n \rightarrow x$ in the supremum metric, $x_n(p) \rightarrow x(p)$ pointwise on $[0, \bar{p}]$. We now show that $H(p | F_n) \rightarrow H(p | F)$ a.e. on $[0, \bar{p}]$. By theorem 11.1.2 in [4, p. 304], $F_n \rightarrow F$ in the Prohorov metric iff $F_n(p) \rightarrow F(p)$ at all $p \in \mathbf{R}$ where F is continuous. Since F is nondecreasing, its discontinuities are at most countable. We therefore have only to show that $F_n(\tilde{p}) \rightarrow F(\tilde{p})$. Choose any p_0 such that $\tilde{p} < p_0 < 0$. Since F is continuous at p_0 (it's locally constant there), $F_n(p_0) \rightarrow F(p_0)$. But $F_n(p_0) = F_n(\tilde{p})$ and $F(p_0) = F(\tilde{p})$. The sequence $\{1_{[0, r_n]}\}$ of indicator functions converges pointwise to $1_{[0, r]}$ except possibly at $p = r$, hence a.e. So $x_n(p) H(p | F_n) 1_{[0, r_n]}(p) \rightarrow x(p) H(p | F) 1_{[0, r]}(p)$ pointwise a.e. Finally, $x_n(p) H(p | F_n) 1_{[0, r_n]}(p) \leq g(p)$, so we can apply the dominated convergence theorem. The differentiability claim follows from one of the fundamental theorems of calculus. The rest is clear. ■

Proof of Proposition 2

We first prove Γ is continuous. Let $(r_n, x_n, F_n) \rightarrow (r, x, F)$. If $r < \bar{p}$, then $r_n < \bar{p}$ after finitely many terms. The result then follows by continuity of γ . If $r > \bar{p}$, then $r_n > \bar{p}$ after finitely many terms. As $(x_n, F_n) \rightarrow (x, F)$, $\gamma(z | x_n, F_n) \rightarrow \gamma(z | x, F)$ uniformly on $[0, \bar{p}]$,

by corollary 5.4 in [10, p. 287]. Hence, $\gamma(\bar{p} | x_n, F_n) \rightarrow \gamma(\bar{p} | x, F)$ and the result follows. If $r = \bar{p}$, then convergence occurs for r_n on both sides of r .

To prove the measurability of $r(s, x, F)$, let $\{\Gamma_n\}$ be a sequence of functions $\Gamma_n : \mathbf{R}_+ \times \mathcal{X} \times \mathcal{D}_{<1} \rightarrow \mathbf{R}$ defined by $\Gamma_n(r | x, F) = \Gamma(r | x, F) - (1/[n(r+1)])$. Each Γ_n is clearly continuous and increasing in r . For each n , define $r_n : \mathcal{X} \times \mathbf{R}_+ \times \mathcal{D}_{<1} \rightarrow \mathbf{R}_+$ as the unique solution $r_n(x, s, F)$ to the equation $\Gamma_n(z | x, F) = s$ in the variable z .

Lemma 1. *For each n , r_n is continuous.*

Proof. Let $(x_k, s_k, F_k) \rightarrow (x, s, F)$. We must show that $r_n^k \equiv r_n(x_k, s_k, F_k) \rightarrow r_n \equiv r_n(x, s, F)$. As $(x_k, F_k) \rightarrow (x, F)$, $\Gamma_n(z | x_k, F_k) \rightarrow \Gamma_n(z | x, F)$ uniformly on \mathbf{R}_+ . By definition, $s = \Gamma_n(r_n | x, F)$ and $s_k = \Gamma_n(r_n^k | x_k, F_k)$. Since Γ_n is increasing and continuous, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|\Gamma_n(z | x, F) - \Gamma_n(r_n | x, F)| < \delta$ implies $z \in (r_n - \epsilon, r_n + \epsilon)$. For k sufficiently large, $|\Gamma_n(z | x_k, F_k) - \Gamma_n(z | x, F)| < \delta/2$ for all $z \geq 0$ by uniform convergence. Since $s_k \rightarrow s$, $|\Gamma_n(r_n^k | x_k, F_k) - \Gamma_n(r_n | x, F)| < \delta/2$ for sufficiently large k . By the triangle inequality, $|\Gamma_n(r_n^k | x, F) - \Gamma_n(r_n | x, F)| < \delta$ so $r_n^k \in (r_n - \epsilon, r_n + \epsilon)$ for sufficiently large k . ■

