Outside Bonds versus Inside Bonds: A Modigliani-Miller Type Result for Liquidity Constrained Economies

Authors	Aleksander Berentsen, and Christopher J. Waller
Working Paper Number	2009-056A
Creation Date	October 2009
Citable Link	https://doi.org/10.20955/wp.2009.056
Suggested Citation	Berentsen, A., Waller, C.J., 2009; Outside Bonds versus Inside Bonds: A Modigliani-Miller Type Result for Liquidity Constrained Economies, Federal Reserve Bank of St. Louis Working Paper 2009-056. URL https://doi.org/10.20955/wp.2009.056

Published In	Journal of Economic Theory
Publisher Link	https://doi.org/10.1016/j.jet.2011.06.016

Outside Versus Inside Bonds: A Modigliani-Miller type result for liquidity constrained economies*

Aleksander Berentsen
University of Basel

Christopher Waller
University of Notre Dame

September 2, 2009

Abstract

When agents are liquidity constrained, two options exist – sell assets or borrow. We compare the allocations arising in two economies: in one, agents can sell government bonds (outside bonds) and in the other they can borrow (issue inside bonds). All transactions are voluntary, implying no taxation or forced redemption of private debt. We show that any allocation in the economy with inside bonds can be replicated in the economy with outside bonds but that the converse is not true. However, the optimal policy in each economy makes the allocations equivalent.

JEL-Code: E4, E5

Key-Words: Liquidity, Financial markets, Monetary policy, Search

*The paper has benefitted from comments by the Editor, two anonymous referees, David Andolfatto, Narayana Kocherlakota, Neil Wallace, Randy Wright and participants at several seminar and conference presentations. We thank the Federal Reserve Bank of Chicago, the Federal Reserve Bank of St.Louis, the Kellogg Institute at the University of Notre Dame and the Swiss National Science Foundation for research support.

1 Introduction

In monetary economies, households often face binding liquidity constraints. In such situations, they can acquire additional liquidity by selling assets or by borrowing. Several papers have studied the case where households can sell nominal government bonds (outside bonds) for money while others allow households to borrow money (issue inside bonds) to finance consumption.[1] These different methods for relaxing liquidity constraints raises the following question: following the logic of Modigliani-Miller, do these alternative financing arrangements of household consumption lead to equivalent allocations? Our focus in this paper is to address this question.

Within a common monetary framework, we consider two economies: one in which households trade outside bonds and one in which they trade inside bonds. Two main results emerge from our analysis. First, any allocation in the inside bond economy can be replicated in the outside bond economy. The converse is not true. Second, if monetary policy is set optimally in both economies, then the allocations are the same.

The key assumption for attaining these results is that all trades between private agents and between private agents and the government must be voluntary. This implies that in the inside bond economy, redemption of inside bonds must be voluntary. In the outside bond economy, it means the government cannot impose a lump-sum tax on agents to redeem outstanding government debt. In short, participation constraints must be taken into account.[2]

The key feature that makes the allocations equivalent across the two economies is the costs associated with participating in financial markets. In the inside bond economy, if a household defaults on its debt, it is excluded from trading in the financial market until it repays its debt. In the outside bond economy, we assume the government can charge a fee to participate in the financial market. If a household does not pay the fee, it is excluded from trading in the financial market.[3] In this way, households face a similar participation decision in either economy – as to whether they

[1] Examples of the first method include Kocherlakota (2003), Shi (2005, 2008), Boel and Camera (2007) and Marchesi and Senesi (2007). Examples of the latter include Aiyagari and Williamson (2000), Aykol (2004), Berentsen, Camera and Waller (2007), and Diaz and Perrera-Tallo (2007).

[2] In a recent paper, Kocherlakota (2007) emphasizes that many results in the literature rely on asymmetric collection powers of private and government entities. To eliminate this asymmetry, we assume that all trades must be voluntary. With this assumption we are ensuring that any differences in allocations that arise are not the result of inherent differences in the collection powers across public and private entities.

[3] This idea is motivated in part by Andolfatto (2009) who looks at voluntary payment of fees to receive interest on money.

should incur a cost today (repay loans or pay the fee) to have access to future financial markets.

We show that for an arbitrary money growth rate, the allocation in the inside bond economy can be replicated in the outside bond economy by an appropriate choice of the fee. In general, the converse is not true. Hence, allocations can differ across the two economies. We then show that in the outside bond economy it is optimal to have the government charge the maximum fee – one that just makes an individual indifferent between participating in the financial market or not. Under this policy, the outside bond economy allocation will always be equivalent to the inside bond economy allocation.

At first glance this result seems counter-intuitive; most economists would probably argue that imposing a tax to participate in the financial sector would inhibit trade and lower welfare, not raise it. However, the result is actually quite intuitive. The intuition is as follows: Assume the participation constraint in the outside bond economy is not binding. Then marginally raising the fee does not deter agents from participating in the outside bond market, yet it allows the government to extract money from the economy. This in turn reduces the inflation tax on money and raises its return, which improves welfare.

In short, our results are an application of standard public finance theory: If lump-sum 'taxes' are available to the government, then it is optimal to use them to reduce distortionary taxes. Since the participation fee is effectively a lump-sum tax, the government can improve welfare by using it to the fullest extent and reduce the distortionary tax on money.

The structure of the paper is as follows. Section 2 contains a brief review of related literature. In Section 3, we describe the environment. Section 4 contains an analysis of the economy with outside bonds. Section 5 examines the economy with inside bonds, and Section 6 compares the allocations of the two economies. Section 7 concludes. All proofs are in the Appendix.

2 Related Literature

Our equivalence result is reminiscent of Wallace's (1981) Modigliani-Miller type result for open market operations. In an overlapping generation model, Wallace shows that the method for financing government spending, either by issuing money or holding interest-bearing real assets, does not affect the equilibrium allocation. A critical element for proving his result is that the government has access to lump-sum taxation.

Our equivalence result is also related to recent papers by Kocherlakota (2007) and Hellwig and Lorenzoni (2009). Kocherlakota considers various models of asset trade. In these models, households can trade a privately issued one-period bond, a publicly issued one-period bond, or publicly issued money. He proves that the allocations for these economies are equivalent.[4] As noted by Kocherlakota, it is crucial for these results to hold that the government and private households have the same enforcement powers, implying the government has access to lump-sum taxes and private lenders can force some repayment of loans. Moreover, in Kocherlakota's model money plays no transaction role. We obtain our equivalence result for economies with limited enforcement and for environments where trade requires a medium of exchange.

Hellwig and Lorenzoni assume the same enforcement structure as we do. They compare two economies: one with inside bonds and no enforcement of repayment and the other with unbacked government debt (outside bonds). The latter means that the government cannot force households to pay taxes, and households cannot force the government to redeem debt in real goods. They show that the allocations in the two economies are equivalent – any allocation in the inside bond economy can be replicated in the outside bond economy *and vice versa.* This is driven by the fact that unbacked government debt in Hellwig and Lorenzoni's model is simply fiat money, which means that fiat money and government bonds are identical assets. In our framework money and government debt have different liquidity properties, hence they are not identical assets.

Several further papers are related to what we do here. Kehoe and Levine (2001) compare allocations in a dynamic economy when households can acquire consumption goods in one case by selling their capital holdings and in another case by issuing debt subject to a borrowing constraint. They show that if households are sufficiently patient, the allocations are the same in a deterministic environment, but if they are sufficiently impatient, then the debt constrained allocation leads to a better allocation. However, they study trade in real assets while we analyze trade in nominal assets. Furthermore, they do not examine government policy in their economies whereas we do. Shi (2008) examines the implications of illiquid bonds in a monetary search model where there are legal restrictions preventing bonds from being used as a medium of exchange in some transactions but not in others. The legal restrictions make outside bonds illiquid relative to money. He finds

[4] In an earlier paper, Taub (1994) derived a related equivalence result between money and credit.

that having illiquid bonds can be welfare improving. In Boel and Camera (2006), bonds are illiquid in the sense that there is a transaction fee for converting them into cash. Since households have different discount factors and trading opportunities, for some parameter configurations, there is a welfare improving role for illiquid bonds under the optimal monetary policy. Marchesiani and Senesi (2009) consider an economy where households with idle money holdings can buy illiquid outside bonds. The government finances the interest payment through lump-sum taxes. They show that the opportunity to buy interest bearing bonds is strictly welfare improving because it allows households with idle money to save. Lagos and Rocheteau (2003) study the use of illiquid bonds in a variant of the Lagos-Wright model. They find that under the optimal monetary policy (zero inflation) illiquid bonds are inessential.[5] Finally, the paper is also related to Andolfatto (2009) and Hu, Kennan and Wallace (2009) who analyze the impact of participation constraints on allocations arising in the Lagos-Wright framework.

3 The environment

The basic framework is the divisible money model developed in Lagos and Wright (2005). This model is useful because it allows us to introduce heterogeneous preferences while still keeping the distribution of money balances analytically tractable.[6] Time is discrete, and in each period there are three perfectly competitive markets that open sequentially. The first market is a financial market where agents trade money for bonds. The second market is a goods market where they trade money for market 2 goods. In the third market, agents produce and consume market 3 goods and readjust their portfolios.

The economy is populated by two types of infinitely-lived agents: households and firms. Each type of agent has measure 1. Households consume in market 2 and consume and produce in market 3. Firms produce in market 2 and consume in market 3. A household's consumption utility in market 2 is $\varepsilon u(q)$ where ε is a preference shock and q consumption in market 2, with $u'(q)$, $-u''(q) > 0$ with $u'(0) = +\infty$. The preference shock ε has a continuous distribution $F(\varepsilon)$

[5] Furthermore, there are a number of papers that study the coexistence of money and bonds (e.g. Diaz and Perrera-Tallo (2007), Ferraris and Watanabe (2008), Sun (2007), and Telyukova and Wright (2008)). The key difference to our work is that they never compare the allocative effects of different bonds.

[6] An alternative framework would be Shi (1997) which we could amend with preference and technology shocks to generate the same results.

with support $[0, \varepsilon_H]$, is iid across households, serially uncorrelated and has the expected value $\bar{\varepsilon} = \int_0^{\varepsilon_H} \varepsilon dF(\varepsilon)$.[7] Firms incur a utility cost $c(q_s) = q_s$ from producing q_s units of output in market 2. All trades in market 2 are anonymous, and trading histories in this market are private information, thus no trade credit exists. Hence, there is a role for money, as firms require immediate compensation for their production effort.

Following Lagos and Wright (2005), we assume that households in market 3 receive utility $U(x)$ from x consumption, with $U'(x), -U''(x) > 0$, $U'(0) = \infty$, and $U'(+\infty) = 0$. They can also produce these goods with a constant returns to scale production technology where one unit of the consumption good is produced with one unit of labor h generating one unit of disutility.[8] Firms do not produce in this market but they can consume. Their utility of consuming y satisfies $U(y) = y$. The discount factor across periods is $\beta = (1+r)^{-1} < 1$ where r is the time rate of discount.

3.1 First-best allocation

We assume without loss in generality that the planner treats all firms symmetrically. He also treats all households experiencing preference shock ε symmetrically. Given this assumption, the weighted average of expected steady state lifetime utility of households and firms can be written as follows

$$(1-\beta)\mathcal{W} = U(x) + \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - h_\varepsilon] \, dF(\varepsilon) + y - q_s. \tag{1}$$

where h_ε is hours worked by a $\varepsilon-$household in market 3 and q_ε is consumption of an $\varepsilon-$household in market 2. The planner maximizes (1) subject to the feasibility constraint

$$\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \leq q_s. \tag{2}$$

$$x + y \leq \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) \tag{3}$$

[7] All of our results go through with a non-zero lower bound. Setting the lower bound of ε to zero simplifies the presentation of the results.

