CONVEX DYNAMIC PROGRAMMING WITH (BOUNDED) RECURSIVE UTILITY

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Abstract. We consider convex dynamic programs with general (bounded) recursive utilities. The Contraction Mapping Theorem fails when the utility aggregator does not obey any discounting property. This failure occurs even with traditional aggregators and certainty equivalent specifications. However, the Bellman operator admits a unique fixed point when an interior policy is feasible. This happens because utility values are unique at interior consumption plans and, when an interior perturbation is feasible, drops in utility values can be avoided.

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1. Introduction

In spite of a growing interest for recursive utility (Koopmans [15]) in macroeconomics and finance (see, for instance, Backus et al. [3], Miao [25] and Skiadas [29]), concomitant progress in dynamic programming methods has not occurred in recent years. In this paper, we develop a suitable approach to convex programs for bounded recursive utilities. Acknowledging the failure of the canonical Contraction Mapping Theorem, our technique rests upon the theory of monotone concave operators (Krasnosel'skiĭ [17]). Under certain interiority restrictions, we prove existence of a unique fixed point to the implied Bellman operator.

The traditional theory for additively time-separable utility is grounded on Blackwell discounting condition (see, for instance, Stokey et al. [31, Theorem 3.3], Acemoglu [1, Theorem 6.9] and Stachurski [30, Theorem 6.3.5]). This guarantees that the Bellman operator is a contraction and so provides an efficient procedure for the computation of the value function as unique fixed point. The prevailing theory for recursive utility reproduces the logic of this approach (see, for instance, Lucas and Stokey [21], Becker and Boyd [5] and Miao [25]).

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Though the aggregator is not linear in continuation utility, as in the additive case, it is assumed that a suitable form of Blackwell discounting is still satisfied. The drawback of this is that commonly used aggregators do not obey this condition (and even fail any Lipschitz continuity) and, so, do not fall under the domain of Blackwell discounting. Furthermore, and independently of time preference, when some form of non-expected utility is introduced (as in Kreps and Porteus [18] and Epstein and Zin [13]), certainty equivalent might not satisfy the additivity property required to establish Blackwell discounting, rendering the Contraction Mapping Theorem unavailable even when utility is otherwise time additive.[1]

When the Contraction Mapping Theorem is inapplicable, existence and uniqueness of the fixed point to the Bellman operator have to be established alternatively. Under monotonicity, existence might be proved by appealing to Tarski Theorem. Uniqueness remains unverified and, as examples illustrate, might even fail in some circumstances. This is not a merely speculative concern. Indeed, implications for optimal policy might differ dramatically at distinct fixed points, thus causing ambiguity on positive and normative grounds, as well as complications for comparative statics. Furthermore, when the Bellman operator admits multiple fixed points, these might correspond to multiple utility functions all consistent with the given aggregator, so further contributing to the ambiguity in terms of welfare.

We consider monotone concave (Thompson) aggregators for recursive utility, introduced by Marinacci and Montrucchio [22].[2] One justification for this is that it is the natural hypothesis that ensures monotone concavity of the induced utility function. For convex recursive programs, we construct the canonical Bellman operator. Rather than approaching uniqueness directly, as when one applies (some form of) the Contraction Mapping Theorem, we decompose the argument into two parts. Monotonicity of the Bellman operator is exploited to establish existence of fixed points by convergence of monotone extreme orbits. When lower and upper limits coincide, this yields uniqueness and, on the side, an efficient computational tool. Otherwise, we relocate the multiplicity in the space of utility values and exploit monotone concavity of the aggregator, along with convexity of the feasible set.

The argument unfolds as follows. Suppose that the Bellman operator generates multiple value functions. The optimal plan for the greatest value function is feasible for the least value function. By optimality, the least value dominates the recursive value corresponding to the optimal plan for the greatest value function. This in general creates no inconsistency, because continuation values are determined by the least fixed point. However, when the utility aggregator admits a unique solution in utility, the least value ends up dominating the greatest value, thus yielding a contradiction. Monotone concave utility aggregators deliver

[1] An alternative approach is due to Streufert [32] (see also Osaki and Streufert [26] and, recently, Bich et al. [7]) and is based on the bicovergence property, a limiting condition ensuring that the substitution of consumption levels with the very best or worst outcome has no effect in the remote future. This approach is not, however, suitable for our purposes since we deal with general (bounded) aggregators for which bicovergence typically fails.

[2] In a recent paper, Marinacci and Montrucchio [23] have developed a general Tarski-type approach to monotone concave operators.

unique values at interior consumption plans, as established by Marinacci and Montrucchio [22], although the optimal plan need not be necessarily interior. Under an interiority condition on the feasible set, the plan can be slightly perturbed towards the interior, so revealing the contraction. Intuitively, the multiplicity can only occur at non-interior feasible plans as a drop in the value of the program. This loss of value, however, can be avoided (and so will be) by possibly shifting the feasible plan towards the interior.

We provide a theorem for stochastic recursive utility, so as to encompass some relevant formulations of Kreps and Porteus [18] and Epstein and Zin [13]. In the construction of the convex program, we remain close to Stokey et al. [31]. Preliminarily, however, we present an analysis of recursive utility in the purely deterministic case. The role of this part, which might be of independent interest, is to uncover the source of potential multiplicity and its implications for dynamic programming. When the aggregator generates multiple utility functions, the Bellman operator admits a fixed point for each of the associated value functions.

The paper is organised as follows. In section 2 we introduce monotone concave aggregators and discuss some facts about multiplicity of the recursively generated utility functions in a simple deterministic setting. In section 3 we present the general recursive program, set out the basic assumptions on fundamentals and define the Bellman operator. In section 4 we show that, under a mild interiority assumption, the value function is the unique fixed point for the Bellman operator and that this value function can be computed by iterating the Bellman operator. In section 5 we present a perturbation method with certain implications for computational work. Finally, in section 6 we relate our approach to the previous literature. All proofs are gathered in an appendix.

2. Multiple utilities

We begin with remarking some basic facts about multiplicity of recursive utility functions generated by a given aggregator. To simplify, we consider the purely deterministic case. The analysis is largely inspired by Marinacci and Montrucchio [22], which provides the initial approach to monotone concave (Thompson) aggregators.[3] They show that utility values are uniquely determined at interior consumption plans. We further develop the analysis so as to establish the existence of a unique upper semicontinuous utility function. Despite of this, other utility functions might be consistent with the given monotone concave aggregator, as illustrated by examples. The relevant implication for dynamic programming is that the Bellman operator delivers a value function for each recursive utility function. Thus, in general, multiple utility functions translate into a multiplicity of fixed points for the Bellman operator.

Time is indexed by t on the infinite horizon $\mathbb{T} = \{0, 1, 2, \ldots\}$. The linear space of all bounded real-valued sequences is L, with positive cone L^+, interpreted as the consumption

[3]See also Becker and Rincón-Zapatero [6], Borovička and Stachurski [8], Balbus [4] and Martins-da-Rocha and Vailakis [24] for other contributions in the same vein.

space. The utility aggregator is a map $W : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. A utility function $U : L^+ \to \mathbb{R}^+$ is recursively generated if, for every consumption plan c in L^+,

$$U(c) = W(c_0, U(\sigma c)), \tag{K}$$

where $\sigma : L \to L$ is the shift operator (that is, $(\sigma c)_t = c_{t+1}$ for every t in $\mathbb{T}$), permitting the decomposition $c = (c_0, \sigma c)$ in $L = \mathbb{R} \times L$. Continuity refers to the (relative) product topology on L^+.

We shall discuss under which conditions Koopmans equation (K) admits a unique solution and provide some insight into multiplicity. The traditional approach requires Blackwell discounting condition: for some constant β in $(0, 1)$, given any c in $\mathbb{R}^+$,

$$|W(c, v') - W(c, v'')| \leq \beta |v' - v''|. \tag{D}$$

Under such a condition, Banach Contraction Mapping Theorem guarantees existence of a unique recursively generated utility function (see, among others, Lucas and Stokey [21], Boyd [9], Becker and Boyd [5], Durán [11, 12] and Miao [25]). What happens more in general?

Blackwell discounting fails for common utility aggregators, as documented by Marinacci and Montrucchio [22]. Furthermore, without any restriction on the utility aggregator, uniqueness of the recursively generated utility does not arise naturally. We here assume that the utility aggregator $W : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous monotone concave map such that, for every c in $\mathbb{R}^+$, there is some v in $\mathbb{R}^+$ satisfying $W(c, v) < v$ and, for every c in $\mathbb{R}^{++}$, $W(c, 0) > 0$. Marinacci and Montrucchio [22] refer to such aggregators as Thompson aggregators.[4]

The method developed by Marinacci and Montrucchio [22] consists in studying an auxiliary equation in utility values, rather than solving directly Koopmans equation (K). Adhering to this approach, for a given consumption plan c in L^+, we consider the operator $T_c : L^+ \to L^+$ that is defined, at every t in $\mathbb{T}$, as

$$(T_c v)_t = W(c_t, v_{t+1}). \tag{M}$$

By proving existence of fixed points to this operator, we recover recursively generated utility functions and, in particular, we establish existence of a unique upper semicontinuous utility function. Some preliminary examples show that, in general, overall uniqueness fails. Specifically, the examples document the existence of multiple solutions in terms of utility values (condition (M)). However, this sort of multiplicity is sufficient to separate the least from the greatest utility function in Proposition 1, thus (non-constructively) proving the existence of multiple solutions to Koopmans equation (condition (K)).

[4]More precisely, Marinacci and Montrucchio [22] only require a form of concavity with respect to zero in $\mathbb{R}^+$ for given c in $\mathbb{R}^+$ (see their assumption (W-iii)). Joint concavity is instead necessary for our dynamic programming application, though their weaker requirement would be sufficient for this preliminary analysis.

