Credit and Inflation under Borrower's Lack of Commitment[*]

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Abstract

Here we investigate the existence of credit in a cash-in-advance economy where there are complete markets but for the fact that agents cannot commit to repay their debts. Defectors are banned from the credit market but they can use money balances for saving purposes. Without uncertainty, deflation crowds out credit completely. The equilibrium allocation, however, is efficient if the government deflates at the time preference rate. Efficiency can also be restored with positive inflation. For any non negative inflation rate below the optimal level, the volume of credit and the real interest rate increase with inflation. Our results hold when idiosyncratic uncertainty is introduced and households are sufficiently impatient but in one instance: efficiency cannot be restored if the deflation rate is nearby the rate of time preference. Our numerical examples suggest that the optimal inflation rate is not too large for reasonable levels of patience and risk aversion. Finally, we present a framework where the use of money arises endogenously and show that it is tantamount to our cash-in-advance framework. Our results hold in this modified environment.

Keywords: Monetary policy, existence of credit, Friedman rule, self-enforcing debt, risk sharing.

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1 Introduction

One of the most persistent results in monetary theory is the Friedman rule, that states that the optimal monetary policy entails zero nominal interest rates: that is, the monetary authority should deflate at the time preference rate (in a deterministic steady setting). The key issue is that if agents are forced to use money for transaction and saving purposes they economize inefficiently on the use of money. To fix this monetary inefficiency, the central bank must follow the Friedman Rule—it must eliminate the difference in returns between money and non-monetary assets. Broadly speaking, this result is robust to various market frictions. See, for instance, Kocherlakota (2005) for a survey in this matter.

In this paper we address this old question in a cash-in-advance economy where agents cannot commit to repay their debts. Households do not value leisure. There are only two assets: money and private bonds, which can be used for lending and borrowing purposes. We assume that there is limited commitment as in Kocherlakota (1996), Kehoe and Levine (2001), and Alvarez and Jermann (2000). This implies that there must exist a centralized credit agency that makes all the record keeping involved in borrowing and lending across households. This credit agency is in charge of punishing households that default and make sure that they are seized their bond holdings and are banned from the credit market forever. Defectors, however, keep their money holdings and can save in the form of money balances. That is, the government cannot tax away defectors' monetary balances. As Kocherlakota (1996), Kehoe and Levine (2001), and Alvarez and Jermann (2000), we assume that there is no private information. This implies that the credit agency never lends off so much so that agents prefer defaulting on their debts to participating in the credit market. Thus, although there is no default in equilibrium, the amount of credit will depend on the return to default, which depends on the inflation rate.

We focus on symmetric steady states. We find that, under full commitment, or, alternatively, perfect enforcement, money is super neutral and the first best allocation is achieved: complete consumption smoothing (complete risk sharing). When the government deflates at the rate of time preference, households are indifferent about the composition of their portfo-

lio. This is so because we have assumed that households do not value leisure. Nevertheless, we wanted to stick to this simple setup with only one friction: limited commitment and defaulters can save in the form of money.

We start by studying a deterministic economy where households alternate between high and bad productivity. In this simple setup we show that contracting the monetary base crowds out private credit completely. The reason for this is that, for any negative inflation rate, the return to money is so high that households with debts are better off defaulting on them and self insuring using money than participating in the credit market. Nevertheless, households can achieve the full commitment allocation by using only money if the government deflates at the rate of time preference. This is so because agents know perfectly their future stream of endowments and can accumulate a finite amount of money balances to smooth consumption completely. Next, we show that the amount of credit and the real return to bonds rise with inflation for low levels of inflation. There exists a positive threshold for the inflation rate at which the return to default is so low that the full commitment allocation can be attained and the volume of credit is maximum. Above that threshold the consumption allocation does not change with the level of inflation. This is a result of assuming that leisure is not valued by households.

We extend our analysis to an economy where agents face idiosyncratic uncertainty in their labor productivity. We assume that households can issue contingent bonds. The real return to money, as opposed to that of bonds, is not contingent. We first study the economy assuming that agents have full commitment. Differently from the deterministic case, the full risk sharing allocation cannot be achieved by self insuring using money, since its return is not contingent. Next, we turn to study the case with limited commitment. Defectors behave as agents in Bewley (1983): they can self insure using a non contingent asset. We show that if households are sufficiently impatient there cannot be full risk sharing if the deflation rate is nearby the rate of time preference. This is so because credit is constrained so that households with the highest productivity are indifferent between defaulting on their debts and self insuring with money balances. Thus, the distribution of consumption across agents depends on the distribution of wealth. A consequence of this is that there is no

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equilibrium if the deflation rate is nearby the rate of time preference. As in the case without uncertainty, there exists a threshold for the inflation rate above which the economy can attaint full risk sharing with credit. If households are sufficiently impatient (low discount factor) the threshold is positive. We conduct some numerical examples and show that the predicted optimal inflation rate is not too large, around 5 percent for reasonable parameter values.

It could be argued that we have used an ad-hoc manner to introduce money in our setup. We show that we can derive endogenously the cash-in-advance constraint in a framework that resembles that of Lagos and Wright (2005): households alternate in being consumers and producers. Utility, however, is not transferable and leisure is not valued. We need two assumptions for money to be used in equilibrium: limited commitment and anonymity in the goods market. In other words, households do not have any record keeping technology. Financial institutions can keep track of households' financial histories but not trading histories. Credit agencies act as intermediaries allowing private IUOs to circulate, moreover, they never lend so much so that borrowers are better off defaulting on their debts. We find that this economy can be mapped exactly onto our cash-in-advance framework.

Our paper is related to Aiyagari and Williamson (2000). They use a model economy where agents hold money balances because a random participation constraint prevents them from fully participating in the credit market. Agents, however, can default on their debts, case in which they are banned from the credit market thereafter. As in our case, the Friedman rule is not optimal. Thus, random limited participation plays the same role that the cash-in-advance constraint in our setting. Our paper is also related to Berentsen, Camera, and Waller (2007). They allow for the existence of credit in the Lagos and Wright's (2005) framework. As in our setup, there is limited commitment. They, however, do not allow the government to contract the monetary stock. Thus, they only consider economies with non negative inflation. Their results are similar to that our economy without idiosyncratic uncertainty: credit is unconstrained whenever the inflation rate is above a positive threshold. We, however, can study the case of deflation in the deterministic economy as well the economy with idiosyncratic uncertainty.

Our paper is also related to Hellwig and Lorenzoni (2006). They study the implications of limited commitment in a very similar environment to ours but from the fact that money is not needed for transaction purposes. Moreover, they assume that defectors can hold positive amount of bonds. That is, financial intermediaries only can keep track of negative bond holdings but cannot monitor positive bond holdings. The authors show that the only sustainable policy entails (in the deterministic setting) zero inflation and zero nominal interest rate. This result is very different from ours: in our setup, coexistence of money and credit does not require zero nominal interest rate. In particular, the optimal inflation rate is positive and the nominal interest rate is positive in our deterministic setting. As a result, monetary policy can restore efficiency in our setting.

There are other instances where the Friedman Rule is not the optimal monetary policy. That is the case in environments where the inflation tax is a proxy for other policies or taxes. Levine (1991) argues that if the government cannot set taxes contingent upon individual's marginal utility an expansionary monetary policy amounts to transferring real resources from those with low marginal utility to those with high marginal utility. That is the case in Akyol (2004) where non contingent bonds are an imperfect means of smoothing away idiosyncratic labor risk and wealthy agents hold precautionary money balances. Thus, the inflation tax acts as a redistributive mechanism of wealth that is ex-ante welfare improving. This is also the case in Bhattacharya, Haslag, and Martin (2005) and Williamson (2005). As Kocherlakota (2005) argues, when the government has a complete array of alternative taxes the Friedman Rule arises as the optimal monetary policy. Berentsen, Camera, and Waller (2005), for instance, find that the government should use the Friedman Rule when lump-sum taxes and transfers are available. Berentsen, Rocheteau, and Shi (2007) find similar result in a search monetary model where the Hossios rule eliminates the inefficiencies of bargaining. In our case the government has available those taxes, too, but with one limitation: the government cannot tax discriminate between defaulters and non defaulters. That is, the only punishment to defaulters is perpetual banning from the credit market, but the government cannot tax away their precautionary money balances in a different manner from non defaulters.

Our result is close to that by He, Huang, and Wright (2005). They study the coexistence of fiat money as a medium of exchange and private liabilities issued by institutions resembling banks. They assume that agents can steal money from others. Thus, the government may find optimal to set positive nominal interest rates to discourage illicit behavior. Finally, Green and Zhou (2005) study a monetary economy which is very similar to our economy without credit. There they show that aggregate welfare may rise with inflation for mild levels of inflation because it makes it impossible for anyone to accumulate very large money balances.

The rest of the paper is organized as follows. Section 2 describes the environment. Section 3 studies the equilibrium assuming that agents have full commitment and derive the efficient allocation. In section 4 we turn to study the economy with limited commitment and characterize the optimal monetary policy. Section 5 extends our basic model to allow for idiosyncratic uncertainty and characterize the optimal monetary policy. Here we include some numerical examples. In section 6 we modify our original setup so that money arises endogenously and show that our cash-in-advanced economy can be mapped into this one. Finally, section 7 concludes.

2   The benchmark model economy

Here we describe our benchmark economy. Section 2.1 describes individual preferences and endowments, in sections 2.2 and 2.3 we describe the market arrangements and the government's monetary policy, and sections 2.4, 2.5, and 2.6 state the recursive formulation of stationary equilibrium for this economy.

2.1 Population, preferences, endowments and production possibilities.

Our basic environment is very similar to that in Kehoe and Levine (2001). There is an infinite number of discrete time periods $t = 0, 1, \ldots$. In each period there are two types of households $i = 1, 2$ and a continuum of each type of households with measure one half. There is a single consumption good c. Both types of households derive utility from consumption and do not value leisure. The amount consumed by type i household at period t is denoted as c_t^i. We write lifetime utility as $\sum_{t=0}^{\infty} \beta^t u(c_t)$. The period utility function is twice continuously differentiable with $u'(c) > 0$, satisfies the boundary condition $u'(c) \to \infty$ as $c \to 0$ and has $u''(c) < 0$, and $u'''(c) > 0$. The discount factor satisfies $0 < \beta < 1$.

Households are endowed with one unit of time each period. Each period households receive a shock to their efficiency units of labor $w \in \{w_l, w_h\}$, where $w_h > w_l$. Both shocks are positive. We assume that when households of type 1 receive the shock w_l households of type 2 receive the shock w_h. We start by assuming that productivity alternates between high and low, so if $w_t^i = w_h$ then $w_{t+1}^i = w_l$. The production of the unique consumption good requires labor. The production function is linear, $Y = L$, where L is aggregate labor. We make the following assumption:

Assumption 1. *Trade is welfare improving,*

$$u\left(\frac{w_h + w_l}{2}\right) > u(w_h) + \beta\, u(w_l). \tag{1}$$

This assumption ensures that complete consumption smoothing is strictly preferred to autarky by all households.

2.2 Market arrangements

We assume that households need money for transaction purposes and, as in Svensson (1989), that consumption expenditures are determined by the amount of money balances held at the end of the previous period. In addition to money households can hold private bonds, which can be used for borrowing and lending purposes.

Households have the option of going bankrupt. In that case they are excluded from further participation in the credit market and they are seized their bond holdings. However, they cannot be seized their money holdings and they can save in the form of money after default. Notice that this model requires the existence of a central agency to keep track of who has gone bankrupt, to assure that their bond holdings are seized and that they do not continue to borrow and lend. We assume that financial institutions are perfectly competitive and that they cannot price discriminate in the sense that they charge the same interest rate to all households. Moreover, we assume that there is perfect entry and exit in the financial sector. Formally, this is a model in which households face the incentive compatibility constraint

$$\sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s^i\right) \geq \sum_{s=t}^{\infty} \beta^{s-t} u\left(z_s^i\right), \text{ for all } t \geq 0. \tag{2}$$

This says that in every period, the value of continuing to participate in the credit market is no less than the value of defection. In this setting, the absence of private information implies that no consumer actually goes bankrupt in equilibrium: the credit agency will never lend so much to consumers so that they will choose bankruptcy.[1]

Notice the difference with Kehoe and Levine (2001). In our setup defection does not mean that households turn to autarky.[2] It rather implies that agents only can rely on self insurance by accumulating real money balances. It could seem inconsistent to assume that financial authorities can seize bond holdings but that they cannot seize money holdings. The

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[1]Alvarez and Jermann (2000) show that these incentive compatibility constraints can be reinterpreted as state and agent-specific borrowing limits (or solvency constraints) set by financial institutions. In absence of private information the solvency constraints ensure that agents will not default in equilibrium, since they will never owe so much as to make them choose to default. That is, both modeling choices are equivalent.

[2]This also the case in Kocherlakota (1996), and Alvarez and Jermann (2000).

reason for this assumption is twofold. First, financial authorities can seize bond holdings in the sense that they can refuse to pay bond proceeds to defectors but we assume that they cannot tax away individual's money holdings, only the government can do that. Second, since money is needed for transactions and marginal utility at zero approaches infinity, financial authorities cannot seize all individual money holdings. The crucial assumption, however, is that households can use money for saving purposes after defaulting on their debts, that is, the government cannot tax away all defaulters' savings.

The timing of the model is as follows: people work, consume, they are paid their return to their labor endowments, decide whether to default on their debts or not and decide their next period wealth as well as the composition of their portfolio.

