Probability Weighting, Stop-Loss and the Disposition Effect.

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In this paper we study a continuous-time, optimal stopping model of an asset sale with prospect theory preferences under pre-commitment. We show for a wide range of value and probability weighting functions, including those of Tversky and Kahneman (1992), that the optimal prospect takes the form of a stop-loss threshold and a distribution over gains. It is skewed with a long right tail. This is consistent with both the widespread use of stop-loss strategies in financial markets, and recent experimental evidence. Moreover, our model with probability weighting in tandem with the S-shaped value function makes predictions for the disposition effect which match in magnitude that calculated by Odean (1998).

Keywords: Prospect theory, behavioral economics, disposition effect, investor trading behavior, probability weighting.

JEL Classification: D81, G19, G39

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1 Introduction

There is a growing body of work which shows that individual investors do not always act as maximizers of expected utility. One of the most prominent alternative explanations of individual decision making is Tversky and Kahneman's (1992) prospect theory (PT). PT has several innovations relative to expected utility, including reference levels, risk seeking behavior on losses and probability weighting.[1] Probability weighting has been successfully linked, both theoretically and empirically, to a wide range of financial phenomena.[2] In this paper, we contribute to this broad agenda by showing that in the setting of dynamic models of trading, PT can generate realistic behavior including the use of stop-loss strategies and the desire for right skewness and can match empirically observed levels of the disposition effect. Probability weighting plays a crucial role in our conclusions and we cannot obtain our main results from the reference level, loss aversion and risk-seeking on losses features of prospect theory alone.

We study the behavior of an investor with PT preferences who chooses when to sell an asset. The inclusion of probability weighting introduces new challenges, and to our knowledge this is the first paper to solve a dynamic liquidation model in continuous time for a pre-committing investor under a complete specification of realistic PT preferences which include all the features of Tversky and Kahneman (1992). Our model nests the models of Kyle, Ou-Yang and Xiong (2006) and Henderson (2012) which consider an optimal sale problem

[1] Many experimental and empirical studies (Tversky and Kahneman (1992), Camerer and Ho (1994), Wu and Gonzales (1996), and Polkovnichenko and Zhao (2013)) have found strong support for probability weighting. These studies confirm the inverse-S shape of the weighting function identified by Tversky and Kahneman (1992) which is associated with individuals overweighting the tails of the distribution.

[2] An overview can be found in Barberis (2013). Barberis and Huang (2008) show that, in a financial market where investors evaluate risk according to prospect theory, probability weighting leads to the prediction that the skewness will be priced. This idea has been used to explain low average returns of IPO stocks (Green and Hwang (2012)), the apparent overpricing of out-of-the-money options and the variance premium (Polkovnichenko and Zhao (2013), Baele et al (2017)), the lack of diversification in household portfolios (Polkovnichenko (2005)) and many other puzzles. On an aggregate scale, De Giorgi and Legg (2012) show that probability weighting is useful in generating a large equity premium - and can do so independently of loss aversion (Benartzi and Thaler (1995)). Probability weighting has also been helpful in understanding the popularity of casino gambling (Barberis (2012)).

for an investor with PT value function but no probability weighting, and is a continuous time version of the discrete time binomial model of Barberis (2012).

Our theoretical contribution is to determine the optimal prospect. For a class of preference structures and asset price dynamics, including many popular specifications from the literature, we determine the optimal prospect for an agent who can commit to an optimal sale strategy. Our analysis leads to several predictions which closely match empirical and laboratory findings.

Our first prediction, unique to the literature, is that PT investors should employ trading strategies which are of threshold type on losses, but not on gains.[3] Consider first the behavior on losses. Stop-loss strategies are in widespread use in financial markets and are also found desirable in the laboratory experiments of Fischbacher, Hoffmann and Schudy (2017). Existing theories, both EU and non-EU alike have struggled to justify stop-loss strategies - often they predict that assets are never sold voluntarily at a loss, but rather sales are deferred indefinitely. In contrast, we find that overweighting of extreme losses encourages the investor to stop at a threshold, and sooner than might be predicted in a model without probability weighting.

Now consider the behavior on gains. The vast majority of trading models - including those based on expected utility and those based purely on the S shaped utility function of prospect theory (Henderson (2012), Barberis and Xiong (2012), Ingersoll and Jin (2013) and Magnani (2015b)) - predict investors sell stocks when the price breaches an upper threshold. Our model predicts that the PT investor does not aim for a simple threshold strategy on gains. Instead probability weighting encourages the investor to aim for a long-tailed distribution, placing some mass on extremely high gains precisely because these are the outcomes which are overweighted under the inverse-S shaped probability weighting. Our results are supported by the fact that despite the ubiquity of stop-loss strategies in financial markets, in these same markets, investors appear unwilling to set thresholds which limit their upside. Similar

[3]As PT distinguishes between gains and losses relative to a reference point, and treats them differently, we expect asymmetric treatment of gains and losses, and potentially skew in the optimal prospect and a disposition (or reverse disposition) effect. The claim here is not merely that investor treatment of gains and losses is different, but rather that it is different in character: on one side it is a stop-loss threshold strategy, whereas on the other it is much more sophisticated.

behavior on the non-use of threshold strategies on gains has been found experimentally by Strack and Viefers (2017) and by Fischbacher, Hoffmann and Schudy (2017).

Our second prediction is that the optimal prospect is skewed. For realistic parameter values it is right skewed. It has a point mass on losses, but a long-tailed distribution over gains. The S shaped value function of PT favors a negative skew — small losses are heavily penalized and small gains are well rewarded and so the value function favours prospects consisting of likely small gains together with rare, but larger, losses (Henderson (2012), Ingersoll and Jin (2013)). Conversely, the inverse-S shaped probability weighting favors long right tails and a positive skew. Starting with no probability weighting and a negative skew, we find that as probability weighting increases in strength, the optimal prospect becomes less negatively skewed. Indeed, for levels of probability weighting proposed in the literature, the target distribution of the PT investor is typically right skewed, where we calculate skewness with the Hinkley (1975) quantile based measure. Positive skewness has long been established as an integral part of individual risk preferences in the empirical and experimental literature. For example, Kumar (2009) documents a desire for positive skew in choices of retail investors[4] and Ebert (2015) finds strong supporting evidence in the lab. Furthermore, although PT and skewness have been linked (Barberis and Huang (2008), Spalt (2013)), this paper is the first to make this connection in a rigorous model of the trading behavior of a PT investor. In fact, existing PT models without probability weighting typically lead to a left skewed target distribution rather than a right skew.