[8] As in Lagos and Wright (2005), these assumptions allow us to get a degenerate distribution of money holdings at the beginning of a period. The different utility functions $U(.)$ and $u(.)$ allow us to impose technical conditions such that in equilibrium all agents produce and consume in the last market.

The first-best allocation satisfies

$$\begin{aligned} U'(x^*) &= 1 \text{ and} \\ \varepsilon u'(q_\varepsilon^*) &= 1 \text{ for all } \varepsilon. \end{aligned} \tag{4}$$

These are the quantities chosen by a social planner who could dictate production and consumption.

3.2 Outside bonds versus inside bonds

We analyze equilibria of the model under two different bond markets – a market for outside bonds and one for inside bonds. Outside bonds are nominal government debt obligations, whereas inside bonds are private debt obligations.

Outside bond economy In the outside bond economy, we assume a government exists that controls the supply of fiat currency and issues one-period, nominal bonds. These bonds are perfectly divisible, payable to the bearer and default free.[9] One bond pays off one unit of currency at maturity. The government is assumed to have a record-keeping technology over bond trades and bonds are book-keeping entries – no physical object exists. This implies that households are not anonymous to the government. Nevertheless, despite having a record-keeping technology over bond trades, the government has no record-keeping technology over goods trades.

At time t, the government sells one-period, nominal discount bonds in market 3 and redeems bonds that were sold in $t-1$. At the start of $t+1$, the idiosyncratic shocks ε are revealed. Then households trade bonds and money. The government acts as the intermediary for these trades, recording purchases/sales of bonds and redistributes money.

Private households are anonymous to each other and cannot commit to honor inter-temporal promises. Since bonds are intangible objects, they are incapable of being used as media of exchange in market 2, hence they are illiquid. Since households are anonymous and cannot commit, a household's promise in market 2 to deliver outside bonds to a firm in market 3 is not credible. Consequently, fiat money is essential for trade in market 2.

[9] The government has no incentive to default since it redeems its bonds by printing money at no cost.

Inside bond economy Inside bonds are financial claims on private households, issued in a private bond market. Consequently, issuing inside bonds is equivalent to receiving credit. We assume that a perfectly competitive financial market exists where intermediaries have a record-keeping technology over financial trades. Thus, while households are anonymous to each other, they are not anonymous to financial intermediaries.[10] In market 1 the intermediaries acquire nominal debt obligations from borrowers and issue nominal debt obligations on themselves to depositors, which are securitized by their acquired claims. In market 3 all debt obligations are settled. As with the government, we assume intermediaries can commit to honor their debt obligations. No record-keeping technology exists in the goods market, and promises to repay in the future are not credible, thus no trade credit exists between households and firms in market 2.

Limited enforcement We consider economies where all trades must be voluntary. For the outside bond economy, it means that the government cannot levy taxes on households.[11] For the inside bond economy, it means that repayment of debt must be voluntary – creditors have no power to collect unpaid debts.

For a household, unpaid debt has two consequences. First, it receives no further loans until the debt is repaid. Second, it cannot save by acquiring nominal debt obligations from the financial intermediary, unless it repays any outstanding debt. These two assumptions imply that a household that defaults on its debt is excluded from participating in future financial markets. Thus, repayment of debt is the price for participating in future financial markets. Given these rules, we derive conditions to ensure voluntary redemption and show that this may involve binding borrowing constraints; i.e., credit rationing.

For the outside bond economy, although it cannot tax, the government can charge a participation fee for trading in the bond market. Households that do not pay the fee cannot buy newly issued government bonds nor trade in the secondary bond market. The government can do this because outside bonds are intangible objects, and the trades amongst private households in the secondary

[10] An example is a bank that accepts nominal deposits and makes nominal loans. While the bank knows who it trades with, borrowers do not know the identity of depositors and vice versa.

[11] The inability to impose lump-sum taxes occurs in environments with limited enforcement. In such environments, all trades must be voluntary, and so lump-sum taxes of money are not feasible because the government cannot impose any penalties on the agents. If it could impose such penalties, there would be no role for money since "producers could be forced to produce for households" (Kocherlakota 2003, p. 185).

bond market are executed by the government since it controls the record-keeping technology.[12] As a result, paying the fee is similar to repaying one's debt – it is the price for participating in the financial market.[13]

3.3 Government policy and the money supply process

In this section we describe the evolution of the money stock for each economy. In both economies we assume that the government does not purchase any goods with money issuance or revenues received from bond sales. This is without loss of generality.

Outside bond economy Denote M_t as the per capita money stock and B_t as the per capita stock of newly issued bonds at the end of period t. Fiat currency pays no interest. Then M_{t-1} is the beginning-of-period money stock in period t. Let $\varphi_t M_{t-1}$ denote the nominal fee charged by the government in market 3 of period t to participate in the bond market. We define the nominal fee as being proportional to the aggregate money stock for mathematical ease. If $\varphi_t < 0$, the government collects a positive fee from households to access the bond market and if $\varphi_t > 0$, then the government is actually paying households to use the bond market. The change in the money stock in period t is given by

$$M_t = M_{t-1} + \Omega_t \varphi_t M_{t-1} + B_{t-1} - \rho_t B_t \tag{5}$$

where $\Omega_t \in [0, 1]$ is the measure of households who choose to pay the fee in t. Given our assumptions that the government does not purchase goods or levy taxes, (5) is the government's temporal budget constraint. If $\Omega_t = 1$, then all households pay the fee and φ_t can be interpreted as the fraction of the aggregate money stock that is withdrawn from the economy from payment of fees. The total change in the money stock is comprised of two components: first, the net difference between the cash created to redeem bonds, B_{t-1}, and the net cash withdrawal from selling B_t units of bonds

[12] Note that there can be no pairwise deviations since agents are anonymous and cannot commit to honoring inter-temporal promises. For example, the following deviation is not possible: 1) agent i pays the fee, 2) i collects money from agent j to buy bonds while promising to payback the value of the bonds in market 3. Agent i would always renege on the promise and j cannot force redemption. If such a deviation were possible, then money would not be essential for goods trades in market 2.

[13] One could consider having a fee to access the inside bond market as well. Since we assume free entry of intermediaries in the inside bond economy, this fee would be driven to zero.

at the price ρ_t; second, the cash withdrawn from households who pay the fee φ_t to access the bond markets.[14] A government policy is a sequence $\{M_t, B_t, \varphi_t\}_{t=1}^{\infty}$ that satisfies (5) given the initial values $M_0, B_0 > 0$.

Inside bond economy In the model with inside bonds, the government only controls the amount of fiat currency in the economy. In this case, the government can only inject lump-sum transfers of money, $\tau_t M_{t-1}$, to households. As a result, the money stock evolves as

$$M_t = (1 + \tau_t) M_{t-1}. \tag{6}$$

We assume that these lump-sum transfers of cash are only given to households who participate in the financial markets and the transfer is received in market 3. Since all exchange must be voluntary, a government policy is a sequence $\{\tau_t \geq 0\}_{t=1}^{\infty}$ given an initial value $M_0 > 0$.

4 Outside bonds

In this section, we analyze the economy with outside bonds. For notational ease, variables corresponding to the next period are indexed by $+1$, and variables corresponding to the previous period are indexed by -1. The money price of goods in market 3 is P, implying that the goods price of money in market 3 is $\phi = 1/P$. Let p be the money price of goods in market 2; a the money price of bonds in market 1; and ρ the money price of newly issued bonds in market 3.

4.1 Firm choices

Sellers produce market 2 goods with linear cost $c(q) = q$ and consume in market 3 obtaining linear utility $U(y) = y$. It is straightforward to show that that firms are indifferent as to how much they sell in market 2 if

$$p\phi = 1. \tag{7}$$

[14] We want to emphasize that we do not impose a lump-sum tax. The difference is that the government's lack of enforcement power limits the revenue it can collect with the fee since households do not have to pay the fee. See Andolfato (2009).

Since we focus on a symmetric equilibrium, we assume that all firms produce the same amount. With regard to bond holdings, it is straightforward to show that, in equilibrium, firms are indifferent to holding any bonds if the Fisher equation holds and will hold not bonds if the yield on the bonds does not compensate them for inflation or time discounting. Thus, for brevity of analysis, we assume firms carry no bonds or money from market 3 to the next market 1.

4.2 Household choices

In what follows we first characterize a household's choices under the assumption that it pays the fee φM_{-1} and therefore has access to the financial market. We then characterize the optimal choices for a deviating household that does not pay the fee. This allows us to derive the set of fees for which it is individually rational to participate in the financial market.

Let $V(m, b)$ be the expected value from entering market 3 with m units of fiat money and b units of nominal government bonds at time t. Let q_ε denote the quantity consumed by a type ε household in market 2 and y_ε the quantity of government bonds *bought* by a household of type ε in market 1. Then, in the third market, the problem of a representative household in period $t-1$ is:

$$\begin{aligned} V_{-1}\left(m_{-1}, b_{-1}\right) &= \max_{x_{-}, h_{-}, m, b, \{q_\varepsilon, y_\varepsilon\}} U\left(x_{-1}\right) - h_{-1} \\ &\quad + \beta \int_0^{\varepsilon_H} \left[\varepsilon u(q_\varepsilon) + V\left(m - a y_\varepsilon - p q_\varepsilon, b + y_\varepsilon\right)\right] dF\left(\varepsilon\right) \end{aligned}$$

subject to constraints

$$x_{-1} + \phi_{-1}\left(m + \rho_{-1} b\right) = h_{-1} + \phi_{-1}\left(m_{-1} + b_{-1} + \tau_{-1} M_{-2}\right) \quad (8)$$

$$m - a y_\varepsilon \geq 0 \quad \forall \varepsilon \quad (9)$$

$$b + y_\varepsilon \geq 0 \quad \forall \varepsilon \quad (10)$$

$$m - a y_\varepsilon - p q_\varepsilon \geq 0 \quad \forall \varepsilon \quad (11)$$

Constraint (8) is the $t-1$ budget constraint in market 3; constraints (9) and (10) are the period t short-selling constraints on money and bonds in market 1; while (11) is the period t money constraint for purchasing goods in market 2. Note that households choose m and b in $t-1$ before the realization of the period t shock ε. Given these choices of m and b, households then choose the state-contingent values $\{q_\varepsilon, y_\varepsilon\}$.