2.3 The government

The government injects money in the economy as lump-sum transfers to agents. The aggregate stock of money supply evolves according to the law

$$M_{t+1} = (1 + \theta) M_t, \tag{3}$$

where θM_t equals to the sum of all transfers. We are going to assume that the money growth rate is always finite and greater than or equal to the rate of time preference, $\theta \in [\beta - 1, +\infty)$. It will be useful to express the law of motion of the aggregate supply of real money balances in per capita terms,

$$\mathbf{m}_{t+1} = \frac{1 + \theta}{1 + \varepsilon_{t+1}} \mathbf{m}_t, \tag{4}$$

where $\varepsilon_{t+1} = p_{t+1}/p_t - 1$ is the inflation rate at period $t + 1$.

We denote as T_{t+1}^i the monetary transfer (or tax) that household i receives at the end of period t. The transfer in terms of consumption good at time t is $T_{t+1}^i/p_t = (1 + \varepsilon_{t+1}) \tau_{t+1}^i$.

We assume that money transfers (or taxes) are non redistributive,

$$(1 + \varepsilon_{t+1}) \tau_{t+1}^i = \frac{w_t^i}{\overline{w}} \theta \, \mathbf{m}_t, \tag{5}$$

where $\overline{w} = (w_h + w_l) / 2$ is aggregate production. In the case in which the government chooses to deflate, these lump-sum taxes are equivalent to imposing a flat tax rate equal to $\theta \, \mathbf{m}_t / \overline{w}$. If the government chooses to increase the monetary base these transfers are proportional to real earnings. Using this scheme for injecting or withdrawing money ensures that the money growth rate θ does not affect the distribution of earnings and, therefore, the amount of credit. The reason for this assumption is that the government could set money transfers to redistribute resources across households, leaving them indifferent between trading in the bond market or not. We want to shut down completely this channel to focus only on the effect of inflation through its effect on the return to default.

2.4 The household's problem

The problem solved at period 0 by household of type i is

$$
\begin{aligned}
\text{max} \quad & \sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right) \\
\text{s. t.} \quad & p_t c_t^i + q_t^n B_{t+1}^i + M_{t+1}^i \leq p_t w_t^i + B_t^i + M_t^i + T_{t+1}^i, \\
& p_t c_t^i \leq M_t^i, \\
& \sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s^i\right) \geq \sum_{s=t}^{\infty} \beta^{s-t} u\left(z_s^i\right), \text{ for all } t.
\end{aligned}
\tag{6}
$$

B_t^i denotes nominal bond holdings at the beginning of period t, q_t^n is the nominal price of bonds at time t, M_t^i stands for money balances at the beginning of period t and T_{t+1}^i is the money injected by the government as transfers. The last inequality, $\sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s^i\right) \geq \sum_{s=t}^{\infty} \beta^{s-t} u\left(z_s^i\right)$, is the incentive compatibility constraint. It says that agents never owe so much so that they prefer to default on their debts. It imposes a lower limit on the amount of bonds carried to next period, B_{t+1}^i. That is, financial intermediaries choose the maximum amount of credit so that households are better off repaying their debts than defaulting on

them. The problem solved after defaulting at time t is

$$
\begin{aligned}
\sum_{s=t}^{\infty} \beta^{s-t} u\left(z_s^i\right) = \quad & \max \sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s^i\right) \\
\text{s. t.} \quad & p_s c_s^i + M_{s+1}^i \leq p_s w_s^i + M_s^i + T_{s+1}^i, \\
& p_s c_s^i \leq M_s^i, \\
& M_t^i \text{ given.}
\end{aligned}
\tag{7}
$$

Hereafter we will refer to the household's problem shown in (6) as the "household's problem" or the problem "staying in trade", whereas we will refer to the problem shown in (7) as the "default problem". Likewise, we will refer to $\sum_{s=t}^{\infty} \beta^{s-t} u\left(z_s^i\right)$ as the "utility after default".

2.5 The cash-in-advance constraint and precautionary money balances

Here we write the household's problem in real terms. We denote real money balances at the beginning of period t as m_t^i. The budget constraint and the cash-in-advance constraint can be written as

$$
\begin{aligned}
& c_t^i + q_t b_{t+1}^i + \left(1 + \varepsilon_{t+1}\right) m_{t+1}^i \leq w_t^i + b_t^i + m_t^i + \left(1 + \varepsilon_{t+1}\right) \tau_{t+1}^i, \\
& c_t^i \leq m_t^i,
\end{aligned}
\tag{8}
$$

where $b_t^i = B_t^i / p_t$ is the real amount of bonds at the beginning of period t, $1 + \varepsilon_{t+1} = p_{t+1}/p_t$ is the rate of inflation, and $q_t = \left(1 + \varepsilon_{t+1}\right) q_t^n$ is the real price of bonds. Notice that the difference $m_t^i - c_t^i$ could be thought of as precautionary money balances held at the beginning of period t.[3] We are going to denote it as d_t^i. In real terms the budget constraint and the cash-in-advance constraint can be written as

$$
\begin{aligned}
& \left(1 + \varepsilon_{t+1}\right) c_{t+1}^i + q_t b_{t+1}^i + \left(1 + \varepsilon_{t+1}\right) d_{t+1}^i \leq w_t^i + b_t^i + d_t^i + \left(1 + \varepsilon_{t+1}\right) \tau_{t+1}^i, \\
& d_{t+1}^i \geq 0,
\end{aligned}
\tag{9}
$$

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[3]Using the term precautionary may seem incorrect since there is no uncertainty. Nevertheless, since we will study the case with uncertainty we prefer to abuse language a bit instead changing names.

This formulation shows clearly that choosing m_{t+1}^i, next period real money balances, amounts to choosing next period consumption and next period precautionary real money balances, c_{t+1}^i, and d_{t+1}^i. It also shows that consumption at period $t+1$ depends on the current state. Now we proceed to define a recursive stationary equilibrium.

2.6 Recursive stationary equilibrium

In a steady state the inflation rate is constant and equal to the money growth rate, θ. The aggregate level of production is equal to $\overline{w} = (w_h + w_l)/2$. The per capita amount of money balances, $\mathbf{m}$, is equal to aggregate consumption plus the aggregate precautionary money balances, $\mathbf{m} = \overline{w} + d^1/2 + d^2/2$. Thus, aggregate money balances and prices will only depend on the monetary policy that is completely summarized by θ.

The household's problem

The individual state variable is the variable $x = \{j, b, d\}$, where j denotes the household's productivity level, $j = h$, l. The problem solved by a household of productivity level j stated recursively is

$$
\begin{aligned}
V\left(j, b, d; \theta\right) = \ &\max_{\substack{c \geq 0, \\ b', \\ d' \geq 0}} \left\{u\left(c\right) + \beta V\left(-j, b', d'; \theta\right)\right\} \\
\text{s. t.} \quad &\left(1 + \theta\right) c + q\, b' + \left(1 + \theta\right) d' \leq w_j + b + d + \tfrac{w_j}{\overline{w}} \theta\, \mathbf{m}, \quad\quad (10) \\
&V\left(j, b, d; \theta\right) \geq V_D\left(j, d; \theta\right), \\
&\text{if } j = h, \text{ then } -j = l, \text{ and vice versa,}
\end{aligned}
$$

Notice that the lower limit on the amount of bonds, b', is set implicitly by the incentive compatibility constraint faced next period, $V\left(-j, b', d'; \theta\right) \geq V_D\left(-j, d'; \theta\right)$: the amount of borrowing cannot be so large as to make agents to default on their debt. The utility after

default $V_D\left(j,d;\theta\right)$ satisfies

$$
\begin{aligned}
V_D\left(j,d;\theta\right) = & \max_{\substack{z\geq 0,\\ d'\geq 0}} \left\{u\left(z\right) + \beta\,V_D\left(-j,d';\theta\right)\right\} \\
\text{s. t.} \quad & \left(1+\theta\right)z + \left(1+\theta\right)d' \leq w_j + d + \tfrac{w_j}{\overline{w}}\,\theta\,\mathbf{m}, \\
& \text{if } j=h, \text{ then } -j=l, \text{ and vice versa.}
\end{aligned}
\tag{11}
$$

The incentive compatibility constraint is rewritten as $V\left(j,b,d;\theta\right) \geq V_D\left(j,d;\theta\right)$. Notice that we are using the same notation for the amount of precautionary money balances held either in trade or in default, d. We are abusing notation a bit, since, typically, the amount of money balances will depend on participating in the credit market. We do so to simplify notation and to stress that households can keep their money balances after default. Whenever there is a possibility of confusion we will point out the difference.

Steady state equilibrium

Now we can define a steady state equilibrium with incentive compatibility constraints. Remember that the individual state variable is denoted as x and summarizes the household's productivity and initial amount of bonds and precautionary money balances.

Definition 1. *An incentive compatible steady state equilibrium for this economy is a monetary policy, θ, an aggregate amount of money balances, $\mathbf{m}$, a price for bonds, q, a set of functions $\left\{V\left(x;\theta\right), g^c\left(x;\theta\right),\, g^b\left(x;\theta\right),\, g^d\left(x;\theta\right)\right\}$, and a stationary measure of households μ an such that:*

1. *given μ, q, the monetary policy, θ, and $\mathbf{m}$, the functions $\left\{V\left(x;\theta\right), g^c\left(x;\theta\right),\, g^b\left(x;\theta\right),\right.$ $\left. g^d\left(x;\theta\right)\right\}$ solve the household's problem, given $\left\{V_D\left(j,d;\theta\right),\, f^d\left(j,d;\theta\right),\, f^z\left(j,d;\theta\right)\right\}$, which solve solve the default problem given μ, q, $\mathbf{m}$, and the monetary policy θ,*

2. *markets clear:*

    (a) $\int_X g^c(x)\,d\mu = \int_X w\,d\mu$,

(b) $\int_X g^b(x) \, d\mu = 0,$

(c) $\mathbf{m} = \int_X \left(g^d(x) + g^c(x) \right) d\mu,$

3. and the measure of household is stationary, $\mu(B) = \int_X P(x, B) \, d\mu$, for all $B \subset \mathcal{B}$.

We are going to examine symmetric incentive compatible steady states. In a symmetric steady state the level of consumption of each household only depends on its level of productivity. That is, $c_t^i = c_h$ if $w_t^i = w_h$ and $c_t^i = c_l$ if $w_t^i = w_l$, for all i and t. Thus, the equilibrium bond and precautionary money holdings satisfy $b_j = g^b(-j, b_{-j}, d_{-j}; \theta)$, $d_j = g^d(-j, b_{-j}, d_{-j}; \theta)$, for $j = h, l$, and $b_h + b_l = 0$. Likewise, the after default allocation is defined as $z_t^i = z_h$, $d_t^i = \mathrm{d}_h$ if $w_t^i = w_h$, and $z_t^i = z_l$, $d_t^i = \mathrm{d}_l$, if $w_t^i = w_l$. Finally, we should keep in mind that all consumption levels, credit and money holdings are functions of the money growth rate, θ, but we will omit it unless necessary.

3 Efficiency and full commitment

In this section we investigate the existence of credit when borrowers can commit to repay their debts. We will compare the equilibrium allocation of our benchmark economy with the equilibrium allocation in this case.

3.1 The efficient allocation

The efficient allocation is the one that solves the social planner's problem

$$
\begin{aligned}
\max_{c_t^1, c_t^2} \quad & \tfrac{1}{2} \sum_{t=0}^{\infty} \beta^t u\left(c_t^1\right) + \tfrac{1}{2} \sum_{t=0}^{\infty} \beta^t u\left(c_t^2\right) \\
\text{s. t.} \quad & \tfrac{c_t^1}{2} + \tfrac{c_t^2}{2} = \overline{w}.
\end{aligned}
\tag{12}
$$

Denoting as c_h the consumption level when a household of type i has the high productivity shock and c_l when it is low we can simplify the social planner's problem to the static problem

$$
\begin{aligned}
\max_{c_h, c_l} \quad & \tfrac{1}{2} u\left(c_h\right) + \tfrac{1}{2} u\left(c_l\right) \\
\text{s. t.} \quad & \tfrac{c_h}{2} + \tfrac{c_l}{2} = \overline{w},
\end{aligned}
\tag{13}
$$

where it is easy to see that the efficient allocation entails full risk sharing, $c_h = c_l = \overline{w}$. Thus, efficiency implies that household's consumption is invariant with respect to household's income. Now we turn to the decentralization of the efficient allocation.

3.2 Equilibrium under full commitment

The symmetric efficient allocation can be decentralized under full commitment. Solving the problem of full commitment only requires to drop the incentive compatibility constraint from the household's problem shown in (10) and substitute it by a non Ponzi scheme constraint. Existence of equilibrium requires $1 + \theta \geq \beta$. It is easy to check that, under full commitment, the price of the bond must be equal to the discount factor, β. Since bonds, as opposed to money, can be held in negative amounts the bond price cannot be higher than the price of money, $q \leq 1 + \theta$. The level of consumption is $\overline{w}$, regardless of the household's productivity. The household's portfolio satisfies

$$
(1 + \theta)\, \overline{w} + q\, b_{-j} + (1 + \theta)\, d_{-j} = \left(1 + \theta\, \frac{\mathbf{m}}{\overline{w}}\right) w_j + b_j + d_j, \ j = h,\, l,
\tag{14}
$$

$$
b_h + b_l = 0,
\tag{15}
$$

$$
d_j \geq 0, \ j = h,\, l,
\tag{16}
$$

$$
\overline{w} + \frac{1}{2}\, d_h + \frac{1}{2}\, d_l = \mathbf{m}.
\tag{17}
$$

There are two possible cases. Either $1 + \theta > \beta$, case in which the nominal return to bonds is positive, or the case in which $1 + \theta = \beta$, that is, the nominal interest rate is zero.