One of the most robust trading anomalies in the empirical literature on investor behavior is the disposition effect, which refers to the stylized fact that investors have a higher propensity to sell risky assets with capital gains compared to risky assets with capital losses (Shefrin and Statman (1985)).[5] Odean's (1998) well known study computes the frequency with which

[4]Kumar (2009) (and others) find if investors are offered a set of investment opportunities in stocks, they tend to choose to buy stocks with right skewed distributions rather than symmetric or left skew. Here in our model, the situation is slightly different since the agent can effectively design the payoff whilst the game is in progress and designs it to be right skewed.

[5]The disposition effect has been documented for individual investors by Odean (1998), institutional investors (Grinblatt and Keloharju (2001)) as well as in the real estate market (Genesove and Mayer (2001)) and options markets (Poteshman and Serbin (2003)). Studies have also examined the impact of trading

individual investors sell winners and losers relative to opportunities to sell each and finds gains are realized at a rate around 50% higher than losses. Prospect theory provides a leading candidate explanation of the disposition effect.[6] The well known static intuition from Shefrin and Statman (1985) linking PT to the disposition effect argues that risk seeking over losses under PT encourages investors to continue gambling, whilst risk aversion over gains means investors tend to sell assets which have increased in value.

In an attempt to understand the implications of PT more fully, there has been a recent program in the literature attempting to build rigorous models in a dynamic setting. Despite the clear intuition, it is a challenge for existing prospect theory models to explain the disposition effect (Kyle, Ou Yang and Xiong (2009), Kaustia (2010), Barberis and Xiong (2009, 2012), Henderson (2012), Li and Yang (2013))[7]. Indeed, current models have been unsuccessful. The first difficulty is that PT models without probability weighting (Ingersoll and Jin (2013), Henderson (2012) and Barberis and Xiong (2012)) typically predict two-sided threshold strategies. The second difficulty is that although the convexity over losses and loss aversion do indeed act to encourage the investor to continue gambling in the domain

experience (Feng and Seasholes (2005)) and investor sophistication (Dhar and Zhu (2006), Calvett, Campbell and Sodini (2009)) on the disposition effect. Experimental evidence from the lab (Weber and Camerer (1998) and more recently, Magnani (2015a,b) and Fischbacher, Hoffmann and Schudy (2017)) is also supportive.

[6]Odean (1998) explicitly considers expected utility explanations for the asymmetry across winners and losers based on richer specifications of the investor's problem, finding that portfolio rebalancing, transaction costs, taxes, and rationally anticipated mean reversion cannot explain the observed asymmetry. Weber and Camerer (1998) find that incorrect beliefs concerning mean reversion cannot explain the disposition effect either.

[7]There are theories of the reference point that can potentially generate a disposition effect, for example, a reference point given by a weighted average of recent prices (Weber and Camerer (1998); Odean (1998)), or by investors expectations (Koszegi and Rabin (2006)), Meng and Weng (2018), Magnani (2015b)).

of losses, this effect tends to be too strong.[8,9] In many models the investor rarely (or even never) stops voluntarily at a loss, giving an extreme disposition effect.[10]

Our third prediction is that the inclusion of probability weighting allows prospect theory to deliver a realistic level of the disposition effect. With an inverse-S shaped probability weighting function an investor overweights the extreme outcomes - both good and bad. Overweighting the extremely poor outcomes encourages the investor to stop in the loss region. Overweighting the very good outcomes encourages the investor to continue when in the region of gains. In isolation, therefore, probability weighting would work in the *opposite* direction to the disposition effect - investors stop earlier when the stock is doing badly and hold longer when it is doing well. The important observation we make in this paper is that when probability weighting is used *in tandem* with the other ingredients of PT (S shaped value function, loss aversion) it moderates the level of the disposition effect predicted by the model to give values which are much closer to observed empirical levels. Indeed, we show the model can match Odean's (1998) measure of the disposition effect with realistic parameters.

In static models, and occasionally in their dynamic counterparts, the intuition from prospect theory has been used to justify many of these stylized facts, but typically only one feature at a time. An important paper in this vein is the binomial model of casino gambling

[8]Ingersoll and Jin (2013) study a realization utility model with reference dependent S-shaped preferences and show that consideration of reinvestment improves the range of parameters over which losses are taken. The model of Ingersoll and Jin (2013) gives an improved fit to the disposition effect, but requires considerable adjustments on the TK value functions and how they are applied. First, the value function is applied over rates of return rather than dollar changes, and second, the TK value function is altered so that the marginal utility at the origin is finite. Further, an implausibly high risk seeking parameter is needed to obtain a good fit. To obtain a better fit for plausible parameters Ingersoll and Jin (2013) mix 50-50 realization utility investors with random Poisson traders. Although it gives a better fit, loss-incurring sales are typically the result of events of the exogenous Poisson process, and not deliberate decisions to sell.

[9]Barberis and Xiong (2009) have the reverse problem - their model often predicts the opposite of the disposition effect.

[10]Most of this literature finds the investor never sells at a loss. An exception is Henderson (2012), who shows that under the Tversky and Kahneman (1992) value function, there is a loss threshold at which the investor will sell, but this only occurs for ranges of parameters where the stock has very poor expected returns, ie. where the investor gives up despite her loss aversion and convex preferences. For higher expected returns, an extreme disposition effect still emerges as loss aversion and convexity are dominant forces.

of Barberis (2012). Barberis (2012) applied PT (with TK utility and weighting functions) in a finite horizon binomial tree, which may be viewed as a discrete-time version of our model. [11] He finds that the investor who can commit (and for whom probability weighting is the dominant effect) aims for a right-skewed, stop-loss strategy. However, this investor continues on gains, which means that losses are taken more readily than gains, and the reverse of the disposition effect emerges.