Using the market 3 constraint to eliminate h_{-1} we get the following program

$$\begin{aligned} V_{-1}\left(m_{-1}, b_{-1}\right) &= \max_{x_{-1}, b, m, \{q_\varepsilon, y_\varepsilon\}} U\left(x_{-1}\right) - x_{-1} + \phi_{-1}\left(m_{-1} + b_{-1} + \tau_{-1} M_{-2}\right) - \phi_{-1}\left(m + \rho_{-1} b\right) \\ &\quad + \beta \int_0^{\varepsilon_H} \left[\varepsilon u(q_\varepsilon) + V\left(m - a y_\varepsilon - p q_\varepsilon, b + y_\varepsilon\right)\right] dF\left(\varepsilon\right) \\ &\quad \text{s.t. (9) - (11)} \end{aligned} \tag{12}$$

The envelope conditions are

$$V_{-1}^{m_{-1}}\left(m_{-1}, b_{-1}\right) = V_{-1}^{b_{-1}}\left(m_{-1}, b_{-1}\right) = \phi_{-1} \tag{13}$$

Let $\beta\phi\mu_\varepsilon$, $\beta\phi\theta_\varepsilon$, and $\beta\phi\lambda_\varepsilon$ denote the multipliers on (9), (10), and (11), respectively. Using (7) and (13) the first-order conditions are

$$\begin{aligned} x_{-1} &: \quad 0 = U'\left(x_{-1}\right) - 1 \\ b &: \quad 0 = -\phi_{-1}\rho_{-1} + \phi\beta + \phi\beta \int_0^{\varepsilon_H} \theta_\varepsilon dF\left(\varepsilon\right) \\ m &: \quad 0 = -\phi_{-1} + \phi\beta + \phi\beta \int_0^{\varepsilon_H} \mu_\varepsilon dF\left(\varepsilon\right) + \phi\beta \int_0^{\varepsilon_H} \lambda_\varepsilon dF\left(\varepsilon\right) \\ q_\varepsilon &: \quad 0 = \varepsilon u'(q_\varepsilon) - 1 - \lambda_\varepsilon \qquad \forall\varepsilon \\ y_\varepsilon &: \quad 0 = 1 - a - a\mu_\varepsilon + \theta_\varepsilon - a\lambda_\varepsilon \qquad \forall\varepsilon \end{aligned}$$

It is straightforward to show that in any monetary equilibrium $\mu_\varepsilon = 0$ and $\lambda_\varepsilon > 0$ for all $\varepsilon > 0$. This follows from the fact that households will never sell all their money for bonds nor will they ever carry money (and forgo interest-bearing bonds) that will not be spent on market 2 goods. It then follows from these expressions that the remaining multipliers are

$$\lambda_\varepsilon = \varepsilon u'(q_\varepsilon) - 1 \;\text{ and }\; \theta_\varepsilon = a\varepsilon u'(q_\varepsilon) - 1.$$

The last expression implies that for $\theta_\varepsilon > 0$, the ε household is constrained by its bond holdings; i.e., it sells all of its bonds for money to acquire goods in market 2. When $\theta_\varepsilon = 0$, the ε household trades off the interest payment on the bond to the marginal liquidity value of having an extra dollar in market 2. In short, it may sell some of its bonds but not all of them or it actually buys bonds with some of its extra cash. Whether or not this constraint is binding for all households or only for a fraction of households drives the equilibrium allocation.

Using these expressions in the first-order conditions for b and m and rearranging yields

$$\begin{aligned} \phi_{-1}\rho_{-1}/a &= \phi\beta \int_0^{\varepsilon_H} \varepsilon u'(q_\varepsilon) dF(\varepsilon) && (14) \\ \phi_{-1} &= \phi\beta \int_0^{\varepsilon_H} \varepsilon u'(q_\varepsilon) dF(\varepsilon). && (15) \end{aligned}$$

With regard to consumption in market 3, we get $U'(x) = 1$ in all t. With regard to consumption in market 2, because a household's desired consumption is increasing in ε, there is a critical value for the taste index $\tilde{\varepsilon}$ such that if $\varepsilon \leq \tilde{\varepsilon}$, $\theta_{\tilde{\varepsilon}} = 0$ and if $\varepsilon \geq \tilde{\varepsilon}$, $\theta_{\tilde{\varepsilon}} \geq 0$. For $\varepsilon \leq \tilde{\varepsilon}$, q_ε solves

$$a\varepsilon u'(q_\varepsilon) = 1 \quad \forall \varepsilon \leq \tilde{\varepsilon} \tag{16}$$

If $\varepsilon = \tilde{\varepsilon}$, the critical household sells all its bonds in market 1 and spends all its money in market 2 to acquire $\tilde{q}_{\tilde{\varepsilon}}$ units of goods. It then follows that households with $\varepsilon \geq \tilde{\varepsilon}$ also consume $\tilde{q}_{\tilde{\varepsilon}}$. Accordingly, in market 2 a household's consumption satisfies

$$q_\varepsilon = \begin{cases} u'^{-1}\left[1/(a\varepsilon)\right] & \text{if} \quad \varepsilon \leq \tilde{\varepsilon} \\ u'^{-1}\left[1/(a\tilde{\varepsilon})\right] & \text{if} \quad \varepsilon \geq \tilde{\varepsilon} \end{cases} \tag{17}$$

Note from (16) that for those households that are unconstrained, the marginal utility of consumption is equalized. Given these consumption choices and the pricing conditions, we get the following bond demands:

$$\begin{aligned} &y_\varepsilon \in [-b, m/a] && \text{if} \quad \varepsilon \leq \tilde{\varepsilon} \\ &y_\varepsilon = -b && \text{if} \quad \varepsilon \geq \tilde{\varepsilon}. \end{aligned} \tag{18}$$

4.3 Equilibrium

We focus on symmetric stationary equilibria where households participate in the financial market and money is used as a medium of exchange. Such equilibria meet the following requirements: (i) Households' decisions solve the maximization problem (12); (ii) The decisions are symmetric across all households with the same preference shocks; (iii) The goods and bond markets clear; (iv) All real quantities are constant across time; (v) The law of motion for the stock of money (5) holds in each period.

Point (iv) requires that the real stock of money is constant; i.e., $\phi M_{-1} = \phi_{+1} M$. This implies that $\phi/\phi_{+1} = M/M_{-1} \equiv \gamma$ where γ is the gross steady-state money growth rate.[15] Symmetry requires $m = M_{-1}$ and $b = B_{-1}$. The restriction that there is a positive demand for money and bonds requires that the following pricing relationship holds in equilibrium:

$$\rho_{-1} = a \tag{19}$$

This relationship comes from (14) and (15). It implies that the bond price has to be the same between market 3 and market 1 in period +1. Moreover, in a stationary equilibrium the bonds price a has to be constant. This can be seen for example from (16), where a changing a involves a non-stationary path for consumption. One can show that a constant bond price implies that the bond-money ratio, respectively the growth rates of money and bonds, have to be equal and that φ is constant.[16]

We assume there are positive initial stocks of money M_0 and outside bonds B_0.[17] Assuming that all households pay the fee, $\Omega_t = 1$, the low of motion for money holdings (5) can be written as follows

$$\frac{B_0}{M_0} = \frac{\gamma - (1+\varphi)}{1 - a\gamma} \tag{20}$$

From this equation the government has two independent policy instruments. We study the case where the government chooses the fee φ and the gross growth rate of the money supply γ which requires that the initial bonds ratio satisfies (20).

[15] Note that we consider the beginning-of-period nominal stock of money and deflate it by the end-of-period price of goods.

[16] The proof of this claim is available by request.

[17] Since the assets are nominal objects, the government can start the economy off by one-time injections of cash M_0 and bonds B_0.

Market clearing in market 1 and market 2 requires

$$\int_0^{\varepsilon_H} y_\varepsilon dF(\varepsilon) = 0 \tag{21}$$

$$q_s - \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) = 0 \tag{22}$$

where q_s is aggregate production by firms. Note that since the entire stock of money is held by the households that then spend it all in market 2, aggregate production in market 2 is equal to the real stock of money; i.e., $q_s = \phi M_{-1}$.

Finally, the requirement that households participate in the financial market imposes a lower bound φM_{-1}. In the proof of Proposition 1 we show that the participation constraint requires that the difference between the expected discounted utility of a household that participates and the expected discounted utility of a household that does not participate in the financial market is non-negative. This condition is summarized by the function $P(\tilde{\varepsilon}, a, \gamma) \geq 0$, where P only depends on the gross growth rate of money γ, the endogenous cutoff value $\tilde{\varepsilon}$, and the bonds price a.

The equilibrium can be of two types. Either some households are constrained in market 1; i.e. $\theta_\varepsilon > 0$ for $\varepsilon \geq \tilde{\varepsilon}$, or none are constrained.

Proposition 1 *For the outside bond economy, an unconstrained equilibrium is a policy* (γ, φ) *and endogenous variables* $(a, \tilde{\varepsilon})$ *that satisfy*

$$\frac{\gamma - (1+\varphi)}{\gamma/\beta - (1+\varphi)} \geq \int_0^{\varepsilon_H} \left[1 - u'^{-1}(\varepsilon_H/\varepsilon)\right] dF(\varepsilon) \tag{23}$$

$$\tilde{\varepsilon} = \varepsilon_H \tag{24}$$

$$a = \beta/\gamma \tag{25}$$

$$P(\varepsilon_H, \beta/\gamma, \gamma) \geq 0. \tag{26}$$

Equation (25) is obtained by using $a\varepsilon u'(q_\varepsilon) = 1$ for all ε in (15) while (23) comes from the budget constraint of the household with the largest preference shock. It reflects the fact that an ε_H household must have enough funds to buy q_H where q_H solves $a\varepsilon_H u'(q_H) = 1$.

In order to verify whether an unconstrained equilibrium exists for a given policy (γ, φ), one needs only to check the participation constraint (26) and the equilibrium condition (23). If both hold, the asset price is $a = \beta/\gamma$ and the critical value is $\tilde{\varepsilon} = \varepsilon_H$. All remaining endogenous variables can then be calculated as follows. From (17), consumption satisfies $q_\varepsilon = u'^{-1}\left[\gamma/\left(\beta\varepsilon\right)\right]$ and from (22) production and the real stock of money is $q_s = \phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF\left(\varepsilon\right)$. Finally, from (20), we get a bonds-to-money ratio that is consistent with the equilibrium.

Proposition 2 *For the outside bond economy, a constrained equilibrium is a policy* (γ, φ) *and endogenous variables* $(a, \tilde{\varepsilon})$ *that satisfy*

$$\frac{\gamma - (1+\varphi)}{a^{-1} - (1+\varphi)} = \int_0^{\tilde{\varepsilon}} \left[1 - u'^{-1}\left(\tilde{\varepsilon}/\varepsilon\right)\right] dF\left(\varepsilon\right) \tag{27}$$

$$\tilde{\varepsilon} < \varepsilon_H \tag{28}$$

$$\frac{\gamma a - \beta}{\beta} = \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1\right) dF\left(\varepsilon\right) \tag{29}$$

$$P\left(\tilde{\varepsilon}, a, \gamma\right) \geq 0. \tag{30}$$

Equation (29) is obtained by using $a\varepsilon u'(q_\varepsilon) = 1$ for all $\varepsilon \leq \tilde{\varepsilon}$ and $a\tilde{\varepsilon}u'(\tilde{q}_\varepsilon) = 1$ for all $\varepsilon \geq \tilde{\varepsilon}$ in (15). Equation (27) comes from the budget constraint of the critical household which has preference shock $\tilde{\varepsilon} \leq \varepsilon_H$. It reflects the fact that all households $\varepsilon \geq \tilde{\varepsilon}$ must have enough funds to buy $\tilde{q}_\varepsilon$.

In order to verify whether a constrained equilibrium exists for a given policy φ and γ, one first derives a and $\tilde{\varepsilon}$ by solving (27) and (29). Then one needs to check the participation constraint (30) and the equilibrium condition $\tilde{\varepsilon} < \varepsilon_H$. Other endogenous variables can then be derived from (17), (20) and (22). If (30) is satisfied, then $\Omega = 1$ and all agents participate in the bond market. Otherwise, $\Omega = 0$.