Equilibrium with positive nominal rates

This is the case when the government sets an inflation rate larger than the rate of time preference, $1 + \theta > \beta$. Then, households choose $d_h = d_l = 0$. It is easy to check that households borrow when they have low productivity. When they have high productivity they repay their debts, consume and save for next period. Using (15) and (14) we can see that the maximum amount of debt is

$$b_h = b_h^E(\theta) \equiv -\frac{(1 + \theta)}{1 + \beta} \left(\overline{w} - w_l\right). \tag{18}$$

We will refer to the amount $b_h^E(\theta)$ as the maximum amount of credit given the inflation rate θ. This level of credit is similar to the "natural borrowing limit" discussed by Aiyagari (1994). That is, if the household is allowed to borrow more than $b_h^E(\theta)$ and can commit to repay its debts it will never exhaust the borrowing limit.

Equilibrium with zero nominal rates

This is the case when the government sets the deflation rate equal to the rate of time preference, $1 + \theta = \beta$. In this case the portfolio composition is indeterminate since the real return to bonds is equal to the real return of money. Any composition of bonds and money that satisfy (14)–(17) is optimal, in particular, households could choose to not use bonds and save in the form of money. In this case the amount of money holdings satisfies

$$(1 + \theta)\,\overline{w} + \beta\, d_l = \left(1 + (\beta - 1)\frac{\mathbf{m}}{\overline{w}}\right) w_h + d_h, \tag{19}$$

$$d_j \geq 0,\ j = h,\, l, \tag{20}$$

$$\overline{w} + \frac{1}{2}\, d_h + \frac{1}{2}\, d_l = \mathbf{m}. \tag{21}$$

That is, it is possible to attain full insurance using only money if the government deflates at the time preference rate. We will come back to this issue in section 5, where productivity is stochastic. Finally, notice that there exists a continuum of equilibria, since the portfolio

composition is not determined.

4 The incentive compatible steady state

The section is organized in the following way. We first characterize the default allocation. Next, we characterize incentive compatible symmetric steady states. Finally, we give conditions under which the full commitment allocation is an incentive compatible steady state allocation.

4.1 The default allocation

First we turn to investigate the household's decisions after default. In this case, the problem solved by the household is shown in expression (11). Notice that defectors cannot borrow and they are restricted to use money whose gross real return is $1/(1 + \theta)$. If $(1 + \theta) \geq \beta u' (w_l)/u' (w_h)$, defectors never save, $d_l = d_h = 0$, and their consumption equals their productivity; that is, they are effectively in autarky. For $(1 + \theta) < \beta u' (w_l)/u' (w_h)$ the solution to the default problem is characterized by the following equations,

$$(1 + \theta) \, u' \, (z_l) = \beta \, u' \, (z_h) \,, \tag{22}$$

$$(1 + \theta) \, u' \, (z_h) \geq \beta \, u' \, (z_l) \,, \tag{23}$$

$$\mathrm{d}_j \geq 0, \; j = h, \, l, \tag{24}$$

$$(1 + \theta) \, z_j + (1 + \theta) \, \mathrm{d}_j = \left(1 + \theta \, \frac{\mathbf{m}}{\overline{w}}\right) w_{-j} + \mathrm{d}_{-j}, \; j = h, \, l, \tag{25}$$

Equation (22) implies that the non negativity constraint on precautionary money balances is not binding when the household has high productivity whereas equation (23) says that it may bind in the low productivity state. As a matter of fact, it is easy to show that defectors always choose to save whenever they have high productivity, $d_l > 0$. If $1 + \theta > \beta$ the household chooses $\mathrm{d}_h = 0$. If the government deflates at the rate of time preference the

value of d_h is indeterminate. Finally, utility after default for the household starts with the high productivity state is

$$V_D\left(h, \mathrm{d}; \theta\right) = \frac{u\left(z_l\right) + \beta\, u\left(z_h\right)}{1 - \beta^2}. \tag{26}$$

4.2   Characterization of the symmetric incentive compatible steady state

The symmetric incentive compatible steady state is characterized by the equations

$$q\, u'\left(c_j\right) \geq \beta\, u'\left(c_{-j}\right), \; j = h,\, l, \tag{27}$$

$$V\left(j, b_j, d_j; \theta\right) = \frac{u\left(c_{-j}\right) + \beta\, u\left(c_j\right)}{1 - \beta^2}, \; j = h,\, l, \tag{28}$$

$$V\left(j, b_j, d_j; \theta\right) - V_D\left(j, d_j; \theta\right) \geq 0, \; j = h,\, l, \tag{29}$$

$$\left(1 + \theta\right) u'\left(c_j\right) \geq \beta\, u'\left(c_{-j}\right), \; j = h,\, l, \tag{30}$$

$$d_j \geq 0, \; j = l,\, h, \tag{31}$$

$$\left(1 + \theta\right) c_j + q\, b_j + \left(1 + \theta\right) d_j = \left(1 + \theta\, \frac{\mathbf{m}}{\overline{w}}\right) w_{-j} + b_{-j} + d_{-j}, \; j = l,\, h, \tag{32}$$

$$b_l + b_h = 0, \tag{33}$$

$$\mathbf{m} = \overline{w} + \frac{1}{2}\, d_l + \frac{1}{2}\, d_h. \tag{34}$$

Notice that if the incentive compatibility constraint, $V\left(j, b_j, d_j; \theta\right) \geq V_D\left(j, d_j; \theta\right)$, is not binding when the household has productivity j, the first order condition with respect to b_j, shown in expression (27), must hold with strict equality. The next proposition states the conditions under which precautionary money holdings are zero.

Proposition 1. *For any $1 + \theta > \beta$, at the incentive compatible symmetric steady state equilibrium the amount of precautionary money balances held in the high productivity state is zero, $d_h = 0$.*

Proof: See Appendix A.

This proposition ensures that if $1 + \theta > \beta$ households do not hold any precautionary money balances when they enter the high productivity state. Since only households with high productivity have incentives to default, this proposition implies that, if they defaulted (which does not happen in equilibrium), they would would start out in the default state with zero precautionary money balances, which is the amount optimally chosen by a defector.

4.3 Credit in deflationary economies

Now we can describe some properties of the incentive compatible symmetric steady states. We first focus on steady states when the inflation rate is negative, $\theta < 0$.

Proposition 2. *For any negative rate of inflation, the amount of credit in the symmetric participation constrained equilibrium is zero, $b_h = b_l = 0$.*

Proof. We are going to show that the incentive compatibility constraint is never satisfied for the household with high productivity when inflation is negative. In order to do so we rewrite the problem solved by the household with high productivity in the following way:

$$
\begin{aligned}
\max \quad & u\left(c_l\right) + \beta\, u\left(c_h\right) \\
\text{s. t.} \quad & \left(1+\theta\right) c_j + q\, b_j + \left(1+\theta\right) d_j = \left(1 + \theta\, \tfrac{\mathrm{m}}{\overline{w}}\right) w_{-j} + b_{-j} + d_{-j},\ j = h,\, l, \\
& d_j \geq 0,\ j = h,\, l, \\
& u\left(c_l\right) + \beta\, u\left(c_h\right) \geq u\left(z_l\right) + \beta\, u\left(z_h\right).
\end{aligned}
\tag{35}
$$

Likewise, the problem solved by that household after default can be written as

$$
\begin{aligned}
\max \quad & u\left(z_l\right) + \beta\, u\left(z_h\right) \\
\text{s. t.} \quad & \left(1+\theta\right) z_j + \left(1+\theta\right) \mathrm{d}_j = \left(1 + \theta\, \tfrac{\mathrm{m}}{\overline{w}}\right) w_{-j} + \mathrm{d}_{-j},\ j = h,\, l, \\
& \mathrm{d}_j \geq 0,\ j = h,\, l.
\end{aligned}
\tag{36}
$$

Since $d_l \geq 0$ and $\mathrm{d}_l \geq 0$ we can write the budget constraints associated to each problem in

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the following way

$$(1 + \theta) z_h + z_l = \left(1 + \theta \frac{\mathbf{m}}{\overline{w}}\right) w_l + \frac{1 + \theta \frac{\mathbf{m}}{\overline{w}}}{1 + \theta} w_h + \left[\frac{1}{1 + \theta} - (1 + \theta)\right] \mathrm{d}_h, \qquad (37)$$

$$(1 + \theta) c_h + c_l = \left(1 + \theta \frac{\mathbf{m}}{\overline{w}}\right) w_l + \frac{1 + \theta \frac{\mathbf{m}}{\overline{w}}}{1 + \theta} w_h + \left[\frac{1}{1 + \theta} - (1 + \theta)\right] \mathrm{d}_h +$$
$$\left(1 - \frac{q}{1 + \theta}\right) b_l + \left(\frac{1}{1 + \theta} - q\right) b_h. \quad (38)$$

Since in equilibrium $b_h + b_l = 0$, we can write (38) as

$$(1 + \theta) c_h + c_l = \left(1 + \theta \frac{\mathbf{m}}{\overline{w}}\right) w_l + \frac{1 + \theta \frac{\mathbf{m}}{\overline{w}}}{1 + \theta} w_h + \frac{\theta(1 + q)}{1 + \theta} b_l + \left[\frac{1}{1 + \theta} - (1 + \theta)\right] \mathrm{d}_h. \quad (39)$$

Notice that if there is credit in equilibrium it must be the case that $b_l > 0$. Moreover, since the household keeps its money balances after declaring default, $\mathrm{d}_h = d_h$. Recall that $\mathrm{d}_h = d_h = 0$ for any inflation rate that satisfies $1 + \theta > \beta$. Since inflation is negative, $\theta < 0$, it follows that $(1 + \theta) c_h + c_l < (1 + \theta) z_h + z_l$. That is, any allocation with credit that satisfies all the first order conditions of the symmetric steady state, but the incentive compatibility constraint, is in the interior of the budget set faced by the households after default. This implies that the only incentive compatible symmetric steady state involves no credit, $b_h = b_l = 0$. $\qquad \square$

The consequence is that the only possible equilibrium is one without credit. Households only can use self insurance. Notice that this result depends on the government's ability to tax defectors. If the government could seize their entire precautionary money holdings the result would disappear. Thus, a key issue is why the government cannot tax discriminate between defaulters and non defaulters.

Corollary 1. *For any negative rate of inflation the price of bonds satisfies $q = 1 + \theta$ and the incentive compatible equilibrium allocation is characterized by*

$$(1 + \theta) u'(c_l) = \beta u'(c_h), \qquad (40)$$

$$d_l \geq 0, \ d_h \geq 0, \tag{41}$$

$$(1+\theta)\, c_{-j} + (1+\theta)\, d_{-j} = \left(1 + \theta \frac{\mathbf{m}}{\overline{w}}\right) w_j + d_j, \ j = l, \ h, \tag{42}$$

$$\frac{1}{2}\, c_h + \frac{1}{2}\, c_l = \overline{w}, \tag{43}$$

$$\frac{1}{2}\, d_h + \frac{1}{2}\, d_l = \mathbf{m}. \tag{44}$$

Notice that, differently from the full commitment case, the price of the bond equals the price of money for negative inflation rates. In other words, the nominal interest rate is zero for any negative inflation rate. This is so because high productivity households are indifferent between saving in the form of bonds or in the form of money, consequently, choosing $b_h = 0$ is optimal for them. Households with low productivity cannot borrow because they would default on any amount borrowed in the following period. Thus, the borrowing limit is exactly zero. This restriction is binding if $1 + \theta > \beta$ (low productivity households would like to borrow) and it is not if $1 + \theta = \beta$, since all households can smooth consumption perfectly by selfinsuring using money.

Thus, if there is deflation credit disappears. Nevertheless, if the government contracts the monetary stock at the rate of time preference the equilibrium allocation is efficient in spite of the non existence of credit.

Corollary 2. *The incentive compatible symmetric steady state allocation is efficient if the inflation rate satisfies, $\beta = 1 + \theta$.*

This corollary is an immediate consequence of the characterization of the steady state. Households with high productivity save so much that full insurance is attained. We will see in section 5 that the existence of this equilibrium relies on the existence of complete markets, as is the case here.

4.4 The case of zero inflation

If the inflation rate is zero, $\theta = 0$, the amount of precautionary holdings carried to the high state, d_h, is zero, either in trade or after default. Moreover, by looking at the intertemporal budget constraint faced after default, shown in (37), we can see that the default allocation is feasible, $z_l + z_h = w_l + w_h$. Assumption 1 implies that $u'(w_h) < \beta\, u'(w_l)$. This inequality implies, by strict concavity of the instantaneous utility function, that there is no other feasible allocation that gives at least the default utility. That is, the default consumption allocation is the unique equilibrium allocation in the incentive compatible steady state. Notice that this allocation can be sustained with credit, too, by setting $d_h = d_l = 0$, the price of bonds equal to one, and the amount of debt

$$b_h = -\frac{w_h - z_l}{2}, \tag{45}$$

which is negative since $z_l < w_h$. In this equilibrium households are indifferent between staying in trade and defaulting since they attain exactly the same allocation in both situations. Notice also that the nominal and the real interest rates are zero. Households are rolling over their debts from one period to another and this policy is incentive compatible.[4]

4.5 Credit in inflationary economies

Now we turn to analyze the case of inflation. The next proposition establishes for which level of inflation there is credit and when full insurance can be achieved.

Proposition 3. *There exists a strictly positive level of inflation, $\theta^* > 0$, such that for any positive inflation rate below that threshold, $\theta \in [0, \theta^*)$, the level of consumption at the high state, $c_h = c_h(\theta)$, and the price of the bond, $q = q(\theta)$, are continuous and strictly monotonous*

---

[4]This result is also found by Hellwig and Lorenzoni (2006). In their setup, households only have one asset, private bonds. Defectors are allowed to purchase bonds (they can save) but they cannot issue bonds (they cannot borrow). We obtain the same result that Hellwig and Lorenzoni (2006): the price of bonds must be equal to one. That is, real and nominal return on assets must be zero. In our setup, where money is needed to purchase consumption goods, this equilibrium arises when the inflation rate is equal to zero. We will come back to this issue in section 4.6.

functions of the inflation rate in the interval $[0, \theta^*]$ and satisfy $q(0) = 1$, $q(\theta^*) = \beta$, $c_h(0) > w_l$, and $c_h(\theta^*) = \overline{w}$. When $\theta > \theta^*$ the incentive compatibility constraint is not binding at the symmetric steady state.