Since Γ_n converges uniformly to Γ from below, $\{r_n\}$ converges pointwise to r . Since continuous functions are measurable, and a pointwise limit of measurable functions is measurable, we are done. The rest follows because a composition of measurable functions is measurable. ■

Proof of Proposition 3.

We first prove (ii). Write $D(p | F) = \int_I 1_{R(i, F) \geq p}(i) \frac{x(p | i)}{H(R(i, F) | F)} d\mu_I$, where $1_{R(i, F) \geq p}(i)$ is the indicator function which equals 1 when $R(i, F) \geq p$ and zero otherwise.

Lemma 2. *For each $(p, F) \in [0, \bar{p}] \times \mathcal{D}_{<1}$,*

$$1_{R(i, F) \geq p}(i) \frac{x(p | i)}{H(R(i, F) | F)} \quad (A1)$$

is μ_I -integrable.

Proof. Fix $(p, F) \in [0, \bar{p}] \times \mathcal{D}_{<1}$. We begin by showing that the denominator in (A1) is positive, and the whole expression is bounded above by a positive constant. For those i

such that $R(i, F) \geq \bar{p}$, the denominator is 1 and $g(0)$ is the upper bound. If $R(i, F) < \bar{p}$, consider $\int_{z(F)}^r x(p|i) H(p|F) dp = s$. Since $x(p|i) H(p|F) \leq g(0)$, a lower bound for $R(i, F)$ is determined by $\int_{z(F)}^r g(0) dp = g(0) [r - z(F)] = \bar{s}$ (see assumption 2) or $\bar{r} = z(F) + \frac{\bar{s}}{g(0)}$. Hence, $R(i, F) \geq \bar{r} > z(F)$ and the upper bound is $\frac{g(0)}{H(\bar{r}|F)}$. To finish the proof, we show (A1) is (I, \mathcal{I}) -measurable. By proposition 2, $1_{R(i, F) \geq p}(i)$ is measurable. The denominator in (A1) is measurable, since H and R are measurable. It remains to show that $x(p|i)$ is measurable. Fix any $k \in \mathbf{R}_+$. Let $\mathcal{X}(p) = \{x \in \mathcal{X} | x(p) \leq k\}$. If $\mathcal{X}(p) = \emptyset$, it is closed. Otherwise, let $\{x_n\}$ be a sequence in $\mathcal{X}(p)$ which converges to $x \in \mathcal{X}$. Since $x_n \rightarrow x$, $x_n(p) \rightarrow x(p)$. Since $x_n(p) \leq k$, $x(p) \leq k$, and $\mathcal{X}(p)$ is closed. Hence $\{i \in I | x(p|i) \leq k\} = B^{-1}(\mathbf{R}_+ \times \mathcal{X}(p))$ is measurable. ■

To prove continuity, let $(p_n, F_n) \rightarrow (p, F)$. The expression “for a.a.i” means “for μ_I -almost all i ”. By assumption 1(ii), $x(p_n|i) \rightarrow x(p|i)$ for a.a.i.

Lemma 3. $R(i, F_n) \rightarrow R(i, F)$ for a.a.i.