Example 1 (Multiplicity at $c = 0$)**.** This is a simple example of multiplicity. Consider the utility aggregator

$$W(c, v) = \sqrt{c + \beta v}.$$

Notice that condition (D) certainly fails in this case, as so does Lipschitz continuity. When consumption vanishes, $c = 0$, utility values are solutions to the equation

$$v(v - \beta) = 0.$$

Hence, multiple utility functions cannot be avoided. In particular, by Proposition 1, there are distinct recursively generated utility functions such that $\underline{U}(0) = 0$ and $\bar{U}(0) = \beta$, respectively, where utility is evaluated at the zero consumption plan in L^+.

Example 2 (KDW aggregator (Koopmans et al. [16]))**.** Another more relevant example is given by the Koopmans-Diamond-Williamson aggregator (see Martins-da-Rocha and Vailakis [24]) of the form

$$W(c, v) = \frac{\log(1 + c + \beta v)}{\theta},$$

under the assumption that $\beta > \theta > 0$. It is easy to verify that, when $c = 0$, the utility equation admits distinct values. Indeed, the implied equation is

$$\theta v = \log(1 + \beta v).$$

One solution is $v = 0$. Furthermore, as $\beta > \theta > 0$, for any sufficiently small $v > 0$,

$$\theta < \frac{\log(1 + \beta v)}{v},$$

whereas

$$\theta > \lim_{v \to \infty} \frac{\log(1 + \beta v)}{v} = 0.$$

Hence, another solution exists by the Intermediate Value Theorem. As in the previous example, by Proposition 1, there are at least two distinct recursively generated utility functions.

Example 3 (Multiple utility values)**.** This example is slightly more elaborated. Take the monotone concave aggregator

$$W(c, v) = c + \min\{v, \alpha + \beta v\},$$

where $1 > \beta > 0$ and $\alpha > 1$. Consider the vanishing consumption plan $(\delta^t)_{t \in \mathbb{T}}$ in L^+, where $1 > \delta > 0$. Assuming that $\alpha > 1$ is chosen sufficiently large, the utility equation (M) takes the form

$$v_t = \delta^t + v_{t+1}.$$

This difference equation is solved by the bounded sequence

$$v_t = \delta^t \left(\frac{1}{1 - \delta}\right) + \lambda,$$

where $\lambda \geq 0$ is arbitrarily chosen so as to satisfy the initial boundary condition

$$\left(\frac{1}{1-\delta}\right) + \lambda \leq \alpha.$$

As $\lambda \geq 0$ is arbitrarily chosen in an open interval, there are actually infinitely many recursively generated utility functions.

Proposition 1 (Recursive utility)**.** *Under the stated assumptions, there exists the least (respectively, the greatest) recursively generated monotone utility function* $\underline{U} : L^{+} \to \mathbb{R}^{+}$ *(respectively,* $\bar{U} : L^{+} \to \mathbb{R}^{+}$*) which is lower (respectively, upper) semicontinuous on uniformly bounded intervals. These utility functions coincide at any uniformly interior consumption plan* c *in* L^{+}*. Furthermore, there exists no other recursively generated monotone utility function which is upper semicontinuous on uniformly bounded intervals.*

We now turn into dynamic programming implications. Feasibility is described by a correspondence $G : X \twoheadrightarrow X$ with non-empty values, where X is some non-empty set. This is complemented by a *bounded* return function $c : D \to \mathbb{R}^{+}$, where $D = \{(x, y) \in X \times X : y \in G(x)\}$. The Bellman operator $T : \mathcal{B}^{+}(X) \to \mathcal{B}^{+}(X)$ is defined by

$$(Tv)(x) = \sup_{y \in G(x)} W(c(x, y), v(y)),$$

where $\mathcal{B}(X)$ is the space of bounded real-valued functions on X.

Given an initial state x_0 in X, a feasible plan x in $\mathit{\Pi}(x_0)$ is a sequence of elements of X such that, at every t in $\mathbb{T}$, x_{t+1} lies in $G(x_t)$. Consistently, the space of feasible consumption plans $C(x_0)$ contains any c in L^{+} satisfying, for some feasible plan x in $\mathit{\Pi}(x_0)$, at every t in $\mathbb{T}$,

$$c_t = c(x_t, x_{t+1}).$$

Under such premises, we establish the Principle of Optimality. Notice that the only assumption we retain from the previous analysis is the continuity of the utility aggregator $W : \mathbb{R}^{+} \times \mathbb{R}^{+} \to \mathbb{R}^{+}$. In particular, we only need that, for every bounded set V in $\mathbb{R}^{+}$,

$$\sup_{v \in V} W(c, v) \geq W\left(c, \sup_{v \in V} v\right).$$

Proposition 2 (Principle of optimality)**.** *Given any recursively generated utility function* $U : L^{+} \to \mathbb{R}^{+}$*, the associated value function* v *in* $\mathcal{B}^{+}(X)$ *is a fixed point of the Bellman operator, where*

$$v(x) = \sup_{c \in C(x)} U(c). \tag{*}$$

This proposition suggests that the Bellman operator admits multiple fixed points, one for each recursively generated utility function. We provide an example that this multiplicity is effective under conditions which represent the minimal displacement from Blackwell discounting (condition (D)), even when the utility aggregator satisfies the monotone concavity property.

Example 4 (Multiple value functions)**.** We consider a sort of cake-eating program in the deterministic case. Suppose that $X = [0, 1]$, $G(x) = [0, x]$ and $c(x, y) = x - y$. The utility aggregator is

$$W(c, v) = c + \min\{v, \alpha + \beta v\},$$

where $1 > \beta > 0$ and $\alpha > 1$. Notice that this utility aggregator admits a unitary Lipschitz constant, as

$$|W(c, v') - W(c, v'')| \leq |v' - v''|.$$

Thus, it is the minimal violation of Blackwell discounting (condition (D)). Under the condition that $v(x) \leq \alpha$ for every x in X, a solution to the Bellman equation is given by

$$v(x) = \sup_{y \in G(x)} x - y + v(y).$$

One fixed point is $v(x) = x$, in which case immediate consumption is optimal in the recursive program (though any other plan is also optimal). Another obvious fixed point is, for instance, $v(x) = \alpha x$. Notice that, in this case, postponing consumption is optimal in the recursive program (which is not saying that no consumption is optimal for some recursively generated utility function, because the utility might be discontinuous). It is worth noticing that the largest utility function is given by

$$\bar{U}(c) = \frac{\alpha}{1-\beta} + \sum_{t \in \mathbb{T}} \beta^t c_t,$$

which implies the value function

$$\bar{v}(x) = \frac{\alpha}{1-\beta} + x.$$

Immediate consumption is optimal for this value function.

The peculiarity of this example is that the feasible set does not contain an interior consumption plan. As Marinacci and Montrucchio [22] show, utility values are uniquely determined at interior consumption plans. When the feasible set has an interior plan, multiple utility values are irrelevant for the Bellman operator, because any drop in utility can be avoided by possibly shifting consumption towards the interior. This perturbation requires convexity of the feasible correspondence. The rest of the paper is devoted to establish uniqueness under interiority in a more general framework.

3. Recursive program

3.1. **Uncertainty.** Uncertainty is described by a Markov transition $Q : Z \to \Delta(Z)$, where Z is a compact metric space, endowed with its Borel σ-algebra $\mathcal{Z}$, and $\Delta(Z)$ is the space of probability measures on the measurable space $(Z, \mathcal{Z})$ (see Aliprantis and Border [2, Definition 19.8]). For a given initial probability measure μ_0 in $\Delta(Z)$, the transition induces a Markov process on the canonical probability space $(\Omega, \mathcal{F}, \mu)$, where $\Omega = Z^\infty$ is the product space endowed with its σ-algebra $\mathcal{F} = \mathcal{Z}^\infty$. As usual, a filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ is generated so that, at every t in $\mathbb{T}$, $\mathcal{F}_t$ is the least σ-algebra on Ω with respect to which the natural projection $z^t : \Omega \to Z^{t+1}$ is $\mathcal{Z}^{t+1}$-measurable, where

$$z^t(\omega) = z^t(z_0, \dots, z_t, z_{t+1}, \dots) = (z_0, \dots, z_t).$$

Notice that, as we do not specify the form of the certainty equivalent operator, Markov transitions are relevant only insofar as they identify negligible events.

3.2. **Basic notation.** Given a measurable metric space $(S, \mathcal{S})$, denote by $\mathcal{B}(S)$ the space of all bounded real-valued maps. Let also $\mathcal{M}(S)$ be the space of all bounded measurable maps with values in $\mathbb{R}$, where $\mathbb{R}$ is endowed with its Borel σ-algebra, and $\mathcal{C}(S)$ be the space of all bounded continuous maps with values in $\mathbb{R}$. Notice that, by construction, we have the relation

$$\mathcal{C}(S) \subset \mathcal{M}(S) \subset \mathcal{B}(S).$$

3.3. **Feasibility restrictions.** Production is described by a closed convex set X of some Banach space and a feasible correspondence $G : X \times Z \twoheadrightarrow X$, which is continuous with non-empty compact values. For every fixed z in Z, correspondence G is also convex, that is, $x \mapsto G(x, z)$ has a convex graph in $X \times X$. This representation is complemented by a *bounded* continuous return map $c : D \to \mathbb{R}^+$, where the set D is given by

$$D = \{(x, y, z) \in X \times X \times Z : y \in G(x, z)\}.$$

For every fixed z in Z, the restricted map $(x, y) \mapsto c(x, y, z)$ is concave. The return is interpreted as consumption or instantaneous utility depending on the application.