In general, there is an apparent inconsistency between right skewness and the disposition effect. Right-skewed payoffs with some large gains might be expected to arise from strategies in which the rate of selling in the gain regime is low relative to the rate of selling in the loss regime; the disposition effect is the reverse relationship. One of the main contributions of this paper is to resolve this apparent paradox. The resolution comes from the fact that the strategy on gains is not a pure threshold strategy; most of the time the investor aims for small gains and moderate losses, with only occasional large gains. The occasional large gains generate the positive skew in the optimal prospect, whereas the (more typical) sales at small gains cause a disposition effect. Hence we can have both right skewness and the disposition effect in our model. Probability weighting is key to the optimality of a non-threshold strategy on gains and hence to a theory which simultaneously explains stop-loss, right skewness and the disposition effect.

We close the introduction with some brief remarks concerning our technical contribution. We study the problem facing an investor with PT preferences (and who can precommit) who chooses when to sell an asset in a continuous-state, continuous-time model. The problem can be cast as a infinite-horizon, optimal stopping problem for a diffusion process. The goal is to determine an optimal stopping rule and the optimal target distribution for the stopped process or equivalently the optimal prospect. Underpinning our results is the important progress we make on the form of the optimal prospect. For a wide range of utility and

[11] Barberis (2012) also drew attention to the time inconsistency induced by probability weighting, see also Machina (1989). Today's optimal strategy may not be optimal at future times, and (naive) PT investors may change their minds. In contrast to the investor who can commit, the naive investor changes his mind and continues on losses whilst stopping on gains - a disposition-like behavior but one which is left-skewed and not stop-loss. Moreover, Ebert and Strack (2015) in a continuous time version of the model show a naive PT investor never stops gambling and thus cannot display the disposition effect.

weighting functions and for general price processes we establish that the optimal prospect takes the form of a stop-loss threshold and a distribution over gains.

We emphasize that we provide the first stopping model for a precommitment agent to feature all the components of Tversky and Kahneman (1992) prospect theory. However, the probability weighting aspect of PT induces time-inconsistency. There are several different ways to model the response of investors to time-inconsistency, and in addition to agents who can commit, we may have naive or sophisticated investors. Naive investors are aware of time-inconsistency and continually recalculate their optimal strategy.[12] Sophisticated investors choose an optimal strategy assuming that their future selves will also behave optimally and in a similar fashion. Recently, studies in a continuous time setting show a naive PT agent never stops (Ebert and Strack (2015), and a sophisticated agent never gets started (Ebert and Strack (2017)). By extending the framework of Ebert and Strack (2015) to allow for randomized strategies, Henderson, Hobson and Tse (2017) show that a naive PT investor may voluntarily sell or cease gambling. Taken together, these works highlight the challenges involved in modeling the stopping behavior of PT agents in dynamic settings and demonstrate that each type of agent behaves very differently and certainly deserves its own study.

2 A Model of asset liquidation under Prospect Theory

2.1 Prospect theory preferences and weighting

Under prospect theory, utility is evaluated in terms of gains and losses relative to a reference point, rather than over final wealth. Denote by Z a random variable and by R the reference point or level and let $Y = Z - R$ denote the gain or loss relative to the reference level. Let U be the (continuous, strictly increasing, twice differentiable away from zero) utility or value function defined over the range of Y such that $U(0) = 0$. Under prospect theory, U is concave over gains and convex over losses. It also exhibits loss aversion, whereby a loss has a larger impact than a gain of equal magnitude.

[12] In order to understand the behavior of a naive agent it is necessary to first understand the behavior of an agent who can pre-commit. This is the behavior we determine.

The final ingredient of prospect theory is that the probabilities of extreme events are overweighted where the degree of probability weighting can differ for gain and loss outcomes. Let $w_{\pm} : [0, 1] \mapsto [0, 1]$ be a pair of (continuous, strictly increasing, differentiable) probability weighting functions with $w_{\pm}(0) = 0, w_{\pm}(1) = 1$. Overweighting of small probabilities on extreme events suggests the probability weighting functions $w_{\pm}$ should be inverse-S shaped functions; in particular there exist $q_{\pm}$ such that $w_{\pm}$ is concave on $[0, q_{\pm}]$ and convex on $[q_{\pm}, 1]$.

The prospect theory value of Z is given by (see Kothiyal et al (2011))

$$V(Z) = \int_0^{\infty} w_+(\mathbb{P}(U(Z-R) > y))dy - \int_{-\infty}^{0} w_-(\mathbb{P}(U(Z-R) < y))dy. \tag{1}$$

Our model is a partial equilibrium framework with an infinite horizon. An investor holds an asset whose price at time t is given by P_t. The investor can sell or liquidate the asset at any time in the future. At the liquidation time τ[13] of their choice, the investor receives the payoff $Z \equiv P_\tau$ and compares it to their reference level R, which may be the breakeven level or price paid for the asset.[14] The investor uses narrow framing in her evaluations of prospects, so that selling decisions are taken in isolation and without reference to other components of the investor's portfolio. Moreover, our focus is on investors who can commit to follow an investment strategy.

The goal of the investor is to choose the best time τ to sell the asset to maximize the PT value: i.e. to find

$$\sup_{\tau} V(P_\tau), \tag{2}$$

Note that if $w_{\pm}(p) = p$ so that there is no probability weighting, then $V(P_\tau) = \mathbb{E}[U(P_\tau - R)]$ and we recover the model of Henderson (2012) (see also Kyle, Ou-Yang and Xiong (2006)). Note also that from (1) and (2) it follows that the problem depends on τ only through the law of P_τ and two stopping times which yield the same probability distribution for the stopped process will be valued as equal by the investor with prospect theory preferences. Hence the problem of finding the optimal stopping rule can be solved via a two-stage process, first find the optimal prospect and then find a stopping rule for which the law of the stopped process

[13] The liquidation time τ must be a stopping time.