Proof: See Appendix A.

We have seen in the previous section 4.4 that for $\theta = 0$ the default allocation gives higher utility than the full commitment allocation to the household with high productivity, $u(z_l) + \beta\, u(z_h) > (1 + \beta)\, u(\overline{w})$. Since the utility yielded by the default allocation is strictly decreasing with θ, there exists a positive level of inflation, θ^*, for which the default allocation gives exactly the same level of utility that the full commitment allocation. This implies that, for any level of inflation above that threshold, $\theta \geq \theta^*$, the full commitment allocation is incentive compatible. The price of bonds is equal to β and the amount of credit is given by expression (18). The nominal interest rate is equal to $(1 + \theta)\,/\beta - 1$, which is positive.

For any level of inflation in the interval $\theta \in [0, \theta^*)$ the incentive compatibility constraint is binding for the high productivity household. In other words, the low productivity household is credit constrained, which implies that the volume of credit is lower than the amount of credit in the case of full commitment. As the inflation rate approaches θ^*, utility after default decreases. This implies that households prefer to stay in trade for a larger amount of debt. Thus, credit increases and the price of bonds decreases. Note also that for any $\theta \in (0, \theta^*]$ the price of bonds is lower than the inflation factor, $q < 1 + \theta$. This implies that the nominal interest rate is positive. The real interest rate is positive and increases with inflation for any $\theta \in (0, \theta^*]$.

Any inflation beyond θ^* has no real effect on the economy. This is so because we have assumed that leisure is not valued. If we had assumed otherwise the results would change slightly. In such a case, the optimal monetary policy under full commitment would be the Friedman rule, too. In absence of full commitment, our Proposition 2 would still hold for $\theta = 1 + \beta$ but it is not clear that it would hold for any negative inflation rate because the optimal inflation rate rate would also depend on the elasticity of substitution between consumption and leisure, and not only on the discount factor.

Notice the importance of limited commitment. In the case of full commitment, inflation has no effects on the consumption allocation because inflation does not alter the amount of credit. With limited commitment, however, inflation reduces the return to default, which, in its turn, affects the volume of credit. It could be argued that our result depends on the assumed scheme of money transfers. As discussed before, if the government could use taxes to punish defectors, the link between credit and inflation investigated here would vanish and, therefore, the optimal deflation rate is the rate of time preference. Thus, it is critical the fact that government cannot tax discriminate between defaulter and non defaulters. We also have assumed that the government uses proportional taxes (or transfers) to withdraw or to inject money in the economy. If taxes (or transfers) were lump-sum the government would effectively be redistributing resources from low productivity households to high productivity households. Our results, though, would go through.[5]

We have found that households can attain full insurance by only using money if the government deflates at the rate of time preference, $1 + \theta = \beta$. This is so because households know exactly their future stream of productivity and can accumulate a finite amount of money to self insure perfectly. We will see in section 5 that this result vanishes in the presence of idiosyncratic uncertainty.

4.6 The role of banning defectors from the credit market and the cash-in-advance constraint

We have assumed that defaulters only can save in the form of money. This is equivalent to assuming that the credit agency has the ability to track positive bond holdings as well as negative positions in bonds. Alternatively, we could have assumed that the credit agency only has the ability to monitor negative bond holdings. For credit to exist in this alternative model economy the price of bonds would have to be equal to one, $q = 1$ (see Appendix B). Since the real return to money has to be no less than the real return to bonds, $1 + \theta \geq q$, the

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[5]A lump-sum tax in this framework is equivalent to an income tax rate equal to $\theta \, \mathbf{m}_t / w_t^i$, which is regressive. This reinforces our result since this tax policy shifts resources from low productivity households to high productivity households, which have the incentives to default on their debts.

existence of credit requires the inflation rate to be not negative. Any non negative inflation rate, $\theta \geq 0$, would be optimal since the cash-in-advance constraint prevents money from disappearing. Households, though, would not keep any precautionary money balances. In this environment households are rolling their debts over time and such policy is sustainable for any $\theta \geq 0$. Thus, the level of the nominal interest rate does not affect the equilibrium, since the real interest rate is zero for any non negative inflation rate. As opposed to our benchmark economy, inflation has no further effect on the equilibrium allocation. This is so because inflation does not affect the return to default (defectors do not hold precautionary money balances) and, therefore, it does not affect the volume of credit. Thus, in our setup the resulting equilibrium depends on the monitoring technology that financial institutions have access to.

This result brings some insights about the role of the cash-in-advance constraint. Suppose that money is not needed for transaction purposes and that defectors can purchase (but not issue) private bonds. Coexistence of money and bonds dictates $q = 1 + \theta$ and existence of credit, as we have seen, $q = 1$. That is, zero nominal and real interest rate. On the contrary, suppose that money is not needed for transaction purposes but defectors cannot purchase private bonds. Then, only defectors would hold money. Since there is no default in equilibrium, coexistence of credit and money dictates zero nominal interest rate and existence of credit $q = 1$.[6] In either case monetary policy cannot restore efficiency. Thus, the presence of the cash-in-advance constraint coupled with the banks' ability to track positive bond holdings and seize those of defectors allows the government to set a real return to money lower than the return to bonds and restore efficiency without money disappearing. In section 6 we describe a set of market and informational frictions that allow the cash-in-advance constraint to arise in equilibrium.

In the following section we extend our analysis to the case in which households are subject to idiosyncratic uncertainty. Hereafter we keep assuming that financial institutions can keep track of positive bond holdings and seize those of defectors. The main departure from the current setting is that private bonds are contingent whereas money is not.

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[6]If there were default in equilibrium and defectors could not purchase bonds, this result would not necessarily hold.

5 A stochastic environment

We modify our physical environment so that agents' productivity can be represented as an idiosyncratic shock. We assume that productivity at period t is drawn from the set $\mathbf{W} = \{w_1, \ldots, w_n\}$, where $w_i < w_{i+1}$, for any $i = 1, \ldots, n$, and $w_1 > 0$. The shock is Markov with transition matrix $\Pi = (\pi_{ij})_{n \times n}$. The associated stationary distribution of productivities is the vector $(\eta_1, \ldots, \eta_n)$. We assume that there are complete markets but for the fact that households cannot commit to repay their debts. That is, households trade Arrow securities in bonds. Notice, however, that money balances are not contingent. The following assumption ensures that trade is always welfare improving,

Assumption 2.

$$(u(\overline{w}), \ldots, u(\overline{w})) H > (u(w_1), \ldots, u(w_n)) H, \tag{46}$$

where H is the matrix $(I - \beta \Pi)^{-1}$ and $\overline{w} =< \eta, (w_1, \ldots, w_n) >$.

This assumption says that households prefer to stay in trade rather than going to autarky, even if they start out with the highest level of productivity. The following assumption will simplify our analysis,

Assumption 3. *The elements of the transition matrix Π satisfy $\pi_{ij} \in (0, 1)$ for all i, j.*

This assumption implies that the transition matrix has a unique ergodic set and it is the matrix itself. As in our benchmark economy, default entails being banned from the credit market. Defaulters only can save using money balances.

5.1 The household's problem

Here we state the recursive formulation of the household's problem in real terms. Appendix C describes the household's problem in the sequential formulation and the derivation of the budget constraint in real terms as well as the recursive equilibrium definition.

The individual state variable is the variable $x = \{j, b, d\}$, where j denotes the household's productivity level. The problem solved by a household of productivity level j is

$$
V\left(j, b, d; \theta\right) = \max_{c_i \geq 0} \sum_{i=1}^{n} \pi_{ji} \left[u\left(c_i\right) + \beta V\left(i, b'_i, d'_i; \theta\right)\right]
$$

s. t.
$$
\left(1 + \theta\right) c_i + \sum_{s=1}^{n} q\left(j, s\right) b'_s + \left(1 + \theta\right) d'_i \leq w_j \left(1 + \theta \frac{\mathrm{m}}{\mathrm{w}}\right) + b + d, \text{ for all } i,
$$

$$
u\left(c_i\right) + \beta V\left(i, b'_i, d'_i; \theta\right) \geq V_D\left(j, i, d; \theta\right), \text{ for all } i,
$$

$$
d'_i \geq 0, \text{ for all } i.
$$

(47)

where $V_D\left(j, i, d; \theta\right)$ satisfies

$$
V_D\left(j, i, d; \theta\right) = \max_{z_i \geq 0} \left\{u\left(z\right) + \beta \sum_{\kappa=1}^{n} \pi_{i\kappa} V_D\left(i, \kappa, d'; \theta\right)\right\}
$$

s. t.
$$
\left(1 + \theta\right) z + \left(1 + \theta\right) d' \leq w_j \left(1 + \theta \frac{\mathrm{m}}{\mathrm{w}}\right) + d,
$$

(48)

$$
d' \geq 0.
$$

Let us discuss briefly why the household's problem is written this way. We have argued that choosing next period money balances, m_{t+1}, amounts to choosing the sum of next period consumption and precautionary money balances, $c_{t+1} + d_{t+1}$. Notice that money is not a contingent asset. Thus, at the beginning of period $t + 1$, the sum $c_{t+1} + d_{t+1}$ is always the same and equal to the previously decided amount of money balances, m_{t+1}, but each item, c_{t+1} and d_{t+1}, could depend on the state faced by the household at period $t + 1$. This is why the problem is written as if the household faced a sequence of budget constraints, one per each productivity state tomorrow. These budget constraints are linked by the amount of contingent bonds carried by households from period t to period $t + 1$.

Let us turn now to the default problem. Defectors cannot purchase nor sell bonds. This means that they cannot shift resources from one state to another. This is why the problem faced by a defector is equivalent to solving one maximization problem per each possible state at period $t + 1$, as shown in expression (48). See Appendix C for details. Notice that defectors behave like Bewley (1983) type of agents: they only can use a non contingent asset for self insurance purposes. This is the main difference with the deterministic case. As in

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the previous section, we call the constraint

$$u\left(c_i\right) + \beta\,V\left(i, b_i', d_i'; \theta\right) \ge V_D\left(j, i, d; \theta\right), \quad \text{for all } j, i, \text{ for all } d \in \mathrm{R}_+, \text{ given } \theta, \tag{49}$$

the incentive compatibility constraint.

5.2   Efficiency and full commitment

As in the case of certainty, the efficient allocation is the one that maximizes the social planner's problem and implies full risk sharing, $c_t = \overline{w}$, for all households.

Let us now turn to the decentralization of the full risk sharing allocation under full commitment. The household's problem is the one shown in (47) but for the incentive compatibility constraint shown in (49). As in the deterministic case, we will focus our attention to symmetric steady states, which are characterized by the following set of equations:

$$q(j, s) \sum_{\kappa=1}^{n} \pi_{j\kappa} u'\left(c_\kappa\right) = \beta\,\pi_{js} \sum_{\kappa=1}^{n} \pi_{s\kappa} u'\left(c_\kappa\right), \text{ for all } j, \ s = 1, \ldots, n, \tag{50}$$

$$\left(1+\theta\right) u'\left(c_s\right) \ge \beta \sum_{\kappa=1}^{n} \pi_{s\kappa} u'\left(c_\kappa\right), \text{ for all } s = 1, \ldots, n, \tag{51}$$

$$d_s \ge 0, \text{ for all } s = 1, \ldots, n, \tag{52}$$

$$\sum_{j=1}^{n} \eta_j\,b_j = 0, \tag{53}$$

$$\mathbf{m} = \overline{w} + \sum_{j=1}^{n} \eta_j\,d_j, \tag{54}$$

$$\left(1+\theta\right)\left(c_i + d_i\right) + \sum_{s=1}^{n} q\left(j, s\right) b_s = w_j\left(1 + \theta\frac{\mathbf{m}}{\overline{w}}\right) + b_j + d_j, \text{ for all } j, \ i = 1, \ldots, n. \tag{55}$$

Here the fact that the real return to money is not contingent poses a problem that is summarized on the following proposition.

Proposition 4. *In a symmetric steady the consumption allocation is the efficient allocation,*

$c_j = \overline{w}$, for all j.

Proof: See Appendix C.

This is a consequence of the return of money not being contingent. If consumption varies across states the amount of money changes across states, too. The fact that the return to money is not contingent would imply that the amount of precautionary money balances and consumption would depend on the household's wealth. Thus, in a symmetric steady state consumption is constant across states. By feasibility, symmetry implies full risk sharing. Notice that this result does not depend on households ability to commit to repay their debts. It depends on the existence of idiosyncratic uncertainty and on the fact that money is the non contingent asset. The following two corollaries are a consequence of consumption being constant across states.

Corollary 3. *In a symmetric steady the amount of precautionary money balances are constant across states, $d_j = d$, for all j.*

Corollary 4. *If the inflation rate is higher than the rate of time preference, $1 + \theta > \beta$, the amount of precautionary money balances at the symmetric steady state is zero, $d_j = 0$, for all $j = 1, \ldots, n$.*

Now we can characterize the symmetric equilibrium under full commitment.

