First publ. in: Journal of Economic Theory ; 146 (2011), 5. - pp. 2165-2168

http://dx.doi.org/10.1016/j.jet.2011.06.008

Comment on "Trees and extensive forms" [J. Econ. Theory 143 (1) (2008) 216–250]

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JEL classification: C72; D70

We correct the formulation of one of the properties in [C. Alós-Ferrer, K. Ritzberger, Trees and extensive forms, J. Econ. Theory 143 (1) (2008) 216–250]. The correction concerns property (EDP.iii′) in Section 6.2 (p. 239) and affects results in that final section only. This property, which determines when an Extensive Decision Problem (EDP) is called an Extensive Form (EF), is misstated in the paper. The correct formulation is as follows:

(EDP.iii′) for all $y, y' \in N$, if $y \cap y' = \emptyset$ then there are $x \in X$, $i \in J(x)$ and $c, c' \in C_i$ such that $x \in P(c) \cap P(c')$, $y \subseteq x \cap c$, $y' \subseteq x \cap c'$, and $c \cap c' = \emptyset$.

This property is the one actually used and proved in Proposition 7, Proposition 9, Theorem 5, and Corollary 4. All those results remain true as stated, with (EDP.iii′) having the meaning stated here. The proofs (with minor, straightforward adaptations) remain as in the paper.

Propositions 5 and 8(b), which state that violations of non-uniqueness follow from violations of (EDP.iii′), are incorrect for the corrected version of (EDP.iii′). The following result replaces these two propositions.

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Proposition 5. *Consider a game tree T with available choices. If T is not selective, then the perfect information EDP $\Pi(T)$ fails outcome uniqueness.*

The "only if" direction of Theorem 6 relied on Proposition 5 and needs to be (slightly) reformulated by changing the hypothesis that (T, C) is an EDP to the hypothesis that it is an EF:

Theorem 6. *An EF (T, C) satisfies* (A1) *and* (A2) *if and only if the (rooted) game tree $T = (N, \supseteq)$ is regular, weakly up-discrete, and coherent.*

The proof of the "if" implication remains essentially unchanged. The only caveat is that Proposition 9 refers to the corrected version of (EDP.iii′) provided here and hence Theorem 5, which requires this version, can be used. To see the "only if" direction, note that Proposition 7 implies that T is selective. Hence it is also regular by Proposition 6(a). By Proposition 10 (see below) the game $\Pi(T)$ fulfills (A1). By Corollary 2, every EDP defined on T satisfies (A1). Theorem 3 then implies that T is up-discrete, hence weakly up-discrete and coherent by Corollary 3. This argument does not make use of Proposition 5.

In what follows notation is as in [1]. All numbered results and definitions also follow the order in that paper.

1. Details

Proposition 7 establishes that (EDP.iii′) in the version given above is equivalent to (EDP.ii′) and selectiveness. (This is not true for the version incorrectly stated in the paper.) Proposition 9 establishes that weak up-discreteness implies (EDP.ii′) and, hence, for weakly up-discrete trees (EDP.iii′) is equivalent to selectiveness (which is a property of the tree alone). These results hold true for the version of (EDP.iii′) given here.

Propositions 5 and 8(b) are not true for the corrected version of (EDP.iii′) (see Example 14 below). There is a common mistake in the proofs of Propositions 5 and 8(b) that is as follows. The construction of a strategy s selecting both w and w' fails when, for a node x with $w \in x$ but $w' \notin x$, $s_i(x)$ is required to pick up the choice leading to w. For a different node x' with $w' \in x'$ but $w \notin x'$, $s_i(x')$ will be required to pick up the choice leading to w'. However, it might be the case that x and x' belong to the same information set of player i, in which case an incompatibility arises.

This problem cannot appear under perfect information. Therefore, the statement remains true for the game $\Pi(T)$. In this case, it follows from Proposition 7 that (EDP.iii′) reduces to selectiveness, because (EDP.ii′) is always fulfilled for $\Pi(T)$. The resulting property coincides with the original formulation of Proposition 8(a). This is the modified version of Proposition 5 as given above, which replaces the original versions of Propositions 5 and 8.

Consider the class of weakly up-discrete trees. This includes the class on which every EDP is everywhere playable. By Theorem 5, Proposition 9, and Proposition 5 above, a weakly up-discrete tree T is selective if and only if every EDP (T, C) satisfies outcome uniqueness.

Proposition 10 in Appendix A is correct as stated also with the version of (EDP.iii′) given here. But the proof contains the same mistake pointed out for Propositions 5 and 8(b). The correct proof is as follows.

Proof of Proposition 10. Suppose for some history h there is a strategy s' for $\Pi(T)$ that does not induce an outcome after h. Let h' be a maximal chain in $U^h(s')$ (the undiscarded nodes after h)

and $W(h') = \bigcap_{x \in h'} x$. Fix a play w as follows. If h' has a minimum z (which then cannot be a terminal node by hypothesis), let $w \in s'(z)$. Otherwise, let $w \in W(h')$. If $U^h(s') = \emptyset$ (and hence there is no such chain h'), fix an arbitrary $w \in W(h)$.