[14] In common with the prospect theory models we compare to, we consider a fixed reference level. We set $R = P_0$.

attains that prospect. The novel part of our results is in the first step. For the second step, we can use the theory of Skorokhod embeddings. This theory tells us how to find stopping rules which attain the given law and that, in general, there are many different stopping rules which are all optimal. The exception is the case where the original prospect is a distribution over one or two points (as is the case in classical stopping problems without probability weighting). Then there is a unique optimal stopping rule which is the first exit time from an interval.

Suppose the price process P is a time-homogeneous diffusion with state space which is an interval with endpoints $a_J < b_J$.[15] Let s be the scale function for P so that $X = s(P)$ is a (local) martingale.[16] Set $v(x) = U(s^{-1}(x) - R)$. The following proposition converts the problem from an optimization over stopping times to an optimization over prospects. The change of scale means that the optimization takes place over a simple space of target distributions. These distributions have zero mean, because X has zero expected return.

Proposition 1. *Suppose $s(a_J) > -\infty$. The investor's objective can be rewritten as to find*

$$\sup_X \left(\int_{1-\mathbb{P}(X>0)}^1 v(G_X(u)) w'_+(1-u)du + \int_0^{\mathbb{P}(X<0)} v(G_X(u)) w'_-(u)du \right), \tag{3}$$

where the supremum is taken over random variables (or prospects) X with mean X_0 and support on $[s(a_J), s(b_J)]$. Here G_X is the quantile[17] function of X. If X^* is an optimal prospect, then there exists a stopping time τ^* such that X_{τ^*} has the same law as X^*. Moreoever any stopping time τ^* constructed such that X_{τ^*} has the same law as X^* is optimal.

If X^* is the optimizer for this problem, then the optimal prospect P^* has CDF $F_{P^*}(p) = F_{X^*}(s(p))$. Moreover P_{τ^*} has the same law as P^*.

[15] The main model of interest is to take P to be geometric Brownian motion. Then $P_t = P_0 e^{\sigma B_t + (\kappa - \frac{1}{2}\sigma^2)t}$ with expected return $\kappa \geq 0$. In this case $a_J = 0$ and $b_J = \infty$ and the scale function is $s(p) = p^\beta - P_0^\beta$ where $\beta = 1 - \frac{2\kappa}{\sigma^2}$. Then $X_t = s(P_t)$ is a translation of a geometric Brownian motion and has zero expected return. Note that $s(a_J) = s(0) = -P_0^\beta > -\infty$.

[16] See Henderson (2012) or Xu and Zhou (2013) for use of the same transformation in a related context.

[17] If F_X is the cumulative distribution function (CDF) of X so that $F_X(x) = \mathbb{P}(X \leq x)$ then the quantile function G_X is the inverse of F_X.

2.2 The main result

Our first theorem describes the optimal prospect.

Theorem 1. *Suppose the value function and probability weighting functions are those of Tversky and Kahneman (1992), and suppose that the price P is geometric Brownian motion. Suppose that the parameter values are such that the problem is well-posed.*

Then the optimal prospect has a distribution which consists of a point mass in the loss regime at some level L and a point mass at some level A in the gains regime, together with a continuous distribution on the unbounded interval (A, ∞).

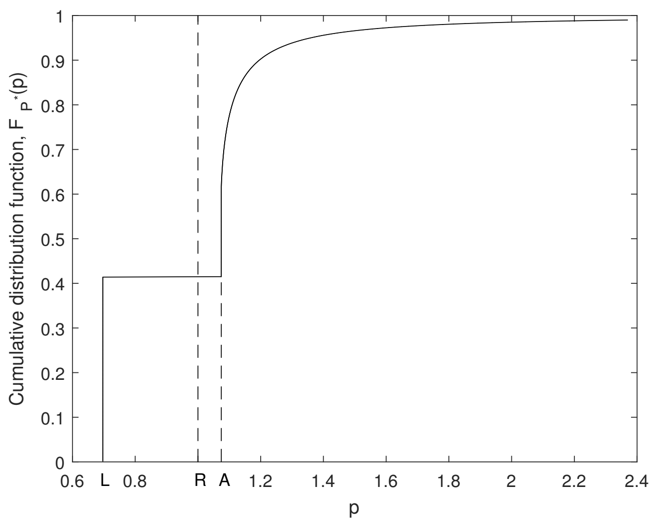


Figure 1: The cumulative distribution function (CDF) of the optimal prospect in the model with Tversky and Kahneman value and weighting functions (see Section 5). The optimal distribution consists of a single loss threshold at L together with a distribution over the gains region. Here we take the reference level $R = 1$ and initial asset price $P_0 = 1$. Other parameters are $\alpha_+ = 0.5$, $\alpha_- = 0.9$, TK probability weighting parameters $\delta_+ = \delta_- = 0.7$, loss aversion parameter $k = 1.25$, $\beta = 1 - 2\kappa/\sigma^2 = 0.9$ (where κ is the asset's expected return and σ the volatility of asset returns). The base case TK model and parameters are described in detail in Section 5.

Theorem 1 is a special case of a more general result (Theorem 2 in the Appendix A.3) which has the same conclusion, but holds under a wide class of model specifications. The precise form of the optimal prospect is given in Proposition 9 in Appendix A.5.3.

Our main technical contribution is to solve for the optimal prospect (for an agent who can commit) for a wide class of prospect theory specifications including those of Tversky and Kahneman (1992) and for asset prices following a time-homogeneous diffusion. For an S-shaped value function and inverse-S shaped probability weighting functions we take a sequential optimization approach and split the problem into two subproblems - a concave maximization (via a Lagrangian) over gains and a convex maximization over losses. Each of these subproblems can be analysed using the results of Xu and Zhou (2013) for one-sided (gains or losses) optimal stopping problems. Then, under some additional assumptions on the value and probability weighting functions[18] which we can show hold in the base case model of Tversky and Kahneman (1992) and for a wide set of value and weighting functions used in the literature, we can solve for the optimal prospect with both gains and losses by finding the optimal allocation of mass over gains and losses subject to a mean constraint. The full development of our technical results is contained in Appendix A with the proofs given in Appendix B and C.