Corollary 5. *The unique symmetric steady state in the full commitment economy is characterized by*

$$q(j,s) = \pi_{js}\beta, \text{ for all } j, \, s = 1, \ldots, n. \tag{56}$$

$$c_j = \overline{w}, \text{ for all } j = 1, \ldots, n. \tag{57}$$

$$\sum_{j=1}^{n} \eta_j \, b_j = 0, \tag{58}$$

$$(1 + \theta) \, \overline{w} + \theta \, d + \sum_{s=1}^{n} q\,(j,s)\, b_s = w_j \left(1 + \theta \frac{\mathbf{m}}{\overline{w}}\right) + b_j, \text{ for all } j = 1, \ldots, n, \tag{59}$$

$$\mathbf{m} = \overline{w} + d. \tag{60}$$

Notice that the price of a bond that yields one unit of consumption good at every state is β, as in the certainty case. Notice also that money is dominated by bonds since households can borrow as much as they want and their return is contingent. Hence, households can never be better off using money than bonds. Nevertheless, the household's portfolio is indeterminate if the government deflates at the time preference rate, $1 + \theta = \beta$.

It is easy to check that the household with the highest productivity holds a negative amount of bonds. Hence, in the limited commitment case the high productivity household will have incentives to default.

5.3 The incentive compatible steady state and the optimal monetary policy

In this section we start by discussing the existence of symmetric incentive compatible steady states and then we turn to describe the conditions under which the optimal monetary policy entails a money growth rate strictly higher than the preferences discount rate.

Proposition 5. *If there exists a symmetric incentive compatible steady state it is the full commitment symmetric steady state.*

Proof. Proposition 4 tells us that in a symmetric steady state the consumption allocation is the full risk sharing allocation, regardless of the existence of either full or limited commitment. Thus, if the full risk sharing allocation is not incentive compatible there is no other symmetric incentive compatible steady state. $\square$

The question that arises is for which range of the inflation rate, given the discount factor, the full risk sharing allocation is incentive compatible. Hence, characterizing the conditions under which the full risk sharing allocation is the symmetric incentive compatible steady state is equivalent to characterizing the optimal monetary policy.

To characterize the optimal inflation rate we use a guess and verify method. We guess that the full risk sharing allocation is the incentive compatible steady state allocation. Then we turn to characterize the inflation rate for which the incentive compatibility is exactly binding for the household with the highest productivity state. Without loss of generality we assume hereafter that households do not hold any precautionary money balances. We start by showing that utility after default, $V_D(j, i, d; \theta)$ is decreasing with the inflation rate.

Lemma 1. $V_D(j, i, d; \theta)$ *is decreasing in* θ.

(Proof: See Appendix C). Notice that utility after default approaches utility in autarky as the inflation rate becomes arbitrarily large, since the real return to money goes to zero. The next Lemma shows that there exist a threshold for the inflation rate such that, for any inflation rate lower than the threshold, utility after default is strictly higher than the utility of the full commitment allocation.

Lemma 2. *If* $\theta \leq -\frac{\overline{w} - w_1}{w_n - w_1}$, *then* $V_D(n, i, d; \theta) > \frac{u(\overline{w})}{1 - \beta}$, *for all* i, *for all* $\beta > 0$.

(Proof: See Appendix C). Notice that this result does not rely on the discount factor. It rather depends on the dispersion of the distribution of productivities. Notice that the closer the mean productivity, $\overline{w}$, is to the minimum productivity, w_1, the larger the range of the deflation rates for which the lemma holds. That is, the more skewed the distribution the smaller the absolute value of $\frac{\overline{w} - w_1}{w_n - w_1}$. Hence, the larger the range of θ for which Lemma 2 holds. The next Corollary tells us that we can always find a discount factor for which the threshold level of inflation found in Lemma 2 satisfies $1 + \theta = \beta$.

Corollary 6. *If* $\beta \leq 1 - \frac{\overline{w} - w_1}{w_n - w_1}$ *the full commitment allocation does not satisfy the incentive compatibility constraint if* $1 + \theta = \beta$.

The next proposition is more general and states the conditions under which the full commitment allocation does not satisfy the incentive compatibility constraint even if the government sets a deflation rate different from the rate of time preference.

Proposition 6. *Given* $\beta > 0$, *there is* $\overline{\theta}$ *such that if* $\theta < \overline{\theta}$ *then the full commitment allocation does not satisfy the incentive compatibility constraint.*

Proof. Assumption 2, Lemma 1 and Lemma 2 imply that there exist $\overline{\theta}$ such that $V_D\left(n, n, 0, \overline{\theta}\right) = u\left(\overline{w}\right) / (1 - \beta)$. Hence, using Lemma 1, $V_D\left(n, n, 0, \theta\right) = u\left(\overline{w}\right) / (1 - \beta)$ for any $\theta < \overline{\theta}$. $\qquad\square$

Corollary 7. *Given $\beta > 0$, for any $\theta < \overline{\theta}$, the full commitment allocation is not the incentive compatible steady state allocation.*

The next proposition establishes, for any given inflation rate, the range of the discount factor for which the full commitment allocation does not satisfy the incentive compatibility constraint.

Proposition 7. *For any inflation rate $\widetilde{\theta} \in \mathbf{R}$ there is an associated discount factor, $\beta\left(\widetilde{\theta}\right)$, such that if $\beta < \beta\left(\widetilde{\theta}\right)$ and $\theta \leq \widetilde{\theta}$, the full commitment allocation does not satisfy the incentive compatibility constraint.*

(Proof: See Appendix C). The next Corollary tells us that it is possible to find a discount factor such that the associated full commitment allocation does not satisfy the incentive compatibility constraint for any non positive inflation rate.

Corollary 8. *There is $\beta\left(0\right)$ such that if $\beta < \beta\left(0\right)$ and $\theta \leq 0$, the full commitment allocation does not satisfy the incentive compatibility constraint.*

Now we turn to characterize the degree of impatience, the level of β, for which the full commitment allocation is not incentive compatible when the government follows the Friedman rule.

Proposition 8. *There is $\beta^* > 0$ such that for all $\beta < \beta^*$ the full commitment allocation does not satisfy the incentive compatibility if $1 + \theta = \beta$.*

Proof. If follows from Corollary 6 that for any $\beta \leq 1 - \frac{\overline{w} - w_1}{w_n - w_1}$ that $V_D\left(n, n, 0, 1 - \beta\right) > u\left(\overline{w}\right) / (1 - \beta)$. However, it could be the case that for all $\beta < 1$ the inequality $V_D\left(n, n, 0, 1 - \beta\right) > u\left(\overline{w}\right) / (1 - \beta)$ holds, case in which $\beta^* = 1$. If this were not the case, continuity of $V_D\left(n, n, 0, 1 - \beta\right)$ with respect to the discount factor that there is $\beta^* > 0$ such that for all $\beta < \beta^*$ $V_D\left(n, n, 0, 1 - \beta\right) > u\left(\overline{w}\right) / (1 - \beta)$ $\qquad\square$

Corollary 9. *If $\beta < \beta^*$ the full commitment allocation is not a symmetric incentive compatible steady state allocation under the Friedman rule.*

Now we can state a non existence result.

Proposition 9. *If $\beta < \beta^*$ there is no equilibrium if the inflation rate satisfies $1 + \theta = \beta$.*

Proof. We prove it by contradiction. Let us suppose that there exists an equilibrium. It follows from Proposition 8 that the full commitment allocation does not satisfy the incentive compatibility constraint. Hence, consumption depends on the individual state variable $x_t = \{j_t, j_{t-1}, d_{t-1}, b_{t-1}\}$. Using the first order condition with respect to precautionary money holdings, the convexity of the marginal utility, $u'''(.) > 0$, and the fact that consumption varies across states we find that $u'(c(x_t)) \geq E\left[u'(c(x_{t+1}))/x_t\right] > u'\left(E\left[c(x_{t+1})/x_t\right]\right)$, which implies that consumption is growing in expected terms, $E\left[c(x_{t+1})/x_t\right] > c(x_t)$. Adding across households, $\int_{X_{t+1}} \left[c_{t+1}(x_{t+1})\right] d\mu_{t+1} = \int_{X_t} E\left[c(x_{t+1})/x_t\right] d\mu_t > \int_{X_t} c_t(x_t) d\mu_t = \int_{X_t} w(x_t) d\mu_t = \int_{X_{t+1}} w(x_{t+1}) d\mu_{t+1}$, we find that the equilibrium allocation is not feasible. Thus, we incur in a contradiction and, hence, equilibrium cannot exist. $\qquad\square$

Proposition 10. *If $\beta < \beta^*$ there is $\theta^* > \beta - 1$ such that if $\theta \geq \theta^*$ the full commitment allocation satisfies the incentive compatibility constraint and, therefore, the equilibrium allocation is optimal.*

Proof. Lemmas 1 and 2, Assumption 2, and the continuity of $V_D(j, i, d, \theta)$ with respect to θ ensure that there exists $\theta^* > \beta - 1$ such that $V_D(n, n, 0, \theta) = \frac{u(\overline{w})}{1-\beta}$. It follows from Proposition 8 that $\theta^* > \beta - 1$. $\qquad\square$

Thus, we have shown that if households are sufficiently impatient, the optimal inflation rate is larger than the discount rate. That is, the nominal interest rate is positive. The question about the optimal inflation rate is, thus, quantitative.

5.4 Some numerical examples

We have seen in the previous section that if households are sufficiently impatient the optimal monetary policy entails an inflation rate greater than the preferences discount rate. This result implies that the question about the level of the optimal inflation rate is quantitative. Here we conduct some numerical exercises to illustrate our results.

Using data from the Panel Study of Income Dynamics (PSID), Aiyagari (1994) argues that a first-order autoregression closely matches the time series properties of the log of annual earnings, with a range of 0.23 to 0.53 for the first-order serial correlation coefficient, ρ, and a standard deviation in unconditional log of earnings, σ_w, of 20 to 40 percent,

$$\ln\left(w_{t+1}\right) = (1-\rho)\ln\left(\overline{w}\right) + \rho \ln\left(w_t\right) + \sigma_w \left(1-\rho^2\right)^{1/2} \epsilon_t, \ \epsilon_t - \text{Normal(0,1)}. \tag{61}$$

We consider three possible values for the standard deviation, $\sigma_w \in \{0.2, 0.3, 0.4\}$, and two possible values for the correlation coefficient, $\rho \in \{0.23, 0.53\}$. We have assumed an instantaneous utility function $u(c) = \left(c^{1-\sigma} - 1\right)/(1-\sigma)$ with $\sigma = 2$.

In our first exercise we have set the discount factor, β equal to 0.94. We have computed the expected utility associated to full commitment. We have also computed the expected utility after default for the household with the highest utility yesterday and today, the period when it declares default, assuming that it holds zero precautionary money balances at that moment. We have computed it for various levels of the inflation rate. This is shown in Figure 1. The first panel of Figure 1 shows both expected utilities when we assume that the standard deviation of unconditional productivity is $\sigma_w = 0.2$. Notice that the optimal inflation rate, the rate for which the expected utility associated to full commitment is exactly equal to the expected utility after default, is negative and it increases with the degree of productivity autocorrelation, ρ. This is so because, the larger the persistence in productivity, the higher the expected utility of defaulting with the highest productivity. The second panel shows both expected utilities assuming that the process for productivity is more volatile, $\sigma_w = 0.3$. There we see that the optimal inflation rate is about 7 percent for $\rho = 0.23$ and rises to almost 15 percent for $\rho = 0.53$. If the standard deviation of productivity is $\sigma_w = 0.4$,

the optimal inflation rate is above 20 percent. Notice that if the government sets the optimal inflation rate the implied real interest rate in equilibrium is $1/q - 1 = 1/\beta - 1$, which is 6.38 percent in this case.
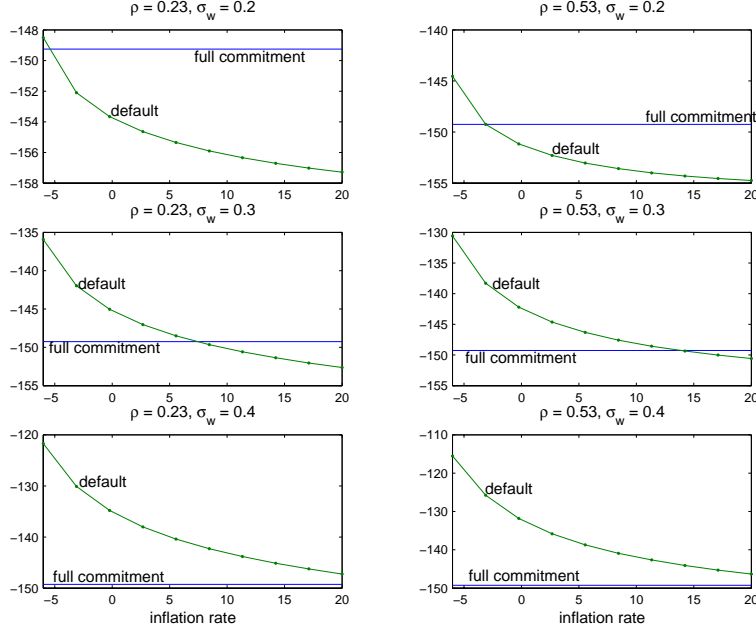

Figure 1: The log of productivity w follows a first order autoregressive process with autocorrelation ρ and standard deviation σ_w. The instantaneous utility function is $u(c) = \left(c^{1-\sigma} - 1\right)/(1 - \sigma)$, $\sigma = 2$, and $\beta = 0.94$. "full commitment refers to expected utility in the full commitment allocation and "default" refers to expected utility of the household with the highest productivity yesterday and today that defaults today with zero precautionary money balances.