Proof. We consider only those i such that $R(i, F) \neq \bar{p}$, which is a set of full measure by assumption 1(i). Let x_i be i 's demand function. There are two cases. If $Z(x_i) \leq z(F)$ then $\Gamma(r|x_i, F) = 0$ for all $r \in [0, \bar{p}]$, and is increasing with slope 1 thereafter. Since $s(i) > \bar{s} > 0$ for a.a.i, $R(i, F) > \bar{p}$ for almost all such i . Now suppose $z(F) < Z(x_i)$. We cannot have $R(i, F) = Z(x_i)$ unless $Z(x_i) = \bar{p}$ (see Figure 1), which we have ruled out. So either $z(F) < \bar{r} \leq R(i, F) < Z(x_i)$ or $R(i, F) > \bar{p}$. Since $\Gamma(r|x_i, F)$ is increasing on $[\bar{r}, Z(x_i))$ and (\bar{p}, ∞) , and $\Gamma(r|x_i, F_n)$ converges uniformly to $\Gamma(r|x_i, F)$ on those intervals, the result follows. ■

Lemma 4. $H(R(i, F_n) | F_n) \rightarrow H(R(i, F) | F)$ for a.a.i.

Proof. In the proof of proposition 1, we showed that $F_n(\tilde{p}) \rightarrow F(\tilde{p})$, so all we need to do is show that $F_n(R(i, F_n)) \rightarrow F(R(i, F))$. Since F has at most countably many discontinuities, we may enumerate them as $\{w_k\}$. For each k , the set of all i such that $R(i, F) = w_k$ is null by assumption 1(i). Since a countable union of null sets is null, the set of all i such that F is discontinuous at $R(i, F)$ is also null. Fix an i such that F is continuous at $R(i, F)$ and $R(i, F_n) \rightarrow R(i, F)$. Fix $\epsilon > 0$. Since F is continuous at $R(i, F)$, there is an interval

(r_1, r_2) such that $R(i, F) \in (r_1, r_2)$ and $|F(r) - F(R(i, F))| < \frac{\epsilon}{6}$ for all $r \in (r_1, r_2)$. By choosing r_1 and r_2 closer to $R(i, F)$, if necessary, we may assume F is continuous at r_1 and r_2 , and the previous inequality holds on $[r_1, r_2]$. Let \mathbf{N} be the set of positive integers. Since $F_n(r_1) \rightarrow F(r_1)$ and $F_n(r_2) \rightarrow F(r_2)$, choose $N_1 \in \mathbf{N}$ such that $|F_n(r_1) - F(r_1)| < \frac{\epsilon}{3}$ for all $n \geq N_1$ and $|F_n(r_2) - F(r_2)| < \frac{\epsilon}{3}$ for all $n \geq N_2$. Since $R(i, F_n) \rightarrow R(i, F)$, choose $N_3 \in \mathbf{N}$ such that $R(i, F_n) \in (r_1, r_2)$ for all $n \geq N_3$. Let $N_4 = \max\{N_1, N_2, N_3\}$. For all $n \geq N_4$, $F(r_1) - \frac{\epsilon}{3} < F_n(r_1) \leq F_n(R(i, F_n)) \leq F_n(r_2) < F(r_2) + \frac{\epsilon}{3}$. Furthermore, $F(r_1) \leq F(R(i, F)) \leq F(r_2)$. So $|F_n(R(i, F_n)) - F(R(i, F))| < [F(r_2) + \frac{\epsilon}{3}] - [F(r_1) - \frac{\epsilon}{3}] < \epsilon$. ■

Lemma 5. $1_{R(i, F_n) \geq p_n}(i) \rightarrow 1_{R(i, F) \geq p}(i)$ for a.a.i. In fact, $1_{R(i, F_n) \geq p_n}(i) = 1_{R(i, F) \geq p}(i)$ after finitely many terms for a.a.i.