3.4. **Recursive utility.** Recursive preferences are given as a utility aggregator and a certainty equivalent operator. The *utility aggregator* $W : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous monotone concave map such that, for every c in $\mathbb{R}^+$, there is some v in $\mathbb{R}^+$ satisfying $W(c, v) < v$ and, for every c in $\mathbb{R}^{++}$, $W(c, 0) > 0$. The *certainty equivalent operator* $M^* : \mathcal{M}^+(Z) \to \mathcal{M}^+(Z)$ is such that, for every measurable bounded map f in $\mathcal{M}^+(Z)$, we have that $(M^* f)$ is a measurable bounded map in $\mathcal{M}^+(Z)$. The certainty equivalent is a monotone map such that $(M^* \eta) = \eta$ for every constant map η in $\mathcal{M}^+(Z)$. It is also restricted by the following assumptions.

Assumption 1 (Subhomogeneity)**.** *For every f in $\mathcal{M}^+(Z)$, given any λ in $(0, 1)$,*

$$\lambda(M^* f) \leq (M^* \lambda f).$$

Remark 1 (Subhomogeneity)**.** Consider the certainty equivalent operator defined by

$$(M^* f)(z) = \phi^{-1}\left(\int_Z \phi(f(\bar{z}))\, Q(z)(d\bar{z})\right),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing smooth map. Subhomogeneity (Assumption 1) is satisfied whenever relative risk aversion is (weakly) increasing, as proved by Marinacci and Montrucchio [22, Appendix C].

Assumption 2 (Feller Property)**.** *For every f in $\mathcal{C}^+(Z)$, (M^*f) is also in $\mathcal{C}^+(Z)$.*

Assumption 3 (Negligible events)**.** *For every (f', f'') in $\mathcal{M}^+(Z) \times \mathcal{M}^+(Z)$,*

$$(M^*f')(z) = (M^*f'')(z) \ \ \text{if } Q(z)(\{\bar{z} \in Z : f'(\bar{z}) = f''(\bar{z})\}) = 1.$$

Assumption 4 (Monotone continuity)**.** *For any weakly decreasing (increasing) sequence $(f^n)_{n\in\mathbb{N}}$ in $\mathcal{C}^+(Z)$ pointwise converging to f in $\mathcal{M}^+(Z)$, the sequence $(M^*f^n)_{n\in\mathbb{N}}$ in $\mathcal{C}^+(Z)$ pointwise converges to (M^*f) in $\mathcal{M}^+(Z)$.*

Assumption 5 (Uniform continuity)**.** *For any sequence $(f^n)_{n\in\mathbb{N}}$ in $\mathcal{C}^+(Z)$ uniformly converging to f in $\mathcal{C}^+(Z)$, the sequence $(M^*f^n)_{n\in\mathbb{N}}$ in $\mathcal{C}^+(Z)$ uniformly converges to (M^*f) in $\mathcal{C}^+(Z)$.*

Remark 2 (Uniform continuity)**.** The only role of Assumption 5 is to establish the Feller property for the extended certainty equivalent operator (Lemma 1). As an alternative, the statement in Lemma 1 could be regarded as a primitive hypothesis and Assumption 5 could be dispensed with. This is, for instance, the approach taken in Marinacci and Montrucchio [22, Assumption (A-v)]. Notice that, for Assumption 5, it will be sufficient that, for every f in $\mathcal{C}^+(Z)$,

$$(M^*(f+\eta)) - (M^*f) \leq \eta,$$

where η denotes any constant map in $\mathcal{M}^+(Z)$. This is the so-called constant subadditivity property (*e.g.*, Marinacci and Montrucchio [22, Assumption (M-iii)]).

We can naturally extend the primitive certainty equivalent to the space $\mathcal{M}^+(X \times Z)$, where X is endowed with its Borel σ-algebra $\mathcal{X}$. The proof reproduces classical arguments (for instance, Stokey et al. [31, Lemma 9.5] or Stachurski [30, Lemma 11.2.3]). Notice that any jointly measurable function is separately measurable (see Aliprantis and Border [2, Theorem 4.48]).

Lemma 1 (Feller property)**.** *The extended certainty equivalent operator $M : \mathcal{M}^+(X \times Z) \to \mathcal{B}^+(X \times Z)$ satisfies the Feller property, that is, it carries $\mathcal{C}^+(X \times Z)$ into $\mathcal{C}^+(X \times Z)$, where*

$$(Mf)(x,z) = (M^*f(x,\cdot))(z).$$

3.5. **Bellman operator.** The Bellman operator $T : \mathcal{M}^+(X \times Z) \to \mathcal{B}^+(X \times Z)$ is given as

$$(Tv)(x,z) = \sup_{y \in G(x,z)} W(c(x,y,z), (Mv)(y,z)),$$

where the certainty equivalent operator $M : \mathcal{M}^+(X \times Z) \to \mathcal{B}^+(X \times Z)$ is extended according to the principle in Lemma 1. We here prove basic properties by means of well-established arguments.

Lemma 2 (Bellman operator)**.** *The Bellman operator $T : \mathcal{M}^+(X \times Z) \to \mathcal{B}^+(X \times Z)$ is a monotone map satisfying $T(\mathcal{C}^+(X \times Z)) \subset \mathcal{C}^+(X \times Z)$.*

Remark 3 (Measurability)**.** It is worth noticing that, because of a well-known potential failure of measurability, the Bellman operator maps $\mathcal{M}^+(X \times Z)$ into $\mathcal{B}^+(X \times Z)$. The Feller Property (Assumption 2) guarantees that $\mathcal{C}^+(X \times Z)$ is invariant for the Bellman operator. Its unique fixed point will be in this invariant space under an additional Interiority Hypothesis (Assumption 6).

3.6. **Plans.** To avoid complications related to the existence of measurable selections, we enlarge the space of feasible plans. Our approach simplifies the analysis and makes the paper self-contained.[5] Notice that this diversion only serves in intermediate steps: the optimal correspondence would in fact admit a measurable selection (as in Stokey et al. [31, Theorem 9.6]).

A plan is simply a sequence of maps which depend on observed partially histories of shocks. In particular, it is equivalent to endow Z with its *discrete* σ-algebra $\mathcal{P}$ (the power set, including the empty set). Thus, at every t in $\mathbb{T}$, $\mathcal{G}_t$ is the least σ-algebra on Ω with respect to which the natural projection $z^t : \Omega \to Z^{t+1}$ is $\mathcal{P}^{t+1}$-measurable. This generates a more permissive filtration $(\mathcal{G}_t)_{t\in\mathbb{T}}$ on the measurable space $(\Omega, \mathcal{G})$, where $\mathcal{G} = \mathcal{P}^\infty$. Given an initial state (x_0, z_0) in $X \times Z$, a *feasible plan* is a $(\mathcal{G}_t)_{t\in\mathbb{T}}$-adapted process (x, y) with values in $(X \times X, \mathcal{X} \times \mathcal{X})$ such that, at every t in $\mathbb{T}$,

$$y_t(\omega) \in G(x_t(\omega), z_t(\omega))$$

and

$$x_{t+1}(\omega) = y_t(\omega).$$

The space of all feasible plans, beginning from the initial state (x_0, z_0) in $X \times Z$, is $\Pi(x_0, z_0)$. An initial state (x_0, z_0) in $X \times Z$ is *trivial* if, for every z in Z, $G(x_0, z) = \{x_0\}$ and $c(x_0, x_0, z) = 0$.

4. Uniqueness

We here show that, under an additional assumption, the Bellman operator admits a unique fixed point. This requires, for any non-trivial initial condition, the existence of a feasible plan generating strictly positive utility uniformly along any path. It seems a fairly unrestrictive requirement ruling out situations in which depreciation or large volatility preclude to secure a minimum value in consumption out of initial resources. We shall provide examples of validity of such a condition. Failure occurs in Example 4. To prevent a misunderstanding, we also clarify that our assumption does not imply an interior optimal plan (see Example 6).

Interiority holds true except for negligible events. To capture this, we say that an event E in $\mathcal{F}$ is *exhaustive* from an initial state z_0 in Z if $z^0(E) = z_0$ and, at every t in $\mathbb{T}$, $\mathcal{F}_t(E)$

[5] An alternative route would rely on sophisticated measurable selection theorems (for instance, Wagner [34, Theorem 9.1(iii)] or Burgess and Maitra [10, Theorem A]). However, our method requires an appeal to Tarski Theorem and it is unclear whether this would be feasible in the alternative approach.

is a measurable set in $\mathcal{F}_t$ such that, for any ω in $\mathcal{F}_t(E)$,

$$Q(z_t(\omega))\left(\left\{z \in Z : \left(z^t(\omega), z\right) \in \mathcal{F}_{t+1}(E)\right\}\right) = 1,$$

where

$$\mathcal{F}_t(E) = \cap \{E_t \in \mathcal{F}_t : E \subset E_t\}.^{6}$$

An exhaustive set is invariant for the Markov transition, up to negligible events, when the process is initiated from state z_0 in Z.

Example 5 (Negligible events)**.** Suppose that $Z = \{u, d\}$ and that the Markov transition is cyclical, that is, it deterministically moves into the other state from each state. The single cyclic orbit beginning from a given state forms an exhaustive event, though Ω contains all infinite sequences of states $\{u, d\}$.