In our second exercise, shown in Figure 2, we have assumed that households are a a little bit more patient, $\beta = 0.96$. In this case if $\sigma_w = 0.2$ the expected utility associated to full commitment is higher than the utility of defaulting. This is why we do not report that case, because the optimal inflation rate is the rate of time preference. If the standard deviation of productivity is $\sigma_w = 0.3$ the optimal inflation rate is now about 0 and 1 percent, depending on the autocorrelation of the productivity process. Finally, if $\sigma_w = 0.4$, the optimal inflation rate is about 10 percent. Finally, notice that if the government sets the optimal inflation rate the implied real interest rate in equilibrium is $1/q - 1 = 1/\beta - 1$, which is 4.17 percent in this case.

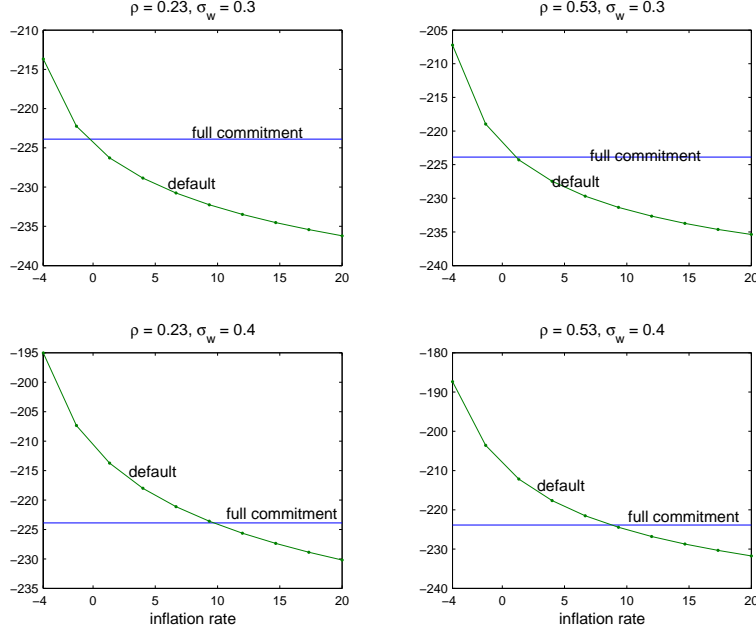Some conclusions can be drawn from this exercise. The higher the patience, the lower

Figure 2: The log of productivity w follows a first order autoregressive process with autocorrelation ρ and standard deviation σ_w. The instantaneous utility function is $u(c) = \left(c^{1-\sigma} - 1\right)/(1 - \sigma)$, $\sigma = 2$, and $\beta = 0.96$. "full commitment refers to expected utility in the full commitment allocation and "default" refers to expected utility of the household with the highest productivity yesterday and today that defaults today with zero precautionary money balances.

the optimal inflation rate. Larger dispersion in productivity, or higher autocorrelation, imply a larger optimal inflation rate. Moreover, it looks like for reasonable parameter values the implied optimal inflation rate is not too large, although is very sensitive to the type of idiosyncratic uncertainty considered and the degree of patience. The range of values obtained is comparable to those obtained by Aiyagari and Williamson (2000) and Akyol (2004), for instance. Finally we should keep in mind that we have abstracted from any distortion that the inflation tax may have, for instance, on leisure. Had we assumed that households valued leisure we would have obtained a lower inflation rate for any discount factor and level of idiosyncratic risk. Moreover, the optimal inflation rate would depend also on the elasticity of substitution between consumption and leisure.

6 A reinterpretation of the Cash-In-Advance framework

It could be argued that the cash-in-advance (CIA) framework introduces money in an ad-hoc manner and asked whether our analysis goes through in frameworks where the use of money arises endogenously. For this reason we modify our original setting to derive the CIA constraint endogenously. Let us assume that the world is populated by a measure 2 of households. There are two types of the same size. Type I households produce at even periods and consume at odd periods. Type II households produce at odd periods and consume at even periods. There is only one perishable consumption good and utility is not transferable. Within each type there are two subtypes of equal measure. When subtype $(j, 1)$ produces w_h units of consumption good, the other subtype produces w_l units of the consumption good, and viceversa. At each period t there are two subperiods, day and night. Households can use money and private bonds. The latter can be used for borrowing purposes. During the day consumers and producers meet in a Walrasian market. The key features of the model are that agents cannot commit to repay their debts and there is anonymity in the goods market.

There is a government that controls the supply of money. The government injects (or withdraws) money by transferring (or taxing) producers in the same manner than in section 2: transfers or taxes are proportional to producers' productivity.

A credit market opens at night. Financial intermediation is done by perfectly competitive firms. We denominate these firms, indistinctly, banks, financial intermediaries or credit agencies. The financial intermediaries act as intermediaries so that private IOUs can circulate. Banks lend money to borrowers, who, in exchange, sell private IOUs to the bank. Lenders give money to the credit agency, which sells a bank IOU to the lenders. The agency always commits to redeem its IOUs and cannot issue more IOUs than the amount of money deposited by lenders. For this process to work we assume that banks have a technology that allows record keeping. This record keeping, however, is limited to financial histories and not available to track trading histories in the goods market. Agencies that operate this record

keeping technology can do so at zero cost. If a household defects, its bond holdings are seized and it is banned from the credit market forever. Specifically, banks keep track of defectors and do not sell them any IOU thereafter. Defectors, however, keep their money holdings and can use money for saving purposes after default.

The timing of the model is the following: First, consumers and producers meet in the day market and exchange goods for assets. At night credit agencies open. Households decide whether to keep participating in the credit market. (This is equivalent to saying that households decide whether to repay their debts or to default on them). Households decide the composition of their portfolio for next period.

The market structure described above implies that money is the only asset exchanged for goods in the day market. The reason is the following: producers will never accept an IOU issued by a consumer since households do not have any record keeping technology. It could be thought, though, that producers would accept a bank IOU held by a consumer. If the producer is a defaulter (it cannot participate in the credit market) he will never accept the bank IOU since it will never be redeemed to him. If the producer is not a defaulter, the bank neither redeems its IOU to the producer because the bank cannot record trading histories and the goods market is anonymous. Moreover, in absence of record keeping of trading histories, the consumer can always claim that he is the claimant of the bank IOU. Since consumption has already taken place, there is no way of checking whether the transaction has taken place or not. This is true even if the bank knows the household's level of productivity each period.[7],[8]

Let us focus on a particular household of subtype i that is a producer at time t. It takes in the day market the money exchanged by the goods they have produced, $p_t\,w_t^i$. At night,

---

[7]Assuming that credit agencies cannot keep record of trading histories and anonymity in the goods markets implies that bank IOUs are illiquid, as assumed by Kocherlakota (2002, 2003), although he refers to government bonds (which are in positive supply).

[8]This framework is somewhat close to that of Berentsen, Camera, and Waller (2007). The main differences are: there is no uncertainty about the periods when households are producers and consumers, utility is not transferable, and there are no search frictions in the goods market.

it faces the budget constraint

$$q_t^n B_{t+1}^i + M_{t+1}^i \leq p_t w_t^i + B_t^i + M_t^i + T_{t+1}^i. \tag{62}$$

Next period the household is a consumer. Then, in the day market it faces the constraint

$$p_{t+1} c_{t+1}^i \leq M_{t+1}^i, \tag{63}$$

whereas at night it faces the constraint

$$p_{t+1} c_{t+1}^i + q_{t+1}^n B_{t+2}^i + M_{t+2}^i \leq B_{t+1}^i + M_{t+1}^i. \tag{64}$$

We write everything in real terms, assuming a constant rate of inflation,

$$q_t b_{t+1}^i + (1+\theta) m_{t+1}^i \leq w_t^i + b_t^i + m_t^i + (1+\theta) \tau_{t+1}^i, \tag{65}$$

$$c_{t+1}^i \leq m_{t+1}^i, \tag{66}$$

$$c_{t+1}^i + q_{t+1} b_{t+2}^i + (1+\theta) m_{t+2}^i \leq b_{t+1}^i + m_{t+1}^i. \tag{67}$$

Let us call $d_{t+1} = m_{t+1}^i - c_{t+1}^i$. Notice that $d_t = m_t^i$ since the households does not consume at time t. Thus we can rewrite the constraints in the following way,

$$q_t b_{t+1}^i + (1+\theta) (c_{t+1} + d_{t+1}) \leq w_t^i + b_t^i + d_t^i + (1+\theta) \tau_{t+1}^i, \tag{68}$$

$$d_{t+1}^i \geq 0, \tag{69}$$

$$q_{t+1} b_{t+2}^i + (1+\theta) m_{t+2}^i \leq b_{t+1}^i + d_{t+1}^i. \tag{70}$$

Assuming that money transfers are not redistributive, $(1+\theta)\tau_{t+1}^i = \frac{w_i}{w} \theta \mathbf{m}$, the house-

hold's problem written recursively is the following:

$$V^p(i, b, d; \theta) = \max_{\substack{c \geq 0, \\ b', \\ d' \geq 0}} \{u(c) + \beta V^c(i, b', d'; \theta)\}$$

s. t. $\quad (1 + \theta) c + q b' + (1 + \theta) d' \leq w_i + b + d + \frac{w_i}{\overline{w}} \theta \mathbf{m}$, $\qquad$ (71)

$\quad V^p(i, b, d; \theta) \geq V_D^p(i, d; \theta)$,

$\quad$ if $i = h$, then $-i = l$, and vice versa,

where $V^p(i, b, d; \theta)$ is the value function of the households that produces goods today with productivity i. $V^c(i, b', d'; \theta)$ denotes the value function next period, when it becomes a consumer,

$$V^c(i, b, d; \theta) = \max_{\substack{b', \\ d' \geq 0}} \{\beta V^p(-i, b', d'; \theta)\}$$

s. t. $\quad q b' + (1 + \theta) d' \leq b + d$ $\qquad$ (72)

$\quad V^c(i, b, d; \theta) \geq V_D^c(i, d; \theta)$,

$\quad$ if $i = h$, then $-j = l$, and vice versa.

The value functions after default are

$$V_D^p(i, d; \theta) = \max_{\substack{c \geq 0, \\ d' \geq 0}} \{u(c) + \beta V_D^c(i, d'; \theta)\}$$

s. t. $\quad (1 + \theta) c + (1 + \theta) d' \leq w_i + d + \frac{w_i}{\overline{w}} \theta \mathbf{m}$, $\qquad$ (73)

$\quad$ if $i = h$, then $-i = l$, and vice versa,

$$V_D^c(i, d; \theta) = \max_{d' \geq 0} \{\beta V_D^p(-i, d'; \theta)\}$$

s. t. $\quad (1 + \theta) d' \leq d$, $\qquad$ (74)

$\quad$ if $i = h$, then $-i = l$, and vice versa.

Lemma 3. *The incentive compatibility constraint is never binding for consumers,* $V^c(i, b, d; \theta) \geq V_D^c(i, d; \theta)$, *for all* $b \in \mathbf{R}$, $d \geq 0$, $i = h, l$.

Proof. We sketch here the proof of the Lemma. If $b \geq 0$ households are never better off defaulting regardless of being producers or consumers. If $b < 0$ the consumer is indifferent

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between defaulting today and setting $d' = d/(1+\theta)$ and setting $d' = d/(1+\theta)$ and $b' = b/q$. That is, they are indifferent between defaulting today and rolling over the debt to the next period. Therefore, $V^c(i, b, d; \theta) \geq \beta V_D^p(-i, d/(1+\theta); \theta) = V_D^c(i, d; \theta)$. Hence, the incentive compatibility constraint is never binding for consumers. $\square$

Thus, we can write the household's problem in the following way:

$$
V^p(i, b, d; \theta) = \max_{\substack{c \geq 0, b', b'', \\ d' \geq 0, d'' \geq 0}} \{u(c) + \beta^2 V^p(-i, b'', d''; \theta)\}
$$

s. t.
$$(1+\theta) c + q b' + (1+\theta) d' \leq w_i + b + d + \frac{w_i}{\overline{w}} \theta \mathbf{m},$$
$$q b'' + (1+\theta) d'' \leq b' + d',$$
$$V^p(i, b, d; \theta) \geq V_D^p(i, d; \theta),$$
if $i = h$, then $-i = l$, and vice versa, (75)

It is easy to check that, in equilibrium $q \leq 1+\theta$. In the case in which $q < 1+\theta$ the household chooses $d' = 0$. If $q = 1 + \theta$ the household is indifferent between money and bonds. Thus, we can further rewrite the household's problem as

$$
V^p(i, b, d; \theta) = \max_{\substack{c \geq 0, \\ d' \geq 0}} \{u(c) + \beta^2 V^p(-i, b', d'; \theta)\}
$$

s. t.
$$(1+\theta) c + q^2 b' + q (1+\theta) d' \leq w_i + b + d + \frac{w_i}{\overline{w}} \theta \mathbf{m},$$
$$V^p(i, b, d; \theta) \geq V_D^p(i, d; \theta),$$
if $i = h$, then $-i = l$, and vice versa. (76)

The default problem collapses to

$$
V_D^p(i, d; \theta) = \max_{\substack{c \geq 0, \\ d' \geq 0}} \{u(c) + \beta^2 V_D^p(-i, d'; \theta)\}
$$

s. t.
$$(1+\theta) c + (1+\theta)^2 d' \leq w_i + d + \frac{w_i}{\overline{w}} \theta \mathbf{m},$$
if $i = h$, then $-i = l$, and vice versa. (77)

The symmetric incentive compatible steady state of this economy is the same of that of our benchmark economy analyzed in section 4. We could extend this model assuming that

productivity is stochastic and follows a Markov process as we did in section 5 and our results would still hold.

Thus, the cash-in-advance constraint arises because trading in the goods market is anonymous and no agent can keep track of households' trading histories.