For a particular set of parameter values, the optimal prospect is displayed graphically in Figure 1. Note that the reference level and the initial asset price are both set equal to 1. The base case TK model and parameters are described in detail in Section 5. We display the cumulative distribution function (CDF). The distribution on losses is a point mass. (In this example the location of this mass is strictly positive, although this need not be the case). The investor places just over 0.4 of the probability onto the single loss threshold L which is about 0.7, that is, at a 30% loss relative to the reference level. The remainder of the probability is distributed over the gains. There is probability of just under 0.3 at A which is about 1.1, ie. at a 10% gain relative to the reference level. The rest of the probability is distributed above this point, tailing off at around 2.4. (More precisely, the level 2.4 represents the upper 99th percentile of the distribution.)

[18] A key point is that although the full problem involves a delicate interaction between the value and probability weighting functions, these assumptions apply separately to the value and weighting functions.

2.3 Implications of the main result

Theorem 1 has several implications for the behavior of commitment-based investors which we highlight here, and with the exception of the final implication, discuss in more detail in later sections.

First, the investor follows a strategy which treats gains and losses differently. This is consistent with the premises of PT which treat gains and losses separately, and in different ways.

Second, all the mass on losses in the optimal prospect is concentrated at a single point. This means that the corresponding strategy must be stop loss. Conversely, the optimal prospect for a PT investor includes a long-tailed distribution on gains. This means that the optimal strategy is not take-gain, and the optimal strategy need not be of threshold type.

Third, the optimal prospect is skewed. When there is no probability weighting the optimal prospect is left-skewed. However, we show below that as probability weighting increases the optimal target distribution becomes right-skewed.

Fourth, given the optimal target distribution we can calculate a model-based version of Odean's disposition ratio. For reasonable parameter values we find a good fit to the empirical results of Odean (1998). Probability weighting plays a key role and is necessary for a good match between model-based values of the disposition effect and empirical results.

Finally, the optimal prospect is never a single point mass. A point mass at the current price of the asset would signify that if the investor were to buy the asset for the current price, then the investor would choose to sell it instantly. More pertinently, in such a scenario the investor would not buy the asset in the first place. As we find that the optimal prospect is always non-trivial, we conclude that the agent would always choose to buy the asset for it's current price, in the knowledge that under narrow framing, and with the reference level set to be the price paid, buying the asset yields a strictly positive value under prospect theory preferences. For this reason we express the investor's problem in terms of an initial endowment of the asset and the desire to select an optimal liquidation strategy.[19]

[19]In Barberis's (2012) discrete model the situation is different. He finds that for some parameter combinations the investor may choose to sell instantly (or, in his framework, to never gamble in the casino). The difference between our results and those of Barberis is driven by the distinction between discrete and

3 PT and investor behavior

3.1 PT trading and stop-loss strategies

Figure 1 shows that a prospect theory investor with S-shaped utility, loss aversion and probability weighting will trade to achieve a distribution over gains but will desire a stop-loss threshold over losses. This difference in how the investor trades gains and losses matches very well how investors behave in financial markets. Stop-loss strategies are in widespread usage in practice but stop-gain or take-gain strategies are much rarer. This mismatch in the useage of stop-loss versus take-gain has been found in a recent experimental study of Fischbacher, Hoffmann and Schudy (2017). In a laboratory experiment where participants can actively buy and sell assets, Fischbacher, Hoffmann and Schudy (2017) investigate whether participants make use of additional opportunities to pre-specify stop-loss and take-gain thresholds at which assets are automatically sold. Participants with access to automatic limits had a significant increase in the frequency of realized losses but no change in realized gains. This was due to participants setting loss limits closer to the current asset price, i.e. deliberately setting a stop-loss, but setting the gain threshold at levels which are unlikely to be breached.

Despite the fact that stop-loss strategies are "ubiquitous" in financial markets, they are not that easily justified by financial theory (Kaminski and Lo (2007)). In our PT model, all the mass on losses in the optimal prospect is concentrated at a single point. This concentration of mass arises from the probability weighting and the overweighting of extreme losses. It also means that the strategy is stop-loss. The loss threshold may be at zero (in which case, since geometric Brownian motion approaches zero but does not hit zero, the agent defers selling indefinitely), or when probability weighting is strong enough, it may be at a strictly positive level.

Standard expected utility settings can predict a stop-gain threshold at which an investor

continuous models. In the binomial model there is a smallest gamble size, and if loss aversion is too great, then it can be optimal to not gamble (i.e. to never purchase the asset). In continuous models there is no smallest gamble and the probability weighting element of PT is sufficient to ensure that there is always a prospect which is preferred to a unit probability mass at the current wealth. This result, due to Ebert and Strack (2015), underpins their work on naive agents under prospect theory.

should sell but tend to put any lower threshold at $-\infty$ (see Strack and Viefers (2017)), or in our case, since the asset has limited liability, at zero. Two exceptions are Henderson (2012) and Ingersoll and Jin (2013). Henderson (2012) studies a problem with S shaped utility but without probability weighting, and finds a strictly positive stop-loss threshold, but only in parameter combinations for which the asset has negative expected return. Ingersoll and Jin (2013) consider a realization utility model (see also Barberis and Xiong (2012)) with a modified value function and no probability weighting in which the problem restarts each time an asset is sold. Ingersoll and Jin find that the opportunity to reset the reference level gives investors an incentive to take losses at strictly positive values. Both Henderson (2012) and Ingersoll and Jin (2013) find that the optimal strategy is stop-loss and take-gain.