Proof. Fix i such that $R(i, F) \neq p$ and $R(i, F_n) \rightarrow R(i, F)$. If $p < R(i, F)$, then $p_n < R(i, F_n)$ after finitely many n . If $R(i, F) < p$, then $R(i, F_n) < p_n$ after finitely many n . ■

We have shown that for a.a.i., $1_{R(i, F_n) \geq p_n}(i) \frac{x(p_n | i)}{H(R(i, F_n) | F_n)}$ is well-defined after finitely many terms, and converges to $1_{R(i, F) \geq p}(i) \frac{x(p | i)}{H(R(i, F) | F)}$. By the dominated convergence theorem, we will be done if we can show that $\frac{x(p_n | i)}{H(R(i, F_n) | F_n)}$ is bounded above by some constant for a.a.i. after finitely many terms. For any n , for those i such that $R(i, F_n) \geq \bar{p}$ the upper bound is $g(0)$. Now choose $z(F) < \hat{r} < \bar{r}$ such that F is continuous at \hat{r} , so $H(\hat{r} | F_n) \rightarrow H(\hat{r} | F)$. Fix $\epsilon > 0$. We may assume $0 < H(\hat{r} | F) - \epsilon$. Choose $N_1 \in \mathbf{N}$ such that $0 < H(\hat{r} | F) - \epsilon < H(\hat{r} | F_n) < H(\hat{r} | F) + \epsilon$ for all $n \geq N_1$. Choose any i such that $R(i, F_n) < \bar{p}$ and $R(i, F_n) \rightarrow R(i, F)$. Choose $\epsilon_2 > 0$ and $N_2 \in \mathbf{N}$ such that $\hat{r} < R(i, F) - \epsilon_2 < R(i, F_n) < R(i, F) + \epsilon_2$ for all $n \geq N_2$. Let $N_3 = \max\{N_1, N_2\}$. For $n \geq N_3$, $0 < H(\hat{r} | F) - \epsilon < H(\hat{r} | F_n) \leq H(R(i, F_n) | F_n)$. Hence, for all $n \geq N_3$, $\frac{x(p_n | i)}{H(R(i, F_n) | F_n)} < \frac{g(0)}{H(\hat{r} | F) - \epsilon}$, which completes the proof of (ii).

To prove (i), write (9) as $D(p | F) = x(p) N \int_{r \in [0, \bar{p}]} 1_{p \leq r \leq \bar{p}}(r) q(\gamma(r | x, F)) x(r) dr$. Let K be an upper bound for q . Viewed as a function of r , the integrand is a.e. continuous and bounded above by $Kg(0)$, therefore integrable. For any r_0 , $\gamma(r_0 | x, F_n) \rightarrow \gamma(r_0 | x, F)$, so $q(\gamma(r_0 | x, F_n)) \rightarrow q(\gamma(r_0 | x, F))$ since q is continuous. The proof that $1_{p_n \leq r \leq \bar{p}} \rightarrow 1_{p \leq r \leq \bar{p}}$ pointwise a.e. is similar to that for lemma 5 above. Finally, $1_{p_n \leq r \leq \bar{p}}(r) q(\gamma(r | x, F_n)) x(r)$

is also bounded above by $Kg(0)$, so the dominated convergence theorem completes the proof. ■

Proof of Proposition 4.

We first show that π is continuous. Let $(p_n, F_n, C_n) \rightarrow (p, F, C)$. Suppose $p \neq \tilde{p}$. Then after finitely many terms, $p_n \in [0, \bar{p}]$. We have $D(p_n | F_n) \rightarrow D(p | F)$ under either set of assumptions. By theorem 5.3 in [10, p. 287], the evaluation map is continuous, so $C_n(D(p_n | F_n)) \rightarrow C(D(p | F))$, and we are done. If $p = \tilde{p}$ then $p_n = \tilde{p}$ after finitely many terms. By corollary 5.4 in [10, p. 287], the function $\mathcal{C} \rightarrow \mathcal{P}$ defined by $C \mapsto \pi(p, F, C)$ is continuous. Hence, the composition with the measurable function S is measurable. ■

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Figure 1. The Marginal Benefit γ of Another Search.