7 Final comments

In this paper we have shown that the optimal monetary policy involves setting an inflation rate larger than the associated to the Friedman rule whenever households cannot commit to repay their debts and defaulters only can save in the form of money. Moreover, the optimal inflation rate allows the economy to attain the efficient allocation. This is so because inflation lowers the return to default and, therefore, rises the real interest rate and the amount of credit. The first implication of this result is that the issue about the optimal inflation rate is quantitative: we know that is larger than the associated to the Friedman rule if agents are sufficiently impatient but its level depends, among other things, on the agents' degree of patience and the risk they face.

We have illustrated this result in a rather extreme economy. For instance, we have assumed that default is punished with perpetual exclusion from the credit market. If the exclusion entailed a finite number of periods, the return to default would be higher and, hence, the optimal inflation rate would be even higher than in our current setup. On the contrary, had we assumed that leisure is valued the optimal inflation rate would be lower than in our current setup and would also depend on the elasticity between consumption and leisure. Notice that these arguments are quantitative in nature and do not alter the first result: the Friedman rule is no longer optimal. The numerical exercises we have conducted suggest that our model economy does not deliver too high optimal inflation rates.

Our result depends critically on the government being unable to tax discriminate between agents. If the government could impose discretionary taxes on agents it could perfectly mimic a credit market and render it unnecessary. Moreover, the government could tax defaulters so

that they could be reduced to autarky. In that case the Friedman rule would be optimal. This is why we have tied our analysis assuming that government taxes (in the case of deflation) or transfers (if there is inflation) are proportional. In this way the real value of earnings does not depend on the inflation rate. Only the real return to money is affected. Thus, our results stems solely from the effect that the inflation rate has on the return to default.

In our framework defectors only can save in the form of money balances. Implicitly we have assumed that financial institutions can keep track of all financial histories and prevent defectors from saving in private bonds. This assumption is crucial for our results. If banks only could keep track of negative bond holdings, existence of credit needs of a real interest rate equal to zero. Credit would be invariant with respect to the nominal interest rate and the equilibrium allocation would not be efficient. Thus, allowing banks to monitor positive bond holdings and seize defectors' holdings gives a larger scope to monetary policy.

It could be thought that the cash-in-advance constraint is unnecessary since all our results steam from the fact that inflation reduces the return to default. If we eliminated the cash-in-advance constraint all our results would hold. Households, though, would not hold any money balances in equilibrium since the optimal monetary policy implies a positive nominal interest rate. Therefore, for money and private bonds to coexist their real return would have to be equal (an equal to zero in the certainty case). In such a case the equilibrium would be the same that in the economy where defectors can save in the form of private bonds, which assumes implicitly that banks cannot monitor positive bond holdings. Eliminating the cash-in-advance constraint amounts to assuming that trading histories can be monitored. Thus, we think of our model economy as one where banks cannot monitor households' trading histories—therefore, money is needed for transactions—but they can monitor financial histories—therefore, banks can ban defectors from the credit market.

We have abstracted from government debt in this paper. Introducing government in this framework has no effect if defectors are not allowed to purchase them. If defectors were allowed to purchase unbacked government debt results would change. In such a case, government debt would replace money in defectors' portfolios and the link between inflation and the return to default (and, therefore, the volume of credit) would be severed. Nev-

ertheless, results could be different if government debt is backed by taxation, because the government could equalize the return to government debt to that of money. The question that arises is the sustainable amount of government debt. Furthermore, government debt, as money, cannot insure households against idiosyncratic risk. We leave this analysis for future research.

We have also abstracted from capital accumulation. The critical issue is, then, whether there is collateralized credit, and how collateralized and non collateralized credit coexist, which is beyond the scope of this paper.

Finally, in our model there is no default in equilibrium, contrary to what is observed. As Kehoe and Levine (2006) argue, the incentive compatibility constraints assumed here require complete contingent claims, and, in practice, these claims are implemented not through Arrow securities, but rather through a combination of non-contingent assets and bankruptcy. Thus, bankruptcy is a way of completing markets, as argued by Livshits, MacGee, and Tertilt (2007). Nevertheless, we leave this question for future research.

Appendix

A   The benchmark model economy

Proof of Proposition 1

Proof. If no incentive compatibility constraint is binding, the equilibrium allocation is the full commitment allocation and, since $1 + \theta > \beta$, $d_h = 0$. If the incentive compatibility constraint is binding for both types of households, both types would like borrow. In such a case the equilibrium amount of credit must be zero, that is, we are effectively in the default scenario and $d_h = 0$ since $1 + \theta > \beta$. Let us suppose that only one incentive compatibility constraint is binding in equilibrium. Since $w_h > w_l$ and using the concavity of the instantaneous utility function it must be the case that the incentive compatibility constraint must be binding for the household that holds debt, that is, the household with high productivity. Thus, $q\, u'\,(c_h) > \beta\, u'\,(c_l)$ and $q\, u'\,(c_l) = \beta\, u'\,(c_h)$. If $d_h > 0$ it would imply that $(1 + \theta)u'\,(c_h) = \beta\, u'\,(c_l)$, which implies that $q > 1 + \theta$ and $(1 + \theta)u'\,(c_l) < \beta\, u'\,(c_h)$, which would imply that the household is not maximizing utility. Thus, $d_h = 0$. $\quad\square$

Proof of Proposition 3

Proof. Notice that when the inflation rate is zero, $\theta = 0$, the amount of precautionary holdings carried to the high state, d_h, are zero. Moreover, by looking at (37) we can see that the default allocation is feasible, $z_l + z_h = w_l + w_h$. Assumption 1 implies that $u'(w_h) < \beta\, u'(w_l)$. This implies, by strict concavity of the instantaneous utility function, that there is no other feasible allocation that gives at least the default utility. Thus, $q(0) = 1$ and $c_h(0) > w_l$.

Whenever $q \in [\beta, 1)$ the following system of equation should be satisfied

$$q\, u'\,(c_l) = \beta\, u'\,(c_h)\,, \tag{78}$$

$$c_l + c_h = w_l + w_h, \tag{79}$$

$$(1 + \theta)\, u'\,(z_l) = \beta\, u'\,(z_h)\,, \tag{80}$$

$$(1 + \theta)z_h + z_l = (1 + \theta)w_l + w_h, \tag{81}$$

$$u\,(c_l) + \beta\, u\,(c_h) = u\,(z_l) + \beta\, u\,(z_h)\,. \tag{82}$$

The Hessian of the above function $\Re^5 \to \Re^5$ (for the variables c_l, c_h, z_l, z_h, $1/q$) is non

singular when $q < 1$:

$$\mathcal{H} = \begin{vmatrix} u''(c_l) & -\beta\, u''(c_h)/q & 0 & 0 & -\beta u'(c_h) \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & u''(z_l) & -\frac{\beta}{1+\theta}u''(z_h) & 0 \\ 0 & 0 & 1 & (1+\theta) & 0 \\ u'(c_l) & \beta u'(c_h) & -u'(z_l) & -\beta u'(z_h) & 0 \end{vmatrix}. \tag{83}$$

Using (78) and (80) the Hessian can be written as

$$\mathcal{H} = \begin{vmatrix} u''(c_l) & -\beta\, u''(c_h)/q & 0 & 0 & -\beta u'(c_h) \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & u''(z_l) & -\frac{\beta}{1+\theta}u''(z_h) & 0 \\ 0 & 0 & 1 & (1+\theta) & 0 \\ u'(c_l) & q\, u'(c_l) & -u'(z_l) & -(1+\theta)\,u'(z_l) & 0 \end{vmatrix}, \tag{84}$$

and finally,

$$\mathcal{H} = \beta u'(c_h)u'(c_l)\,[1-q]\left[u''(z_l)\,(1+\theta) + \frac{\beta}{1+\theta}\,u''(z_h)\right] < 0. \tag{85}$$

Applying the Cramer rule,

$$\frac{\partial q}{\partial \theta} = \mathcal{H}^{-1} \begin{vmatrix} u''(c_l) & -\beta u''(c_h)/q & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & u''(z_l) & -\frac{\beta}{1+\theta}u''(z_h) & \frac{\beta}{(1+\theta)^2}u'(z_h) \\ 0 & 0 & 1 & (1+\theta) & z_h \\ u'(c_l) & q\, u'(c_l) & -u'(z_l) & -u'(z_l)\,(1+\theta) & 0 \end{vmatrix} < 0, \tag{86}$$

$$\frac{\partial c_h}{\partial \theta} = \mathcal{H}^{-1} \begin{vmatrix} u''(c_l) & -\beta u''(c_h)/q & 0 & 0 & 0 \\ u''(c_l) & 0 & 0 & 0 & -\beta u'(c_h) \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta}{(1+\theta)^2}u'(z_h) & u''(z_l) & -\frac{\beta}{1+\theta}u''(z_h) & 0 \\ 0 & z_h & 1 & (1+\theta) & 0 \\ u'(c_l) & 0 & -u'(z_l) & -u'(z_l)\,(1+\theta) & 0 \end{vmatrix} > 0. \tag{87}$$

It is easy to check, using the default budget constraint and recalling assumption 1 that

$$\lim_{\theta\to\infty} V_D(h,0;\theta) = u(w_h) + \beta\, u(w_l) < (1+\beta)u\,(\overline{w}) = V\,(h,d;\beta-1). \tag{88}$$

Whenever $\theta = 0$ we know that the only participation constrained steady state is the default allocation,

$$V_D(h,d_h;0) = V(h,d_h;0) > V\,(h,d_h;\beta-1). \tag{89}$$

Since V and V_D are both continuous functions there exists θ^* such that $V_D\,(h,d_h;\theta^*) = V\,(h,d_h;\theta^*)$. It follows from the fact that the Hessian of the function that determines the

equilibrium is non singular that θ^* is unique.

$\square$

B   Defaulters can purchase bonds

The default's problem is very similar to the household's problem shown in (10) but for the incentive compatibility constraint that must be substituted by a non negativity constraint on the amount of bonds carried to next period. Since we are focusing on symmetric steady states, we denote as a_h and a_l the amount of bonds held when the household has high and low productivity, respectively. In a symmetric equilibrium, the consumption allocation chosen by a household with high productivity in trade solves the problem shown in (35). In the case of default the allocation chosen solves the problem

$$
\begin{aligned}
\max \quad & u\left(z_l\right) + \beta\, u\left(z_h\right) \\
\text{s. t.} \quad & (1+\theta)\, z_l + q\, a_l + (1+\theta)\, \mathrm{d}_l = \left(1 + \theta\, \tfrac{\mathbf{m}}{w}\right) w_h + a_h + \mathrm{d}_h, \\
& (1+\theta)\, z_h + q\, a_h + (1+\theta)\, \mathrm{d}_h = \left(1 + \theta\, \tfrac{\mathbf{m}}{w}\right) w_l + a_l + \mathrm{d}_l, \\
& a_j \geq 0, \\
& \mathrm{d}_j \geq 0.
\end{aligned}
\tag{90}
$$

We can write the intertemporal budget constraint in each case in the following way,

$$
z_l + q\, z_h = \frac{1 + \theta\, \frac{\mathbf{m}}{\overline{w}}}{1+\theta}\, w_h + q\, \frac{1 + \theta\, \frac{\mathbf{m}}{\overline{w}}}{1+\theta}\, w_l + \frac{1 - q^2}{1+\theta}\, a_h + \frac{q - (1+\theta)}{1+\theta}\, \mathrm{d}_h,
\tag{91}
$$

$$
c_l + q\, c_h = \frac{1 + \theta\, \frac{\mathbf{m}}{\overline{w}}}{1+\theta}\, w_h + q\, \frac{1 + \theta\, \frac{\mathbf{m}}{\overline{w}}}{1+\theta}\, w_l + \frac{1 - q^2}{1+\theta}\, b_h + \frac{q - (1+\theta)}{1+\theta}\, d_h.
\tag{92}
$$

The following proposition establishes the portfolio allocation of defaulters:

Proposition 11. *For any* $1+\theta > \beta$*, the amount of precautionary money balances and bonds held in the high productivity state are zero,* $a_h = \mathrm{d}_h = 0$*.*

Proof. If the non negativity constraint is not binding for both a_l and a_h the consumption allocation is the full commitment allocation. Since $1 + \theta > \beta$ this implies that $\mathrm{d}_h = \mathrm{d}_l = 0$ and, therefore $a_h < 0$, which contradicts the fact that defaulters cannot borrow. Thus, the non negativity constraint must bind for some productivity state. Since $w_h > w_l$ and utility is concave, the constraint must bind only when the household has low productivity. This implies that $a_h = 0$. Since the constraint is not binding for the household with high productivity, $q\, u'\left(z_l\right) = \beta\, u'\left(z_h\right)$. If $\mathrm{d}_h > 0$ the non negativity constraint is not binding for d_h this would imply, using the first order conditions with respect to money and bonds, that $q > 1 + \theta$ and $(1 + \theta)\, u'\left(z_l\right) = \beta\, u'\left(z_h\right)$, which contradicts the fact that the defaulter is maximizing utility. Thus, $\mathrm{d}_h = 0$. $\square$

For a high productivity households that is evaluating whether to default or not, $d_h = \mathrm{d}_h$. Thus, the comparison of both budgets constraints, (91) and (92), shows clearly that the

bond price must be equal to 1. The reason is that if $q > 1$ the default allocation would be in the interior of the budget constraint in trade, since $b_h < 0$. Hence, the household with high productivity would get strictly higher utility in trade than in default. Thus, none would default. But, if such were the case, the equilibrium allocation would be the full commitment allocation and the bond price would be equal to the discount factor, $q = \beta$ and we would arrive to a contradiction. On the other hand, if $q < 1$, the trade allocation would be in the interior of the default budget set. There would not be credit in equilibrium. Therefore, $q = 1$.