An interesting result is the different interest rate prevailing in each equilibrium. In the unconstrained equilibrium, the nominal interest rate satisfies the Fisher equation, $1 + i = \gamma/\beta = (1+\pi)(1+r)$. In the constrained equilibrium, the interest rate on bonds, $1 + i = 1/a < \gamma/\beta$, is lower than the value satisfying the Fisher equation. This implies that bonds in the constrained equilibrium are 'bad' stores of value; i.e., no household would buy one in market 3 with the intention of simply holding it to the next market 3. In short, the marginal liquidity value of bonds from relaxing households' cash constraints increases the bonds price and hence reduces its return below the risk-free rate.[18]

[18] Note that an econometrician that would observe the interest rate of an constrained economy would infer that the risk-free rate is too low and conclude that there is a risk-free rate puzzle. A similar point has been made by Lagos

From Propositions 1 and 2 it is evident that the government's choice of γ and φ affects which equilibrium occurs. Given φ, define $\bar{\gamma}(\varphi)$ as the value of γ such that $\tilde{\varepsilon} = \varepsilon_H$ and let $\Theta \equiv 1 - \int_0^{\varepsilon_H} u'^{-1}(\varepsilon_H/\varepsilon)\, dF(\varepsilon)$. We then have the following Proposition.

Proposition 3 *For a given policy* (φ, γ)*, there exists a unique* $1 < \bar{\gamma}(\varphi) < \infty$ *if* $\beta > \Theta$*. If* $\gamma \geq \bar{\gamma}(\varphi)$ *and* $P(\varepsilon_H, a, \gamma) \geq 0$*, then a unique unconstrained equilibrium exists, and if* $\gamma \leq \bar{\gamma}(\varphi)$ *and* $P(\tilde{\varepsilon}, a, \gamma) \geq 0$*, a unique constrained equilibrium exists.*

The essence of this proposition is that for sufficiently low inflation rates, high ε households will face binding constraints on bond sales, and so $a\varepsilon u'(q_\varepsilon) > 1$. In contrast, for sufficiently high inflation rates, all households are constrained, implying $a\varepsilon u'(q_\varepsilon) = 1$ for all ε.

Essential Illiquid Bonds Note that, if $a < 1$, then illiquid outside bonds are essential since they improve the allocation relative to the money-only economy. This follows from two features of the equilibrium allocation. First, at $a = 1$, from (16) we have $\varepsilon u'(q_\varepsilon) = 1$, so unconstrained households are consuming the first-best quantity while constrained households are away from the first-best. By reducing a marginally, the consumption of the unconstrained households falls since they sell some of their real balances for interest-bearing bonds. But the first-order welfare loss from this reduction in consumption is zero due to standard envelope arguments. By shifting real balances to constrained households, their consumption increases and since they are away from the first-best consumption, this generates a first-order welfare gain. Second, from (29), we see that a reduction in a from 1 causes $\tilde{\varepsilon}$ to increase. This means fewer households are constrained, so the marginal utility of consumption is equated across more households. Thus, the distribution of consumption is improved. As a result, these two effects imply that welfare is higher when bonds are illiquid and $a < 1$.

Thus, raising γ marginally above 1 makes $a < 1$ and generates a welfare gain. In short, by creating illiquid, interest-bearing bonds, households with idle cash can trade them for bonds and reduce their exposure to the inflation tax. As a result the real value of money increases as does consumption and welfare. This confirms that Kocherlakota's (2003) result can be extended to stationary, inflationary economies.

(2008).

5 Inside Bonds

In this section, we analyze the model with inside bonds. In market 1, low ε households can use their idle cash balances to acquire nominal bonds from the financial intermediary, which are redeemed in market 3. High ε households can issue nominal bonds to the financial intermediary and redeem them in market 3. Inside bonds are perfectly divisible, and one inside bond pays off 1 unit of fiat currency in market 3. Let a denote the market 1 price of these inside bonds.[19]

5.1 Household choices

Let $V(m,y)$ be the expected value from entering market 3 with m units of fiat money and y units nominal bonds at time t. Let q_ε denote the quantity consumed by a type ε household in market 2 and y_ε the quantity of inside bonds *bought* by a household of type ε in market 1. Let b denote the maximal amount of bonds that a household can issue in market 1. Then, in the third market, the problem of a representative household in period $t-1$ is:

$$\begin{aligned} V_{-1}\left(m_{-1},y_{-1}\right) &= \max_{x_{-1},m,\{q_\varepsilon,y_\varepsilon\}} U\left(x_{-1}\right)-x_{-1}-\phi_{-1}m+\phi_{-1}\left(m_{-1}+y_{-1}\right)+\tau_{-1}M_{-2} \\ &\quad +\beta\int_0^{\varepsilon_H}\left[\varepsilon u(q_\varepsilon)+V\left(m-ay_\varepsilon-pq_\varepsilon,y_\varepsilon\right)\right]dF\left(\varepsilon\right) \end{aligned} \tag{31}$$

subject to constraints

$$m-ay_\varepsilon \geq 0 \quad \forall\varepsilon \tag{32}$$

$$b+y_\varepsilon \geq 0 \quad \forall\varepsilon \tag{33}$$

$$m-ay_\varepsilon-pq_\varepsilon \geq 0 \quad \forall\varepsilon \tag{34}$$

Constraint (32) is the period t short-selling constraint on money; constraint (33) is the borrowing constraint; while (34) is the period t money constraint for purchasing goods in market 2. Note that households choose m in $t-1$ before the realization of the period t shock ε. Given the choice of m households then choose the state-contingent values $\{q_\varepsilon, y_\varepsilon\}$. Except for the choice of outside bonds in market 3, the two maximization problems (12) and (31) are equivalent. Consequently, the

[19] One-period contracts are optimal here due to the quasi-linearity of preferences. In short, linearity of utility in hours worked means there are no welfare gains from smoothing market 3 labor across time to repay current debt.

first-order conditions (15)-(18) continue to hold in the inside bond economy.

Nevertheless, problem (31) differs in one important aspect from problem (12). In the inside bond economy, the borrowing constraint (33) limits the amount of credit that a household can get. Although the households take this constraint as exogenous, in equilibrium it is endogenously determined. The corresponding constraint in the outside bond economy is the short-selling constraint (10). The crucial difference is that in the outside bond economy b is a choice variable of the household while in the inside bond economy b is determined by the financial intermediary which calculates the maximal loan that a household is willing to pay back in market 3.

5.2 Stationary equilibria

We focus on symmetric stationary equilibria where households participate in the financial market and money is used as medium exchange. Such an equilibrium meets the following requirements: (i) Households' decisions solve the maximization problems specified above; (ii) The decisions are symmetric across all households with the same preference shocks; (iii) The goods and bond markets clear; (iv) All real quantities are constant across time. (v) The government budget constraint (6) holds in each period.

As for the outside bonds economy, point (iv) requires that the real stock of money is constant implying $\phi/\phi_{+1} = M/M_{-1} = (1+\tau) \equiv \gamma$. Symmetry requires $m = M_{-1}$. Market clearing in market 1 and market 2 requires (21) and (22) to hold. Note also that since the entire stock of money is held by the households that then spend it all in market 2, aggregate production in market 2 is equal to the real stock of money; i.e., $q_s = \phi M_{-1}$.

Finally, the requirement that households participate in the financial market imposes a lower bound on y_ε. In the proof of Proposition 3 we show that an equilibrium requires that the difference between the expected discounted utility of a household that repays and the expected discounted utility of a household that does not repay is non-negative. This condition is summarized by the function $R(\tilde{\varepsilon}, a, \gamma) \geq 0$, where R only depends on policy γ, the endogenous cutoff value $\tilde{\varepsilon}$, and the bonds price a.

The equilibrium can be of two types. Either some households are constrained in market 1 or none are constrained.

Proposition 4 *For the inside bond economy, an unconstrained equilibrium is a policy γ and endogenous variables $(a, \tilde{\varepsilon})$ that satisfy*

$$\tilde{\varepsilon} = \varepsilon_H \tag{35}$$

$$a = \beta/\gamma \tag{36}$$

$$R(\varepsilon_H, \beta/\gamma, \gamma) \geq 0. \tag{37}$$

Equation (36) is obtained by using $a\varepsilon u'(q_\varepsilon) = 1$ for all ε in (15) while (37) comes from the ε_H household's budget constraint. It reflects the fact that an ε_H household must have enough funds to buy q_H.

Proposition 5 *For the inside bond economy, a constrained equilibrium is a policy γ and endogenous variables $(a, \tilde{\varepsilon})$ that satisfy*

$$\tilde{\varepsilon} < \varepsilon_H \tag{38}$$

$$\frac{\gamma a - \beta}{\beta} = \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1\right) dF(\varepsilon) \tag{39}$$

$$R(\tilde{\varepsilon}, a, \gamma) = 0. \tag{40}$$

Equation (39) is obtained by using $a\varepsilon u'(q_\varepsilon) = 1$ for all $\varepsilon \leq \tilde{\varepsilon}$ and $a\tilde{\varepsilon}u'(\tilde{q}_\varepsilon) = 1$ for all $\varepsilon \geq \tilde{\varepsilon}$ in (15). Equation (40) comes from the budget constraint of the critical household with preference shock $\tilde{\varepsilon}$. It reflects the fact that all households $\varepsilon \geq \tilde{\varepsilon}$ must have enough funds to buy $\tilde{q}_\varepsilon$.

In any equilibrium with $a < 1$, inside bonds are essential. The reasoning is the same as in Berentsen, Camera and Waller (2007); Iinterest bearing inside bonds allow households to earn interest on money. This makes money more valuable, thereby raising ϕ and consumption.

As was the case in the outside bond economy, the nominal interest rate satisfies the Fisher equation, $1 + i = \gamma/\beta = (1 + \pi)(1 + r)$ in the unconstrained equilibrium. When households are credit-constrained, the interest rate on inside bonds, $1 + i = 1/a < \gamma/\beta$, is lower than the value satisfying the Fisher equation. In short, when households are credit-constrained, interest rates have to be low to induce repayment. This result is similar to that found by Alvarez and Jermann (2000) and Hellwig and Lorenzoni (2009).

6 Inside vs outside bonds

We can now state the main proposition of the paper.

Proposition 6 *For a given value of γ, if an equilibrium exists in the inside bond economy, then φ can be chosen such that an equivalent equilibrium allocation exists in the outside bond economy. The converse is not true.*

The reason for this result is as follows. In the inside bond economy, the money growth rate is the only policy instrument. Thus any equilibrium that exists in this economy for a given value of γ can be replicated in the outside bond economy for the same γ by choosing φ in an appropriate manner. If the choice of γ forces the inside bond economy to be borrowing-constrained ($\theta_\varepsilon > 0$ for some households), then by charging the maximum fee the government makes the participation constraint (30) binding in exactly the same way that the borrowing constraint (40) is binding.

The multiplicity of policy instruments in the outside bond economy is what drives the converse part of the proposition – in the inside bond economy the government only has γ as a policy instrument to affect the allocation. So in general, it is not possible to replicate the allocation occurring in the outside bond economy via a choice of γ alone.

The main point of the proposition is that by choosing policies in a particular way, the allocations in the two economies are equivalent. The remaining question is whether or not the optimal policy in each economy generates equivalent allocations.

6.1 Optimal Policy

There are two inefficiencies in this economy that policy must try to overcome. First, when $\tilde{\varepsilon} < \varepsilon_H$, there is an inefficient allocation of consumption across households since some households are constrained while others are not. As a result, the marginal utilities of consumption are not equalized. This is an extensive margin inefficiency. Second, due to the time cost of holding money, the quantities consumed by all households are inefficiently low if $\gamma > \beta$. This is an intensive margin inefficiency.