Assumption 6 (Interiority)**.** *Given any non-trivial initial state (x_0, z_0) in $X \times Z$, there exists a feasible plan (x, y) in $\Pi(x_0, z_0)$ such that, for some exhaustive event E in $\mathcal{F}$ from initial state z_0 in Z,*

$$\liminf_{t \in \mathbb{T}} \inf_{\omega \in \mathcal{F}_t(E)} c(x_t(\omega), y_t(\omega), z_t(\omega)) > \epsilon > 0.$$

Remark 4 (Eventually interior)**.** The interiority assumption only requires that a utility value be secured eventually in finite time. Hence, an arbitrarily long initial phase might be used to restore productive capacity or wealth, so as to ensure a perpetual minimum value in utility.

Example 6 (Non-interior optimal plan)**.** Consider again the utility aggregator given by

$$W(c, v) = c + \min\{v, \alpha + \beta v\},$$

where $1 > \beta > 0$ and $\alpha > 1$. Suppose that $X = [0, 1]$, $G(x) = \left[0, \rho^{-1}x\right]$ and $c(x, y) = x - \rho y$, where $1 > \rho > \beta$. Assumption 6 is satisfied because, for any initial $x_0 > 0$ in X, a constant consumption $c_t = (1 - \rho)x_0 > 0$ can be sustained in all periods t in $\mathbb{T}$. However, the only solution to the Bellman operator (apart from multiple values at the trivial initial state $x = 0$ in X) is given by

$$v^*(x) = \frac{\alpha}{1 - \beta} + x.$$

The only optimal policy is thus $g(x) = 0$ for every x in X.

Notice that, in establishing the validity of Assumption 6, it is not necessary to verify measurability of the feasible plan. To further help applicability, we provide simple conditions under which Assumption 6 holds true. These are substantially more restrictive than necessary, although they are satisfied in the stochastic optimal growth program (Example 7) and in the optimal saving program (Example 8).

[6] Throughout the analysis, with some abuse of notation, we identify a measurable set E_t in $\mathcal{F}_t$ with its truncation $z^t(E_t)$ in $\mathcal{Z}^{t+1}$. This map is indeed a bijection.

Lemma 3 (Operational criterion)**.** *Under the hypothesis that X is the positive cone of some Euclidean space, Assumption 6 holds true when:*

(a) *for each z in Z, the return function $c : D \to \mathbb{R}^+$ is strictly concave;*

(b) *the feasible correspondence $G : X \times Z \twoheadrightarrow X$ is such that*

$$0 \in \bigcap_{z \in Z} G(0, z);$$

(c) *for every non-trivial (x_0, z_0) in $X \times Z$, there is a non-zero x in $G(x_0, z_0)$ such that*

$$x \in \bigcap_{z \in Z} G(x, z).$$

Example 7 (Stochastic single-sector growth)**.** Suppose that the bounded production function is $f : X \times Z \to X$, where $X = \mathbb{R}^+$. The production function is smoothly concave with $f(0, z) = 0$ for every z in Z. The feasible correspondence is

$$G(x, z) = [0, f(x, z)],$$

whereas consumption is determined according to

$$c(x, y, z) = f(x, z) - y,$$

which is a bounded continuous concave map (because production is bounded). Suppose that the production function satisfies

$$\lim_{x \to 0} \inf_{z \in Z} \nabla_x f(x, z) > 1.$$

Under this condition, for any given non-trivial (x_0, z_0) in $X \times Z$, there is a sufficiently small $x_0 > x > 0$ such that $f(x, z) > x$ for every z in Z. Indeed, concavity implies

$$\inf_{z \in Z} \nabla_x f(x, z) x \leq \nabla_x f(x, z) x \leq f(x, z).$$

Hence, Assumption 6 is satisfied.

Example 8 (Optimal saving)**.** The feasible correspondence is given as

$$G(x, z) = [0, e(z) + \rho(z) x],$$

whereas consumption is determined by the budget restriction, so that

$$c(x, y, z) = c(e(z) + \rho(z) x - y),$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ is a (non-trivial) bounded concave continuous map (better interpreted as instantaneous utility in this case). Here, $e : Z \to \mathbb{R}^+$ is the labor income and $\rho : Z \to \mathbb{R}^+$ is the stochastic return on savings. Assumption 6 is trivially satisfied whenever $\inf_{z \in Z} e(z) > 0$.

We now establish basic uniqueness exploiting interiority, which is the major contribution of this paper. The intuition is more conveniently understood in a deterministic environment. Suppose that the Bellman operator admits ordered fixed points. For some initial condition, take an optimal recursive plan for the dominant value function (such a plan exists if the value function is upper semicontinuous). This implies, at every t in $\mathbb{T}$,

$$\bar{v}_t = W\left(\bar{c}_t, \bar{v}_{t+1}\right),$$

where $\bar{c}$ in L^+ is the consumption plan and $\bar{v}$ in L^+ is the sequence of utility values recursively generated by the dominant fixed point. The same plan is feasible under the dominated fixed point. By optimality, this delivers, at every t in $\mathbb{T}$,

$$\underline{v}_t \geq W\left(\bar{c}_t, \underline{v}_{t+1}\right),$$

where $\underline{v}$ in L^+ is the sequence of utility values recursively generated by the dominated fixed point. In terms of the operator given by condition (M), $\bar{v} \leq (T_{\bar{c}}\bar{v})$ and $\underline{v} \geq (T_{\bar{c}}\underline{v})$. When the operator admits a unique fixed point, these conditions permit to locate it unambiguously, so that

$$\bar{v}_0 \leq U\left(\bar{c}\right) \leq \underline{v}_0,$$

thus yielding a contradiction. However, in general, values in utility are not unambiguously identified, unless the consumption plan is interior. Convexity of the feasible set permits a perturbation towards the non-empty interior, which restores uniqueness of utility values and, hence, delivers the contradiction.

Lemma 4 (Basic uniqueness)**.** *Under the Interiority Hypothesis (Assumption 6), the Bellman operator does not admit fixed points $\underline{v}$ and $\bar{v}$ in $\mathcal{M}^+(X \times Z)$, with $\bar{v} : X \times Z \to \mathbb{R}$ being upper semicontinuous, such that, for some non-trivial (x_0, z_0) in $X \times Z$,*

$$\underline{v}\left(x_0, z_0\right) < \bar{v}\left(x_0, z_0\right).$$

We next prove that the limits of lower and upper orbits are fixed points of the Bellman operator. This only requires well-established arguments (see, for instance, Streufert [32, 33] and Rustichini [28]). The proof of upper convergence is comparatively more involved.

Lemma 5 (Lower convergence)**.** *For any map $\underline{\eta}$ in $\mathcal{C}^+(X \times Z)$ such that $\left(T\underline{\eta}\right) \geq \underline{\eta}$, the map $\underline{v}$ in $\mathcal{M}^+(X \times Z)$ is a fixed point of the Bellman operator, where*

$$\underline{v}\left(x, z\right) = \lim_{n \in \mathbb{N}} \left(T^n \underline{\eta}\right)\left(x, z\right).$$

Lemma 6 (Upper convergence)**.** *For any map $\bar{\eta}$ in $\mathcal{C}^+(X \times Z)$ such that $(T\bar{\eta}) \leq \bar{\eta}$, the map $\bar{v}$ in $\mathcal{M}^+(X \times Z)$ is a fixed point of the Bellman operator, where*

$$\bar{v}\left(x, z\right) = \lim_{n \in \mathbb{N}} \left(T^n \bar{\eta}\right)\left(x, z\right).$$

We are now in the condition of proving that the Bellman operator admits a (substantially) unique fixed point. The only residual multiplicity arises from the potentially multiple values in utility when consumption vanishes. This multiplicity can be regarded of no allocative relevance, because it only occurs when the only feasible plan is trivial. To the purpose of convergence, let $\underline{\eta}$ in $\mathcal{C}^{+}(X \times Z)$ be the constant map corresponding to the *largest* solution to

$$v = W(0, v).$$[7]

Similarly, let $\bar{\eta}$ in $\mathcal{C}^{+}(X \times Z)$ be any constant map greater than or equal to the *only* solution to

$$v = W(\kappa, v),$$

where $\kappa = \sup_{(x,y,z) \in D} c(x, y, z)$.[8]

Proposition 3 (Uniqueness)**.** *Under the Interiority Hypothesis (Assumption 6), the Bellman operator admits a unique fixed point in the interval* $[\underline{\eta}, \bar{\eta}] \subset \mathcal{M}^{+}(X \times Z)$*. This fixed point* v^{*} *lies in* $\mathcal{C}^{+}(X \times Z)$*. Furthermore, for every* v *in* $[\underline{\eta}, \bar{\eta}]$*, the orbit* $(T^{n} v)$ *in* $[\underline{\eta}, \bar{\eta}]$ *converges uniformly to* v^{*} *on any compact set* $K \subset X \times Z$*.*

5. A perturbation method

We here reconsider the Bellman equation *without* imposing interiority (Assumption 6). For any sufficiently small $\epsilon > 0$, we introduce the perturbed utility aggregator $W_{\epsilon} : \mathbb{R}^{+} \times \mathbb{R}^{+} \to \mathbb{R}^{+}$ given by

$$W_{\epsilon}(c, v) = W_{0}(c, v) + \epsilon. \tag{$\dagger$}$$

We show that the Bellman operator admits a unique fixed point for the perturbed utility aggregator. Furthermore, as $\epsilon > 0$ vanishes, the value function approaches the greatest fixed point of the unperturbed program.

