When Are Signals Complements or Substitutes?*

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July 31, 2012

*We are grateful to Geoffroy de Clippel, Drew Fudenberg, Margaret Meyer, Asher Wolinsky, a referee, and an associate editor for their comments. Tilman Börgers thanks the Center for Mathematical Studies in Economics and Management Science at Northwestern University for the hospitality offered while he was working on this paper. Angel Hernando-Veciana gratefully acknowledges support by the Spanish Ministry of Science through ECO2009-10531. Daniel Krähmer gratefully acknowledges support by the German Science Foundation (DFG) through SFB/TR 15.

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Abstract

The paper introduces a notion of complementarity (substitutability) of two signals which requires that in all decision problems each signal becomes more (less) valuable when the other signal becomes available. We provide a general characterization which relates complementarity and substitutability to a Blackwell comparison of two auxiliary signals. In a setting with a binary state space and binary signals, we find an explicit characterization that permits an intuitive interpretation of complementarity and substitutability. We demonstrate how these conditions extend to more general settings. We also illustrate the implications of our concepts for three economic applications: information disclosure in auctions, information aggregation through voting, and polarization of beliefs.

Keywords: Complementarity, substitutability, value of information, Blackwell ordering.

JEL Classification No.: C00, C44, D81, D83

1 Introduction

Suppose that two signals are available to a decision maker, and that each signal contains some information about an aspect of the world that is relevant to the decision maker's choice. In this paper we ask under which conditions these two signals are *substitutes*, and under which conditions they are *complements*. Roughly speaking, we call signals substitutes if the incentive to acquire one signal decreases as the other signal becomes available, and we call them complements if this incentive increases as the other signal becomes available.

The incentives to acquire signals depend on the decision for which the information will be used. When we call signals complements or substitutes in this paper, then we mean that the conditions described above are satisfied *in all decision problems*. Our notions of substitutability and complementarity of signals are therefore in the spirit of Blackwell's [6] comparison of the informativeness of signals, because they do not refer to any particular decision problem, but only to the joint distribution of signals, conditional on the various possible states of the world. An example of complements in the sense of our definition are two signals one of which communicates the state of the world using some code, and the second signal provides the code, where the code is independent of the state. An example of substitutes in the sense of our definition are completely correlated signals.

In addition to introducing new and general notions of substitutability and complementarity of signals, the main contribution of this paper is to identify conditions for the joint distribution of signals that are necessary or sufficient for these signals to be substitutes or complements. In some special cases these conditions are very simple, and straightforward to check. We also show that there are more examples of complements and substitutes, and in the case of complements also more robust examples, than those given in the previous paragraph. Finally, a sequence of applications will demonstrate the importance of complementarity and substitutability relations among signals in economic contexts.

We begin our analysis by establishing that two signals are complements (resp. substitutes) if and only if, among two other signals that are derived from the two original signals, one dominates the other in the sense of Blackwell [6], that is, is more valuable in all decision problems. This observation is key for our analysis, because it allows us to reduce the problem of determining whether two signals are complements (resp. substitutes) to the problem of determining whether among two other signals one Blackwell dominates the other. We can then use well-known characterizations of Blackwell dominance to determine whether two signals are complements (resp. substitutes).

It is well-known that Blackwell comparisons are qualitatively different in the case of two states, and in the case of three or more states, with the case of two states being easier to study (Blackwell and Girshick [7], Section 12.4). For this reason we obtain stronger results for the case of two states than for the case of more states. If we assume not only that there are only two states, but also that there are only two possible realizations, then we can provide necessary and sufficient conditions for signals to be complements or substitutes.

With two states, and two possible realizations per signal, signals are complements if and only if there are a state and a realization of each signal so that if received by themselves, each realization increases the probability of the state in comparison to the prior, yet if received together, the two signal realizations decrease the probability of the state. We refer to this as "meaning reversal:" the meaning of each realization is reversed when received together with the realization of the other signal.

An example of meaning reversal is in Dow and Gorton [9]. A technology company is observed by two analysts. One analyst learns whether the company's lead engineer is leaving the company to create an independent competitor. The other analyst learns whether the technology that the engineer is working on is likely to succeed. If the technology is likely to succeed and the engineer stays, then this is good news for the company's value. If the technology is likely to fail, and the engineer leaves, that is also good news because the company is likely to stay dominant in its market. However, the remaining cases are bad news about the company's value, because either a competitor with a promising technology is created, or because a dubious project will be continued further. The interpretation of each analyst's signal may be reversed by the other analyst's signal.

The reversal result that we have just illustrated will be shown in this paper for the setting with two states and two realizations per signal only. With two states, but more than two realizations per signal, we obtain in the appendix a related sufficient condition for signals to be complements. For the case of more than two states, or more than two realizations per signal, it is, as we show in this paper under some additional assumptions, necessary for complementarity of signals that the meaning of the realization of one signal can be reverted by a realization of the other signal. This condition is not, in general, sufficient.

Returning to the setting with two states and two realizations per signal, we prove that a property that is related to perfect correlation is necessary and sufficient for signals to be substitutes. This property is weaker than perfect correlation, however. Roughly speaking, it requires that conditional on observing certain realizations of one signal, the other signal does not provide further information to the decision maker. In the general setting, with arbitrarily many states and signal realizations, a similar condition is necessary, but not sufficient for signals to be substitutes.

While our general investigation of complements and substitutes is formulated in a setting with a single decision maker, we also show in applications where different agents observe different signals, that complementarity and substitutability of the signals may drive important properties of the agents' strategic interaction. First, we revisit the question when public disclosure of information by a seller increases the expected revenue from a first price, common value auction.[1] We demonstrate that a seller whose information is complementary to the agents' private information raises the agents' informational advantage, and therefore lowers expected revenue, by releasing his own information, whereas releasing substitute information raises expected revenue. This is similar to, but different from, a result due to Milgrom and Weber [18]. They use a notion of complementarity and substitutability that is specific to the auction model, and that is not directly related to ours. In fact, the assumptions of their model, if adapted to our auction model, rule out that signals are complements in our sense.

In a second economic application we show that the ability of weighted majority voting to efficiently aggregate information depends on whether voters' private signals are substitutes or complements. In our final application we investigate when the public disclosure of information may lead to increased polarization of agents' beliefs. Building on definitions of polarization of beliefs due to Kondor [12] and Andreoni and Mylovanov [1], we show that polarization is closely related to complementarity of public and private signals.

Section 2 provides general definitions and explains the connection between substitutability, complementarity, and Blackwell comparisons. Section 3 is about the special case that there are only two states of the world. Section 4 considers an arbitrary number of states of the world. Applications are in Section 5. Section 6 concludes. An appendix contains some results, proofs, and details left out in the main body of the paper.

[1] [17] has shown that this is the case when bidders' signals are affiliated.

2 Definitions

The state of the world is a random variable $\tilde{\omega}$ with realizations ω in a finite set Ω which has at least two elements. The probability distribution of $\tilde{\omega}$ is denoted by π. Without loss of generality we assume that each state in Ω occurs with the same probability: $\pi(\omega) = 1/|\Omega|$ for all $\omega \in \Omega$.[2] Two signals are available: $\tilde{s}_1$ with realizations s_1 in the finite set S_1 where S_1 has at least two elements, and $\tilde{s}_2$ with realizations s_2 in the finite set S_2 where S_2 also has at least two elements. We assume without loss of generality that $S_1 \cap S_2$ is empty. The joint distribution of signals $\tilde{s}_1$ and $\tilde{s}_2$ conditional on the state being $\omega \in \Omega$ is denoted by $p_{12,\omega}$. The probability assigned by this distribution to some realization $(s_1, s_2) \in S_1 \times S_2$ is denoted by $p_{12,\omega}(s_1, s_2)$. The unconditional distribution of $(\tilde{s}_1, \tilde{s}_2)$ is denoted by $\bar{p}_{12}$ and is given by: $\bar{p}_{12}(s_1, s_2) = \sum_{\omega \in \Omega} p_{12,\omega}(s_1, s_2)\pi(\omega)$ for all $(s_1, s_2) \in S_1 \times S_2$. The probability distribution on Ω conditional on observing signal realization $(s_1, s_2) \in S_1 \times S_2$ (where $\bar{p}_{12}(s_1, s_2) > 0$) is denoted by q_{s_1,s_2} and is given by:

$$q_{s_1,s_2}(\omega) = \pi(\omega)\frac{p_{12,\omega}(s_1, s_2)}{\bar{p}_{12}(s_1, s_2)} \text{ for all } \omega \in \Omega. \tag{1}$$

For $i = 1, 2$ the marginal distribution of signal $\tilde{s}_i$ conditional on the state being $\omega \in \Omega$ is denoted by $p_{i,\omega}$. The probability assigned by this distribution to some realization $s_i \in S_i$ is denoted by $p_{i,\omega}(s_i)$. For $i = 1, 2$ the unconditional distribution of $\tilde{s}_i$ is denoted by $\bar{p}_i$ and it is given by: $\bar{p}_i(s_i) = \sum_{\omega \in \Omega} p_{i,\omega}(s_i)\pi(\omega)$ for all $s_i \in S_i$. Without loss of generality we assume that $\bar{p}_i(s_i) > 0$ for all $s_i \in S_i$. For $i = 1, 2$ the probability distribution on Ω conditional on observing signal realization $s_i \in S_i$ is denoted by q_{s_i} and is given by:

$$q_{s_i}(\omega) = \pi(\omega)\frac{p_{i,\omega}(s_i)}{\bar{p}_i(s_i)} \text{ for all } \omega \in \Omega. \tag{2}$$

Our first objective is to define when the two signals are substitutes or complements. To do so, we need some auxiliary definitions.

Definition 1. *A* decision problem *is a pair (A, u) where A is some finite set of actions and u is a utility function: $u : A \times \Omega \to \mathbb{R}$.*

Definition 2. *For given decision problem (A, u):*

[2] Our results would not be different if the prior was any other distribution with support Ω. This follows from the relation between our analysis and the Blackwell comparison of signals that is pointed out in Proposition 1 below, and from the fact that the Blackwell comparison of signals is independent of the prior as long as the prior has full support.

- *The* value of not having any signal *is:*

$$V_\emptyset(A,u) \equiv \max_{a\in A} \sum_{\omega\in\Omega} \left(u(a,\omega)\pi(\omega)\right). \tag{3}$$

- *For* $i \in \{1,2\}$ *the* value of having signal $\tilde{s}_i$ alone *is:*

$$V_i(A,u) \equiv \sum_{s_i\in S_i} \bar{p}_i(s_i) \max_{a\in A} \sum_{\omega\in\Omega} (u(a,\omega)q_{s_i}(\omega)). \tag{4}$$

- *The* value of having both signals *is:*

$$V_{12}(A,u) \equiv \sum_{s_1\in S_1}\sum_{s_2\in S_2} \bar{p}_{12}(s_1,s_2) \max_{a\in A} \sum_{\omega\in\Omega} (u(a,\omega)q_{s_1,s_2}(\omega)). \tag{5}$$

We can now introduce our notions of substitutes and complements of signals:

Definition 3. *Signal* $\tilde{s}_i$ *is a* substitute *for signal* $\tilde{s}_j$ *if for all decision problems* (A,u) *we have:*

$$V_j(A,u) - V_\emptyset(A,u) \geq V_{12}(A,u) - V_i(A,u). \tag{6}$$

Definition 4. *Signal* $\tilde{s}_i$ *is a* complement *for signal* $\tilde{s}_j$ *if for all decision problems* (A,u) *we have:*

$$V_{12}(A,u) - V_i(A,u) \geq V_j(A,u) - V_\emptyset(A,u). \tag{7}$$

Note that the inequalities in Definition 3 and 4 remain true when the indices i and j are swapped. This makes clear that substitutability and complementarity are symmetric notions: if signal 1 is a substitute for signal 2, then signal 2 is a substitute for signal 1, and the same is true for complements.

For a simple interpretation of the inequalities in Definitions 3 and 4 suppose that the decision maker's not explicitly modeled *overall* utility is additive in the expected utility from decision problem (A,u) and money. Then the inequalities in Definitions 3 and 4 compare the decision maker's willingness to pay for signals in different scenarios. For example, the inequality in Definition 3 says that the willingness to pay for signal $\tilde{s}_j$ is larger if signal $\tilde{s}_i$ is not available than if it is available. It seems natural to call signals substitutes in this case.[3]

[3]Without postulating the existence of money, and additive utility, one could interpret the inequalities

We can obtain an alternative way of writing the definition of complements or substitutes by defining two auxiliary signals, $\tilde{s}_S$ and $\tilde{s}_C$. The signal $\tilde{s}_S$ can be described as follows. An unbiased coin is tossed. If "heads" comes up, the decision maker is informed about the realization of $\tilde{s}_1$. If "tails" comes up, the decision maker is informed about the realization of $\tilde{s}_2$. Therefore, for a given decision problem (A, u), the value of having signal $\tilde{s}_S$ is given by:

$$V_S(A, u) = \frac{1}{2} \cdot V_1(A, u) + \frac{1}{2} \cdot V_2(A, u). \tag{8}$$

The second auxiliary signal, $\tilde{s}_C$, is constructed as follows. An unbiased coin is tossed. If "heads" comes up, the decision maker is informed about the realizations of $\tilde{s}_1$ and $\tilde{s}_2$. If "tails" comes up, the decision maker receives no information. Therefore, for a given decision problem (A, u), the value of having signal $\tilde{s}_C$ is given by

$$V_C(A, u) = \frac{1}{2} \cdot V_{12}(A, u) + \frac{1}{2} \cdot V_\emptyset(A, u). \tag{9}$$

The following result is a simple re-writing of the definition of substitutes and complements. We omit the proof.