We find that whether our model predicts a strictly positive threshold at which the asset is sold (and voluntary realization of losses) or whether our model predicts losses are only taken when the asset price falls to zero (which cannot happen in finite time in our geometric Brownian motion model) depends on the degree of probability weighting. In Figure 2 we plot the cumulative distribution functions (CDFs) of the optimal prospect for four different values of the probability weighting parameter. When there is no probability weighting, or when probability weighting is weak (δ is close to 1) the optimal prospect has an atom at zero (and the investor never sells at a loss). With no probability weighting, the convexity of the S shaped utility on losses encourages the investor to continue gambling. It is only when probability weighting is sufficiently strong (δ far enough from 1), and the overweighting of large losses incentivizes against taking extreme losses, that we see a prospect with an atom in the interval $(0, 1)$. For our base parameters the critical value is when δ is about 0.74.[20]

[20]Typically, we find that the convexity of the value function on losses means that a loss threshold of zero is preferred to a very small loss threshold. However, the prospect value is not a unimodal function of the parameters, and there is another local maximum in the interior of the domain. Which of these gives the global maximum depends on the level of probability weighting. At the critical value of δ there is a discontinuity in the optimal prospect. A similar discontinuity arises in the model of Ingersoll and Jin (2013), see Figure 3 of that paper.

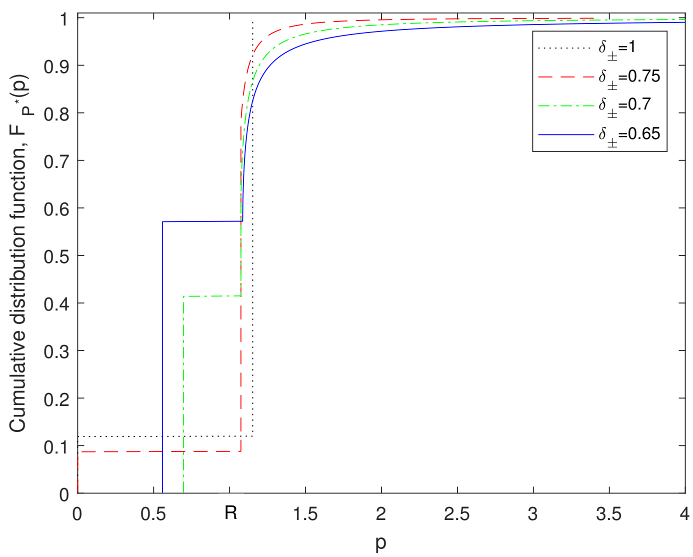


Figure 2: Cumulative distribution function (CDF) plots for the optimal prospect in the model with Tversky and Kahneman value and weighting functions, for different values of probability weighting parameter $\delta = \delta_+ = \delta_-$. For weak probability weighting ($\delta > 0.74$) the mass on losses is at zero, but for sufficiently strong probability weighting ($\delta < 0.74$) the mass on losses is at a strictly positive level. We set the reference level $R = 1$ and initial asset price $P_0 = 1$. Other base parameters are $\alpha_+ = 0.5$, $\alpha_- = 0.9$, loss aversion parameter $k = 1.25$, $\beta = 1 - 2\kappa/\sigma^2 = 0.9$ (where κ is the asset's expected return and σ the volatility of asset returns). The base case TK model and parameters are described in Section 5.

3.2 PT trading and non-threshold strategies on gains

Classical EU models (and also PT models without probability weighting such as Henderson (2012) and Ingersoll and Jin (2013)) typically tend to predict that investors stop at a threshold on gains.[21] In contrast, we find that the optimal prospect for a PT investor includes a long-tailed distribution on gains, see Figure 2. This fundamental difference in behavior with regard to losses and gains in our model mirrors very well what we see in the financial markets.

It is worth highlighting that our finding of a long-tailed gains distribution holds under very general model assumptions including the popular TK utility and weighting functions. The long tail on gains arises from the impact of the probability weighting on gains. Extreme gains are overweighted: this promotes behaviors which sometimes generate large gains.

Since the investor stops at more than one value of gains, the optimal stopping rule cannot be a pure threshold strategy. Indeed, although there is a unique optimal prospect, there are many optimal stopping rules, and it is possible for an investor who is behaving optimally to stop at a gain level at which they have previously decided that it is optimal to continue. In this way our PT model predicts non-threshold stopping rules on gains, a prediction which is supported by recent evidence (Strack and Viefers (2017), Fischbacher, Hoffmann and Schudy (2017)). Strack and Viefers (2017) conduct an experiment in a sophisticated asset selling task whereby subjects played sixty-five rounds during which they could sell their stock. In each round they observe a path of the market price which follows a random walk with positive expected return. Since their subjects receive zero payoff if they wait, their experiment is only concerned with gains. Strack and Viefers (2017) present evidence that players do not play cut-off or threshold strategies over gains - they do not behave time-consistently within rounds 75% of the time, and visit the same price level three times on average before stopping at it. In their study of the impact of automatic selling devices on experimental trading behavior, Fischbacher, Hoffman and Schudy (2017) find that participants tend to set any upper limit further away from the current price than any lower limits and use the upper limit less

[21] In a binomial model with TK probability weighting, Barberis (2012) finds the investor who can precommit tends to continue on gains. He, Hu, Obłój and Zhou (2015) extend the binomial model to an infinite horizon but only in the case of concave (power) weighting functions rather than inverse-S shaped functions.

frequently. In fact, there is no significant difference in the proportion of gains realized in their treatments with and without automatic limits, implying that the upper limits are not being used.

Our research also has implications for the design of future experimental studies. Magnani (2015b)'s recent experimental evidence to support the disposition effect is predicated on the behavior of subjects being well approximated by threshold rules. Our theoretical findings, combined with Strack and Viefer's experimental observations, point to threshold-type behavior providing an incomplete description of individual behavior.

3.3 PT trading and skewness

Prospect theory and skewness have been heavily linked in the extant literature. When we use the Hinkley (1975) quantile-based measure of skewness we find that the skewness is sensitive to the parameter values (of the value function, weighting function, and price process) and the model may predict left- or right-skewness depending on these values. The impact of the S-shaped value function in isolation is to encourage investors to aim for small gains at the risk of occasionally incurring a large loss. This leads to a left-skewed optimal prospect. The impact of the probability weighting works in the opposite direction. On gains, the probability weighting acts to encourage prospects with some very large values, and a distribution with a long right-tail. On losses the overweighting of the largest losses is an incentive to avoid the worst cases. Thus an increase in the degree of probability weighting (decrease in δ) makes the skew of the optimal prospect more positive. Either the value function effect or the probability weighting effect may dominate, but the stronger the probability weighting, the more right-skewed the prospect.