Lemma 7 (Perturbation)**.** *For any sufficiently small* $\epsilon > 0$*, the perturbed Bellman operator* $T_{\epsilon} : \mathcal{M}^{+}(X \times Z) \to \mathcal{B}^{+}(X \times Z)$ *admits a unique fixed point* v_{ϵ} *in* $\mathcal{C}^{+}(X \times Z)$*.*

We finally provide an existence theorem for the unperturbed program by considering the limit. The advantage of this approach is prominently computational. The fixed point of the perturbed operator can be computed as the monotone limit of the orbit $(T_{\epsilon}^{n} 0)_{n \in \mathbb{N}}$ in $\mathcal{C}^{+}(X \times Z)$ converging to v_{ϵ} in $\mathcal{C}^{+}(X \times Z)$. A second monotonically decreasing orbit delivers the greatest fixed point for the unperturbed program.

Proposition 4 (Limit)**.** *The Bellman operator* $T_{0} : \mathcal{M}^{+}(X \times Z) \to \mathcal{B}^{+}(X \times Z)$ *admits the greatest (upper semicontinous) fixed point* v_{0} *in* $\mathcal{M}^{+}(X \times Z)$ *given by*

$$v_{0}(x, z) = \lim_{\epsilon \to 0} v_{\epsilon}(x, z).$$

---

[7] The largest solution exists by monotone concavity, because $W(0, \eta) < \eta$ for some η in $\mathbb{R}^{+}$.

[8] Notice that monotone concavity implies $\bar{\eta} \geq W(c, \bar{\eta})$ for every c in $\mathbb{R}^{+}$ such that $\kappa \geq c$.

6. Comments

We briefly relate our approach to the previous literature. In particular, as this is the most delicate step in the argument, we reconsider the issue of unique utility values generated by the given aggregator. We explain how alternative restrictions in previous studies ensure such a uniqueness. To simplify, we present our discussion in the simple bounded deterministic environment, although in a form that can be immediately extended to uncertainty and unbounded processes.

For a consumption plan c in L^{+}, suppose that condition (M) admits distinct solutions $\bar{v}$ and $\underline{v}$ in L^{+}. For every n in $\mathbb{N}$, $\bar{v} = (T_c^n \bar{v})$ and $\underline{v} = (T_c^n \underline{v})$, which deliver

$$|\bar{v}_0 - \underline{v}_0| = \left|(T_c^n \bar{v})_0 - (T_c^n \underline{v})_0\right|.$$

Monotonicity implies

$$|\bar{v}_0 - \underline{v}_0| \leq \left|(T_c^n \bar{v} \vee \underline{v})_0 - (T_c^n \bar{v} \wedge \underline{v})_0\right|.$$

Thus, uniqueness ultimately obtains whenever

$$\text{(T)} \qquad \liminf_{n\to\infty} \left|(T_c^n \bar{v} \vee \underline{v})_0 - (T_c^n \bar{v} \wedge \underline{v})_0\right| = 0.$$

This is a sort of transversality-like condition, according to the terminology of Le Van and Morhaim [19], Rincón-Zapatero and Rodríguez-Palmero [27], Le Van and Vailakis [20] and Kamihigashi [14].

For a time-additive aggregator, $W(c, v) = u(c) + \beta v$, condition T takes the form

$$\liminf_{t\to\infty} \beta^t \left|\bar{v}_{t+1} - \underline{v}_{t+1}\right| = 0.$$

This is enforced by the fact that utility values are uniformly bounded and, more indirectly, that the Bellman operator acts on the linear space of bounded functions. In the unbounded case, utility values need be restricted in terms of rate of growth by imposing conditions on the relevant feasible set.

The concept of biconvergence introduced by Streufert [32, 33] and Osaki and Streufert [26] for recursive utility serves a similar purpose.[9] Their lower convergence requires

$$\lim_{n\to\infty} (T_c^n \underline{e})_0 = U(c),$$

whereas upper convergence imposes

$$\lim_{n\to\infty} (T_c^n \bar{e})_0 = U(c),$$

where $\bar{e}$ and $\underline{e}$ in L^{+} satisfy $\bar{e} \geq \bar{v} \vee \underline{v} \geq \bar{v} \wedge \underline{v} \geq \underline{e}$. These restrictions are sufficient to enforce condition T, because

$$\liminf_{n\to\infty} \left|(T_c^n \bar{v} \vee \underline{v})_0 - (T_c^n \bar{v} \wedge \underline{v})_0\right| \leq \liminf_{n\to\infty} \left|(T_c^n \bar{e})_0 - (T_c^n \underline{e})_0\right|.$$

In our case, condition T is enforced by monotone concavity of the utility aggregator at an interior consumption plan. However, it fails for non-interior consumption plans. For

[9] See also Bich et al. [7] for a recent treatment of deterministic programs along this line.

this reason, we complement the argument with a perturbation towards the interior of the feasible set.

Appendix: Proofs

Proof of Proposition 1. By Marinacci and Montrucchio [22, Proposition 1], there is a map $\underline{\phi} : L^+ \to L^+$ (respectively, $\bar{\phi} : L^+ \to L^+$) such that, for every consumption plan c in L^+, $\underline{\phi}(c)$ in L^+ (respectively, $\bar{\phi}(c)$ in L^+) is the least (respectively, the greatest) solution to equation (M), that is, at every t in $\mathbb{T}$,

$$\phi_t(c) = W(c_t, \phi_{t+1}(c)).$$

Furthermore, when c is uniformly interior in L^+, $\underline{\phi}(c) = \bar{\phi}(c)$ (see Marinacci and Montrucchio [22, Theorem 1(ii)]). We now show that these extreme solutions are necessarily recursive.

Take any consumption plan c in L^+. As $\underline{\phi}(\sigma c)$ in L^+ is the smallest solution to equation (M), $\underline{\phi}_1(c) \geq \underline{\phi}_0(\sigma c)$. Supposing that $\underline{\phi}_1(c) > \underline{\phi}_0(\sigma c)$, then equation (M) admits a solution $\phi(c)$ in L^+ such that

$$\phi_0(c) = W\left(c_0, \underline{\phi}_0(\sigma c)\right)$$

and, for every t in $\mathbb{T}$,

$$\phi_{t+1}(c) = \underline{\phi}_t(\sigma c),$$

a contradiction as $\underline{\phi}(c)$ in L^+ is the least solution (in the lattice sense) and $\phi_1(c) < \underline{\phi}_1(c)$. Hence, for every consumption plan c in L^+,

$$\underline{\phi}_0(c) = W\left(c_0, \underline{\phi}_0(\sigma c)\right).$$

The utility function is thus defined by $\underline{U}(c) = \underline{\phi}_0(c)$. A specular argument applies to the greatest solution.

Monotonicity is implied by the fact that the utility aggregator is (weakly) increasing.[10] We verify lower semicontinuity of $\underline{U} : L^+ \to \mathbb{R}^+$ on some interval $[0, \eta]$ in L^+, where $\eta > 0$ is a constant sequence. Take any sequence $(c^n)_{n\in\mathbb{N}}$ in the interval $[0, \eta]$ of L^+ converging to c in the interval $[0, \eta]$ of L^+. For every t in $\mathbb{T}$, define

$$\underline{c}_t^n = \inf_{s\in\mathbb{T}} c_t^{n+s}$$

and

$$\underline{v}_t^n = \inf_{s\in\mathbb{T}} v_t^{n+s},$$

where $v_t^n = \underline{\phi}_t(c^n)$ for every n in $\mathbb{N}$. Notice that, at every t in $\mathbb{T}$, by monotonicity of the utility aggregator,

$$v_t^{n+s} \geq W\left(\underline{c}_t^n, \underline{v}_{t+1}^n\right)$$

and, therefore,

$$\underline{v}_t^n \geq W\left(\underline{c}_t^n, \underline{v}_{t+1}^n\right).$$

[10] Monotonicity can only be established for the least and the greatest utility functions, as it might fail for any other recursively generated utility function.

In the limit, as all sequences are monotonically increasing in the interval $[0, \eta] \times [0, \underline{U}(\eta)]$ in $L^+ \times L^+$ and the utility aggregator is continuous,

$$\lim_{n\in\mathbb{N}} \underline{v}_t^n \geq W\left(\lim_{n\in\mathbb{N}} \underline{c}_t^n, \lim_{n\in\mathbb{N}} \underline{v}_{t+1}^n\right).$$

Noticing that $\lim_{n\in\mathbb{N}} \underline{c}_t^n = \lim_{n\in\mathbb{N}} c_t^n = c_t$, we conclude that, at every t in $\mathbb{T}$,

$$\underline{v}_t \geq W(c_t, \underline{v}_{t+1}),$$

where $\underline{v}_t = \lim_{n\in\mathbb{N}} \underline{v}_t^n$. This implies that the operator $T_c : L^+ \to L^+$ admits a fixed point in $[0, \underline{v}] \subset L^+$. Consequently,

$$\liminf_{n\in\mathbb{N}} \underline{\phi}_t(c^n) = \liminf_{n\in\mathbb{N}} v_t^n = \lim_{n\in\mathbb{N}} \underline{v}_t^n \geq \underline{\phi}_t(c).$$

This establishes lower semicontinuity. A specular argument applies to the greatest recursively generated utility function.