Proposition 1. *(i) Signals $\tilde{s}_1$ and $\tilde{s}_2$ are substitutes if and only if signal $\tilde{s}_S$* Blackwell dominates *signal $\tilde{s}_C$, i.e. in all decision problems (A, u):*

$$V_S(A, u) \geq V_C(A, u). \tag{10}$$

(ii) Signals $\tilde{s}_1$ and $\tilde{s}_2$ are complements if and only if signal $\tilde{s}_C$ Blackwell dominates *signal $\tilde{s}_S$, i.e. in all decision problems (A, u):*

$$V_C(A, u) \geq V_S(A, u). \tag{11}$$

Blackwell and Girshick [7, Theorem 12.2.2.] offer a variety of necessary and sufficient conditions for Blackwell dominance. Proposition 1 allows one to use those conditions to characterize substitutes and complements.

We shall say that signal $\tilde{s}_i$ is "informative" if there is at least one $s_i \in S_i$ such that

in Definitions 3 and 4 using an idea in [22, p. 18]. They argue that inequalities that involve differences of von Neumann Morgenstern utilities reflect differences in the *intensity* of a preference. For example, in the case of Definition 3, this interpretation says that the preference for having signal $\tilde{s}_1$ over not having signal $\tilde{s}_1$ is *more intense* when signal $\tilde{s}_2$ is not present than when it is present. This interpretation of the difference of von Neumann Morgenstern utilities is not universally accepted, however. [15, p. 32] regards this interpretation as a fallacy, whereas [5, p. 67] is sympathetic to this interpretation.

$q_{s_i} \neq \pi$. It is obvious that signals are complements if at least one of the signals is uninformative. In the rest of the paper we shall assume that both signals are informative.

3 The Case of Two States

It is easier to verify Blackwell dominance when there are only two states of the world, and therefore beliefs are one-dimensional, than when there are more than two states of the world, and therefore beliefs are multi-dimensional. The qualitative difference between the one-dimensional case and the case of two or more dimensions is explained in Section 12.4 of Blackwell and Girshick [7]. In the one-dimensional case the convex value function[4] arising from an arbitrary decision problem can be approximated arbitrarily closely by linear combinations of a very simple subclass of piecewise linear, convex functions. No such approximation result is known in the two or more-dimensional case. The relevance of having a dense class of simple value functions is that one can correspondingly restrict attention to a simple class of decision problems when checking Blackwell dominance. A suitable class of decision problems is the set of all two action decision problems where $A = \{T, B\}$ and u is given by Figure 1.

	$\omega = a$	$\omega = b$
T	0	x
B	$1 - x$	0

Figure 1: A two action decision problem

Lemma 1. *In the two states model, signals are complements (substitutes) if and only if they are complements (substitutes) in all two action decision problems given by Figure 1 with* $x \in (0, 1)$.

Proof. The main argument in the proof of Theorem 12.4.1. in Blackwell and Girshick [7] demonstrates that in the two states case a signal $\tilde{s}$ Blackwell dominates another signal $\tilde{s}'$ if and only if $\tilde{s}$ is more valuable than $\tilde{s}'$ in all two action problems of the form shown in Figure 1.[5] We can then apply Proposition 1 to infer Lemma 1. □

[4] Value functions map posterior beliefs into the expected utility that the decision maker obtains when holding those beliefs and choosing optimally. Every decision problem gives rise to a convex value function.

[5] Blackwell and Girshick's proof refers to a decision problem that is as in Figure 1 but with the first

This result motivates our focus in this section on the case of only two states. We also assume that each signal has only two realizations. Without loss of generality we assume that observing α or $\hat{\alpha}$ (resp. β or $\hat{\beta}$) alone raises the decision maker's belief that the state is a (resp. b): $q_\alpha(a) > \pi(a)$ and $q_{\hat{\alpha}}(a) > \pi(a)$. We refer to the model with two states and two realizations per signal if it satisfies this assumption as the "binary-binary" model.

3.1 Substitutes

Proposition 2. *In the binary-binary model, signals are substitutes if and only if the joint realizations $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ each have strictly positive prior probability, and*

$$q_{\alpha,\hat{\alpha}}(a) = \max\{q_\alpha(a), q_{\hat{\alpha}}(a)\}, \text{ and} \tag{12}$$

$$q_{\beta,\hat{\beta}}(b) = \max\{q_\beta(b), q_{\hat{\beta}}(b)\}. \tag{13}$$

Call a realization of a single signal "extreme" if it provides the strongest evidence for state a, or state b, among all four individual signal realizations. The conditions in Proposition 2 say that conditional on an extreme realization of a signal the other signal is not informative. Thus, in the binary-binary model, substitutability amounts to a form of conditional uninformativeness of signals.

Signal distributions that satisfy the conditions of Proposition 2 can be classified into two types. For signal distributions of the first type the two extreme realizations are different realizations of the same signal, whereas for signal distributions of the second type, the two extreme realizations are realizations of two different signals. We illustrate these two types in Figure 2. For each example, we provide two tables which display the two conditional distributions $p_{12,a}$ and $p_{12,b}$. Rows correspond to realizations of signal $\tilde{s}_1$, and columns correspond to realizations of signal $\tilde{s}_2$.

Example 1 illustrates the first type. We show the case in which both extreme signal realizations come from signal $\tilde{s}_1$. It then has to be the case that, conditional on the realization of signal $\tilde{s}_1$, signal $\tilde{s}_2$ is always not informative. This happens if conditional on any realization of signal $\tilde{s}_1$, the likelihood ratios of joint signal realizations are the same for all realizations of signal $\tilde{s}_2$. The corresponding information structure is displayed in

row of payoffs replaced by $(-(1-x), x)$, where $x \in (0,1)$, and the second row of payoffs replaced by $(0,0)$. The same argument that Blackwell and Girshick use can be used to demonstrate that a signal $\tilde{s}$ Blackwell dominates another signal $\tilde{s}'$ if and only if $\tilde{s}$ is more valuable than $\tilde{s}'$ in all two action problems of the form shown in Figure 1.

	$\hat{\alpha}$	$\hat{\beta}$
α	ρ	φ
β	$\mu\varphi'$	$\mu\rho'$

$\omega = a$

	$\hat{\alpha}$	$\hat{\beta}$
α	$\eta\rho$	$\eta\varphi$
β	φ'	ρ'

$\omega = b$

Example 1 (α and β are extreme signal realizations)

	$\hat{\alpha}$	$\hat{\beta}$
α	ρ	0
β	φ	$1-\rho-\varphi$

$\omega = a$

	$\hat{\alpha}$	$\hat{\beta}$
α	ρ'	0
β	φ'	$1-\rho'-\varphi'$

$\omega = b$

Example 2 (α and $\hat{\beta}$ are extreme signal realizations)

Figure 2: Two different types of substitutes

Example 1 where the likelihood ratios are denoted by η and μ which are both less than 1.[6] Note that the case of perfect correlation is a special case of Example 1 where $\varphi = \varphi' = 0$, $\rho = \rho'$ and $\mu\rho' = \eta\rho = 1 - \rho$.

Example 2 illustrates the second type of signal distributions that make signals substitutes. In this type, the two extreme realizations come from different signals. We show the case in which α and $\hat{\beta}$ are the extreme realizations. In this case, signals are substitutes if and only if signal $\tilde{s}_1$ is not informative conditional on $\hat{\beta}$, and signal $\tilde{s}_2$ is not informative conditional on α. It is not hard to see that this is equivalent to the realization $(\alpha, \hat{\beta})$ having zero probability in both states. Accordingly, the information structure is of the form shown in Example 2.[7] Note that perfect correlation is also a special case of Example 2, when $\varphi = \varphi' = 0$ and $\rho' = 1 - \rho$.

We prove the sufficiency of the conditions in Proposition 2 in the appendix. The proof is by calculation, using the fact that according to Lemma 1 we can restrict attention to

[6] Of course, the entries in each table in Example 1 have to sum to one. Moreover, since $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ occur with positive probability, we have $\rho, \rho' > 0$ while $\varphi, \varphi' \geq 0$. Finally, to satisfy our assumption that $\hat{\alpha}$ indicates state a, we need that $\rho + \mu\varphi' \geq \eta\rho + \varphi'$.

[7] In accordance with Proposition 2 we need $\rho, \rho' > 0$ and $\varphi, \varphi' \geq 0$. To satisfy our assumption that α and $\hat{\alpha}$ indicate state a, we need that $\rho \geq \rho'$ and $\rho + \varphi \geq \rho' + \varphi'$. To ensure that α is the strongest signal for state a we need: $\rho\varphi' \geq \rho'\varphi$, and finally, to ensure that $\hat{\beta}$ is the strongest signal for state b we need: $(1-\rho)\varphi' \leq (1-\rho')\varphi$.

the decision problems in Figure 1. The necessity of the conditions in Proposition 2 follows from the more general Corollary 1 below.

3.2 Complements

Proposition 3. *In the binary-binary model, signals are complements if and only if the joint realizations* $(\alpha, \hat{\alpha})$ *and* $(\beta, \hat{\beta})$ *each have strictly positive probability in at least one state, and one of the following conditions holds:*[8]

$$q_{\alpha,\hat{\alpha}}(a) \leq \pi(a), \qquad \textit{or} \tag{14}$$

$$q_{\beta,\hat{\beta}}(b) \leq \pi(b). \tag{15}$$

Inequality (14) says that if the decision maker receives signal $(\alpha, \hat{\alpha})$ the decision maker's posterior probability of state a is less than or equal to the prior $\pi(a)$, even though individually both α and $\hat{\alpha}$ move the decision maker's probability of state a above $\pi(a)$. Inequality (15) is the analogous condition for the signal realization $(\beta, \hat{\beta})$. In both cases, two signals which by themselves move the decision maker's beliefs into one direction, if received together move the decision maker's beliefs into the opposite direction. The "meaning" of these signals is reversed if they are received together.

We prove the sufficiency of the conditions in Proposition 3 in the appendix. We derive the necessity in the next section from a more general result. Example 3 shows a class of complements. If $\nu > \mu$, the signal realizations α and $\hat{\alpha}$ by themselves raise the decision maker's belief that the true state is a. If $\rho \leq \varphi$, then the joint signal realization $(\alpha, \hat{\alpha})$, by contrast, reduces the decision maker's probability that the true state is a or leaves it unchanged.[9]

In the appendix, we generalize the sufficiency part of Proposition 3 to the case of more than two signal realizations.

Remark 1. *Among all pairs of conditional joint distributions of signals* $\tilde{s}_1$ *and* $\tilde{s}_2$ *in the binary-binary model the ones shown in Figure 2 are rare. One way of saying this formally*

[8] One can show that the two conditions are mutually exclusive.

[9] Example 3 captures all conditional joint probability distributions of the two signals in the binary-binary model for which condition (14) holds, and for which in each state the probabilities of the two signal realizations $(\alpha, \hat{\beta})$ and $(\beta, \hat{\alpha})$ are the same. (There are, of course, other conditional joint distributions of the two signals for which signals are complements.) All suitable values for the four parameters in Example 3 can be found by making choices allowed in the following procedure: First pick ν such that $0 < \nu < 1$. Then pick $\mu > 0$ such that $2\nu - 1 \leq \mu < \nu$. Then pick $\varphi \geq 0$ such that $2\nu - 1 \leq \varphi \leq \mu$. Finally, pick $\rho \geq 0$ such that $2\nu - 1 \leq \rho \leq \varphi$.

	$\widehat{\alpha}$	$\widehat{\beta}$
α	ρ	$\nu - \rho$
β	$\nu - \rho$	$1 + \rho - 2\nu$

$\omega = a$

	$\widehat{\alpha}$	$\widehat{\beta}$
α	φ	$\mu - \varphi$
β	$\mu - \varphi$	$1 + \varphi - 2\mu$

$\omega = b$

Figure 3: Example 3 (signals are complements)

is to identify pairs of conditional joint distributions of the two signals with vectors in 8-dimensional Euclidean space, and to endow the set of all joint distributions with the relative Euclidean topology. The set of distributions that are not *like the distributions in Figure 2 is then an open and dense subset of the set of all joint distributions, and is thus generic. This may seem intuitively plausible given how stringent the requirement that defines substitutes is. However, in the same topological sense, complements, although their definition seems equally stringent, are* not *rare. The set of distributions that correspond to complements has an open subset. For example, a small open ball around a pair of full support distributions that satisfy one of the conditions in Proposition 3 as a strict inequality*[10] *is a subset of the set of all distributions that correspond to complements. The stringency of the requirement that defines complements makes this observation intuitively surprising.*

4 The General Case

The main results of this section show that the conditions that are necessary and sufficient for substitutes or complements in the binary-binary model are necessary, but not sufficient, for substitutes or complements in the general model.