C   Idiosyncratic uncertainty

The household's problem

Let us write the household's budget constraint in sequential form. w_t denotes the household current state. This random variable is Markov and takes values in the countable set $\mathbf{W}$. The vector w^t denotes a possible history of events for the household up to time t and $\Pi(w^t)$ is the probability of such an event. The monetary policy is not contingent since there is no aggregate uncertainty and follows the law

$$M_{t+1} = (1+\theta)M_t. \tag{93}$$

This also implies that, in steady state, the price of the consumption good only depends on the inflation rate. The problem solved by a household is

$$\max \quad \sum_{t=0}^{\infty} \beta^t \sum_{t=0}^{\infty} \Pi(w^t) u(c_t^i(w^t))$$

$$\text{s. t.} \quad p_t\, c_t^i(w^t) + \sum_l q_t^n(w^t, w_l)\, B_{t+1}^i(w^t, w_l) + M_{t+1}^i(w^t) \leq$$

$$p_t\, w_t^i + B_t^i(w^t) + M_t^i(w^{t-1}) + T_{t+1}^i(w^t), \tag{94}$$

$$p_t c_t^i(w^t) \leq M_t^i(w^t),$$

$$\sum_{s=t}^{\infty} \beta^{s-t} \sum_{s=t}^{\infty} \Pi(w^s) u(c_s^i(w^s)) \geq \sum_{s=t}^{\infty} \beta^{s-t} \sum_{s=t}^{\infty} \Pi(w^s) u(z_s^i(w^s)), \text{ for all } t.$$

where $\sum_{s=t}^{\infty} \beta^{s-t} \sum_{s=t}^{\infty} \Pi\left(w^s\right) u\left(z_s^i\left(w^s\right)\right)$ is the maximum utility being out of trade (default),

$$\max \quad \sum_{s=t}^{\infty} \beta^{s-t} \sum_{s=t}^{\infty} \Pi\left(w^s\right) u\left(z_s^i\left(w^s\right)\right)$$

$$\text{s. t.} \quad p_s z_s^i\left(w^s\right) + M_{s+1}^i\left(w^s\right) \leq p_s w_s^i + M_s^i\left(w^{s-1}\right) + T_{s+1}^i\left(w^s\right),$$

$$p_s c_s^i\left(w^s\right) \leq M_s^i\left(w^s\right),$$

$$M_t^i\left(w^t\right) \text{ given.}$$

(95)

Denote $b_{t+1}^i\left(w^t, w_l\right) = B_{t+1}^i\left(w^t, w_l\right) / p_{t+1}$ and $q_t\left(w^t, w_l\right) = (1+\theta) q_t^n\left(w^t, w_l\right)$. Furthermore, $m_{t+1}^i\left(z^t\right) = M_{t+1}^i\left(w^t\right) / p_{t+1}$. Notice that the household enters period $t+1$ with a given amount of money, but it can decide to consume a lower amount. In the case in which the cash-in-advance constraint is slack consumption and precautionary money balances are contingent to the household's state in $t+1$,

$$c_{t+1}^i\left(w^t, w_{t+1}\right) + d_{t+1}^i\left(w^t, w_{t+1}\right) = m_{t+1}^i\left(w^t\right), \text{ for all } w_{t+1} \in \mathbf{W}. \tag{96}$$

The household's problem can be written as

$$\max \quad \sum_{t=0}^{\infty} \beta^t \sum_{t=0}^{\infty} \Pi\left(w^t\right) u\left(c_t^i\left(w^t\right)\right)$$

$$\text{s. t.} \quad (1+\theta)\left(c_{t+1}^i\left(w^t, w_{t+1}\right) + d_{t+1}^i\left(w^t, w_{t+1}\right)\right) + \sum_l q_t\left(w^t, w_l\right) b_{t+1}^i\left(w^t, w_l\right) \leq$$

$$w_t^i + b_t^i\left(w^t\right) + d_t^i\left(w^t\right) + (1+\theta)\tau_{t+1}^i\left(w^t\right), \text{ for all } w_{t+1} \in \mathbf{W}, \tag{97}$$

$$d_{t+1}^i\left(w^t, w_{t+1}\right) \geq 0, \text{ for all } w_{t+1} \in \mathbf{W},$$

$$\sum_{t=0}^{\infty} \beta^t \sum_{t=0}^{\infty} \Pi\left(w^t\right) u\left(c_t^i\left(w^t\right)\right) \geq \sum_{t=0}^{\infty} \beta^t \sum_{t=0}^{\infty} \Pi\left(w^t\right) u\left(z_t^i\left(w^t\right)\right).$$

The household's problem after default is

$$\max \quad \sum_{t=0}^{\infty} \beta^t \sum_{t=0}^{\infty} \Pi\left(w^t\right) u\left(z_t^i\left(w^t\right)\right)$$

$$\text{s. t.} \quad (1+\theta)\left(z_{t+1}^i\left(w^t, w_{t+1}\right) + d_{t+1}^i\left(w^t, w_{t+1}\right)\right) \leq$$

$$w_t^i + d_t^i\left(w^t\right) + (1+\theta)\tau_{t+1}^i\left(w^t\right), \text{ for all } w_{t+1} \in \mathbf{W}, \tag{98}$$

$$d_{t+1}^i\left(w^t, w_{t+1}\right) \geq 0, \text{ for all } w_{t+1} \in \mathbf{W}.$$

We can rewrite the household's problem in trade and after default shown in (97) and (98) very easily and obtain their recursive formulation shown in (47) and (48).

Equilibrium

We denote $X = \{\mathbf{W} \times \mathbf{R} \times [0, \overline{d}]\}$ as the set where the individual state variable x can take values. We assume that $\overline{d}$ is sufficiently large. We construct a Markov process for the individual state variables from the Markov process on the shocks and the decision rules of the agents (see Huggett 1993 or Hopenhayn and Prescott 1992 for details). Let $\mathcal{B}$ be the σ-algebra generated in X by, say, the open intervals. A probability measure μ over $\mathcal{B}$ exhaustively describes the economy by stating how many households there are of each type. Let $P(x, B)$ denote the probability that a type $\{x\}$ has of becoming of a type in $B \subset \mathcal{B}$. The function P naturally describes how the economy moves over time by generating a probability measure for tomorrow, μ', given a probability measure, μ, today. The exact way in which this occurs is

$$\mu'(B) = \int_X P(x, B) \, d\mu. \tag{99}$$

To find a steady state, we look for the measure of households μ such that given the prices implied by that measure, households actions reproduce the same measure μ in the following period.

Definition 2. *A participation constrained steady state equilibrium for this economy is a monetary policy, θ, an aggregate amount of money balances $\mathbf{m}$, a set of bond prices, $q(j, s)$, a set of functions $\left\{ V(x; \theta), g^c(x; \theta), g^b(x; \theta), g^d(x; \theta) \right\}$, $\left\{ V_D(j, d; \theta), f^d(j, i, d; \theta), f^z(j, i, d; \theta) \right\}$ and a measure of households μ such that:*

1. *given μ, the monetary policy θ, $q(j, s)$, $\mathbf{m}$, and the functions $\left\{ V(x; \theta), g^c(x; \theta), \right.$ $\left. g^b(x; \theta), g^d(x; \theta) \right\}$ solve the household's problem, shown in (47)*

2. *$\left\{ V_D(j, i, d; \theta), f^d(j, i, d; \theta), f^z(j, i, d; \theta) \right\}$ solve the default problem, shown in (48)*

3. *markets clear,*

   (a) *$\int_X g^c(x) \, d\mu = \int_X w \, d\mu$,*
   (b) *$\int_X g^b(x) \, d\mu = 0$,*
   (c) *$\mathbf{m} = \int_X \left(g^d(x) + g^c(s) \right) d\mu$,*

4. *and the measure of household is stationary, $\mu(B) = \int_X P(x, B) \, d\mu$, for all $B \subset \mathcal{B}$.*

Proof of Proposition 4

Proof. Suppose that there exists a symmetric steady state in which consumption varies across states. Without loss of generality assume that there exist two states, j_1 and j_2, for which $c_{j_2} < c_{j_1}$. Lets us denote as d_{j_1} and d_{j_2} the amount of precautionary savings associated to each state. Using the budget constraint shown in (55) we know that $c_{j_1} + d_{j_1} = c_{j_2} + d_{j_2}$. Therefore, $d_{j_2} > d_{j_1} \geq 0$. Since $d_{j_2} > 0$ the first order condition shown in (51) should hold with strict equality. Since $\beta \leq 1 + \theta$, it follows that

$$u'(c_{j_2}) \leq \sum_{\kappa=1}^{n} \pi_{j_1 \kappa} u'(c_\kappa). \tag{100}$$

Strict concavity of the instantaneous utility function, implies that $u'(c_{j_1}) < u'(c_{j_2})$. Therefore, there must exist j_3 such that $u'(c_{j_2}) < u'(c_{j_3})$, which implies that $d_{j_3} > d_{j_2}$. Following the argument and using Assumption 3, since the set of possible productivity estates is countable we should find $d_{j_1} > d_{j_2}$ and we arrive to a contradiction. Therefore, $c_j = c$ for any productivity state j. Applying clearing market conditions it should be the case that $c_j = \overline{w}$ for all $j = 1, \ldots, n$. □

Proof of Lemma 1

Proof. Let $\theta_2 < \theta_1$, and define

$$\begin{aligned} W_1(j, i, d; \theta_2, \theta_1) = \quad & \max_{z \geq 0} \left\{ u(z) + \beta \sum_{\kappa=1}^{n} \pi_{i\kappa} V_D(i, \kappa, d'; \theta_1) \right\} \\ \text{s. t.} \quad & z + d' \leq w_j + \frac{1}{1+\theta_2} d, \\ & d' \geq 0. \end{aligned} \tag{101}$$

Clearly, $W_1(j, i, d; \theta_2, \theta_1) > V(j, i, d; \theta_1)$. Now define the sequence of functions $\{W_\ell\}$ such that

$$\begin{aligned} W_{\ell+1}(j, i, d; \theta_2, \theta_1) = \quad & \max_{z \geq 0} \left\{ u(z) + \beta \sum_{\kappa=1}^{n} \pi_{i\kappa} W_\ell(i, \kappa, d'; \theta_2, \theta_1) \right\} \\ \text{s. t.} \quad & z + d' \leq w_j + \frac{1}{1+\theta_2} d, \\ & d' \geq 0. \end{aligned} \tag{102}$$

Notice that $V(j, i, d; \theta_2) > W_{\ell+1}(j, i, d; \theta_2, \theta_1)$ and that $W_{\ell+1}(j, i, d; \theta_2, \theta_1) > W_\ell(j, i, d; \theta_2, \theta_1)$, for all ℓ. Hence, it follows that $V(j, i, d; \theta_2) > V(j, i, d; \theta_1)$. □

Proof of Lemma 2

Proof. Let us assume that the household with high productivity has zero money balances at the time of default. If $\theta \leq -\frac{\overline{w} - w_1}{w_n - w_1}$ it is possible to accumulate enough real money balances to ensure the full commitment level of consumption, $\overline{w}$, even if the productivity state is the

lowest possible, w_1, thereafter. That is, the budget constraint at time t is

$$d_{t+1} = w_t + \frac{1}{1+\theta} d_t - c_t. \tag{103}$$

let us assume that at $t = 1$, the period when the household defaults, the productivity is w_n and for any subsequent period $t > 1$ is the lowest, w_1 and assume that $c_t = \overline{w}$. For this plan to be affordable it has to be the case that

$$d_{t+1} = w_1 + \frac{1}{1+\theta} d_t - \overline{w} \geq 0. \tag{104}$$

Assuming that at the period of default the household consumes $\overline{w}$ we get that $d_1 = w_n - \overline{w}$. If $\theta \leq -\frac{\overline{w}-w_1}{w_n-w_1}$ we can easily check that $d_{t+1} \geq 0$, for any $t > 1$. Therefore, $V(n,i,0;\theta) > \frac{u(\overline{w})}{1-\beta}$, for all i, for all $\beta > 0$. Since $V(n,i,d;\theta)$ is increasing in the amount of real money balances, the result follows. $\qquad\square$

Proof of Proposition 7

Proof. Let us take the default function for the household with highest productivity and compare $V_D\left(n,n,0;\overline{\theta}\right) - \frac{u(\overline{w})}{1-\beta} > u(w_n) + \frac{\beta}{1-\beta} u(w_1) - u(\overline{w}) - \frac{\beta}{1-\beta} u(\overline{w}) = u(w_n) - u(\overline{w})$ $+\frac{\beta}{1-\beta}\left(u(w_1) - u(\overline{w})\right)$. Taking limits with respect to the discount factor, $\lim_{\beta\to 0}\left(V_D\left(n,n,0;\widehat{\theta}\right) - \frac{u(\overline{w})}{1-\beta}\right) \geq$ $\lim_{\beta\to 0}\left(u(w_n) - u(\overline{w}) + \frac{\beta}{1-\beta}\left(u(w_1) - u(\overline{w})\right)\right) = u(w_n) - u(\overline{w}) > 0$. If follows from continuity of $V_D(j,i,d;\theta)$ that there exists $\beta\left(\overline{\theta}\right)$ such that its associated discount factor satisfies $V_D\left(n,n,0;\overline{\theta}\right) = \frac{u(\overline{w})}{1-\beta(\overline{\theta})}$ and that for all $\beta < \beta\left(\overline{\theta}\right)$ $V_D\left(n,n,0;\overline{\theta}\right) > \frac{u(\overline{w})}{1-\beta}$. Furthermore, it follows from Lemma 1 that if $\theta \leq \overline{\theta}$ and $\beta \leq \beta\left(\overline{\theta}\right)$, $V_D(n,n,0;\theta) \geq V_D\left(n,n,0;\overline{\theta}\right) > \frac{u(\overline{w})}{1-\beta}$. $\qquad\square$

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