Suppose that $U : L^+ \to \mathbb{R}^+$ is any other recursively generated monotone utility function which is upper semicontinuous on uniformly bounded intervals. Pegging any consumption plan c in L^+, notice that, for every $\epsilon > 0$,

$$U(c + \epsilon\mathbf{1}) = \bar{U}(c + \epsilon\mathbf{1}).$$

This happens because utility values are uniquely determined at interior consumption plans. As the utility function is monotone, upper semicontinuity implies

$$U(c) \leq \lim_{\epsilon\to 0} U(c + \epsilon\mathbf{1}) = \limsup_{\epsilon\to 0} U(c + \epsilon\mathbf{1}) \leq U(c)$$

and

$$\bar{U}(c) \leq \lim_{\epsilon\to 0} \bar{U}(c + \epsilon\mathbf{1}) = \limsup_{\epsilon\to 0} \bar{U}(c + \epsilon\mathbf{1}) \leq \bar{U}(c).$$

This establishes that $U(c) = \bar{U}(c)$. □

Proof of Proposition 2. Notice that, since $c : D \to \mathbb{R}^+$ is bounded by some sufficiently large constant $\eta > 0$, $C(x) \subset [0, \eta] \subset L^+$ for every x in X. Thus,

$$\sup_{c\in C(x)} U(c) \leq \sup_{c\in C(x)} \bar{U}(c) \leq \bar{U}(\eta),$$

where we recall that $\bar{U} : L^+ \to \mathbb{R}^+$ is specified in Proposition 1. Observe that, by (*), for any choice of y in $G(x)$ and any choice of c in $C(y)$,

$$v(x) \geq U(c(x,y), c) = W(c(x,y), U(c)).$$

By continuity of the utility aggregator,

$$\begin{aligned} v(x) &\geq \sup_{c\in C(y)} W(c(x,y), U(c)) \\ &\geq W\left(c(x,y), \sup_{c\in C(y)} U(c)\right) \\ &= W(c(x,y), v(y)). \end{aligned}$$

Hence,

$$v(x) \geq \sup_{y \in G(x)} W(c(x,y), v(y)).$$

To prove the reverse inequality, peg any arbitrary $\epsilon > 0$. By (*), there are y in $G(x)$ and c in $C(y)$ such that

$$\begin{aligned} v(x) &\leq U(c(x,y), c) + \epsilon \\ &= W(c(x,y), U(c)) + \epsilon \\ &\leq W(c(x,y), v(y)) + \epsilon \\ &\leq \sup_{y \in G(x)} W(c(x,y), v(y)) + \epsilon. \end{aligned}$$

As $\epsilon > 0$ is arbitrary,

$$v(x) \leq \sup_{y \in G(x)} W(c(x,y), v(y)),$$

thus proving the claim. □

Proof of Lemma 1. Any map f in $\mathcal{M}^+(X \times Z)$ is dominated by a constant map η in $\mathcal{M}^+(X \times Z)$. Hence, monotonicity implies that $(Mf)(x,z) \leq (M\eta)(x,z) \leq \eta$, so that (Mf) lies in $\mathcal{B}^+(X \times Z)$. Given a map f in $\mathcal{C}^+(X \times Z)$, take any sequence $(x_n, z_n)_{n \in \mathbb{N}}$ in $X \times Z$ converging to (x,z) in $X \times Z$. Notice that, for every n in $\mathbb{N}$,

$$\begin{aligned} |(Mf)(x,z) - (Mf)(x_n,z_n)| &\leq |(Mf)(x,z) - (Mf)(x,z_n)| \\ &\quad + |(Mf)(x,z_n) - (Mf)(x_n,z_n)|. \end{aligned}$$

The first term vanishes by the Feller property (Assumption 2) of the primitive certainty equivalent operator. For every n in $\mathbb{N}$, let

$$\eta_n = \max_{\bar{z} \in Z} |f(x,\bar{z}) - f(x_n,\bar{z})|.$$

As Z is compact, $\lim_{n \in \mathbb{N}} \eta_n = 0$. Hence, by continuity (Assumption 5),

$$\lim_{n \to \infty} \max_{\bar{z} \in Z} |(Mf)(x,\bar{z}) - (Mf)(x_n,\bar{z})| = 0.$$

This proves the claim. □

Proof of Lemma 2. Monotonicity is implied by the fact that the utility aggregator and the certainty equivalent operator are monotone. Applying Berge's Maximum Theorem [2, Theorem 17.31], continuity follows from the Feller Property of the certainty equivalent operator (Assumption 2, jointly with Lemma 1). □

Proof of Lemma 3. The feasible plan consists of the constant choice λx in X, where x in X is given by condition (c) and λ is in $(0,1)$. By strict concavity,

$$c(\lambda x, \lambda x, z) > \lambda c(x,x,z) + (1-\lambda) c(0,0,z).$$

As Z is compact, there is some sufficiently small $\epsilon > 0$ such that $c(\lambda x, \lambda x, z) \geq \epsilon$ for every z in Z. □

Proof of Lemma 4. Pick any non-trivial (x_0, z_0) in $X \times Z$ and its associated exhaustive event E in $\mathcal{F}$ given in Assumption 6. We arbitrarily fix λ in $(0, 1)$ and show that $\underline{v}(x_0, z_0) \geq (1 - \lambda)\,\bar{v}(x_0, z_0)$. This proves the claim as it contradicts $\bar{v}(x_0, z_0) > \underline{v}(x_0, z_0)$. For a better understanding, the proof is articulated into steps.

Step 1 (Preliminary notation). Let L be the space of all bounded real-valued $(\mathcal{G}_t)_{t\in\mathbb{T}}$-adapted processes endowed with its Borel σ-algebra. Furthermore, let $L^+(x_0, z_0)$ be the set of all elements u of L^+ such that, at every t in $\mathbb{T}$, $u_t(\omega) = 0$ whenever ω is not in $\mathcal{F}_t(E)$. Finally, consider the extended certainty equivalent operator $M^{**} : \mathcal{B}^+(Z) \to \mathcal{B}^+(Z)$ defined by

$$(M^{**} f)(z) = \inf_{f^* \in \mathcal{M}^+(Z)} \{(M^* f^*)(z) : f \leq f^*\}.$$

Notice that, when f lies in $\mathcal{M}^+(Z)$, $(M^{**} f) = (M^* f)$ in $\mathcal{M}^+(Z)$. For an adapted process u in $L^+(x_0, z_0)$, the certainty equivalent is computed, at every t in $\mathbb{T}$, for each ω in Ω, according to

$$(M_t u_{t+1})(\omega) = \left(M^{**} u_{t+1}\left(z^t(\omega), \cdot\right)\right)(z_t(\omega)). \tag{†}$$

□

Step 2 (Optimal plan). We now identify a feasible plan that is optimal when continuation values are given by $\bar{v}$ in $\mathcal{M}^+(X \times Z)$. Since $\bar{v}$ in $\mathcal{M}^+(X \times Z)$ is upper semicontinuous, it is the pointwise limit of a weakly decreasing sequence of continuous maps in $\mathcal{C}^+(X \times Z)$ (see Aliprantis and Border [2, Theorem 3.13]). By Assumption 4 on the certainty equivalent operator, along with Aliprantis and Border [2, Lemma 2.41], $(M\bar{v})$ is also an upper semicontinuous map in $\mathcal{M}^+(X \times Z)$. Hence, Weierstrass Theorem (see Aliprantis and Border [2, Theorem 2.43]) implies the existence of a policy function $\bar{g}$ in G such that, for every (x, z) in $X \times Z$,

$$\bar{v}(x, z) = W\left(c\left(x, \bar{g}(x, z), z\right), (M\bar{v})\left(\bar{g}(x, z), z\right)\right).$$

Notice that we do not exploit measurability of this policy function, though it would be ensured by Wagner [34, Theorem 9.1(iii)] or Burgess and Maitra [10, Theorem A]. We indeed prefer to maintain our proof on the ground of primitive principles and to use the more permissive filtration $(\mathcal{G}_t)_{t\in\mathbb{T}}$ on the measurable space $(\Omega, \mathcal{G})$.

Consider the feasible plan $(\bar{x}, \bar{y})$ in $\Pi(x_0, z_0)$ implemented by this policy beginning from (x_0, z_0) in $X \times Z$. This is inductively constructed by setting, at every t in $\mathbb{T}$,

$$\bar{y}_t(\omega) = \bar{g}\left(\bar{x}_t(\omega), z_t(\omega)\right)$$

and

$$\bar{x}_{t+1}(\omega) = \bar{y}_t(\omega).$$

An adapted process for utility values $\bar{u}$ in $L^+(x_0, z_0)$ is determined, at every t in $\mathbb{T}$, as

$$\bar{u}_t(\omega) = \bar{v}\left(\bar{x}_t(\omega), z_t(\omega)\right)\chi_{\mathcal{F}_t(E)}(\omega),$$

where $\chi_F : \Omega \to \{0,1\}$ is the indicator function for event F in $\mathcal{F}$. Notice that, for every x in X, the map $z \mapsto \bar{v}(x,z)$ is an element of $\mathcal{M}^+(Z)$, because $\bar{v}$ lies in $\mathcal{M}^+(X \times Z)$ (see Aliprantis and Border [2, Theorem 4.48]). Therefore, at every t in $\mathbb{T}$, for each ω in $\mathcal{F}_t(E)$,

$$\begin{aligned} (M_t \bar{u}_{t+1})(\omega) &= \left(M^{**} \bar{u}_{t+1}\left(z^t(\omega), \cdot\right)\right)(z_t(\omega)) \\ &= \left(M^{**} \bar{v}(\bar{y}_t(\omega), \cdot)\, \chi_{\mathcal{F}_{t+1}(E)}\left(z^t(\omega), \cdot\right)\right)(z_t(\omega)) \\ &= \left(M^{*} \bar{v}(\bar{y}_t(\omega), \cdot)\, \chi_{\mathcal{F}_{t+1}(E)}\left(z^t(\omega), \cdot\right)\right)(z_t(\omega)) \\ &= \left(M^{*} \bar{v}(\bar{y}_t(\omega), \cdot)\right)(z_t(\omega)) \\ &= (M\bar{v})(\bar{y}_t(\omega), z_t(\omega)), \end{aligned}$$

where we exploit Assumption 3 along with condition (†). As $\bar{v}$ in $\mathcal{M}^+(X \times Z)$ is a fixed point of the Bellman operator and the plan is implemented by the associated optimal policy, it follows that, at every t in $\mathbb{T}$, for each ω in $\mathcal{F}_t(E)$,

$$\begin{aligned} \bar{u}_t(\omega) &= \bar{v}(\bar{x}_t(\omega), z_t(\omega)) \\ &= W\left(c(\bar{x}_t(\omega), \bar{y}_t(\omega), z_t(\omega)), (M\bar{v})(\bar{y}_t(\omega), z_t(\omega))\right) \\ &= W\left(c(\bar{x}_t(\omega), \bar{y}_t(\omega), z_t(\omega)), (M_t \bar{u}_{t+1})(\omega)\right), \end{aligned}$$

that is,

$$\bar{u}_t(\omega) = W\left(c(\bar{x}_t(\omega), \bar{y}_t(\omega), z_t(\omega)), (M_t \bar{u}_{t+1})(\omega)\right). \tag{*}$$