4.1 Substitutes

We showed in the previous section that in the binary-binary model a necessary and sufficient condition for substitutes is that a signal is not informative conditional on the other signal having a realization that induces extreme posteriors. We now show that a similar condition is in general necessary for substitutes. An example in the appendix shows that the condition is not sufficient. For any subset C of a finite-dimensional Euclidean space

[10] With a suitable choice of parameters in Example 3, condition (14) holds as a strict inequality.

we denote by "*co* C" the convex hull of C.

Proposition 4. *If signals are substitutes, then for every* $(s_1, s_2) \in S_1 \times S_2$ *such that* $\bar{p}_{12}(s_1, s_2) > 0$*:*

$$q_{s_1,s_2} \in co\,\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}\,. \tag{16}$$

Proof. By part (i) of Proposition 1, if signals are substitutes, $\tilde{s}_S$ Blackwell dominates $\tilde{s}_C$. By condition (5) of Theorem 12.2.2. in Blackwell and Girshick [7] this means that the posteriors after observing $\tilde{s}_S$ are a mean-preserving spread of the posteriors after observing $\tilde{s}_C$. Therefore, the posteriors after observing $\tilde{s}_C$ are contained in the convex hull of the posteriors after observing $\tilde{s}_S$. This implies Proposition 4. □

Recall that an element of a convex set C is called an "extreme point" of C if it is not a convex combination of at least two different elements of C where each of these elements has strictly positive weight.

Corollary 1. *Suppose signals are substitutes. If for some* $i \in \{1,2\}$ *and some* $s_i^* \in S_i$ *the vector* $q_{s_i^*}$ *is an extreme point of* $co\,\{q_{s_k} | k \in \{1,2\}, s_k \in S_k\}$*, then signal* $\tilde{s}_j$ *(where* $j \neq i$*) is not informative conditional on signal realization* s_i^**.*

Corollary 1 is a generalization of the necessity part of Proposition 2. An example in the appendix shows that the condition in Corollary 1 is in general not sufficient for substitutes, in contrast to the case of Proposition 2.

Proof. Indirect. Suppose $q_{s_i^*,s_j} \neq q_{s_i^*}$ for some $s_j \in S_j$ with $\bar{p}_{12}(s_i^*, s_j) > 0$. By standard properties of posteriors $q_{s_i^*}$ can be written as a convex combination of the vectors $q_{s_i^*,s_j}$ ($s_j \in S_j$). We can infer that $q_{s_i^*,s_j} \neq q_{s_i^*}$ for at least two $s_j \in S_j$ with $\bar{p}_{12}(s_i^*, s_j) > 0$, and that both of these vectors $q_{s_i^*,s_j}$ receive positive weight in the convex combination that makes up $q_{s_i^*}$. By Proposition 4 for every $s_j \in S_j$ with $\bar{p}_{12}(s_i^*, s_j) > 0$ the vector $q_{s_i^*,s_j}$ is an element of $co\,\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$. We have thus inferred that $q_{s_i^*}$ can be written as the convex combination of at least two different elements of $co\,\{q_{s_i} | i \in \{1,2\}\, s_i \in S_i\}$ where each element receives positive weight. This contradicts our assumption that $q_{s_i^*}$ is an extreme point of $co\,\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$. □

One can use Corollary 1 to prove the necessity of the condition in Proposition 2. Indeed, the necessity of the condition in Proposition 2 is an immediate consequence of Corollary 1 once one shows that the signal realizations $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ have strictly positive probability in some state. We omit the elementary proof of this.

4.2 Complements

Earlier we showed that a form of "meaning reversal" is necessary and sufficient for signals to be complements in the binary-binary example. The next result shows that with more than two states or more than two signal realizations, under an additional assumption, meaning reversal is necessary for complements. An example in the appendix shows that the condition is not sufficient. The following result looks formidable. We explain it in the text that follows the result. The proof of the result is in the appendix.

Proposition 5. *Suppose signals are complements. Consider any $r \in \mathbb{R}^{|\Omega|}$. Define $e \equiv r\pi$. If for each $i \in \{1,2\}$ there is a partition (S_i^E, S_i^R) of S_i such that the following three conditions are satisfied:*

(i) For each $i \in \{1,2\}$:

$$e \geq rq_{s_i} \text{ for all } s_i \in S_i^E \text{ and } e > rq_{s_i} \text{ for at least one } s_i \in S_i^E, \tag{17}$$

and

$$rq_{s_i} \geq e \text{ for all } s_i \in S_i^R \text{ and } rq_{s_i} > e \text{ for at least one } s_i \in S_i^R; \tag{18}$$

(ii) For each $k \in \{E, R\}$ there is at least one $(s_1, s_2) \in S_1^k \times S_2^k$ such that

$$\bar{p}_{12}(s_1, s_2) > 0; \tag{19}$$

(iii) For each $(k, \ell) \in \{(E, R), (R, E)\}$:

$$e \geq rq_{s_1,s_2} \text{ for all } (s_1, s_2) \in S_1^k \times S_2^\ell \text{ with } \bar{p}_{12}(s_1, s_2) > 0, \tag{20}$$

or

$$rq_{s_1,s_2} \geq e \text{ for all } (s_1, s_2) \in S_1^k \times S_2^\ell \text{ with } \bar{p}_{12}(s_1, s_2) > 0; \tag{21}$$

then

$$rq_{s_1,s_2} \geq e \text{ for some } (s_1, s_2) \in S_1^E \times S_2^E \text{ with } \bar{p}_{12}(s_1, s_2) > 0, \tag{22}$$

or

$$e \geq rq_{s_1,s_2} \text{ for some } (s_1, s_2) \in S_1^R \times S_2^R \text{ with } \bar{p}_{12}(s_1, s_2) > 0. \tag{23}$$

Lines (22) and (23) show that a form of meaning reversal is necessary for complementarity. To interpret the result suppose the decision maker wants to learn from the signals whether the expected utility of a risky action R whose payoffs are given by the vector r is larger or smaller than the expected utility from a safe action E that yields payoff e in all states. Assume that r and e are such that with the prior belief π the decision maker is indifferent between the two actions. We denote the set of realizations of signal $\tilde{s}_i$ which imply a posterior belief for which action E has higher expected utility than action R by S_i^E, and we denote the set of realizations of signal $\tilde{s}_i$ which imply a posterior belief for which action R has higher expected utility than action E by S_i^R. Beliefs for which the decision maker is indifferent can be assigned arbitrarily to one of these two sets.

Signal realizations in S_i^E by themselves indicate that the expected value rq_{s_i} is not larger than e. But according to (22) for some joint realization where both realizations are in S_i^E we have (almost[11]) the reverse: $rq_{s_1,s_2} \geq e$. In the same way, (23) is a form of meaning reversal. At least one of these two meaning reversals must occur according to Proposition 5.

Note, however, that the meaning reversal is necessary only if conditions (19), (20) and (21) hold. Among these, (19) is a mild regularity condition. The remaining two conditions are more restrictive. They refer to the case that the decision maker receives "mixed messages" from the two signal. There are two possible types of mixed messages: the first type is when s_1 is in S_1^E but s_2 is in S_2^R; the second type is when s_1 is in S_1^R but s_2 is in S_2^E. The conditions require that for each of the two types of mixed signals one can say unambiguously which signal is "stronger," irrespective of the specific realization of the signals. Thus either for all mixed realizations of the first type the expected value of action E is at least as large as that of action R, and hence signal $\tilde{s}_1$ is stronger, or for all mixed realizations of the first type the expected value of action R is at least as large as that of action E, and hence signal $\tilde{s}_2$ is stronger. An analogous condition needs to hold for all mixed realizations of the second type, but it is not necessary that the same signal is stronger for mixed realizations of both types.

In the appendix we show how to use Proposition 5 to derive the necessity part of Proposition 3.

5 Applications

We present three applications of our analysis. These applications will be models of asymmetric information, in which two signals that are either complements or substitutes are

[11]Ignoring the possibility of indifference.

observed by two different agents, who then interact in a game. We focus on the binary-binary model, and use the notation from that model. For each application we also briefly explain what we know about possible generalizations of our results to the case of more than two states and signal realizations.

5.1 Information Disclosure in Auctions

Milgrom and Weber [17, 18] have presented models of common value auctions in which an auctioneer can increase her expected revenue by publicly disclosing her own information about the object that she is selling. In Milgrom and Weber [18] it is argued that this result always holds if the auctioneer's information is in a substitute relation with the information that bidders hold privately, but that it need not hold if the auctioneer's information is complementary to bidders' private information. The intuition for this is as follows: in common value auctions, bidders can gain expected surplus if they hold private information that, if shared with other bidders, would change those bidders' expected values of the object. By publicly disclosing information that is in a substitute relation with some bidders' private information, the auctioneer reduces these bidders' informational advantage, and thus increases her expected revenue. By publicly disclosing information that is a complement to some bidders' private information, the auctioneer increases these bidders' information advantage, and thus reduces her expected revenue.

Milgrom and Weber's [18] notion of complements and substitutes for signals is tailored to their particular model. In this subsection we re-consider Milgrom and Weber's result using the definitions of complements and substitutes developed in this paper. Note that our definitions are not directly related to Milgrom and Weber's. Indeed, their definitions are only meaningful under the assumption that the expected value of the object conditional on both signals, and conditional on the bidder's private signal alone, are both increasing in the realization of the bidder's private signal. One can easily deduce from our Proposition 3 that this assumption, if adapted to our auction model, rules out complements in our sense.

As we mentioned above, we focus on the binary-binary model. We assume that the state indicates the common value of the good. Without loss of generality, we assume that the common value is one if the state is a, and zero if the state is b. Bidder 1 observes the realization of $\tilde{s}_1$, bidder 2 has no private information, and the auctioneer observes the realization of $\tilde{s}_2$. The auction is a first price auction with minimum bid zero. The winner's utility is the difference between the common value of the good and the winner's bid, and the loser's utility is zero. We study how the auctioneer's expected revenue varies if the auctioneer commits to disclosing the realization of signal $\tilde{s}_2$ publicly to both bidders before bidders submit their bids.

In the appendix we construct equilibria in mixed strategies of the auction without and with information disclosure. Conditional on public information, the expected utility of the bidder without private information is zero, and the bidder with private information has zero expected utility when observing a signal realization that makes him less optimistic about the value of the good. Otherwise, his expected utility is equal to the difference between the expected value conditional on all information and the expected value conditional on the public information. One can derive the auctioneer's expected revenue from the observation that it is equal to the difference between the unconditional expected value of the good and the sum of the bidders' expected utilities. Without public disclosure of information, expected revenue is given by:

$$\pi(a) - \bar{p}(\alpha)\left(q_\alpha(a) - \pi(a)\right). \tag{24}$$

With public disclosure of information the auctioneer's expected revenue is:

$$\pi(a) - \bar{p}(\bar{\sigma}(\hat{\alpha}), \hat{\alpha})(q_{\bar{\sigma}(\hat{\alpha}),\hat{\alpha}}(a) - q_{\hat{\alpha}}(a)) - \bar{p}(\bar{\sigma}(\hat{\beta}), \hat{\beta})(q_{\bar{\sigma}(\hat{\beta}),\hat{\beta}}(a) - q_{\hat{\beta}}(a)), \tag{25}$$

where for every realization s_2 of $\tilde{s}_2$ we define:

$$\bar{\sigma}(s_2) = \arg \max_{\{s_1 : \bar{p}(s_1, s_2) > 0\}} q_{s_1,s_2}(a). \tag{26}$$

Using these expressions, we can prove:

Proposition 6. *If signals are complements, information disclosure does not increase the auctioneer's expected revenue. If signals are substitutes, information disclosure does not decrease the auctioneer's expected revenue.*

Proof. Suppose signals are complements. We want to show that (25) is not more than (24). Since the prior is a convex combination of the posteriors, each of the two terms subtracted in (25) is non-negative by definition of $\bar{\sigma}(s_2)$. Thus, it is sufficient to show that one of the two terms subtracted in (25) is at least as large as the term subtracted in

(24):

$$\bar{p}(\bar{\sigma}(\hat{\alpha}),\hat{\alpha})(q_{\bar{\sigma}(\hat{\alpha}),\hat{\alpha}}(a)-q_{\hat{\alpha}}(a)) \geq \bar{p}(\alpha)(q_\alpha(a)-\pi(a)), \quad \text{or} \tag{27}$$

$$\bar{p}(\bar{\sigma}(\hat{\beta}),\hat{\beta})(q_{\bar{\sigma}(\hat{\beta}),\hat{\beta}}(a)-q_{\hat{\beta}}(a)) \geq \bar{p}(\alpha)(q_\alpha(a)-\pi(a)). \tag{28}$$

Now suppose that inequality (14) holds. We show (28). Note that (14) implies that $\bar{\sigma}(\hat{\beta})=\alpha$ because the prior is a convex combination of the posteriors and $q_\alpha(a)>\pi(a)$. Thus, we can show (28) as follows:

$$\begin{aligned}
\bar{p}(\alpha,\hat{\beta})(q_{\alpha,\hat{\beta}}(a)-q_{\hat{\beta}}(a)) &= \bar{p}(\alpha,\hat{\beta})(q_{\alpha,\hat{\beta}}(a)-q_\alpha(a))+\bar{p}(\alpha,\hat{\beta})(q_\alpha(a)-q_{\hat{\beta}}(a)) \\
&= \bar{p}(\alpha,\hat{\alpha})(q_\alpha(a)-q_{\alpha,\hat{\alpha}}(a))+\bar{p}(\alpha,\hat{\beta})(q_\alpha(a)-q_{\hat{\beta}}(a)) \\
&\geq \bar{p}(\alpha,\hat{\alpha})(q_\alpha(a)-\pi(a))+\bar{p}(\alpha,\hat{\beta})(q_\alpha(a))-\pi(a)) \\
&= \bar{p}(\alpha)(q_\alpha(a)-\pi(a)).
\end{aligned} \tag{29}$$

The equalities in the first and fourth line are obvious. The first term in the sum in the first line is the same as the first term in the sum in the second line because $\bar{p}(\alpha,\hat{\beta})(q_{\alpha,\hat{\beta}}(a)-q_\alpha(a)=\bar{p}(\alpha,\hat{\alpha})(q_\alpha(a)-q_{\alpha,\hat{\alpha}}(a))$ is equivalent to: $\bar{p}(\alpha)q_\alpha(a)=\bar{p}(\alpha,\hat{\alpha})q_{\alpha,\hat{\alpha}}(a)+\bar{p}(\alpha,\hat{\beta})q_{\alpha,\hat{\beta}}(a)$, which is true by the equality of the expected posterior and the prior. The inequality in the third line follows from (14) and the assumption $q_{\hat{\beta}}(a)\leq\pi(a)$. This proves the claim when (14) holds.