To demonstrate the role of probability weighting on skewness in our model, we calculate a measure of skewness for the optimal distribution under our base model with Tversky and Kahneman value and weighting functions, as detailed in Section 5 below. We use the robust, tail or quantile based measure of skewness of Hinkley (1975) (see Ebert and Hilpert (2015), Green and Hwang (2012) and Conrad, Dittmar, and Ghyssels (2013)):

$$\Gamma(0.99) = \frac{F^{-1}(0.99) + F^{-1}(0.01) - 2F^{-1}(1/2)}{F^{-1}(0.99) - F^{-1}(0.01)} = \frac{G(0.99) + G(0.01) - 2G(1/2)}{G(0.99) - G(0.01)}$$

where F is the cumulative distribution function and G is the quantile function. Note that skewness is an attempt to summarize the shape of a distribution in a single statistic, which is often a difficult task. $\Gamma(0.99)$ depends only on the quantiles at 0.01, 0.5 and 0.99.

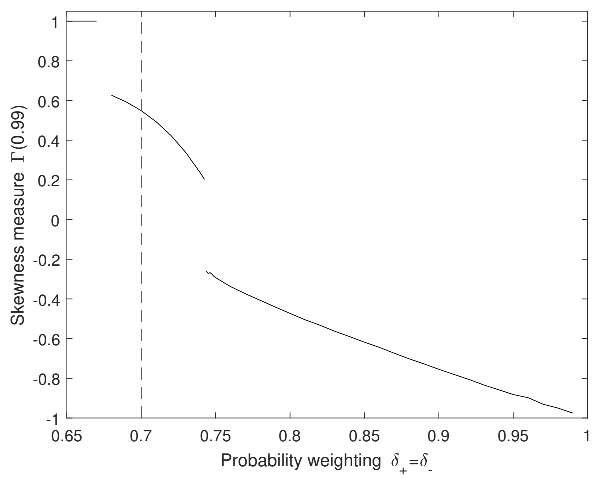


Figure 3: Skewness measure for the optimal distribution in the model with Tversky Kahneman value and weighting functions. The skewness measure $\Gamma(0.99)$ is plotted for varying values of the probability weighting parameter $\delta_{\pm}$. The vertical dashed line indicates the base parameter value of $\delta_{\pm} = 0.7$. Other base parameters are $\alpha_{+} = 0.5$, $\alpha_{-} = 0.9$, loss aversion parameter $k = 1.25$, $\beta = 0.9$, reference level $R = 1$ and $P_0 = 1$.

In Figure 3 we plot skewness, as measured by $\Gamma(0.99)$, across different levels of the probability weighting parameter $\delta_{\pm}$. The corresponding optimal distributions for the same parameter choice (with varying δ) were displayed earlier in Figure 2. We first observe that skewness can be positive or negative, and can take values over the full range of +1 and -1, depending on the level of probability weighting.

Without probability weighting, PT investors take small gains frequently, with some occasional large losses (Henderson (2012), Ingersoll and Jin (2013)). This typically leads to a left or negatively skewed distribution. In particular, $F^{-1}(0.01)$ is zero, whereas both $F^{-1}(0.5)$ and $F^{-1}(0.99)$ are equal and both just above P_0 (the agent follows a two-sided threshold strategy). It follows $\Gamma(0.99) = -1$ when $\delta_{\pm} = 1$. With an S shaped utility and no probability weighting, investors prefer left skewed return distributions.

Once probability weighting is included, the skewness measure is no longer -1. Then the investor does not follow a two-sided threshold and looks for a long-tailed distribution on gains. This tail gets larger as $\delta_{\pm}$ decreases, although for $\delta_{\pm}$ close to one, the return distribution remains negatively skewed. For $\delta_{\pm}$ greater than about 0.74 the optimal prospect includes an atom at zero and the skewness statistic is negative. However, at $\delta_{\pm} \sim 0.74$ the optimal prospect undergoes a step change and the mass on losses moves from zero to a strictly positive level. This leads to a jump in the skewness statistic, which now becomes positive. As $\delta_{\pm}$ decreases further, the right tail on the optimal prospect becomes larger and $F^{-1}(0.99)$ and the skewness increase further from about 0.2 to 0.6. Now the investor is taking losses of moderate size, regular small gains, and occasional large gains.

The second jump in the skewness statistic occurs when the total mass on losses reaches 0.5 and is an artifact of Hinkley's measure of skewness. For values of $\delta_{\pm}$ below about 0.65, $F^{-1}(0.01) = F^{-1}(0.5) < P_0 < F^{-1}(0.99)$ and $\Gamma(0.99) = +1$. The optimal prospect now places more than half the mass on losses and the skewness measure simplifies in a way which does not depend on either the location of this mass, nor on the location of the point $F^{-1}(0.99)$ describing the size of the right tail. Nonetheless, as probability weighting increases, the size of this right tail increases, see Figure 2, even if this change cannot be captured in the skewness statistic.

To summarize, as the strength of probability weighting increases the investor's return distribution changes from left or negatively skewed to right or positively skewed and the right tail becomes fatter. Most of this change is captured in the skewness statistic.

4 Explaining the disposition effect: Odean's measure

The disposition effect is a tendency for investors to sell winners sooner than losers. Odean (1998) quantifies this tendency by comparing the frequency at which winners are sold relative to the opportunities to sell winners with the corresponding frequency for losses. In this section we translate the empirical Odean measure into a model-based *disposition ratio*. An important attribute of our definition is that our model-based variable depends on the optimal prospect, but not on the stopping rule used to generate that prospect. As a result we can use the form of the optimal prospect implicit in Theorem 1 to calculate the disposition ratio.

Odean (1998) compares the proportion of gains realized (PGR) to the proportion of losses realized (PLR) by 10 000 individual investors with accounts at a discount brokerage firm over a six year period. Each time a stock is sold, the price of all unsold stocks in the investors' portfolio are checked and it is recorded if they are trading at a gain, loss or neither on that day. The PGR (PLR) is the number of times a gain (loss) is realized as a fraction of the total number of times a gain (loss) could have been realized. Odean (1998) reports PGR=0.148 and PLR = 0.098, giving a disposition ratio of 1.51, or equivalently, investors realize gains at a 50% higher rate than losses. Using data over a different time period, Dhar and Zhu (2006) obtain a slightly higher ratio of 2.06.