□

Step 3 (Bounds to utility values). We now perturb the plan $(\bar{x}, \bar{y})$ in $\Pi(x_0, z_0)$, so as to ensure interiority, and construct adapted processes for utility values corresponding to a lower and an upper bound. Consider the plan (x, y) in $\Pi(x_0, z_0)$ given in Assumption 6 and define the perturbed plan $\left(\underline{x}, \underline{y}\right)$ in $\Pi(x_0, z_0)$ as, at every t in $\mathbb{T}$,

$$\underline{x}_t(\omega) = \lambda x_t(\omega) + (1-\lambda)\bar{x}_t(\omega),$$

and

$$\underline{y}_t(\omega) = \lambda y_t(\omega) + (1-\lambda)\bar{y}_t(\omega).$$

This is feasible because the correspondence $G : X \times Z \twoheadrightarrow X$ is convex. Also, let the adapted process $\underline{u}$ in $L^+(x_0, z_0)$ be determined as

$$\underline{u}_t(\omega) = \underline{v}\left(\underline{x}_t(\omega), z_t(\omega)\right) \chi_{\mathcal{F}_t(E)}(\omega). \tag{**}$$

Arguing as in the previous case, as $\underline{v}$ in $\mathcal{M}^+(X \times Z)$ is a fixed point of the Bellman operator and the plan is not necessarily optimal, condition (**) implies that, at every t in $\mathbb{T}$, for each ω in $\mathcal{F}_t(E)$,

$$\underline{u}_t(\omega) \geq W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), (M_t \underline{u}_{t+1})(\omega)\right). \tag{L}$$

Furthermore, using concavity of the return function $c : D \to \mathbb{R}^+$ and monotone concavity of the utility aggregator $W : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, by Assumption 1, condition (*) implies, at every t in $\mathbb{T}$, for each ω in $\mathcal{F}_t(E)$,

$$\begin{aligned} W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), \left(M_t(1-\lambda)\bar{u}_{t+1}\right)(\omega)\right) &\geq \\ W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), (1-\lambda)\left(M_t\bar{u}_{t+1}\right)(\omega)\right) &\geq \\ \lambda W\left(c\left(x_t(\omega), y_t(\omega), z_t(\omega)\right), 0\right) + & \\ (1-\lambda) W\left(c\left(\bar{x}_t(\omega), \bar{y}_t(\omega), z_t(\omega)\right), \left(M_t\bar{u}_{t+1}\right)(\omega)\right) &\geq (1-\lambda)\bar{u}_t(\omega), \end{aligned}$$

that is,

$$\text{(U)} \qquad (1-\lambda)\bar{u}_t(\omega) \leq W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), \left(M_t(1-\lambda)\bar{u}_{t+1}\right)(\omega)\right).$$

We are now ready to derive a contradiction. □

Step 4 (Unique utility values). Arguing as in Marinacci and Montrucchio [22], we show that utility values are uniquely determined and this implies a reversal of the upper and the lower bound (see Step 5). Consider the operator $T^* : L^+(x_0, z_0) \to L^+(x_0, z_0)$ that is defined, at every t in $\mathbb{T}$, for each ω in Ω,

$$(T^*u)_t(\omega) = W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), \left(M_t u_{t+1}\right)(\omega)\right) \chi_{\mathcal{F}_t(E)}(\omega).$$

This operator is monotone and admits a unique fixed point. Indeed, existence follows from Tarski Theorem. Suppose that u^* and u^{**} in $L^+(x_0, z_0)$ are two distinct fixed points and, at no loss of generality, assume that $u^{**} \leq u^*$ does not hold true. By monotonicity and concavity, at every t in $\mathbb{T}^*$, for each ω in Ω,

$$\begin{aligned} \lambda W(\epsilon, 0)\chi_{\mathcal{F}_t(E)}(\omega) &\leq \lambda W\left(c\left(x_t(\omega), y_t(\omega), z_t(\omega)\right), 0\right)\chi_{\mathcal{F}_t(E)}(\omega) \\ &\quad + (1-\lambda) W\left(c\left(\bar{x}_t(\omega), \bar{y}_t(\omega), z_t(\omega)\right), 0\right)\chi_{\mathcal{F}_t(E)}(\omega) \\ &\leq W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), 0\right)\chi_{\mathcal{F}_t(E)}(\omega) \\ &= (T^*0)_t(\omega) \\ &\leq (T^*u^*)_t(\omega) \\ &= u_t^*(\omega), \end{aligned}$$

where $\epsilon > 0$ is given in Assumption 6 and the set $\mathbb{T}^*$ contains all but finitely many periods in $\mathbb{T}$. Consider the largest ψ in $[0,1]$ such that, at every t in $\mathbb{T}^*$, for each ω in Ω,

$$\psi u_t^{**}(\omega) \leq u_t^*(\omega).$$

Notice that $\psi = 0$ is ruled out because we have established that, at every t in $\mathbb{T}^*$, for each ω in Ω,

$$\lambda W(\epsilon, 0)\chi_{\mathcal{F}_t(E)}(\omega) \leq u_t^*(\omega).$$

Furthermore, assuming that $\psi = 1$ on $\mathbb{T}^*$, backward induction delivers $\psi = 1$ on the entire $\mathbb{T}$ by monotonicity of operator $T^* : L^+(x_0, z_0) \to L^+(x_0, z_0)$, a circumstance which is ruled out by hypothesis. By monotone concavity, along with Assumption 1, there is ψ^* in $(\psi, 1)$ such that, at every t in $\mathbb{T}^*$, for each ω in Ω,

$$\begin{aligned}
\psi^* u_t^{**}(\omega) &\leq \psi u_t^{**}(\omega) + (1-\psi)\lambda W(\epsilon, 0)\chi_{\mathcal{F}_t(E)}(\omega) \\
&\leq \psi W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), \left(M_t u_{t+1}^{**}\right)(\omega)\right)\chi_{\mathcal{F}_t(E)}(\omega) \\
&\quad + (1-\psi) W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), 0\right)\chi_{\mathcal{F}_t(E)}(\omega) \\
&\leq W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), \psi\left(M_t u_{t+1}^{**}\right)(\omega)\right)\chi_{\mathcal{F}_t(E)}(\omega) \\
&\leq W\left(c\left(\underline{x}_t(\omega), \underline{y}_t(\omega), z_t(\omega)\right), \left(M_t \psi u_{t+1}^{**}\right)(\omega)\right)\chi_{\mathcal{F}_t(E)}(\omega) \\
&= \left(T^*\psi u^{**}\right)_t(\omega) \\
&\leq \left(T^* u^*\right)_t(\omega) \\
&= u_t^*(\omega),
\end{aligned}$$

a contradiction. Hence, the operator admits a unique fixed point u^* in $L^+(x_0, z_0)$. □

Step 5 (Reversal of bounds). By condition (L), operator T^* maps the interval $[0, \underline{u}] \subset L^+(x_0, z_0)$ into itself and, hence, by Tarski Theorem, $u^* \leq \underline{u}$. By condition (U), for some sufficiently large constant adapted process η in $L^+(x_0, z_0)$, operator T^* maps the interval $[(1-\lambda)\bar{u}, \eta] \subset L^+(x_0, z_0)$ into itself and, hence, by Tarski Theorem, $u^* \geq (1-\lambda)\bar{u}$. It follows that $(1-\lambda)\bar{v}(x_0, z_0) \leq \underline{v}(x_0, z_0)$. □

This completes the proof. □

Proof of Lemma 5. By Lemma 2, $\left(T^n \underline{\eta}\right)_{n\in\mathbb{N}}$ is a weakly increasing orbit in $\mathcal{C}^+(X \times Z)$. Fix any (x, z) in $X \times Z$. Notice that, by monotonicity of the Bellman operator,

$$(T\underline{v})(x, z) \geq \left(T^{n+1}\underline{\eta}\right)(x, z),$$

which implies

$$(T\underline{v})(x, z) \geq \underline{v}(x, z).$$

To show the reverse inequality, given any $\epsilon > 0$, choose any y in $G(x, z)$ such that

$$(T\underline{v})(x, z) \leq W(c(x, y, z), (M\underline{v})(y, z)) + \epsilon.$$

Using continuity (Assumption 4) of the certainty equivalent operator, this implies that

$$\begin{aligned}
-\epsilon + (T\underline{v})(x, z) &\leq W(c(x, y, z), (M\underline{v})(y, z)) \\
&= W\left(c(x, y, z), \lim_{n\in\mathbb{N}}\left(M\left(T^n\underline{\eta}\right)\right)(y, z)\right) \\
&= \lim_{n\in\mathbb{N}} W\left(c(x, y, z), \left(M\left(T^n\underline{\eta}\right)\right)(y, z)\right) \\
&\leq \lim_{n\in\mathbb{N}}\left(T^{n+1}\underline{\eta}\right)(x, z) \\
&= \underline{v}(x, z).
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, this establishes the claim. □