If inequality (15) holds, we can show (27). Indeed, one can show using (twice) that the prior is a convex combination of the posteriors and using that $q_\beta(a)<\pi(a)<q_{\hat{\alpha}}(a)$ that (15) implies that $\bar{\sigma}(\hat{\alpha})=\alpha$. Hence, by using again the equality of the expected posterior and the prior, we can write the left-hand side of (27) as:

$$\bar{p}(\alpha,\hat{\alpha})\left(q_{\alpha,\hat{\alpha}}(a)-q_{\hat{\alpha}}(a)\right)=\bar{p}(\beta,\hat{\alpha})\left(q_{\hat{\alpha}}(a)-q_{\beta,\hat{\alpha}}(a)\right)=\bar{p}(\beta,\hat{\alpha})\left(q_{\beta,\hat{\alpha}}(b)-q_{\hat{\alpha}}(b)\right). \tag{30}$$

By a similar argument, the right-hand side of (27) is equal to $\bar{p}(\beta)\left(q_\beta(b)-\pi(b)\right)$. Thus, (27) is equivalent to:

$$\bar{p}(\beta,\hat{\alpha})\left(q_{\beta,\hat{\alpha}}(b)-q_{\hat{\alpha}}(b)\right)\geq\bar{p}(\beta)\left(q_\beta(b)-\pi(b)\right). \tag{31}$$

To establish (31), we can now proceed in the same way as we did in (29), with α being replaced by β, and $\hat{\alpha}$ being replaced by $\hat{\beta}$.

The proof for the case that signals are substitutes is in the appendix. □

The difficulty in extending Proposition 6 beyond the binary-binary case is that the expected value comparisons in Proposition 6 require knowledge of the complete distribution of posteriors induced by the joint signal realizations. However, the general necessary conditions for substitutes and complements for the case with an arbitrary number of states and signals realizations as stated in Propositions 4, Corollary 1, and Proposition 5 only imply restrictions on a small subset of posteriors. Therefore, the necessary conditions developed in the previous section are too weak to extend Proposition 6 to the general case.

5.2 Information Aggregation Through Voting[12]

Weighted majority voting over two alternatives has strong efficiency properties when the role of voting is to aggregate different voters' preferences and voters' types are independent (Schmitz and Tröger [21], Azreli and Kim [4]). When voters have identical preferences, and voting serves the purpose of information aggregation, on the other hand, it is known that weighted majority voting may fail to produce efficient information aggregation. By "efficient information aggregation" we mean that the best decision is made given the combined information of all voters.[13] One reason why weighted majority voting may fail to efficiently aggregate information is that the weights assigned to different voters and the majority requirement may be wrong. The weights and majority requirement that are suitable for efficient information aggregation are very sensitive to the environment. When different weights or majority requirements are used, efficient information aggregation may fail, as exemplified by the voting equilibrium described in Lemma 1 in Austen-Smith [3]. In this section we show another way in which weighted majority voting may fail to aggregate information efficiently: when different voters hold complementary private information, there need not be *any* weights and majority requirement that allow efficient information aggregation. The intuition for this finding is that an additive method for aggregating votes does not allow a vote to reverse meaning, which is required for efficient information aggregation if signals are complements, and hence sometimes reverse meaning.[14]

[12] The example in this subsection is based on Lemma 1 in [8]. We thank Kata Bognar for allowing us to describe her result here.

[13] Note that we are concerned here with the efficiency of information aggregation for a fixed and finite number of voters. There is also a literature (e.g. [10]) investigating the efficiency of weighted majority voting in aggregating information when the number of voters tends to infinity.

[14] [20] considers a model of common value voting with endogenous information acquisition, and shows how the voting rule may induce the signals of different voters to be complementary.

We consider a model with two agents: $i \in \{1,2\}$. Agent i privately observes the realization of signal $\tilde{s}_i$. The agents have to choose one from a set of two candidates: $A = \{\mathcal{A}, \mathcal{B}\}$. Both agents have the same utility. The utility from choosing each of the two candidates depends on the state in the way indicated in the table below:

	$\omega = a$	$\omega = b$
$\mathcal{A}$	x	0
$\mathcal{B}$	0	$1-x$

where $x \in (0,1)$.

The two agents play the following game: each agent is allocated a weight $w_i \in (0,1)$. The sum of the weights is: $w_1 + w_2 = 1$. Agents simultaneously and independently choose how to allocate the weight between the two candidates. Denote by $w_i^{\mathcal{A}} \geq 0$ the weight that agent i assigns to candidate $\mathcal{A}$ and denote by $w_i^{\mathcal{B}} \geq 0$ the weight that agent i assigns to candidate $\mathcal{B}$. We require: $w_i^{\mathcal{A}} + w_i^{\mathcal{B}} = w_i$ for $i \in \{1,2\}$. The candidate for whom the sum of the assigned weights is largest wins. In case of a tie, one of the two candidates is chosen randomly. A pure strategy for agent 1 is a function $\sigma_1 : \{\alpha, \beta\} \to [0, w_1]$, where $\sigma_1(s_1)$ is the weight that agent 1 assigns to candidate $\mathcal{A}$ if observing signal realization s_1. Analogously, a pure strategy for agent 2 is a function $\sigma_2 : \{\hat{\alpha}, \hat{\beta}\} \to [0, w_2]$.[15]

Definition 5. *The utility function parameterized by x, and the joint distribution of signals $p_{12,a}, p_{12,b}$* allow efficient information aggregation through voting *if and only if there is a pair of pure strategies (σ_1, σ_2) such that*

$$q_{\alpha,\hat{\alpha}}(a) > 1 - x \;\Rightarrow\; \sigma_1(\alpha) + \sigma_2(\hat{\alpha}) > 0.5, \tag{32}$$

$$q_{\alpha,\hat{\alpha}}(a) < 1 - x \;\Rightarrow\; \sigma_1(\alpha) + \sigma_2(\hat{\alpha}) < 0.5, \tag{33}$$

and analogous conditions hold for the three other combinations of signal realizations.

Note that any strategy pair that has the properties in Definition 5 leads to expected utility maximizing choices conditional on all signal realizations, and is therefore a Bayesian Nash equilibrium of the voting game (McLennan [16]). There may be other Bayesian Nash equilibria of this game, but we shall ignore them.

[15] We allow voters to divide their weight among candidates. Thus, implicitly, we allow abstention.

Proposition 7. *(i) If signals are complements, then there exists $x \in (0,1)$ such that the utility function parameterized by x and the joint distribution of signals do not allow efficient information aggregation through voting.*

(ii) If signals are substitutes, then for all $x \in (0,1)$ the utility function parameterized by x and the joint distribution of signals allow efficient information aggregation through voting.

Proof. (i) Suppose that signals are complements, and that inequality (14) holds (the case in which (15) holds is analogous.) That the prior is a convex combination of the posteriors means that (14), $q_\alpha(a) > \pi(a)$, and $q_{\hat{\alpha}}(a) > \pi(a)$ imply that $q_{\alpha,\hat{\beta}}(a) > \pi(a)$ and $q_{\beta,\hat{\alpha}}(a) > \pi(a)$. These inequalities, together with $q_\beta(a) < \pi(a)$ imply that $q_{\beta,\hat{\beta}}(a) < \pi(a)$. Now pick $x < 1 - \pi(a)$. Then, for efficient information aggregation through voting to be possible, we have to have:

$$\sigma_1(\alpha) + \sigma_2(\hat{\alpha}) < 0.5 \text{ (because } q_{\alpha,\hat{\alpha}}(a) < 1 - x\text{);} \tag{34}$$

$$\sigma_1(\beta) + \sigma_2(\hat{\beta}) < 0.5 \text{ (because } q_{\beta,\hat{\beta}}(a) < 1 - x\text{).} \tag{35}$$

Adding these two inequalities, we obtain:

$$\sigma_1(\alpha) + \sigma_2(\hat{\alpha}) + \sigma_1(\beta) + \sigma_2(\hat{\beta}) < 1. \tag{36}$$

This implies that at least one of the sums $\sigma_1(\alpha)+\sigma_2(\hat{\beta})$ and $\sigma_1(\beta)+\sigma_2(\hat{\alpha})$ must be strictly less than 0.5, which contradicts efficient information aggregation, as $q_{\alpha,\hat{\beta}}(a) > 1 - x$ and $q_{\beta,\hat{\alpha}}(a) > 1 - x$ if x is sufficiently close to $1 - \pi(a)$.

(ii) Suppose that signals are substitutes. If the two extreme realizations referred to in Proposition 2 are realizations of the same signal $\tilde{\sigma}_i$, then efficient information aggregation can be achieved by the pair of strategies where voter i allocates all weight w_i to candidate $\mathcal{A}$ if i observes the signal realization α (resp. $\hat{\alpha}$) and to candidate $\mathcal{B}$ if i observes the signal realization β (resp. $\hat{\beta}$). Voter $j \neq i$ allocates $w_j/2$ votes to each of the two candidates.

Next consider the case that the two extreme realizations referred to in Proposition 2 are realizations of two different signals. Without loss of generality assume that we are in the case of Example 2. If the optimal choice conditional on the two extreme signal realizations is the same, then voting weights can be allocated arbitrarily. Now suppose that the optimal candidate conditional on α is $\mathcal{A}$, whereas conditional on $\hat{\beta}$ it is $\mathcal{B}$. If the optimal candidate conditional on $(\beta, \hat{\alpha})$ is $\mathcal{A}$, then only voter 2's signal matters for the optimal decision. If the optimal candidate conditional on $(\beta, \hat{\alpha})$ is $\mathcal{B}$, then only voter 1's signal matters for the optimal decision. Let i denote the voter whose signal realization determines the optimal decision. By adopting the strategy described in the previous paragraph, information can be efficiently aggregated through voting. $\square$

The impossibility of information aggregation in the complementarity case results from the simultaneity of voting. With sequential voting, the first voter can always use her vote to signal her observation, but at the same time leave the decision to the second voter. With more than two states and signal realizations part (i) of Proposition 7 can be shown using the same proof as above if one can find a partition of signal realizations that satisfies the conditions in Proposition 5. By contrast, the proof of part (ii) of Proposition 7 does not apply to the case of more than two states and signal realizations, because general necessary conditions for substitutes as stated in Proposition 4 and Corollary 1 imply restrictions only on the joint signal realizations that induce extreme beliefs, but do not restrict the ordering of beliefs which are not extreme. Therefore, we do not know whether Proposition 7 remains true in that case.

5.3 Polarization and Disagreement[16]

In this section we show that complementarity of signals is closely related to the phenomenon that two agents whose initial beliefs disagree, may come to disagree even more when confronted with additional public evidence. In an influential psychological study, Lord et al. [14] show that experimental subjects' attitudes towards the deterrent efficacy of capital punishment become more polarized when subjects are exposed to the same, mixed empirical evidence on deterrent effects. Lord et al. interpret their findings as evidence for biased information processing. In contrast, Kondor [12] and Andreoni and Mylovanov [1] have recently described information structures where increased polarization following the public disclosure of additional evidence is consistent with Bayesian updating.[17] In these authors' models agents hold opposing private information about how to interpret the public signal. For example, they may disagree about how reliable or neutral the public evidence is. This means that the agents' private and the public information are complements in our sense.