Keeping in mind these two inefficiencies, we now characterize the optimal policy in both economies.

Proposition 7 *In either economy, it is optimal to set γ such that $\tilde{\varepsilon} < \varepsilon_H$.*

The proof is a straightforward application of the envelope theorem. In the unconstrained equilibrium, the marginal utility of consumption is equalized across all households. It then follows that the only inefficiency is from q_ε being too low when $\gamma > \beta$. Conjecture there is a value $\gamma = \bar{\gamma}$ such that $\tilde{\varepsilon} = \varepsilon_H$. Now consider a marginal reduction in γ from $\bar{\gamma}$ causing $\tilde{\varepsilon} < \varepsilon_H$. The first-order loss from reducing $\tilde{\varepsilon}$ below ε_H is zero, while there is a first-order gain from lowering inflation and raising q_ε for all households. Hence, it is optimal to choose γ such that $\tilde{\varepsilon} < \varepsilon_H$.

Given that it is optimal to have some households constrained, we have the following

Proposition 8 *It is optimal to charge the maximum fee in the outside bond economy making the households' participation constraints binding. Consequently, the allocations are the same in both economies for all γ. It then follows that the optimal value of γ is the same in both economies.*

It is optimal for the government to charge the highest possible fee in the outside bond economy. The reason is simple. Suppose the fee was such that the participation constraint is not binding. Then it is possible to raise φ and lower γ such that the government budget constraint still holds and the participation constraint is satisfied. By lowering γ, the cost of holding real balances is lower, thereby raising q_ε for all households. As we stated earlier, this is just an application of standard public finance theory – if lump-sum 'taxes' are available to the government, then it is optimal to use them to reduce distortionary taxes. Since the participation fee is effectively a lump-sum tax, the government can improve welfare by using it to the fullest extent and reduce the distortionary tax on money.

By making the household's participation constraint binding for $\gamma < \bar{\gamma}(\varphi_{\max})$, the solutions to (23)-(26) also solve (35)-(37) for the unconstrained economies and the solutions to (27)-(30) solve (38)-(40) for the constrained economies.

The key point of this proposition is that even if a policy (φ, γ) can generate a different allocation in the outside bond economy, those allocations are Pareto inferior to the ones achieved in the inside bond economy.

7 Conclusion

When households are liquidity-constrained, two options exist to relax this constraint: sell assets or issue debt. We have analyzed and compared the welfare properties of these two options in a model where households can either issue nominal inside bonds or sell nominal outside bonds. The key assumption of our analysis is the absence of collection powers by private households and the government. The following results emerged from our analysis. First, for any positive inflation rate, bonds are essential in both economies, and thus generate societal benefits. Second, any allocation attained in the economy with inside bonds can be replicated in the economy with outside bonds. The converse is not true. Finally, under the optimal policies, the allocations in the two economies are the same as are the optimal money growth rates. We also showed that the key element responsible for these two economies to have equivalent allocations is a cost to participating in financial markets. Thus, in a manner similar to the results of Hellwig and Lorenzoni (2009), Andolfatto (2009) and Hu, Kennan and Wallace (2009), participation constraints have serious ramifications for analyzing allocations arising in monetary models.

8 Appendix

Allocation for a household that does not participate in the financial market

For many of the proofs that follow we need to know the allocation of an agent who does not participate in the financial market. Throughout the Appendix we indicate the choice variables of a deviating household by a "^".

It is straightforward to show that the quantities consumed by an agent who does not participate in the financial market satisfy (41) and that the first-order condition for the choice of money holdings satisfies (42):

$$\hat{q}_\varepsilon = \begin{cases} u'^{-1}(1/\varepsilon) & \text{if} \quad \varepsilon \leq \hat{\varepsilon} \\ u'^{-1}(1/\hat{\varepsilon}) & \text{if} \quad \varepsilon \geq \hat{\varepsilon} \end{cases} \tag{41}$$

$$\phi_{-1} = \phi\beta \int_0^{\varepsilon_H} \varepsilon u'(\hat{q}_\varepsilon) dF(\varepsilon), \tag{42}$$

where $0 \leq \hat{\varepsilon} \leq \varepsilon_H$ is the critical cutoff for a household that does not participate and $\hat{q}_\varepsilon$ are the quantities it consumes. Dividing (42) by $\phi\beta$ and using (41), we can write (42) as follows:

$$\frac{\gamma - \beta}{\beta} = \int_{\hat{\varepsilon}}^{\varepsilon_H} (\varepsilon/\hat{\varepsilon} - 1) \, dF(\varepsilon).$$

The right-hand side is decreasing in $\hat{\varepsilon}$ and approaches ∞ as $\hat{\varepsilon} \to 0$. The left-hand side is a constant larger than 0 for $\gamma > \beta$. Accordingly, for any $\gamma > \beta$ there exists a unique $\hat{\varepsilon}(\gamma) < \varepsilon_H$. Finally, note that a deviator brings in $\phi\hat{m} = \hat{q}_{\hat{\varepsilon}}$ units of money into a period and that expected consumption $\int_0^{\varepsilon_H} \hat{q}_\varepsilon(\gamma) \, dF(\varepsilon)$ and expected utility $\int_0^{\varepsilon_H} \varepsilon u[\hat{q}_\varepsilon(\gamma)] \, dF(\varepsilon)$ depend via $\hat{\varepsilon}(\gamma)$ on γ only. We will use these results to derive the participation constraints in the following proofs.

Proof of Proposition 1. The proof involves two steps. We first derive the participation constraint. We then derive the equilibrium conditions (23) and (25).

STEP 1: Participation constraint

To derive the participation constraint, consider a household of type ε that enters mkt 3 in t and that pays the fee in every period for all t. Its expected payoff in mkt 3 is

$$EV = U(x^*) - h_\varepsilon + \frac{\beta}{1-\beta} \left\{ \int_0^{\varepsilon_H} \varepsilon u(q_\varepsilon) \, dF(\varepsilon) + U(x^*) - Eh \right\},$$

where h_ε are hours worked in the current period in mkt 3 if it pays the fee and Eh is expected hours worked in future periods. Suppose a household deviates by not paying the fee in the current and all future periods (we could also use the one-step deviation principle to arrive at the same participation constraint). Since $\widehat{x} = x^*$, a deviator's expected discounted utility is

$$E\widehat{V} = U(x^*) - \widehat{h}_\varepsilon + \frac{\beta}{1-\beta}\left\{\int_0^{\varepsilon_H} \varepsilon u(\widehat{q}_\varepsilon)\, dF(\varepsilon) + U(x^*) - E\widehat{h}\right\}.$$

It then follows that the participation constraint satisfies $EV \geq E\widehat{V}$ which requires

$$h_\varepsilon - \widehat{h}_\varepsilon \leq \frac{\beta}{1-\beta}\int_0^{\varepsilon_H} \left[\varepsilon u(q_\varepsilon) - \varepsilon u(\widehat{q}_\varepsilon)\right] dF(\varepsilon) + \frac{\beta}{1-\beta}\left(E\widehat{h} - Eh\right). \tag{43}$$

Deriving h_ε: On the equilibrium path, an ε household arrives in mkt 3 with $m - ay_\varepsilon - pq_\varepsilon$ money and $b + y_\varepsilon$ bonds that payoff one unit of money. It leaves mkt 3 with m_{+1} money and b_{+1} bonds. Accordingly, current hours worked on the equilibrium path are

$$h_\varepsilon = x^* + \phi(m_{+1} + ab_{+1}) - \phi\tau M_{-1} - \phi(m - ay_\varepsilon - pq_\varepsilon) - \phi(b + y_\varepsilon). \tag{44}$$

Deriving $\hat{h}_\varepsilon$: On the equilibrium path, an ε household arrives in mkt 3 with $m - ay_\varepsilon - pq_\varepsilon$ money and $b + y_\varepsilon$ bonds. If the household deviates by not paying the fee, it leaves mkt 3 with $\hat{m}_{+1}$ money and no bonds. Accordingly, current hours worked by a deviator are

$$\hat{h}_\varepsilon = x^* + \phi\hat{m}_{+1} - \phi(m - ay_\varepsilon - pq_\varepsilon) - \phi(b + y_\varepsilon).$$

The difference in current hours worked $h_\varepsilon - \hat{h}_\varepsilon$ is

$$h_\varepsilon - \hat{h}_\varepsilon = -\phi\tau M_{-1} + \phi(m_{+1} + ab_{+1}) - \phi\hat{m}_{+1}. \tag{45}$$

Deriving $E(h)$: To derive $E(h)$ we integrate (44) to get

$$Eh = \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) = x^* + \phi(m_{+1} + ab_{+1} - m - b - \tau M_{-1}) + \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon),$$

since market clearing implies $\int_0^{\varepsilon_H} y_\varepsilon dF(\varepsilon) = 0$ and $\phi p = 1$. In equilibrium, $m_{+1} = M$ and $b_{+1} = B$.

Using the government's budget constraint (5), and market clearing $q_s = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$ yields

$$Eh_\varepsilon = \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) = x^* + q_s.$$

Deriving $E\left(\hat{h}\right)$: In the future a deviator holds $\hat{m} - p\hat{q}_\varepsilon$ units of money arriving in mkt 3 and leaves with $\hat{m}_{+1}$. A deviator's mkt 3 hours are then

$$\widehat{h}_\varepsilon = x^* + \phi(\hat{m}_{+1} - \hat{m}) + \hat{q}_\varepsilon.$$

So its expected hours worked are

$$E\widehat{h} = \int_0^{\varepsilon_H} \widehat{h}_\varepsilon dF(\varepsilon) = x^* + \phi(\hat{m}_{+1} - \hat{m}) + \hat{q}_s.$$

where $\hat{q}_s \equiv \int_0^{\varepsilon_H} \hat{q}_\varepsilon dF(\varepsilon)$. Thus the difference in expected hours worked is

$$E\widehat{h} - Eh = \phi(\hat{m}_{+1} - \hat{m}) + \hat{q}_s - q_s. \tag{46}$$

Maximal fee: Using (45) and (46), we can write the participation constraint (43) as follows

$$-\phi\varphi M_{-1} + \phi(m_{+1} + ab_{+1}) - \phi\hat{m}_{+1} \leq \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\beta}{1-\beta}\phi(\hat{m}_{+1} - \hat{m}),$$

where $\Psi(q_\varepsilon, \hat{q}_\varepsilon) \equiv \int_0^{\varepsilon_H} \{[\varepsilon u(q_\varepsilon) - q_\varepsilon] - [\varepsilon u(\hat{q}_\varepsilon) - \hat{q}_\varepsilon]\} dF(\varepsilon)$. Use the deviator's critical consumption $\phi\hat{m} = \hat{q}_{\hat{\varepsilon}}$ to get

$$-\phi\varphi M_{-1} \leq -\phi(m_{+1} + ab_{+1}) + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\gamma-\beta}{1-\beta}\hat{q}_{\hat{\varepsilon}}. \tag{47}$$

Finally, use the critical agent's budget constraint to substitute $\phi(m_{+1} + ab_{+1})$ by $\gamma\tilde{q}_\varepsilon$ to get the maximal fee $\Phi_{\max}$

$$-\phi\varphi M_{-1} \leq \Phi_{\max} \equiv -\gamma\tilde{q}_\varepsilon + \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\gamma-\beta}{1-\beta}\hat{q}_{\hat{\varepsilon}}.$$