Proof of Lemma 6. By Lemma 2, $(T^n \bar{\eta})_{n\in\mathbb{N}}$ is a weakly decreasing orbit in $\mathcal{C}^+(X \times Z)$. Fix any (x,z) in $X \times Z$. Notice that, by monotonicity of the Bellman operator,

$$(T\bar{v})(x,z) \leq \left(T^{n+1}\bar{\eta}\right)(x,z),$$

which implies

$$(T\bar{v})(x,z) \leq \bar{v}(x,z).$$

To show the reverse inequality, for every n in $\mathbb{N}$, choose any y_n in $G(x,z)$ such that

$$\left(T^{n+1}\bar{\eta}\right)(x,z) = W\left(c(x,y_n,z), \left(M\left(T^n\bar{\eta}\right)\right)(y_n,z)\right).$$

Let y in $G(x,z)$ be an accumulation point of the sequence $(y_n)_{n\in\mathbb{N}}$ in $G(x,z)$. As in Streufert [32] and Rustichini [28], using the Feller Property (Assumption 2) and continuity (Assumption 4) of the certainty equivalent operator, this implies that

$$\begin{aligned}
(T\bar{v})(x,z) &\geq W\left(c(x,y,z), (M\bar{v})(y,z)\right) \\
&= \lim_{n\in\mathbb{N}} W\left(c(x,y,z), \left(M\left(T^n\bar{\eta}\right)\right)(y,z)\right) \\
&= \lim_{n\in\mathbb{N}} \left(\lim_{m\in\mathbb{N}} W\left(c\left(x,y_{k(m)},z\right), \left(M\left(T^n\bar{\eta}\right)\right)\left(y_{k(m)},z\right)\right)\right) \\
&\geq \lim_{n\in\mathbb{N}} \left(\liminf_{m\in\mathbb{N}} W\left(c\left(x,y_{k(m)},z\right), \left(M\left(T^{n\vee k(m)}\bar{\eta}\right)\right)\left(y_{k(m)},z\right)\right)\right) \\
&= \liminf_{m\in\mathbb{N}} W\left(c\left(x,y_{k(m)},z\right), \left(M\left(T^{k(m)}\bar{\eta}\right)\right)\left(y_{k(m)},z\right)\right) \\
&= \liminf_{m\in\mathbb{N}} \left(T^{k(m)+1}\bar{\eta}\right)(x,z) \\
&= \bar{v}(x,z).
\end{aligned}$$

This completes the proof. □

Proof of Proposition 3. The lower orbit $\left(T^n\underline{\eta}\right)_{n\in\mathbb{N}}$ in $\left[\underline{\eta},\bar{\eta}\right]$ converges to a fixed point $\underline{v}$ in $\mathcal{M}^+(X \times Z)$, which is a lower semicontinuous map. The upper orbit $(T^n\bar{\eta})_{n\in\mathbb{N}}$ in $\left[\underline{\eta},\bar{\eta}\right]$ converges to a fixed point $\bar{v}$ in $\mathcal{M}^+(X \times Z)$, which is an upper semicontinuous map. Furthermore, $\underline{v} \leq \bar{v}$. At a trivial (x_0,z_0) in $X \times Z$, coincidence follows from

$$\bar{v}(x_0,z_0) \leq \underline{\eta} \leq \underline{v}(x_0,z_0).$$

By Lemma 4, the established coincidence of values at non-trivial (x_0,z_0) in $X \times Z$ proves the claim. This also shows that the fixed point v^* is a continuous map in $\mathcal{C}^+(X \times Z)$. Convergence is implied by Dini's Theorem (see Aliprantis and Border [2, Theorem 2.66]). □

Proof of Lemma 7. By Lemmas 5 and 6 (which hold true even when Assumption 6 is not imposed), there are a lower semicontinuous fixed point $\underline{v}_\epsilon$ in $\mathcal{M}^+(X \times Z)$ and an upper semicontinuous fixed point $\bar{v}_\epsilon$ in $\mathcal{M}^+(X \times Z)$. Furthermore, any other fixed point v_ϵ in

$\mathcal{M}^+(X \times Z)$ lies in the interval $[\underline{v}_\epsilon, \bar{v}_\epsilon]$. Finally, notice that, for every v in $\mathcal{M}^+(X \times Z)$, given any λ in $(0,1)$,

$$\begin{aligned} W_\epsilon\left(c\left(x,y,z\right),\left(M\lambda v\right)\left(y,z\right)\right) &\geq W_\epsilon\left(c\left(x,y,z\right),\lambda\left(Mv\right)\left(y,z\right)\right) \\ &\geq \lambda W_\epsilon\left(c\left(x,y,z\right),\left(Mv\right)\left(y,z\right)\right)+\left(1-\lambda\right)W_\epsilon\left(c\left(x,y,z\right),0\right) \\ &\geq \lambda W_\epsilon\left(c\left(x,y,z\right),\left(Mv\right)\left(y,z\right)\right)+\left(1-\lambda\right)W_\epsilon\left(0,0\right), \end{aligned}$$

where y is taken in the feasible set $G(x,z)$. Here we use concavity of the utility aggregator and the subhomogeneity of the certainty equivalent operator (Assumption 1). This implies

$$\begin{aligned} \left(T_\epsilon\lambda v\right)\left(x,z\right) &= \sup_{y\in G(x,z)} W_\epsilon\left(c\left(x,y,z\right),\left(M\lambda v\right)\left(y,z\right)\right) \\ &\geq \lambda \sup_{y\in G(x,z)} W_\epsilon\left(c\left(x,y,z\right),\left(Mv\right)\left(y,z\right)\right)+\left(1-\lambda\right)W_\epsilon\left(0,0\right) \\ &= \lambda\left(T_\epsilon v\right)\left(x,z\right)+\left(1-\lambda\right)W_\epsilon\left(0,0\right). \end{aligned}$$

Let λ in $(0,1)$ be the largest value satisfying $\lambda\bar{v}_\epsilon \leq \underline{v}_\epsilon$. For every (x,z) in $X \times Z$, monotonicity, along with the perturbation (†), implies

$$\begin{aligned} \left(1-\lambda\right)\epsilon+\lambda\bar{v}_\epsilon\left(x,z\right) &\leq \left(1-\lambda\right)W_\epsilon\left(0,0\right)+\lambda\left(T_\epsilon\bar{v}_\epsilon\right)\left(x,z\right) \\ &\leq \left(T_\epsilon\lambda\bar{v}_\epsilon\right)\left(x,z\right) \\ &\leq \left(T_\epsilon\underline{v}_\epsilon\right)\left(x,z\right) \\ &= \underline{v}_\epsilon\left(x,z\right). \end{aligned}$$

By revealing a contradiction, this proves coincidence of the extreme fixed points, thus establishing the claim. □

Proof of Proposition 4. The sequence $(v_\epsilon)_{\epsilon>0}$ in $\mathcal{C}^+(X \times Z)$ is weakly decreasing (because $v_\epsilon = \lim_{n\in\mathbb{N}}(T_\epsilon^n 0)$) and, hence, v_0 in $\mathcal{M}^+(X \times Z)$ is its well-defined (upper semicontinuous) pointwise limit. Notice that, for every v in $\mathcal{M}^+(X \times Z)$, $(T_\epsilon v) = (T_0 v) + \epsilon$. Monotonicity implies

$$v_\epsilon = (T_\epsilon v_\epsilon) = (T_0 v_\epsilon) + \epsilon \geq (T_0 v_0) + \epsilon,$$

thus showing that $v_0 \geq (T_0 v_0)$. To prove the reverse inequality, fix any (x,z) in $X \times Z$. For every sufficiently small $\epsilon > 0$, choose any y_ϵ in $G(x,z)$ such that

$$v_\epsilon\left(x,z\right) = W_\epsilon\left(c\left(x,y_\epsilon,z\right),\left(Mv_\epsilon\right)\left(y_\epsilon,z\right)\right).$$

Let y in $G(x,z)$ be an accumulation point of the sequence $(y_\epsilon)_{\epsilon>0}$ in $G(x,z)$. As in Streufert [32] and Rustichini [28], using the Feller Property (Assumption 2) and continuity (Assumption 4) of the certainty equivalent operator, this implies that

$$\begin{aligned}
(T_0 v_0)(x,z) &\geq W_0\left(c(x,y,z),(Mv_0)(y,z)\right) \\
&= W_0\left(c(x,y,z),\lim_{\epsilon\to 0}(Mv_\epsilon)(y,z)\right) \\
&= \lim_{\epsilon\to 0} W_0\left(c(x,y,z),(Mv_\epsilon)(y,z)\right) \\
&= \lim_{\epsilon\to 0}\lim_{\eta\to 0} W_0\left(c(x,y_\eta,z),(Mv_\epsilon)(y_\eta,z)\right) \\
&\geq \lim_{\epsilon\to 0}\liminf_{\eta\to 0} W_0\left(c(x,y_\eta,z),(Mv_\eta)(y_\eta,z)\right) \\
&= \liminf_{\eta\to 0} W_0\left(c(x,y_\eta,z),(Mv_\eta)(y_\eta,z)\right) \\
&= \liminf_{\eta\to 0} v_\eta(x,z)-\eta \\
&= v_0(x,z).
\end{aligned}$$

Supposing that the operator in the limit admits another fixed point v in $\mathcal{M}^+(X\times Z)$, then $(T_\epsilon v)=v+\epsilon\geq v$, which is sufficient to show that $v_\epsilon\geq v$ and, hence, $v_0\geq v$. □

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