Consider now an arbitrary EF (T, C). By Proposition 7 T is selective. We construct a strategy profile in (T, C) as follows. For all $y \in N$ such that $w \in y \subseteq W(h)$ and $i \in J(y)$, choose $s_i(y)$ such that $y \cap [\bigcap_{i \in J(y)} s_i(y)] = s'(y)$. This is possible by (EDP.ii′) and (EDP.iv). If $y \in h'$, then of course $s'(y) = \gamma(y, w)$. For any other node, specify the strategy profile s arbitrarily. Notice that we determine s only along a play, which is possible by (EDP.iv). We claim that $w \notin R_s^h(w)$. For, if it were, by construction of s, we would obtain $w \in R_{s'}^h(w)$ for $\Pi(T)$, a contradiction.

Let $w' \in W(h)$ with $w' \neq w$. By selectiveness there exists $x \in X$ such that $w, w' \in x$ and $\gamma(x, w) \cap \gamma(x, w') = \emptyset$ (by Proposition 1(a)). Notice that, necessarily, $x \subseteq W(h)$. There are two possibilities. If $x \notin h'$, $x \in D^h(s')$ for $\Pi(T)$ which implies by construction that $x \in D^h(s)$ for (T, C). Hence $w' \notin R_s^h(w')$. If $x \in h'$, then $s'(x) = \gamma(x, w) \neq \gamma(x, w')$. Since $s'(x) = x \cap [\bigcap_{i \in J(x)} s_i(x)]$, it follows that $w' \notin R_s^h(w')$. Since $w' \in W(h)$ was arbitrary, we conclude that s does not induce an outcome in (T, C). □

Finally, Corollary 5 remains true when EDP is replaced by EF in its formulation:

Corollary 5.

(a) *If an EF satisfies* (A1) *and* (A2)*, then so does every EF with the same tree.*
(b) *An EF satisfies* (A1) *and* (A2) *if and only if its tree is regular and up-discrete. Furthermore, the EDP is then everywhere playable.*

2. Example

There is another mistake in the computation of the perfect information choices in Example 7 of [1], the differential game as introduced in Example 2. The correct expression is as follows: For a play $g \in x_t(f)$,

$$\gamma\big(x_t(f), g\big) = \big\{h \in W \bigm| \exists \tau > t \text{ such that } h|_{[0,\tau)} = g|_{[0,\tau)}\big\}.$$

Notice that the choices $c_t(f, a)$ given in Example 7 still define an EDP. We still refer to the game with choices $c_t(f, a)$ as the differential game. This EDP, which is used in Examples 10, 12, and 13, is different from $\Pi(T)$ for this tree. In Example 13, this gives rise to another mistake. In fact, the tree of the differential game is selective. To see this, let $f, g \in W$ with $f \neq g$. If $f(0) \neq g(0)$, then $\gamma(W, f) \neq \gamma(W, g)$. If $f(0) = g(0)$, let $t^* = \sup\{\tau > 0 \mid f|_{[0,\tau)} = g|_{[0,\tau)}\}$. Then $x_{t^*}(f) = x_{t^*}(g)$ and $\gamma(x_{t^*}(f), f) \neq \gamma(x_{t^*}(f), g)$.

The point of Example 13 was to provide a counterexample to the converse of Proposition 6(a). Yet, this is already accomplished by Example 4.

The differential game actually fails (EDP.ii′) and hence (by Proposition 7) also the corrected version of (EDP.iii′). On the other hand, the game $\Pi(T)$ based on the same tree is, in fact, an EF. The comment after the statement on Proposition 7 needs to be adjusted accordingly (pp. 240–241).

We now provide a common counterexample to the statements of Propositions 5 and 8(b) under the corrected formulation of (EDP.iii′) given here.

Example 14. Let T be the tree of the differential game as in Example 2 of [1]. Consider an EDP (T, C) based on this tree as follows. There is a continuum of players, $I = \mathbb{R}_+$. Each player chooses an action $a \in A$. Player t is the only player who plays at time t. All nodes at period t belong to the same information set, i.e. no player ever learns any previous decision. That is, the choices of player t are of the form $c_t(a) = \{f \in W \mid f(t) = a\}$. The set of nodes where player t is active is the "slice" $X_t = \{x_t(f) \mid f \in W\}$. Further, each such slice is the only information set of the corresponding player, where all choices of the form $c_t(a)$ are available, $P(c_t(a)) = X_t$ for all $a \in A$.

This game is just the "cascading information sets" version of the normal-form game where each player in $I = \mathbb{R}_+$ chooses an action $a \in A$. The strategy of player t is simply an action $a \in A$ and the outcomes (plays) of the game are simply functions $f : \mathbb{R}_+ \to A$. Hence, (A1) and (A2) follow immediately.

Recalling the expression of $\gamma(x_t(f), g)$ given above, it is immediate that this EDP fails (EDP.ii′), hence (by Proposition 7) also (EDP.iii′). Since outcome uniqueness (A2) is satisfied, this shows that Propositions 5 and 8(b) as stated in [1] do not hold with the corrected version of (EDP.iii′).

References

[1] C. Alós-Ferrer, K. Ritzberger, Trees and extensive forms, J. Econ. Theory 143 (1) (2008) 216–250.