Confining ourselves again to the case of the binary-binary model, we assume that there are two agents who privately each observe a realization of a conditionally independent

[16]We thank Christian Hellwig for pointing us to the example discussed in this subsection.
[17]For other approaches that rationalize belief disagreement, see the references in [1].

version of signal 1, potentially inducing different private beliefs about the state. Then, the realization of signal 2 is publicly disclosed, leading decision makers to update their private beliefs.

Now suppose one agent has observed realization α of signal 1, raising his belief that the state is a; and the other agent has observed realization β (of a conditionally independent version) of signal 1, decreasing his belief that the state is a. Thus, the agents disagree. We are interested in how their disagreement evolves after signal 2 is released. We discuss the notions of disagreement proposed by Kondor [12] and Andreoni and Mylovanov [1]. Kondor offers the following notion of disagreement:

Definition 6. *(i) Signals exhibit* polarization with respect to $s_2 \in \{\hat{\alpha}, \hat{\beta}\}$ *if observing s_2 induces more extreme beliefs: $q_{\beta,s_2}(a) < q_\beta(a)$ and $q_\alpha(a) < q_{\alpha,s_2}(a)$.*
(ii) Signals exhibit belief swap with respect to $s_2 \in \{\hat{\alpha}, \hat{\beta}\}$ *if observing s_2 swaps the order of beliefs: $q_{\beta,s_2}(a) > q_{\alpha,s_2}(a)$.*[18]

We now show that complements display polarization and belief swap.

Proposition 8. *(i) If signals are complements, then they exhibit polarization with respect to one and belief swap with respect to the other realization of signal 2.*
(ii) If signals are substitutes, then they do not exhibit polarization or belief swap with respect to any realization of signal 2.

Proof. (i) Suppose signals are complements and (14) holds (the case in which (15) holds is analogous.) One can show using that the prior is a convex combination of the posteriors that (I) (14) and $q_\alpha(a) > \pi(a)$ imply that $q_{\alpha,\hat{\beta}}(a) > q_\alpha(a)$; (II) (14) and $q_{\hat{\alpha}}(a) > \pi(a)$ imply that $q_{\beta,\hat{\alpha}}(a) > q_{\hat{\alpha}}(a) > \pi(a)$; and (III) $q_{\beta,\hat{\alpha}}(a) > \pi(a)$ and $q_\beta(a) < \pi(a)$ imply that $q_{\beta,\hat{\beta}}(a) < q_\beta(a)$. The conclusions of (I) and (III) mean that signals display polarization with respect to $\hat{\beta}$ and the conclusion of (II) and (14) that signals display belief swap with respect to $\hat{\alpha}$. This establishes part (i).

(ii) Suppose signals are substitutes. We distinguish three cases. The first case is that the two extreme realizations referred to in Proposition 2 are from signal 1, then we have for all $s_2 \in \{\hat{\alpha}, \hat{\beta}\}$ that $q_{\alpha,s_2}(a) = q_\alpha(a)$ and $q_{\beta,s_2}(a) = q_\beta(a)$, or one of the realizations $(\alpha, \hat{\beta})$ or $(\beta, \hat{\alpha})$ has zero probability. Thus, there is neither polarization nor belief swap with respect to s_2. The second case is that the two extreme realizations referred to in

[18] If one of the realizations mentioned in the definition has zero probability, we say that polarization or belief swap do not occur.

Proposition 2 are from signal 2, then we have for all $s_2 \in \{\hat{\alpha}, \hat{\beta}\}$ that $q_{\alpha,s_2}(a) = q_{\beta,s_2}(a)$, or one of the realizations $(\alpha, \hat{\beta})$ or $(\beta, \hat{\alpha})$ has zero probability. This implies that there is neither polarization nor belief swap with respect to s_2. Finally, suppose the extreme realizations referred to in Proposition 2 are from different signals. Suppose realizations α and $\hat{\beta}$ are extreme. (The other case is analogous.) Then $q_{\alpha,\hat{\alpha}}(a) = q_\alpha(a) \geq q_{\beta,\hat{\alpha}}(a)$. This implies that there is no polarization or belief swap with respect to $s_2 = \hat{\alpha}$. Moreover, since the realization $(\alpha, \hat{\beta})$ has zero probability, polarization and belief swap with respect to $\hat{\beta}$ do not occur. □

Next, we turn to the notion of disagreement proposed by Andreoni and Mylovanov [1] who define disagreement in terms of disagreement about the optimal course of action. They consider a two action decision problem where agents are indifferent between the actions at the prior π. Agents privately observe conditionally independent versions of signal 1, and then observe a public signal 2. Disagreement is said to occur when agents who have privately observed different realizations of signal 1 prefer different actions even after observing the public signal. When signal 1 is decisive in the sense that the beliefs induced by itself are so extreme that no realization of signal 2 would induce an action change, then disagreement trivially occurs. Therefore, the possibility of disagreement is primarily interesting when signal 1 is not decisive.

We now investigate Andreoni and Mylovanov's notion of disagreement in the binary–binary model. We say that signal 1 is *not decisive* if for at least one realization of signal 1 there is one realization of signal 2 that can sway the agent and induce a different action. Moreover, disagreement about the optimal action means that after the release of signal 2, one agent's belief is larger and the other agent's belief is smaller than the prior.

Definition 7. *(i) Signal 1 is* not decisive *if there is $s_1 \in \{\alpha, \beta\}$ and $s_2 \in \{\hat{\alpha}, \hat{\beta}\}$ so that $(q_{s_1}(a) - \pi(a))(q_{s_1,s_2}(a) - \pi(a)) < 0$.*
(ii) Signal 2 induces action disagreement *with respect to $s_2 \in \{\hat{\alpha}, \hat{\beta}\}$ if after observing s_2, agents prefer different actions:*[19]

$$(q_{\alpha,s_2}(a) - \pi(a))(q_{\beta,s_2}(a) - \pi(a)) \leq 0. \tag{37}$$

The next result shows that action disagreement with respect to *all* realizations of signal 2 is implied by complementarity and, in fact, characterizes complementarity when

[19] [1] distinguish between "strict" and "weak" action disagreement depending on whether the inequality is strict or weak. For simplicity, we only consider weak disagreement.

signal 1 is not decisive.

Proposition 9. *(i) If signals are complements, then signal 2 induces action disagreement with respect to $s_2 = \hat{\alpha}$ and $s_2 = \hat{\beta}$.*
(ii) Let signal 1 be not decisive. If signal 2 induces action disagreement with respect to $s_2 = \hat{\alpha}$ and $s_2 = \hat{\beta}$, then signals are complements.

Proof. (i) Suppose signals are complements and (14) holds (the case in which (15) holds is analogous.) As we have already shown in the proof of Proposition 8, (14) implies that $q_{\beta,\hat{\alpha}}(a) > \pi(a)$, $q_{\alpha,\hat{\beta}}(a) > q_\alpha(a)$ and $q_{\beta,\hat{\beta}}(a) < q_\beta(a)$. The first inequality and (14) mean that (37) follows for $s_2 = \hat{\alpha}$, and the second and third inequalities together with $q_\beta(a) < \pi(a) < q_\alpha(a)$ mean that (37) also follows for $s_2 = \hat{\beta}$.

(ii) Suppose signals are not complements, that is, $q_{\alpha,\hat{\alpha}} > \pi(a)$ and $q_{\beta,\hat{\beta}} < \pi(a)$. Because signal 1 is not decisive, this implies that $q_{\alpha,\hat{\beta}} < \pi(a)$ or $q_{\beta,\hat{\alpha}} > \pi(a)$. In the first case, (37) is violated for $s_2 = \hat{\beta}$, and in the second case, (37) is violated for $s_2 = \hat{\alpha}$, a contradiction. □

Turning to substitutes, note because complementarity and substitutability exclude each other in the binary–binary model, the second part of the previous lemma makes clear that if signal 1 is not decisive, substitutes do not induce action disagreement.

For the case with more than two signal realizations yet still two states[20], part (i) of Proposition 9 can be generalized if one can find a partition of signal realizations that satisfies the conditions of Proposition 5. In that case, disagreement occurs in the sense that there are at least two realizations s_1' and s_1'' of signal 1 and a realization s_2 of signal 2 so that if the two agents have privately observed s_1' and s_1'' of (conditionally independent versions) of signal 1, then signal 2 induces disagreement with respect to s_2, i.e., $(q_{s_1',s_2} - \pi(a))(q_{s_1'',s_2} - \pi(a)) \leq 0$.[21] By contrast, the necessary conditions in Proposition 5 are too weak to imply polarization and belief swap.

For the same reason as in the previous subsection, our general necessary conditions for substitutes in the case with two or more signal realizations imply too little restrictions on

[20] It is not fully obvious how to best extend the notions of belief swap, polarization, or disagreement to the case with more than two states.

[21] To see this, let r in Proposition 5 correspond to the action that pays 1 in state b and 0 in state a. Thus, $e = \pi(a)$. Now consider some $(s_1', s_2) \in S_1^E \times S_2^E$ for which, say, (22) holds, i.e., $rq_{s_1',s_2} - e = q_{s_1',s_2} - \pi(a) \geq 0$ Then there are two cases. Either there is a realization $s_1'' \in S_1^E$ which, together with observation s_2 induces action E: $rq_{s_1'',s_2} - e = q_{s_1'',s_2} - \pi(a) \leq 0$. Then the claim holds for the realizations s_1', s_1'' and s_2. Or, all realizations $s_1 \in S_1^E$ induce action R when observed together with s_2: $rq_{s_1,s_2} - e = q_{s_1,s_2} - \pi(a) \geq 0$. Then condition (iii) in Proposition 5 implies that all realizations $s_1'' \in S_1^R$, when observed together with s_2 induce action E: $rq_{s_1'',s_2} - e = q_{s_1'',s_2} - \pi(a) \leq 0$. Hence, in this case the claim holds for these realizations s_1''.

posterior beliefs to rule out that in the general case substitutability of signals is consistent with belief swap, polarization, or disagreement for some realizations.

The conclusion that emerges from this analysis is that complementarity of signals can rationalize the psychological evidence on belief disagreement. Whether rational or biased, if polarization aggravates desirable consensus finding in committees or public debate, then providing additional information might be normatively problematic. In contrast to psychological accounts of polarization, our analysis points out that what might be problematic is not the release of public information in general, but in particular the release of public information which is complementary to agents' private information.

6 Conclusion

This paper has provided some insights into the nature of substitutability and complementarity relations among signals, and into the strategic importance of these relations. Further development of the general statistical theory should also open up the possibility of more applied work. Our most general conditions for substitutability and complementarity in the case that there are more than two states are only necessary, not sufficient, and therefore give us only a partial description of signals that are substitutes or complements. As the necessary condition for substitutes is obviously very restrictive, whereas the necessary condition for complements is not obviously as restrictive, perhaps the most intriguing open question is how large the class of complements is if there are more than two states.

Many pairs of signals are neither complements nor substitutes if our definitions are used. This is because our definitions of these terms require certain conditions to be true in *all* decision problems. This is in the spirit of Blackwell's comparison whose ordering of signals is incomplete. More signals will satisfy the conditions for being substitutes or complements if we restrict attention to smaller classes of decision problems. In the context of Blackwell's original work this line of investigation has been pursued by Lehmann [13], Persico [19], Athey and Levin [2] and Jewitt [11]. A similar research agenda is feasible in our context.

Complementarity of signals may also matter when agents acquire signals sequentially. In this case, the second signal may be acquired when the agent already knows the realization of the first signal. By contrast, in our setting, each signal is acquired without knowing the realization of the other signal. Extending our results to a setting where agents evaluate signals knowing the realization of other signals is another project for future work.

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Appendix

Proof of the Sufficiency Part of Proposition 2

We only consider the case in which the realization α provides the strongest individual evidence for state a: $q_\alpha(a) \geq q_{\hat{\alpha}}(a)$. The other case can be dealt with analogously. There are two further cases: we first consider the case in which β provides the strongest individual evidence for b: $q_\beta(b) \geq q_{\hat{\beta}}(b)$. In this case, conditions (12) and (13) become:

$$q_{\alpha,\hat{\alpha}}(a) = q_\alpha(a), \quad q_{\beta,\hat{\beta}}(a) = q_\beta(a). \tag{38}$$

We now argue that signal $\tilde{s}_2$ does not affect the decision maker's belief if he has observed signal $\tilde{s}_1$. Indeed, if the realization $(\alpha, \hat{\beta})$ has strictly positive probability in some state, then since $q_\alpha(a)$ is a convex combination of $q_{\alpha,s_2}(a)$, $s_2 \in \{\hat{\alpha}, \hat{\beta}\}$, the left equality above implies that $q_{\alpha,\hat{\beta}}(b) = q_\alpha(b)$. Moreover, if $(\alpha, \hat{\beta})$ has zero probability in all states, then clearly the decision maker maintains his belief after having observed the realization α with probability 1. In sum, we have shown that the probability that a realization of signal $\tilde{s}_2$ changes the decision maker's belief if realization α of signal $\tilde{s}_1$ has been observed is zero. Symmetrically, the probability that a realization of signal $\tilde{s}_2$ changes the decision maker's belief if realization β of signal $\tilde{s}_1$ has been observed is zero. But this means that the marginal value of signal $\tilde{s}_2$, if signal $\tilde{s}_1$ is available, is zero in all decision problems. Hence, signals are substitutes.