Participation constraint: To derive the participation constraint, use (20) to replace $-\phi\varphi M_{-1}$ in

(47) to get

$$\phi\left(M_{-1}+B_{-1}\right)-\phi\gamma\left(M_{-1}+aB_{-1}\right)\leq-\gamma\tilde{q}_{\varepsilon}+\frac{\beta\Psi\left(q_{\varepsilon},\hat{q}_{\varepsilon}\right)}{1-\beta}+\frac{\gamma-\beta}{1-\beta}\hat{q}_{\hat{\varepsilon}}.$$

From the critical agent's budget we have $\tilde{q}_{\varepsilon}=\phi M_{-1}+a\phi b\leq\phi M_{-1}+a\phi B_{-1}$. Since $\gamma\tilde{q}_{\varepsilon}-\gamma\phi\left(M_{-1}+aB_{-1}\right)\leq 0$, a sufficient condition for the participation constraint to hold is

$$\phi M_{-1}+\phi B_{-1}\leq\frac{\beta\Psi\left(q_{\varepsilon},\hat{q}_{\varepsilon}\right)}{1-\beta}+\frac{\gamma-\beta}{1-\beta}\hat{q}_{\hat{\varepsilon}}.$$

Finally, replace ϕB_{-1} from the budget constraint of the critical agent $\tilde{q}_{\varepsilon}=\phi M_{-1}+a\phi B_{-1}$ and replace ϕM_{-1} by $\int_0^{\varepsilon_H}q_{\varepsilon}dF\left(\varepsilon\right)$ to get

$$0\leq P\left(\tilde{\varepsilon},a,\gamma\right)\equiv\left(1-a\right)\int_0^{\varepsilon_H}q_{\varepsilon}dF\left(\varepsilon\right)+\frac{a\beta\Psi\left(q_{\varepsilon},\hat{q}_{\varepsilon}\right)}{1-\beta}+\frac{a\left(\gamma-\beta\right)}{1-\beta}\hat{q}_{\hat{\varepsilon}}-\tilde{q}_{\varepsilon}. \tag{48}$$

Since from (17) q_{ε} depends on a only, and, as shown in Step 2, $\widehat{q}_{\varepsilon}$ depends on γ only, the right-hand side can be summarized by the function $P\left(\tilde{\varepsilon},a,\gamma\right)$ which depends on γ, the asset price a, and the critical cutoff value $\tilde{\varepsilon}$ only.

Finally, in the unconstrained equilibrium we have $a=\beta/\gamma$ and $\tilde{\varepsilon}=\varepsilon_H$ and so we get (26):

$$0\leq P\left(\varepsilon_H,\beta/\gamma,\gamma\right)\equiv\frac{\gamma-\beta}{\beta}\int_0^{\varepsilon_H}q_{\varepsilon}dF\left(\varepsilon\right)+\frac{\beta\Psi\left(q_{\varepsilon},\hat{q}_{\varepsilon}\right)}{1-\beta}+\frac{\gamma-\beta}{1-\beta}\hat{q}_{\hat{\varepsilon}}-\frac{\gamma}{\beta}q_H.$$

Thus, the participation constraint is satisfied if $P\left(\varepsilon_H,\beta/\gamma,\gamma\right)\geq 0$.

STEP 2: Equilibrium conditions

To derive (25), divide (15) by $\phi\beta$, substitute ϕ_{-1}/ϕ by γ, and substitute $\varepsilon u'(q_{\varepsilon})$ by $1/a$ to get

$$\gamma a/\beta=1.$$

Equilibrium condition (23) is derived from ε_H household's budget constraint $pq_H\leq M_{-1}+aB_{-1}$. If we multiply it by ϕ, we can write it as follows:

$$q_H\leq\phi M_{-1}\left(1+a\frac{B_{-1}}{M_{-1}}\right).$$

We next use the government's budget constraint (5) to substitute B_{-1}/M_{-1} to get the following expression:

$$q_H \leq \phi M_{-1} \left(\frac{1 - a(1+\tau)}{1 - a\gamma} \right).$$

Use the market clearing condition (22) to substitute $\phi M_{-1} = q_s = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$, divide by q_H, and rearrange to get

$$\frac{a[\gamma - a(1+\varphi)]}{1 - a(1+\varphi)} \geq \int_0^{\varepsilon_H} (1 - q_\varepsilon / q_H) \, dF(\varepsilon).$$

Finally, use (17) to substitute all q_ε and substitute a by β/γ to get (23):

$$\frac{\gamma - (1+\varphi)}{\gamma/\beta - (1+\varphi)} \geq \int_0^{\varepsilon_H} \left[1 - u'^{-1}(\varepsilon_H/\varepsilon)\right] dF(\varepsilon).$$

■

Proof of Proposition 2. The proof involves two steps. We first derive the participation constraint. We then derive the equilibrium conditions (27) and (29).

STEP 1: Participation constraint

The derivation of the participation constraint is equal to STEP 1 of the previous proof. From (48), the participation constraint satisfies (30):

$$0 \leq P(\tilde{\varepsilon}, a, \gamma) \equiv (1-a) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) + \frac{a\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{a(\gamma - \beta)}{1-\beta} \hat{q}_{\hat{\varepsilon}} - \tilde{q}_\varepsilon.$$

Since from (17) q_ε depends on a only, and $\widehat{q}_\varepsilon$ depends on γ only, the right-hand side can be summarized by the function $P(\tilde{\varepsilon}, a, \gamma)$ which depends on γ, the asset price a, and the critical cutoff value $\tilde{\varepsilon}$ only.

STEP 1: Equilibrium conditions

To derive (29), divide (15) by $\phi\beta$, substitute ϕ_{-1}/ϕ by γ, and use (17) to substitute $u'(q_\varepsilon)$ to get:

$$\gamma/\beta = \int_0^{\tilde{\varepsilon}} (1/a) \, dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} (1/a)(\varepsilon/\tilde{\varepsilon}) \, dF(\varepsilon).$$

Finally, multiply by a and rewrite it to get (29).

The equilibrium condition (27) is derived from the critical household's budget constraint $p\tilde{q}_\varepsilon =$

$M_{-1} + aB_{-1}$. If we multiply the budget constraint by ϕ, we can write it as follows

$$\tilde{q}_{\varepsilon} = \phi M_{-1}\left(1 + a\frac{B_{-1}}{M_{-1}}\right).$$

We next use the government's budget constraint (5) to substitute $B_{-1}/M_{-1} = \frac{\gamma-(1+\tau)}{1-a\gamma}$ to get

$$\tilde{q}_{\varepsilon} = \phi M_{-1}\left(\frac{1-a\left(1+\varphi\right)}{1-a\gamma}\right).$$

Use the market clearing condition $\phi M_{-1} = \int_0^{\varepsilon_H} q_{\varepsilon} dF\left(\varepsilon\right)$ to substitute ϕM_{-1} and divide by $\tilde{q}_{\varepsilon}$to get

$$1 = \left[\frac{1-a\left(1+\varphi\right)}{1-a\gamma}\right]\left[\int_0^{\tilde{\varepsilon}}\left(q_{\varepsilon}/\tilde{q}_{\varepsilon}-1\right)dF\left(\varepsilon\right)+1\right].$$

Use (17) to substitute all q_{ε} and rearrange to get (27):

$$\frac{\gamma-\left(1+\varphi\right)}{1/a-\left(1+\varphi\right)} = \int_0^{\tilde{\varepsilon}}\left[1-u'^{-1}\left(\tilde{\varepsilon}/\varepsilon\right)\right]dF\left(\varepsilon\right).$$

■

Proof of Proposition 3. The proof involves two steps. First, we derive $\bar{\gamma}\left(\tau\right)$. Then we show existence and uniqueness of the equilibrium.

STEP 1: Derivation of $\bar{\gamma}\left(\tau\right)$

If all households are unconstrained, we have $pq_{\varepsilon} \leq M_{-1} + aB_{-1}$ for all ε. Then, $\bar{\gamma}\left(\tau\right)$ is the value of γ that solves $pq_H = M_{-1} + aB_{-1}$. In the proof of Proposition 1, we have shown that we can write $pq_H \leq M_{-1} + aB_{-1}$ as follows

$$\frac{\gamma-\left(1+\varphi\right)}{\gamma/\beta-\left(1+\varphi\right)} \geq \Phi \tag{49}$$

where $\Theta \equiv \int_0^{\varepsilon_H}\left[1-u'^{-1}\left(\varepsilon_H/\varepsilon\right)\right]dF\left(\varepsilon\right)$ is a constant with $\Theta \in [0,1]$. The left-hand side is increasing in γ and equal to 0 at $\gamma = \varphi + 1$. Moreover, it approaches β for $\gamma \to \infty$. Accordingly, there exists a unique $1 + \varphi < \bar{\gamma}\left(\varphi\right) < \infty$ that solves (49) if $\beta > \Phi$. If $\beta < \Phi$, the unconstrained equilibrium does not exist.

The critical value $\bar{\gamma}(\varphi)$ satisfies

$$\bar{\gamma}(\varphi) = \frac{\beta(1+\varphi)(1-\Phi)}{\beta-\Phi}.$$

Accordingly, for a given policy (φ, γ) if $\gamma \geq \bar{\gamma}(\varphi)$, the asset price a and the cutoff value $\tilde{\varepsilon}$ satisfy Proposition 1. If $\gamma \in [1, \bar{\gamma}(\varphi)]$, then the asset price a and the cutoff value $\tilde{\varepsilon}$ satisfy Proposition 2.

Consider $\gamma \geq \bar{\gamma}(\varphi)$. Then $\tilde{\varepsilon} = \varepsilon_H$ and $a = \beta/\gamma$ so existence and uniqueness follows trivially.

Consider $\gamma \leq \bar{\gamma}(\varphi)$. For convenience, we replicate the two equations that solve for the asset price a and the critical value $\tilde{\varepsilon}$:

$$\frac{\gamma-(1+\varphi)}{a^{-1}-(1+\varphi)} = \int_0^{\tilde{\varepsilon}} \left[1-u'^{-1}(\tilde{\varepsilon}/\varepsilon)\right] dF(\varepsilon) \tag{50}$$

$$\frac{a\gamma-\beta}{\beta} = \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}}-1\right) dF(\varepsilon). \tag{51}$$

Equation (51) is decreasing in $(\tilde{\varepsilon}, a)$ space with $a = \beta/\gamma$ at $\tilde{\varepsilon} = \varepsilon_H$. Equation (50) is increasing in $(\tilde{\varepsilon}, a)$ space. Moreover, at $\tilde{\varepsilon} = \varepsilon_H$, we have $a \geq \beta/\gamma$. To see this, evaluate (50) at $\tilde{\varepsilon} = \varepsilon_H$ to get

$$\frac{\gamma-(1+\varphi)}{a^{-1}-(1+\varphi)} = \Theta.$$

If we solve this equation for a, we get

$$a = \frac{\Theta}{\gamma+(\Theta-1)(1+\varphi)}.$$

Then $a \geq \beta/\gamma$ implies

$$\gamma \leq \bar{\gamma}(\tau) = \frac{\beta(1+\varphi)(1-\Theta)}{\beta-\Theta},$$

which is true since by assumption $\gamma \leq \bar{\gamma}(\varphi)$. Hence, for $\gamma \leq \bar{\gamma}(\varphi)$ there exists a unique $(a, \tilde{\varepsilon})$ that solves (50) and (51) with $a \in [\beta/\gamma, 1]$ and $\tilde{\varepsilon} \leq \varepsilon_H$. ∎

Proof of Proposition 4. The proof involves two steps. We first derive the maximal loan that a household can get. We then derive the equilibrium conditions (36) and (37).