We next consider the case $q_{\hat{\beta}}(b) \geq q_\beta(b)$. In this case, conditions (12) and (13) become:

$$q_{\alpha,\hat{\alpha}}(a) = q_\alpha(a), \quad q_{\beta,\hat{\beta}}(b) = q_{\hat{\beta}}(b). \tag{39}$$

We first argue that this implies

$$p_{12,a}(\alpha, \hat{\beta}) = p_{12,b}(\alpha, \hat{\beta}) = 0. \tag{40}$$

Indeed, suppose the contrary were true. Then because for i, j, $q_{s_i}(a)$ is a convex combination of $q_{s_i,s_j}(a)$, $s_j \in S_j$, (39) would imply that $q_{\alpha,\hat{\beta}}(a) = q_\alpha(a)$, and $q_{\alpha,\hat{\beta}}(a) = q_{\hat{\beta}}(a)$, a contradiction to our assumption that realization α indicates state a and realization $\hat{\beta}$ indicates state b.

We now demonstrate that signals are substitutes. Suppose first that the realization $(\beta, \hat{\alpha})$ has zero probability in all states. Then (40) implies that signals are perfectly correlated. Therefore, the probability that a realization of one signal changes the decision maker's belief if the other signal is available is zero. Hence, signals are substitutes.

Suppose next that $(\beta, \hat{\alpha})$ has strictly positive probability in some state. (39) together with the fact that for i, j, $q_{s_i}(a)$ is a convex combination of $q_{s_i,s_j}(a)$, $s_j \in S_j$ and the assumption that α provides the strongest and $\hat{\beta}$ the weakest individual evidence for state a implies the ordering:

$$q_{\beta,\hat{\beta}}(a) \;=\; q_{\hat{\beta}}(a) \;\leq\; q_{\beta}(a) \;\leq\; q_{\beta,\hat{\alpha}}(a) \;\leq\; q_{\hat{\alpha}}(a) \;\leq\; q_{\alpha}(a) \;=\; q_{\alpha,\hat{\alpha}}(a). \tag{41}$$

We now use Lemma 1 to demonstrate that signals are substitutes. By Lemma 1, it is sufficient to verify that signals are substitutes in all two action problems of Figure 1 for all $x \in (0, 1)$. We show that for any x there is a signal $\tilde{s}_i$ so that $V_{12}(A, u) - V_i(A, u) = 0$ holds for the two action decision problem (A, u) with parameter x.

- $x \leq q_{\beta,\hat{\beta}}(a)$ or $x \geq q_{\alpha,\hat{\alpha}}(a)$: Then all realizations of $(\tilde{s}_1, \tilde{s}_2), \tilde{s}_1, \tilde{s}_2$ induce the same optimal action, so that $V_{12}(A, u) - V_1(A, u) = V_{12}(A, u) - V_2(A, u) = 0$.

- $x \in (q_{\beta,\hat{\beta}}(a), q_{\beta,\hat{\alpha}}(a)]$: Then the probability that a realization of signal $\tilde{s}_1$ moves the decision maker's belief when realization $\hat{\beta}$ of signal $\tilde{s}_2$ has been observed is zero, and no realization of signal $\tilde{s}_1$ changes the optimal action if realization $\hat{\alpha}$ of signal $\tilde{s}_2$ has already been observed. Therefore, $V_{12}(A, u) - V_2(A, u) = 0$.

- $x \in (q_{\beta,\hat{\alpha}}(a), q_{\alpha,\hat{\alpha}}(a)]$: Then the probability that a realization of signal $\tilde{s}_2$ moves the decision maker's belief if realization α of signal $\tilde{s}_1$ has been observed is zero, and no realization of signal $\tilde{s}_2$ changes the optimal action if realization β of signal $\tilde{s}_1$ has already been observed. Therefore, $V_{12}(A, u) - V_1(A, u) = 0$.

Proof of the Sufficiency Part of Proposition 3

We begin with the observation that the conditions in Proposition 3 imply that all signal realizations have strictly positive prior probability. Suppose, for example, (14) were true and $\bar{p}_{12}(\alpha, \hat{\beta}) = 0$. Then $q_{\alpha}(a) = q_{\alpha,\hat{\alpha}}(a) \leq \pi(a)$ which would contradict our assumption that $q_{\alpha}(a) > \pi(a)$. The argument can be completed by repeating this step a number of times.

By Lemma 1, it suffices to verify complementarity for all two action problems described in Figure 1. Below, we shall assume that $x \leq 0.5 = \pi(a)$. If $x \leq 0.5$, then it is optimal under the prior belief to choose B. We shall assume that $q_\beta(a) < x$ and $q_{\hat\beta}(a) < x$ so that after observing β or $\hat\beta$ it is strictly optimal to choose T. If this were not true, at least one of the signals would by itself never provide a strict incentive to switch away from the action that maximizes expected utility under the prior, and thus this signal by itself would have zero value. Signals would then trivially be complements.

A signal has positive value by itself if it sometimes induces the decision maker to switch to T, and the value of the signal is the expected utility increase arising from these switches. If the decision maker attaches probability $q(a) < x$ to state a, and switches from B to T, then the increase in expected utility is:

$$(1 - q(a))x - q(a)(1 - x) = x - q(a). \tag{42}$$

Observing a second signal realization sometimes induces the decision maker to switch back from T to B. If some signal observation induces the decision maker to hold beliefs $q(a) > x$, and to switch from T to B, then the increase in expected utility is:

$$q(a)(1 - x) - (1 - q(a))x = q(a) - x. \tag{43}$$

Building on these considerations, we can now calculate for the two action decision problem (A, u) that corresponds to the parameter value x:

$$\begin{aligned} V_2(A,u) - V_\emptyset(A,u) &= \bar p_2(\hat\beta)[x - q_{\hat\beta}(a)] \\ &= \bar p_{12}(\beta,\hat\beta)[x - q_{\beta,\hat\beta}(a)] + \bar p_{12}(\alpha,\hat\beta)[x - q_{\alpha,\hat\beta}(a)]. \end{aligned} \tag{44}$$

The first line uses the assumption $q_{\hat\beta}(a) < x$. The first and the second line are equal because the expected value of the posterior belief after observing both signal realizations, taking expected values over the realizations of signal 1, is the posterior belief after observing the realization of signal 2 only. We next compute the marginal value of signal $\tilde s_2$ when signal $\tilde s_1$ is available:

$$\begin{aligned} V_{12}(A,u) - V_1(A,u) &= \bar p_{12}(\beta,\hat\alpha)[q_{\beta,\hat\alpha}(a) - x]^+ + \bar p_{12}(\beta,\hat\beta)[q_{\beta,\hat\beta}(a) - x]^+ \\ &\quad + \bar p_{12}(\alpha,\hat\alpha)[x - q_{\alpha,\hat\alpha}(a)]^+ + \bar p_{12}(\alpha,\hat\beta)[x - q_{\alpha,\hat\beta}(a)]^+. \end{aligned} \tag{45}$$

Here, we use for any real number z the following notation: $z^+ \equiv z$ if $z \geq 0$, and $z^+ \equiv 0$ if $z < 0$. We have also made use of our assumption $q_\beta(a) < x$.

We now prove first that (14) implies that signals are complements. Condition (14) implies that $q_{\beta,\hat{\alpha}}(a) > 0.5 > x$ because otherwise we could not have $q_{\hat{\alpha}}(a) > 0.5 = \pi(a)$. Thus,

$$V_{12}(A,u) - V_1(A,u) \geq \bar{p}_{12}(\beta,\hat{\alpha})[q_{\beta,\hat{\alpha}}(a) - x]. \tag{46}$$

Therefore, we obtain for the difference:

$$\begin{aligned} & V_{12}(A,u) - V_1(A,u) - (V_2(A,u) - V_\emptyset(A,u)) \\ \geq\ & \bar{p}_{12}(\beta,\hat{\alpha})[q_{\beta,\hat{\alpha}}(a) - x] + \bar{p}_{12}(\beta,\hat{\beta})[q_{\beta,\hat{\beta}}(a) - x] + \bar{p}_{12}(\alpha,\hat{\beta})[q_{\alpha,\hat{\beta}}(a) - x]. \end{aligned} \tag{47}$$

Now we add and subtract $\bar{p}_{12}(\alpha,\hat{\alpha})[q_{\alpha,\hat{\alpha}}(a) - x]$ on the right-hand side. Using the fact that $\sum_{(s_1,s_2)\in S_1\times S_2} \bar{p}_{12}(s_1,s_2)q_{s_1,s_2}(a) = \pi(a) = 0.5$, the right-hand side of (47) becomes equal to

$$0.5 - x - \bar{p}_{12}(\alpha,\hat{\alpha})[q_{\alpha,\hat{\alpha}}(a) - x] \geq 0.5 - x - \bar{p}_{12}(\alpha,\hat{\alpha})[0.5 - x] \geq 0. \tag{48}$$

The first inequality follows because $q_{\alpha,\hat{\alpha}}(a) \leq 0.5$ by (14). The second inequality follows because $x \leq 0.5$ and since $\bar{p}_{12}(\alpha,\hat{\alpha}) < 1$. This establishes that (14) implies that signals are complements.

We next prove that (15) implies that signals are complements. Condition (15) implies: $q_{\beta,\hat{\beta}}(a) \geq \pi(a) = 0.5 \geq x$, and hence we have:

$$V_{12}(A,u) - V_1(A,u) \ \geq\ \bar{p}_{12}(\beta,\hat{\beta})[q_{\beta,\hat{\beta}}(a) - x] + \bar{p}_{12}(\alpha,\hat{\beta})[x - q_{\alpha,\hat{\beta}}(a)]^+. \tag{49}$$

Thus,

$$\begin{aligned} & V_{12}(A,u) - V_1(A,u) - (V_2(A,u) - V_\emptyset(A,u)) \\ \geq\ & \bar{p}_{12}(\beta,\hat{\beta})[q_{\beta,\hat{\beta}}(a) - x] + \bar{p}_{12}(\alpha,\hat{\beta})[x - q_{\alpha,\hat{\beta}}(a)]^+ \\ & + \bar{p}_{12}(\beta,\hat{\beta})[q_{\beta,\hat{\beta}}(a) - x] + \bar{p}_{12}(\alpha,\hat{\beta})[q_{\alpha,\hat{\beta}}(a) - x] \geq 0. \end{aligned} \tag{50}$$

The sum in (50) is non-negative since $q_{\beta,\hat{\beta}}(a) \geq \pi(a) = 0.5 \geq x$ by (15), and because the sum of the second and the fourth term is always non-negative. Thus we have again shown that signals are complements.

A Sufficient Condition for Complements with Two States and Many Signal Realizations

We generalize the sufficiency part of Proposition 3 to obtain a sufficient condition for complementarity in the case when signals have arbitrarily many realizations. Let $\underline{s}_i$ (resp. $\bar{s}_i$) be the realization of signal $\tilde{s}_i$, which provides the weakest (resp. strongest) support for state a: $q_{\underline{s}_i}(a) = \min_{s_i} q_{s_i}(a)$ and $q_{\bar{s}_i}(a) = \max_{s_i} q_{s_i}(a)$. Let

$$x \in X \equiv \left(\max\{q_{\underline{s}_1}(a), q_{\underline{s}_2}(a)\}, \min\{q_{\bar{s}_1}(a), q_{\bar{s}_2}(a)\}\right), \tag{51}$$

that is, x is larger than the smallest posterior probability of a that is induced by any realization of a single signal, and smaller than the largest posterior probability of a induced by any realization of a single signal. We partition the set S_i of realizations of signal $\tilde{s}_i$ into two subsets, depending on whether they induce posterior beliefs $q_{s_i}(a)$ that are smaller or larger than x:

$$S_i^{\beta}(x) = \{s_i \in S_i \mid q_{s_i}(a) \leq x\}, \quad S_i^{\alpha}(x) = \{s_i \in S_i \mid q_{s_i}(a) > x\}. \tag{52}$$

Now imagine that, instead of observing each realization of signal $\tilde{s}_i$, the decision maker only observes whether a realization is in one of the two partitions. This amounts to observing a signal with two realizations. We call this binary signal $\tilde{t}_i(x)$ and denote the realization of $\tilde{t}_i(x)$ by $t_i^{\beta}(x)$ if $s_i \in S_i^{\beta}(x)$ and by $t_i^{\alpha}(x)$ if $s_i \in S_i^{\alpha}(x)$.

Proposition 10. *In the two state case, if for all $x \in X$ the signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are complements, then the signals $\tilde{s}_1$ and $\tilde{s}_2$ are complements.*

Proof. We denote the expected utility that the decision maker receives when maximizing expected utility in some arbitrary decision problem (A, u) after observing the realization of $\tilde{t}_i(x)$ by $V_{i,x}(A, u)$ and we denote the expected utility that the decision maker receives when maximizing expected utility in decision problem (A, u) after observing the joint realization $(\tilde{t}_1(x), \tilde{t}_2(x))$ by $V_{12,x}(A, u)$. Let the auxiliary signals $\tilde{t}_C(x)$ and $\tilde{t}_S(x)$ be defined analogously to $\tilde{s}_C$ and $\tilde{s}_S$, and denote the expected utility that the decision maker receives when maximizing expected utility in decision problem (A, u) after observing these signals by $V_{C,x}(A, u)$ and $V_{S,x}(A, u)$.

By Lemma 1 it is sufficient to verify complementarity for the two action problem of Figure 1 for all $x \in (0, 1)$. For $x \notin X$, there is at least one signal $\tilde{s}_i$ which is not informative. Hence, signals are obviously complements. Let $x \in X$, and let (A, u) for the purposes of this proof be the corresponding two action decision problem. By Proposition 1, it is sufficient to show that $V_C(A, u) \geq V_S(A, u)$.

To demonstrate this, we begin with two observations. The first observation is that $V_i(A, u) = V_{i,x}(A, u)$. This is so since in the two action problem at hand, all that matters for the decision maker's optimal action after observing realization s_i of signal $\tilde{s}_i$ is whether the posterior belief $q_{s_i}(a)$ is smaller or larger than x. But this is precisely the information provided by signal $\tilde{t}_i(x)$. We omit the formal proof. The second observation is that, evidently, the signal $(\tilde{s}_1, \tilde{s}_2)$ is (weakly) more informative than the signal $(\tilde{t}_1(x), \tilde{t}_2(x))$. Hence, $V_{12}(A, u) \geq V_{12,x}(A, u)$. Using these two observations, we can deduce:

$$
\begin{aligned}
V_C(A,u) &= 0.5V_{12}(A,u) + 0.5V_{\emptyset}(A,u) \\
&\geq 0.5V_{12,x}(A,u) + 0.5V_{\emptyset}(A,u) \\
&= V_{C,x}(A,u) \\
&\geq V_{S,x}(A,u) \\
&= 0.5V_{1,x}(A,u) + 0.5V_{2,x}(A,u) \\
&= 0.5V_1(A,u) + 0.5V_2(A,u) \\
&= V_S(A,u),
\end{aligned} \tag{53}
$$

where the inequality in the fourth line follows because by assumption $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are complements. This proves the claim. □

Proof of Proposition 5

Indirect. Assume for some $r \in \mathbb{R}^{|\Omega|}$ and $e \in \mathbb{R}$ there were partitions (S_i^E, S_i^R) (for $i \in \{1, 2\}$) that satisfy the conditions (i)-(iii) of the Proposition, but neither (22) nor (23) were true. Consider the decision problem with two actions, R and E, where the payoff of action R in state ω is given by the ω-th component of r, and the payoff of action E is equal to e in all states of the world. For an arbitrary belief q the expected payoff of action R is rq, and the expected payoff of E is e. By assumption, the prior π is such that $r\pi = e$, that is, the agent is indifferent between the two actions based on the prior. We shall show that the signals are not complements in this decision problem. For the remainder of this proof, (A, u) will denote this particular decision problem.

Suppose for $(k, \ell) = (E, R)$ condition (20) were true, and for $(k, \ell) = (R, E)$ condition

(21) were true. Together with the assumption that neither (22) nor (23) are true, we can deduce that, conditional on observing any joint signal realization (s_1, s_2), one optimal action for the decision maker is E whenever $s_1 \in S_1^E$, independent of the realization of signal $\tilde{s}_2$, and R whenever $s_1 \in S_1^R$, again independent of the realization of signal $\tilde{s}_2$. Therefore, $V_{12}(A,u) - V_1(A,u) = 0$. On the other hand, the strict inequalities in conditions (17) and (18), applied to $i = 2$, imply that $V_2(A,u) - V_\emptyset(A,u) > 0$. Thus, signals are not complements. The case that for $(k,\ell) = (E,R)$ condition (21) is true, and for $(k,\ell) = (R,E)$ condition (20) is true, is analogous, with the roles of signals 1 and 2 swapped.

Now consider the case that for both admissible (k,ℓ) condition (20) holds. We shall calculate $V_1(A,u) - V_\emptyset(A,u)$ and $V_{12}(A,u) - V_2(A,u)$. To calculate these value differences we recall that a positive marginal value from a signal arises only when the signal changes the decision maker's optimal choice. As the prior makes the decision maker indifferent, we can pick the decision maker's choice when holding the prior as is convenient for our proof. We pick it to be R. Then we have:

$$\begin{aligned} V_1(A,u) - V_\emptyset(A,u) &= \sum_{s_1 \in S_1^E} \bar{p}_1(s_1)(e - rq_{s_1}) && (54) \\ &= \sum_{s_1 \in S_1^E} \sum_{\substack{s_2 \in S_2: \\ \bar{p}_{12}(s_1,s_2)>0}} \bar{p}_{12}(s_1,s_2)(e - rq_{s_1,s_2}), && (55) \end{aligned}$$

where the second line equals the first because the expected value of the posterior belief conditional on the realizations of both signals is the posterior belief conditional on the realization of signal $\tilde{s}_1$. Focusing again on signal realizations that change the set of optimal choices for the decision maker we also calculate:

$$V_{12}(A,u) - V_2(A,u) = \sum_{s_1 \in S_1^E} \sum_{\substack{s_2 \in S_2^R: \\ \bar{p}_{12}(s_1,s_2)>0}} \bar{p}_{12}(s_1,s_2)(e - rq_{s_1,s_2}). \quad (56)$$

This equation follows from the assumption that (20) holds for both admissible (k,ℓ) and that neither (22) nor (23) are true. Subtracting (56) from (55), we find:

$$\begin{aligned}&V_1(A,u) - V_\emptyset(A,u) - (V_{12}(A,u) - V_2(A,u))\\ =& \sum_{s_1 \in S_1^E} \sum_{\substack{s_2 \in S_2^E: \\ \bar{p}_{12}(s_1,s_2)>0}} \bar{p}_{12}(s_1,s_2)(e - rq_{s_1,s_2}).\end{aligned} \tag{57}$$

By condition (19), applied to $k = E$, in Proposition 5, the sum on the right-hand side of the last equality is over at least one pair (s_1, s_2). Moreover, because (22) does not hold for any $(s_1,s_2) \in S_1^E \times S_2^E$, this sum is negative, and therefore signals are not complements. The remaining case, when for both admissible (k,ℓ) condition (21) holds, is analogous, with the optimal choice under the prior taken to be E. □

Proof of the Necessity Part of Proposition 3

We use the following corollary to Proposition 5.

Corollary 2. *If signals are complements, then for every signal realization (s_1', s_2') with $\bar{p}_{12}(s_1', s_2') > 0$ we have:*

$$\pi \in co\left\{q_{s_1,s_2} | (s_1,s_2) \in S_1 \times S_2 \setminus \{(s_1', s_2')\} \text{ and } \bar{p}_{12}(s_1,s_2) > 0\right\}. \tag{58}$$

Proof. Indirect. Denote the convex hull to which the corollary refers by C and suppose $\pi \notin C$. Then there is a hyperplane through π that does not intersect with C. Let r be the orthogonal vector of this hyperplane, and define $e \equiv r\pi$. We can choose r such that $rq < e$ for all $q \in C$. We now show that with this choice of r and e the necessary condition of Proposition 5 is violated. For $i = 1, 2$ define $S_i^E \equiv S_i \setminus \{s_i'\}$ and $S_i^R \equiv \{s_i'\}$. We first verify conditions (17) and (18) of Proposition 5. Let $i \in \{1, 2\}$ and $j \neq i$. Because for every $s_i \in S_i^E$ and every $s_j \in S_j$ we have: $q_{s_i,s_j} \in C$, we can conclude: $rq_{s_i,s_j} < e$. Because q_{s_i} is a convex combination of q_{s_i,s_j} for $s_j \in S_j$, this implies: $rq_{s_i} < e$, and thus (17) holds. Now consider $q_{s_i'}$. If this belief satisfied: $rq_{s_i'} \leq e$, then we could infer $r\pi < e$, because π is a convex combination of q_{s_i} for $s_i \in S_i$, which contradicts $e = r\pi$. Therefore: $rq_{s_i'} > e$, which verifies (18). Next, we note that (19) holds by construction, and that also by construction (20) is true for both (k,ℓ). On the other hand, (22) and (23) are violated by construction. Thus, Proposition 5 implies that signals are not complements. □

We now use Corollary 2 to derive the necessity part of Proposition 3. We begin by proving that $\bar{p}_{12}(\alpha, \hat{\alpha}) > 0$ and $\bar{p}_{12}(\beta, \hat{\beta}) > 0$. The proof is indirect. Suppose first that both probabilities were zero. Then the signals would be perfectly correlated, and therefore not be complements. Next suppose $\bar{p}_{12}(\alpha, \hat{\alpha}) = 0$ but $\bar{p}_{12}(\beta, \hat{\beta}) > 0$. Because α and $\hat{\alpha}$ occur with strictly positive prior probability probability, we have to have: $\bar{p}_{12}(\alpha, \hat{\beta}) > 0$ and $\bar{p}_{12}(\beta, \hat{\alpha}) > 0$. Because α and $\hat{\alpha}$ indicate that the state is more likely to be a, it must be that $q_{\alpha,\hat{\beta}}(a) > \pi(a)$ and $q_{\beta,\hat{\alpha}}(a) > \pi(a)$. But then the condition of Corollary 2 is violated if we take (s'_1, s'_2) to be $(\beta, \hat{\beta})$. A symmetric argument applies if $\bar{p}_{12}(\alpha, \hat{\alpha}) > 0$ and $\bar{p}_{12}(\beta, \hat{\beta}) = 0$. We conclude that $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ have strictly positive prior probability.

We now prove that $q_{\alpha,\hat{\alpha}}(a) \leq \pi(a)$ or $q_{\beta,\hat{\beta}}(b) \leq \pi(b)$. The proof is indirect. Suppose

$$q_{\alpha,\hat{\alpha}}(a) > \pi(a) \quad \text{and} \quad q_{\beta,\hat{\beta}}(b) > \pi(b). \tag{59}$$

We begin with the case that the two mixed realizations $(\alpha, \hat{\beta})$ and $(\beta, \hat{\alpha})$ both have strictly positive prior probability so that posteriors conditioning on these signal realizations are well-defined. We go through different possible orderings of the posterior beliefs, and show that none of them is compatible with signals being complements. Consider first the following two cases:

$$q_{\alpha,\hat{\beta}}(a) \geq \pi(a) \quad \text{and} \quad q_{\beta,\hat{\alpha}}(a) \leq \pi(a), \tag{60}$$

$$q_{\alpha,\hat{\beta}}(a) \leq \pi(a) \quad \text{and} \quad q_{\beta,\hat{\alpha}}(a) \geq \pi(a). \tag{61}$$

Condition (60) together with (59) implies that in the decision problem of Figure 1 with $x = 0.5 = \pi(a)$, which we shall denote by (A, u) in this proof, the marginal value of signal $\tilde{s}_2$ conditional on signal $\tilde{s}_1$ is zero for both signal realizations of signal $\tilde{s}_1$. Thus, $V_{12}(A, u) - V_1(A, u) = 0$, and signals are not complements (note that $V_2(A, u) - V_\emptyset(A, u) > 0$ by the assumption that signal $\tilde{s}_2$ is informative and $x = 0.5$.) For ordering (61) the argument is the same with the roles of signals 1 and 2 swapped.

We are left with the orderings:

$$q_{\alpha,\hat{\beta}}(a) > \pi(a) \quad \text{and} \quad q_{\beta,\hat{\alpha}}(a) > \pi(a), \tag{62}$$

$$q_{\alpha,\hat{\beta}}(a) < \pi(a) \quad \text{and} \quad q_{\beta,\hat{\alpha}}(a) < \pi(a). \tag{63}$$

If (62) holds in combination with (59), the necessary condition in Corollary 2 is violated if we choose $(s'_1, s'_2) = (\beta, \hat{\beta})$, and if (63) holds in combination with (59), the necessary

	α_2	σ_2	σ_2'	β_2
α_1	ρ	0	0	0
σ_1	0	φ	$\lambda\varphi$	0
σ_1'	0	$\lambda\varphi$	φ	0
β_1	0	0	0	$\lambda\rho$

$\omega = a$

	α_2	σ_2	σ_2'	β_2
α_1	$\lambda\rho$	0	0	0
σ_1	0	$\lambda\varphi$	φ	0
σ_1'	0	φ	$\lambda\varphi$	0
β_1	0	0	0	ρ

$\omega = b$

Figure 4: Example 4 (signals are substitutes if $2\varphi \leq \rho$ and complements if $2\varphi \geq \rho$)

condition in Corollary 2 is violated if we choose $(s_1', s_2') = (\alpha, \hat{\alpha})$.