STEP 1: Maximal loan

To derive the maximal loan, consider a household of type ε that enters mkt 3 in t and repays the loan in every period for all t. Its expected payoff in mkt 3 is

$$EV = U(x^*) - h_\varepsilon + \frac{\beta}{1-\beta}\left\{\int_0^{\varepsilon_H} \varepsilon u(q_\varepsilon)\, dF(\varepsilon) + U(x^*) - Eh\right\},$$

where h_ε are hours worked in the current period in mkt 3 if it repays the loan and Eh is expected hours worked in future periods. Suppose a household deviates by not repaying in the current and all future periods (we could also use the one-step deviation principle to arrive at the same participation constraint). Since $\widehat{x} = x^*$ a deviator's expected discounted utility is

$$E\widehat{V} = U(x^*) - \widehat{h}_\varepsilon + \frac{\beta}{1-\beta}\left\{\int_0^{\varepsilon_H} \varepsilon u(\widehat{q}_\varepsilon)\, dF(\varepsilon) + U(x^*) - E\widehat{h}\right\}.$$

It then follows that the participation constraint satisfies $EV \geq E\widehat{V}$ which requires

$$h_\varepsilon - \widehat{h}_\varepsilon \leq \frac{\beta}{1-\beta}\int_0^{\varepsilon_H} \left[\varepsilon u(q_\varepsilon) - \varepsilon u(\widehat{q}_\varepsilon)\right] dF(\varepsilon) + \frac{\beta}{1-\beta}\left(E\widehat{h} - Eh\right). \tag{52}$$

Deriving h_ε: On the equilibrium path, an ε household arrives in mkt 3 with $m - ay_\varepsilon - pq_\varepsilon$ money and y_ε bonds. It receives the transfer τM_{-1} and it leaves mkt 3 with m_{+1} money. Accordingly, current hours worked on the equilibrium path are

$$h_\varepsilon = x^* + \phi m_{+1} - \tau M_{-1} - \phi\left[m - ay_\varepsilon - pq_\varepsilon\right] - \phi y_\varepsilon. \tag{53}$$

Deriving $\hat{h}_\varepsilon$: On the equilibrium path, an ε household arrives in mkt 3 with $m - ay_\varepsilon - pq_\varepsilon$ money and y_ε bonds. If the household deviates by not repaying the loan, it leaves mkt 3 with $\hat{m}_{+1}$. Note that it gets no lump-sum transfer from the government.[20] Accordingly, current hours worked by a deviator are

$$\hat{h}_\varepsilon = x^* + \phi\hat{m}_{+1} - \phi\left[m - ay_\varepsilon - pq_\varepsilon\right].$$

The difference in current hours worked $h_\varepsilon - \hat{h}_\varepsilon$ is

$$h_\varepsilon - \hat{h}_\varepsilon = \phi\left(m_{+1} - \hat{m}_{+1}\right) - \tau\phi M_{-1} - \phi y_\varepsilon. \tag{54}$$

[20] We could assume that all lump-sum transfers are paid out in mkt 1 so that a necessary requirement to get the transfers is participation in financial markets. This assumption would generate the same borrowing constraint.

Deriving $E(h)$: To derive $E(h)$ we integrate (53) to get

$$Eh = \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) = x^* + \phi[m_{+1} - m - \tau M_{-1}] + \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon).$$

since market clearing implies $\int_0^{\varepsilon_H} y_\varepsilon dF(\varepsilon) = 0$ and $\phi p = 1$. In equilibrium $m_{+1} = M$. Using the government budget constraint (6), and market clearing $q_s = \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$ yields

$$Eh = \int_0^{\varepsilon_H} h_\varepsilon dF(\varepsilon) = x^* + q_s.$$

Deriving $E\left(\hat{h}\right)$: In the future a deviator holds $\hat{m} - p\hat{q}_\varepsilon$ units of money arriving in mkt 3 and leaves the market with $\hat{m}_{+1}$. A deviator's mkt 3 hours are then

$$\widehat{h}_\varepsilon = x^* + \phi(\hat{m}_{+1} - \hat{m}) + \hat{q}_\varepsilon.$$

So his expected hours worked are

$$E\widehat{h} = \int_0^{\varepsilon_H} \widehat{h}_\varepsilon dF(\varepsilon) = x^* + \phi(\hat{m}_{+1} - \hat{m}) + \hat{q}_s.$$

where $\hat{q}_s \equiv \int_0^{\varepsilon_H} \hat{q}_\varepsilon dF(\varepsilon)$. Thus the difference in expected hours worked is

$$E\widehat{h} - Eh = \phi(\hat{m}_{+1} - \hat{m}) + \hat{q}_s - q_s. \tag{55}$$

Maximal Loan: Using (54) and (55), we can write the borrowing constraint (52) as follows

$$\phi(m_{+1} - \hat{m}_{+1}) - \phi y_\varepsilon - \tau M_{-1} \leq \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\beta}{1-\beta}\phi(\hat{m}_{+1} - \hat{m}).$$

Use the deviator's critical consumption $\hat{q}_{\hat{\varepsilon}} = \phi\hat{m}$ to get

$$\phi m_{+1} - \phi y_\varepsilon - \tau M_{-1} \leq \frac{\beta\Psi(q_\varepsilon, \hat{q}_\varepsilon)}{1-\beta} + \frac{\gamma-\beta}{1-\beta}\hat{q}_{\hat{\varepsilon}}.$$

Then, since $\phi m = \phi m_{+1} - \tau M_{-1}$ the maximal loan ϕb satisfies

$$-\phi y_\varepsilon \leq \phi b \equiv -\phi m + \frac{\beta \Psi\left(q_\varepsilon, \hat{q}_\varepsilon\right)}{1-\beta} + \frac{\gamma - \beta}{1-\beta} \hat{q}_{\hat{\varepsilon}}. \tag{56}$$

STEP 2: Equilibrium conditions

To derive (36), divide (15) by $\phi\beta$, substitute ϕ_{-1}/ϕ by γ, and substitute $\varepsilon u'(q_\varepsilon)$ by $1/a$ to get

$$\gamma a / \beta = 1.$$

To derive (37) note that in the unconstrained equilibrium, the ε_H household must have enough funds to pay for its consumption; i.e., $q_H \leq \phi m + a\phi b$. Use (56) to substitute ϕb to get

$$q_H \leq (1-a)\,\phi m + \frac{a\beta \Psi\left(q_\varepsilon, \hat{q}_\varepsilon\right)}{1-\beta} + \frac{\gamma - \beta}{1-\beta} \hat{q}_{\hat{\varepsilon}}.$$

Finally, replace $\phi m = \phi M_{-1} = \int_0^{\varepsilon_H} q_\varepsilon dF\left(\varepsilon\right)$

$$0 \leq (1-a) \int_0^{\varepsilon_H} q_\varepsilon dF\left(\varepsilon\right) + \frac{a\beta \Psi\left(q_\varepsilon, \hat{q}_\varepsilon\right)}{1-\beta} + \frac{a\left(\gamma - \beta\right)}{1-\beta} \hat{q}_{\hat{\varepsilon}} - q_H.$$

Since from (17) q_ε depends on a only, and, as shown in Step 2, $\hat{q}_\varepsilon$ depends on γ only, the right-hand side can be summarized by the function $R\left(\varepsilon_H, a, \gamma\right)$ which depends on policy γ and asset price a only.

Finally, in the unconstrained equilibrium we have $a = \beta/\gamma$ and $\tilde{\varepsilon} = \varepsilon_H$ and so we have

$$0 \leq R\left(\varepsilon_H, \beta/\gamma, \gamma\right) \equiv \frac{\gamma - \beta}{\beta} \int_0^{\varepsilon_H} q_\varepsilon dF\left(\varepsilon\right) + \frac{\beta \Psi\left(q_\varepsilon, \hat{q}_\varepsilon\right)}{1-\beta} + \frac{\gamma - \beta}{1-\beta} \hat{q}_{\hat{\varepsilon}} - \frac{\gamma}{\beta} q_H. \tag{57}$$

Thus, the participation constraint is satisfied if $R\left(\varepsilon_H, \beta/\gamma, \gamma\right) \geq 0$. ■

Proof of Proposition 5. The proof involves two steps. We first derive the maximal loan that a household can get. We then derive the equilibrium conditions (39) and (40).

STEP 1: Maximal loan

The derivation of the maximal loan is equal to STEP 1 of the previous proof. From (56), the maximal loan satisfies

$$\phi b \equiv -\phi m + \frac{\beta \Psi\left(q_{\varepsilon}, \hat{q}_{\varepsilon}\right)}{1-\beta} + \frac{\gamma - \beta}{1-\beta} \hat{q}_{\hat{\varepsilon}}. \tag{58}$$

STEP 2: Equilibrium conditions

To derive (39), divide (15) by $\phi\beta$, substitute ϕ_{-1}/ϕ by γ, and substitute $\varepsilon u'(q_{\varepsilon})$ by $1/a$ to get

$$\frac{\gamma a - \beta}{\beta} = \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1\right) dF\left(\varepsilon\right).$$

To derive (40) note that in the constrained equilibrium, the ε_H household's budget constraint holds with equality; i.e., $q_H = \phi m + a\phi b$. Use (56) to substitute ϕb to get

$$q_H = \left(1-a\right)\phi m + \frac{a\beta \Psi\left(q_{\varepsilon}, \hat{q}_{\varepsilon}\right)}{1-\beta} + \frac{\gamma - \beta}{1-\beta} \hat{q}_{\hat{\varepsilon}}.$$

Finally, replace $\phi m = \phi M_{-1} = \int_0^{\varepsilon_H} q_{\varepsilon} dF\left(\varepsilon\right)$ to get (40):

$$0 = R\left(\tilde{\varepsilon}, a, \gamma\right) \equiv \left(1-a\right) \int_0^{\varepsilon_H} q_{\varepsilon} dF\left(\varepsilon\right) + \frac{a\beta \Psi\left(q_{\varepsilon}, \hat{q}_{\varepsilon}\right)}{1-\beta} + \frac{a\left(\gamma - \beta\right)}{1-\beta} \hat{q}_{\hat{\varepsilon}} - q_H.$$

■

Proof of Proposition 6. Consider a policy γ such that an unconstrained equilibrium exists in the inside bond economy. Then the equilibrium allocation satisfies

$$\tilde{\varepsilon} = \varepsilon_H \text{ and } a = \beta/\gamma,$$

with $R\left(\tilde{\varepsilon}, a, \gamma\right) \geq 0$. Assume the same γ and the same real allocation is implemented in the outside bond economy. Then, by comparing $R\left(\tilde{\varepsilon}, a, \gamma\right)$ with $P\left(\tilde{\varepsilon}, a, \gamma\right)$ it straightforward to see that

$$R\left(\varepsilon_H, a, \gamma\right) = P\left(\varepsilon_H, \beta/\gamma, \gamma\right) \geq 0.$$

Finally, the value for φ in the outside bond economy that is consistent with this equilibrium must satisfy

$$\frac{\gamma - \left(1+\varphi\right)}{\gamma/\beta - \left(1+\varphi\right)} \geq \int_0^{\tilde{\varepsilon}} \left[1 - u'^{-1}\left(\tilde{\varepsilon}/\varepsilon\right)\right] dF\left(\varepsilon\right).$$

Now consider a policy γ such that a constrained equilibrium exists in the inside bond economy.