It remains to discuss the cases in which at least one of $(\alpha, \hat{\beta})$ and $(\beta, \hat{\alpha})$ does not have strictly positive prior probability. Suppose first that both realizations $(\alpha, \hat{\beta})$ and $(\beta, \hat{\alpha})$ have zero prior probability. This means that signals are perfectly correlated and therefore the marginal value of a signal when the other signal is available is zero. Hence, signals are not complements. Suppose next that $(\alpha, \hat{\beta})$, but not $(\beta, \hat{\alpha})$ has zero probability. If $q_{\beta,\hat{\alpha}}(a) \leq \pi(a)$, then the same argument as for ordering (60) can be used, and if $q_{\beta,\hat{\alpha}}(a) \geq \pi(a)$, the same argument as for ordering (61) can be used. For the remaining case that $(\beta, \hat{\alpha})$, but not $(\alpha, \hat{\beta})$ has zero probability, the argument is analogous. $\square$

A Counterexample

In this subsection we present an example that shows that the conditions in Proposition 4 for substitutes and Proposition 5 for complements are only necessary, but not sufficient. The example also shows that the sufficient conditions for complements in Proposition 10 are not necessary for complements.

Example 4 is shown in Figure 4.[22] The example is a two state example: $\Omega = \{a, b\}$. Each individual signal $\tilde{s}_i$ has two informative realizations: α_i, β_i, and two not informative realizations: σ_i, σ_i'. Among all individual and joint signal realizations, the posterior belief that the state is a can take on only three values: it equals $1/(1+\lambda) > 1/2$ for the realizations $\alpha_i, (\sigma_1, \sigma_2), (\sigma_1', \sigma_2')$; it equals $1/2$ for the realizations σ_i, σ_i'; and it equals $\lambda/(1+\lambda) < 1/2$ for the realizations $\beta_i, (\sigma_1, \sigma_2'), (\sigma_1', \sigma_2)$.

Lemma 2. *In Example 4 signals are substitutes if $2\varphi \leq \rho$ and complements if $2\varphi \geq \rho$.*

[22] To ensure that all probabilities are non-negative and sum to one, we have to choose the parameters $\rho, \varphi, \lambda \in (0, 1)$ such that $(1+\lambda)(\rho + 2\varphi) = 1$.

Proof. Individually, a signal is informative with probability $(1+\lambda)\rho$. If it is informative, it induces the same posteriors as a signal with likelihood ratios $1/\lambda$ and λ. Therefore, the marginal value of an individual signal is the same as the marginal value of a signal with likelihood ratios $1/\lambda$ and λ multiplied by the probability $(1+\lambda)\rho$.

Conditional on being informative, signals are perfectly correlated. Therefore, if one signal is available and is informative, then the other signal's marginal value is zero. On the other hand, if one signal is available and not informative, the other signal induces the same posteriors as a signal with likelihood ratios $1/\lambda$ and λ. Therefore, the marginal value of a signal given the other signal is already available is the same as the marginal value of a signal with likelihood ratios $1/\lambda$ and λ multiplied by the probability that the other signal is not informative, which is $1-(1+\lambda)\rho$.

It follows that signals are substitutes if and only if $1-(1+\lambda)\rho \leq (1+\lambda)\rho$, and signals are complements if and only if $1-(1+\lambda)\rho \geq (1+\lambda)\rho$. With $(1+\lambda)(\rho+2\varphi)=1$, these conditions are equivalent to $2\varphi \leq \rho$ resp. $2\varphi \geq \rho$. □

We shall now show that the example satisfies, for *all* parameter combinations, the necessary conditions in Proposition 4 for substitutes and Proposition 5 for complements. We shall thus show that neither set of conditions is sufficient. Consider first the conditions in Proposition 4. The realizations of signal $\tilde{s}_i$ which individually induce the most extreme posteriors are α_i and β_i. Conditional on such an extreme realization, signals are perfectly correlated. In particular, once an extreme realization is observed, no realization of the other signal changes the decision maker's belief. This means that the necessary condition for substitutes in Proposition 4 is met for both signals $\tilde{s}_i$. However, for $2\varphi > \rho$, signals are not substitutes.

Next, we show that the example satisfies all conditions of Proposition 5. It is easy to see that for any r and e for which some partition of S_1 and S_2 satisfies condition (i) of Proposition 5, the equation $rq \geq e$ is equivalent to $q(a) \geq 0.5$ or $q(a) \leq 0.5$. Without loss of generality we assume it is equivalent to $q(a) \geq 0.5$. For each of the two sets S_i there are four partitions that satisfy condition (i) of Proposition 5. We must have $\alpha_i \in S_i^R$ and $\beta_i \in S_i^E$, but σ_i and σ_i' can each be allocated to either of the two sets. This yields 16 pairs of partitions, all of which satisfy condition (ii) of Proposition 5. One can check that condition (iii) is violated by the two pairs of partitions for which σ_i and σ_i' are both in S_i^E for some $i \in \{1,2\}$ and σ_j and σ_j' are both in S_j^R for $j \neq i$. Ignoring these two cases, one can check that in all other cases there is some meaning reversal. For example, if $S_1^R = \{\alpha_1, \sigma_1\}$, $S_1^E = \{\sigma'_1, \beta_1\}$, $S_2^R = \{\alpha_2, \sigma'_2\}$, and $S_2^E = \{\sigma_2, \beta_2\}$, then meaning reversal occurs for the signal realizations (σ_1, σ'_2). This shows that the example satisfies the necessary condition for complementarity in Proposition 5. However, for $2\varphi < \rho$, signals are not complements.

The example also demonstrates that the sufficient condition in Proposition 10 for complementarity is not necessary. To see this, pick some x such that $\lambda/(1+\lambda) < x < 0.5$, and note that $S_i^\alpha(x) = \{\alpha_i, \sigma_i, \sigma'_i\}$ and $S_i^\beta(x) = \{\beta_i\}$ for $i = 1, 2$. The information structure for the derived signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ is shown in Figure 5. Observe that $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are perfectly correlated and therefore are not complements.

	$t_2^\alpha(x)$	$t_2^\beta(x)$
$t_1^\alpha(x)$	$1-\lambda\rho$	0
$t_1^\beta(x)$	0	$\lambda\rho$

$\omega = a$

	$t_2^\alpha(x)$	$t_2^\beta(x)$
$t_1^\alpha(x)$	$1-\rho$	0
$t_1^\beta(x)$	0	ρ

$\omega = b$

Figure 5: Signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ for Example 4 and partition $\{\{\sigma'_i, \sigma_i, \alpha_i\}, \{\beta_i\}\}$

Equilibrium Bidding in the Auction in Subsection 5.1

We begin with the case with no information disclosure. If there is no information disclosure, it is an equilibrium that the uninformed bidder chooses a bid randomly from the interval $[q_\beta(a), \pi(a)]$ with a cumulative distribution function:

$$G(p) = \frac{q_\alpha(a) - \pi(a)}{q_\alpha(a) - p}, \tag{64}$$

and the informed bidder bids $q_\beta(a)$ when he observes the realization β and chooses a bid randomly from the interval $(q_\beta(a), \pi(a)]$ with a cumulative distribution function:

$$F(p) = \frac{\bar{p}_1(\beta)\,(p - q_\beta(a))}{\bar{p}_1(\alpha)\,(q_\alpha(a) - p)} \tag{65}$$

when he observes the realization α.

To check that the uninformed bidder does not have incentives to deviate, we first show that she is indifferent between any bid in $[q_\beta(a), \pi(a)]$. Her expected utility of bidding

$p \in (q_\beta(a), \pi(a)]$ is equal to:

$$\bar{p}_1(\beta)\left(q_\beta(a) - p\right) + \bar{p}_1(\alpha) F(p) \left(q_\alpha(a) - p\right), \tag{66}$$

which is equal to zero by the definition of F. It is easy to see that a bid equal to $q_\beta(a)$ also gives the uninformed bidder zero expected utility. Finally, it is immediate that the uninformed bidder cannot gain by submitting a bid outside of $[q_\beta(a), \pi(a)]$.

That the informed bidder has no incentive to deviate upon observing β is obvious. If the informed bidder observes α, then his expected utility of bidding a price $p \in (q_\beta(a), \pi(a)]$ is equal to:

$$\left(q_\alpha(a) - p\right) G(p), \tag{67}$$

which is equal to $(q_\alpha(a) - \pi(a))$ by definition of G. Thus, the informed bidder is indifferent between any bid in $(q_\beta(a), \pi(a)]$. It is obvious that the informed bidder cannot gain by submitting bids outside of $(q_\beta(a), \pi(a)]$ when the informed bidder observes α.

The case with information disclosure is analogous, with the expected value of the good conditional on the publicly released signal realization taking the role of $\pi(a)$, provided that bidder 1's observation of $\tilde{s}_1$ provides additional information about the value of the good conditional on the realization of signal 2. If bidder 1's observation of $\tilde{s}_1$ is uninformative conditional on the realization of signal 2, then both bidders bid in equilibrium the expected value of the good conditional on the realization of signal 2.

Proof of Proposition 6 in the Case That Signals are Substitutes

If signals are substitutes, and if the two realizations of signal $\tilde{s}_2$ generate extreme beliefs in the sense of Proposition 2, then by publicly releasing the realization of signal $\tilde{s}_2$ the auctioneer ensures that the two bidders have identical beliefs about the value of the object. The bidders in a first price auction then engage in Bertrand competition, and it is obvious that no other policy can generate larger expected revenue for the auctioneer.

Next suppose that the two realizations of signal $\tilde{s}_1$ generate extreme beliefs in the sense of Proposition 2. We shall only deal with the case in which all joint signal realizations occur with positive probability, i.e. $\overline{p}(s_1, s_2) > 0$ for all $(s_1, s_2) \in \{\alpha, \beta\} \times \{\hat{\alpha}, \hat{\beta}\}$. When some joint signal realizations have zero probability, then the analysis that we present in the final two parts of the proof can be used. For example, $\overline{p}(\alpha, \hat{\beta}) = 0$ implies that $q_{\hat{\beta}}(a) = q_{\beta,\hat{\beta}}(a) = q_\beta(a)$ which means that $\hat{\beta}$ is also an extreme realization. Thus we have a special instance of the case in which α and $\hat{\beta}$ give rise to extreme beliefs. The case $\overline{p}(\beta,\hat{\alpha}) = 0$ is analogous.

If all joint signal realizations occur with positive probability expression (24) is equal to:

$$\pi(a) - \bar{p}(\alpha)(1-\bar{p}(\alpha))\left(q_\alpha(a) - q_\beta(a)\right), \tag{68}$$

which can be shown using the equality of the expected posterior and the prior. Similarly, expression (25) is equal to:

$$\pi(a) - \sum_{s_2 \in \{\hat{\alpha},\hat{\beta}\}} \bar{p}(s_2)\bar{p}(\alpha|s_2)(1-\bar{p}(\alpha|s_2))\left(q_{\alpha,s_2}(a) - q_{\beta,s_2}(a)\right), \tag{69}$$

where $\bar{p}(s_1|s_2) = \bar{p}(s_1, s_2)/\bar{p}(s_2)$. Because signal 2 is uninformative conditional on signal 1, (69) this is the same as:

$$\pi(a) - \sum_{s_2 \in \{\hat{\alpha},\hat{\beta}\}} \bar{p}(s_2)\bar{p}(\alpha|s_2)(1-\bar{p}(\alpha|s_2))\left(q_{\alpha}(a) - q_{\beta}(a)\right). \tag{70}$$

The concavity of $x(1-x)$ implies that this is not less than

$$\pi(a) - \bar{p}(\alpha)\left(1-\bar{p}(\alpha)\right)\left(q_\alpha(a) - q_\beta(a)\right). \tag{71}$$

which is equal to expression (68), as desired.

If signals are substitutes and the signals α and $\hat{\beta}$ give rise to extreme beliefs, as in Example 2, expression (24) can be written as:

$$\pi(a) - \bar{p}(\alpha)(q_\alpha(a) - \pi(a)) = \pi(a) - \bar{p}(\alpha,\hat{\alpha})(q_{\alpha,\hat{\alpha}}(a) - \pi(a)), \tag{72}$$

and expression (25) equals:

$$\pi(a) - \bar{p}(\alpha,\hat{\alpha})(q_{\alpha,\hat{\alpha}}(a) - q_{\hat{\alpha}}(a)), \tag{73}$$

and the former is larger than the latter because $q_{\hat{\alpha}}(a) > \pi(a)$.

If signals are substitutes and the signals β and $\hat{\alpha}$ give rise to extreme beliefs, expression (24) can be written as:

$$\pi(a)-\bar{p}(\alpha)(q_\alpha(a)-\pi(a)) = \pi(a)-\bar{p}(\beta)(\pi(a)-q_\beta(a)) = \pi(a)-\bar{p}(\beta,\hat{\beta})(\pi(a)-q_{\beta,\hat{\beta}}(a)) \tag{74}$$

and expression (25) is equal to:

$$\pi(a) - \bar{p}(\alpha, \hat{\beta})(q_{\alpha,\hat{\beta}}(a) - q_{\hat{\beta}}(a)) = \pi(a) - \bar{p}(\beta, \hat{\beta})(q_{\hat{\beta}}(a) - q_{\beta,\hat{\beta}}(a)), \tag{75}$$

and the former is larger than the latter because $q_{\hat{\beta}}(a) < \pi(a)$. $\square$