Then the equilibrium allocation satisfies

$$\frac{\gamma a - \beta}{\beta} = \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left(\frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) \text{ and } R(\tilde{\varepsilon}, a, \gamma) = 0,$$

with $\tilde{\varepsilon} \leq \varepsilon_H$. Assume that the same real allocation is implemented in the outside bond economy. Then, by comparing $R(\tilde{\varepsilon}, a, \gamma)$ with $P(\tilde{\varepsilon}, a, \gamma)$ it straightforward that

$$R(\tilde{\varepsilon}, a, \gamma) = P(\tilde{\varepsilon}, a, \gamma) = 0.$$

It remains to choose a value for φ in the inside bonds economy such that

$$\frac{\gamma - (1 + \varphi)}{\gamma/\beta - (1 + \varphi)} = \int_0^{\tilde{\varepsilon}} \left[1 - u'^{-1}(\tilde{\varepsilon}/\varepsilon) \right] dF(\varepsilon).$$

Solving for φ yields

$$\varphi = \frac{\gamma - 1 - (\gamma/\beta - 1) \int_0^{\tilde{\varepsilon}} \left[1 - u'^{-1}(\tilde{\varepsilon}/\varepsilon) \right] dF(\varepsilon)}{1 + \int_0^{\tilde{\varepsilon}} \left[1 - u'^{-1}(\tilde{\varepsilon}/\varepsilon) \right] dF(\varepsilon)}.$$

The converse is not true because there are policies (γ, φ) in the outside bond economy that result in allocations that cannot be replicated in the inside bond economy. ■

Proof of Proposition 7. In equilibrium welfare is given by

$$(1 - \beta) \mathcal{W} = \int_0^{\varepsilon_H} \left[\varepsilon u(q_\varepsilon) - q_\varepsilon \right] dF(\varepsilon) + U(x^*) - x^*.$$

Conjecture there is a value $\bar{\gamma}$ such at $\tilde{\varepsilon} = \varepsilon_H$; i.e., the highest household is just constrained. Consider the change in welfare from a marginal increase in γ above $\bar{\gamma}$:

$$(1 - \beta) \left. \frac{d\mathcal{W}}{d\gamma} \right|_{\gamma = \bar{\gamma}} = \int_0^{\varepsilon_H} \left[\varepsilon u'(q_\varepsilon) - 1 \right] \left. \frac{dq_\varepsilon}{d\gamma} \right|_{\gamma = \bar{\gamma}} dF(\varepsilon).$$

At this value $\varepsilon u'(q_\varepsilon) = 1/a$ for all ε, so we have

$$(1 - \beta) \left. \frac{d\mathcal{W}}{d\gamma} \right|_{\gamma = \bar{\gamma}} = \int_0^{\varepsilon_H} \left(\frac{1 - a}{a} \right) \left. \frac{dq_\varepsilon}{d\gamma} \right|_{\gamma = \bar{\gamma}} dF(\varepsilon).$$

Since $a = \beta/\gamma < 1$ at $\gamma = \bar{\gamma}$, the sign of this derivative hinges on the sign of $dq_\varepsilon/d\gamma$. From the household's FOC we have $\varepsilon u'(q_\varepsilon) = \gamma/\beta$ with

$$\frac{dq_\varepsilon}{d\gamma} = \frac{\beta}{\varepsilon u''(q_\varepsilon)} < 0.$$

It then follows that $\left.\frac{d\mathcal{W}}{d\gamma}\right|_{\gamma=\bar{\gamma}} < 0$ so lowering γ to generate $\tilde{\varepsilon} < \varepsilon_H$ is welfare improving. ■

Proof of Proposition 8. Consider the outside bond economy in the constrained equilibrium. Welfare is given by

$$(1-\beta)\mathcal{W} = \int_0^{\tilde{\varepsilon}} [\varepsilon u(q_\varepsilon) - q_\varepsilon]\, dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} [\varepsilon u(q_{\tilde{\varepsilon}}) - q_{\tilde{\varepsilon}}]\, dF(\varepsilon) + U(x^*) - x^*.$$

The FOC wrt to γ yields

$$(1-\beta)\frac{d\mathcal{W}}{d\gamma} = \int_0^{\tilde{\varepsilon}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\gamma} dF(\varepsilon) + \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left[\varepsilon u'(q_{\tilde{\varepsilon}}) - 1\right] \frac{dq_{\tilde{\varepsilon}}}{d\gamma} dF(\varepsilon).$$

For all households we have $a\varepsilon u'(q_\varepsilon) \geq 1$ so the bracketed terms are positive. Thus, the sign of this derivative hinges on the signs of $dq_{\tilde{\varepsilon}}/d\gamma$ and $dq_\varepsilon/d\gamma$. For all $\varepsilon \leq \tilde{\varepsilon}$, $a\varepsilon u'(q_\varepsilon) = 1$ which yields

$$\frac{dq_\varepsilon}{d\gamma} = -\frac{\varepsilon u'(q_\varepsilon)}{a\varepsilon u''(q_\varepsilon)} \frac{da}{d\gamma} \tag{59}$$

$$\frac{dq_{\tilde{\varepsilon}}}{d\gamma} = -\frac{\tilde{\varepsilon} u'(q_{\tilde{\varepsilon}})}{a\varepsilon u''(q_\varepsilon)} \frac{\partial a}{\partial \gamma} - \frac{u'(q_{\tilde{\varepsilon}})}{\varepsilon u''(q_\varepsilon)} \frac{\partial \tilde{\varepsilon}}{\partial \gamma}. \tag{60}$$

If the fee is low enough that the participation constraint is not binding, then using (29) we obtain

$$\frac{\partial a}{\partial \gamma} = \frac{-\gamma}{a} \qquad \frac{\partial \tilde{\varepsilon}}{\partial \gamma} = \frac{-a}{\beta \int_{\tilde{\varepsilon}}^{\varepsilon_H} \frac{\varepsilon}{\tilde{\varepsilon}^2} dF(\varepsilon)},$$

hence

$$\frac{dq_\varepsilon}{d\gamma} = -\frac{\varepsilon u'(q_\varepsilon)}{a\varepsilon u''(q_\varepsilon)} \frac{da}{d\gamma} = \frac{dq_\varepsilon}{d\gamma} = \frac{\varepsilon u'(q_\varepsilon)}{a\varepsilon u''(q_\varepsilon)} \frac{\gamma}{a} < 0$$

$$\frac{dq_{\tilde{\varepsilon}}}{d\gamma} = \frac{\tilde{\varepsilon} u'(q_{\tilde{\varepsilon}})}{a\varepsilon u''(q_\varepsilon)} \frac{\gamma}{a} + \frac{u'(q_{\tilde{\varepsilon}})}{\varepsilon u''(q_\varepsilon)} \frac{a}{\beta \int_{\tilde{\varepsilon}}^{\varepsilon_H} \frac{\varepsilon}{\tilde{\varepsilon}^2} dF(\varepsilon)} < 0.$$

It then follows that raising φ and lowering γ improves welfare. Thus setting φ such that $P(\tilde{\varepsilon}, a, \gamma) = 0$ is optimal. It then follows from Proposition 6 that the allocations are equivalent and thus the optimal value of γ is the same in both economies. ■

References

[1] Aiyagari, R. and S. Williamson (2000). "Money and Dynamic Credit Arrangements with Private Information." *Journal of Economic Theory* 91, 248-279.

[2] Alvarez ,F. and U. Jermann (2000), "Efficiency, Equilibrium, and Asset Pricing with Risk of Default." *Econometrica*, 68, 775-797.

[3] Andolfatto, D. (2009). "Essential Interest-Bearing Money." Mimeo, Simon Fraser University.

[4] Aykol, A. (2004). "Optimal Monetary Policy in an Economy with Incomplete Markets and Idiosyncratic Risk." *Journal of Monetary Economics* 51, 1245-1269.

[5] Berentsen, A., G. Camera and C. Waller (2005). "The Distribution of Money Balances and the Non-Neutrality of Money." *International Economic Review,* 46, 465-487.

[6] Berentsen, A., G. Camera and C. Waller (2007). "Money, Credit and Banking." *Journal of Economic Theory*, 135, 171-194.

[7] Boel, P. and G. Camera (2006). "Efficient Monetary Allocations and the Illiquidity of Bonds." *Journal of Monetary Economics.*

[8] Diaz, A. and Perrera-Tallo (2007). "Credit and Inflation under Borrower's Lack of Commitment," Mimeo, University Carlos III de Madrid.

[9] Ferraris L. and M. Watanabe (2008). "Collateral Secured Loans in a Monetary Economy." *Journal of Economic Theory*, 143, 405-424.

[10] Hellwig, M. (1993)."The Challenge of Monetary Theory." *European Economic Review,* 37, 215-242.

[11] Hellwig, C. and G. Lorenzoni (2008). "Bubbles and Self-Enforcing Debt." *Econometrica* 77, 1137-1164.

[12] Hu, Kennan and Wallace (2009). "Coalition-Proof Trade and the Friedman Rule in the Lagos-Wright Model." Journal of Political Economy, 117, 116-137.

[13] Kehoe, T. and D. Levine (2001). "Liquidity Constrained Markets versus Debt Constrained Markets." *Econometrica*, 69, 575-598.

[14] Kocherlakota, N. (2003). "Societal Benefits of Illiquid Bonds." *Journal of Economic Theory*, 108, 179-193.

[15] Kocherlakota, N. (2008). "Money and Bonds: An Equivalence Theorem." Mimeo, University of Minnesota.

[16] Lagos, R. (2008). "Asset Prices and Liquidity in an Exchange Economy." Mimeo, NYU.

[17] Lagos, R. and G. Rocheteau (2003). "On the Coexistence of Money and Other Assets." Mimeo, Federal Reserve Bank of Cleveland.

[18] Lagos, R. and G. Rocheteau (2005). "Inflation, Output and Welfare." *International Economic Review*, 46, 495-522.

[19] Lagos, R. and R. Wright (2005). "A Unified Framework for Monetary Theory and Policy Evaluation." *Journal of Political Economy*, 113, 463-484.

[20] Marchesiani, A. and P. Senesi (2007). "Money and Nominal Bonds." *Macroeconomic Dynamics* 13, 189-199.

[21] Rocheteau, G. and R. Wright (2004). "Money in Search Equilibrium, in Competitive Equilibrium and in Competitive Search Equilibrium." *Econometrica,* 73, 175-202.

[22] Shi, S. (1997). "A Divisible Search Model of Fiat Money." *Econometrica*, 65, 75-102.

[23] Shi, S. (2005). "Nominal Bonds and Interest Rates." *International Economic Review*, 46, 579-618.

[24] Shi, S. (2008). "Efficiency Improvement from Restricting the Liquidity of Nominal Bonds." *Journal of Monetary Economics* 55, 1025-1037.

[25] Sun, L. (2007). "Sustained Societal Benefit of Illiquid Bonds." Mimeo, University of Toronto.

[26] Taub, B. (1994). "Currency and Credit are Equivalent Mechanisms." *International Economic Review*, 35, 921-956.

[27] Telyukova, I. and R. Wright (2007). "A Model of Money and Credit, with Application to the Credit Card Debt Puzzle." *Review of Economic Studies*, 75, 629-647.

[28] Wallace, N. (1981). "A Modigliani-Miller Theorem for Open-Market Operations." *American Economic Review*, 71, 267-274.