Since we are working in continuous time, to capture the opportunities the investor had to sell at a gain (loss) we calculate the expected amount of time the price spent in the gain (loss) regime before a sale. A model-based measure of the rate of selling at gains (losses), denoted R_G (respectively, R_L) is found by dividing the probability of selling at a gain (loss) by the expected time the price spent above (below) the initial price:

$$R_G = \frac{\mathbb{P}(P_\tau > P_0)}{\mathbb{E}(\int_0^\tau 1_{(P_u > P_0)} du)}, \qquad\qquad R_L = \frac{\mathbb{P}(P_\tau < P_0)}{\mathbb{E}(\int_0^\tau 1_{(P_u < P_0)} du)}$$

where τ is an optimal sale time in the model.

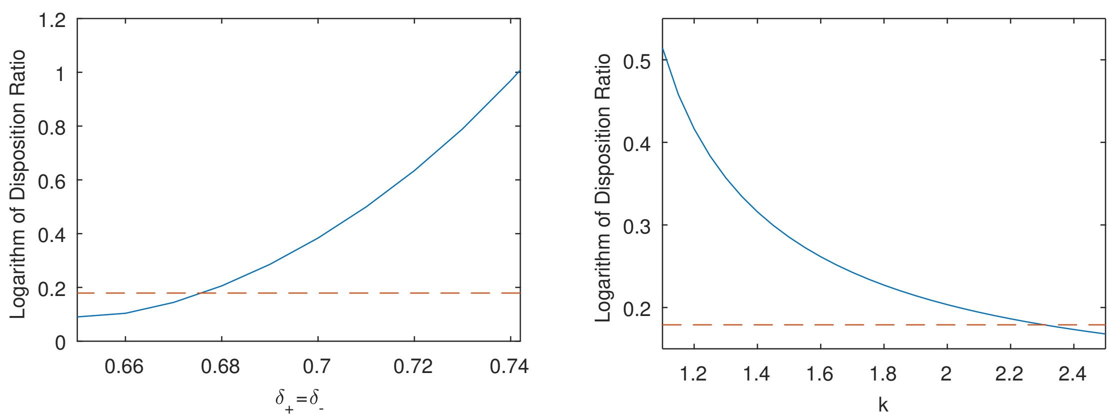
Then, following Henderson (2012) (see also Magnani (2015a)) we define the *disposition ratio* D by

$$D = \frac{R_G}{R_L} = \frac{\mathbb{P}(P_\tau > P_0)}{\mathbb{E}(\int_0^\tau 1_{(P_u > P_0)} du)} \frac{\mathbb{E}(\int_0^\tau 1_{(P_u < P_0)} du)}{\mathbb{P}(P_\tau < P_0)}. \tag{4}$$

This is the continuous time analog of Odean's measure. We say the disposition effect occurs when the ratio D is in excess of one.

Proposition 2. *D depends on the optimal prospect, but not on the stopping rule used to generate that prospect.*

If we assume the investor is a PT value maximizer, then his optimal scaled prospect can be computed from the optimal quantile function (see (A-12) in the Appendix), and we can calculate D without making any assumption about how the investor trades to achieve the optimal distribution. We prove Proposition 2 in a more general setting in Appendix D.



(a) Probability weighting $\delta_+ = \delta_-$ varying

(b) Loss aversion k varying

Figure 4: Base-10 logarithm of the disposition ratio D given in (4). Panel (a) varies probability weighting, keeping other parameters fixed (including loss aversion fixed at $k = 1.25$). Panel (b) varies loss aversion, keeping probability weighting on gains and losses $\delta_\pm$ fixed at 0.7. Other base parameters used are $\alpha_+ = 0.5$, $\alpha_- = 0.9$, $\beta = 0.9$. The reference level is $R = 1$, and the current price is $P_0 = 1$. The horizontal dashed lines mark Odean's disposition estimate of $\log_{10} 1.51 \approx 0.18$.

Figure 4 plots the (base-10 logarithm of the) disposition ratio D against the Tversky and Kahneman (1992) weighting parameter in panel (a), and loss aversion in panel (b). Other parameters are our base values. The horizontal dashed line in each panel represents Odean's PGR/PLR of $\log_{10} 1.51 \approx 0.18$.

We first comment on the extreme case of no probability weighting, $\delta_{\pm} \to 1$. In this case the disposition ratio becomes extremely large, and we recover the model studied in Henderson (2012) where the calibrated disposition measure was much greater than that found in the empirical data. In this case, there is a stop-gain threshold close to the reference level and a stop-loss threshold much further away. This results in many more sales at the gain threshold than the loss threshold, and a too extreme disposition ratio emerges. Ingersoll and Jin (2013) show it is difficult to improve the match to Odean's statistic, even when reinvestment is introduced. In fact, they only improve the match by mixing reference-dependent realization utility traders with random Poisson traders in a 50-50 ratio.

Our main finding is that probability weighting can reduce these extreme values of the disposition effect. Indeed, for realistic values of probability weighting we can recover the magnitude of the values seen in data. In contrast to PT models without probability weighting, our model incorporating weighting can indeed deliver Odean's estimate. In Figure 4 (a), we see that for a probability weighting parameter $\delta_{\pm}$ of about 0.675, we obtain a disposition ratio of about 1.5. In panel (b), holding other parameters fixed and varying loss aversion, we see that for a loss aversion of around 2.25, the disposition ratio is again about 1.5. Dhar and Zhu (2006)'s disposition measure of 2.06 ($\log_{10} 2.06 \approx 0.313$) can be obtained with slightly less probability weighting or a lower level of loss aversion. Thus, probability weighting does indeed help PT explain realistic levels of the disposition effect and we can explain empirical levels of the disposition effect for the group of investors who can commit to their initial plan. It is worth highlighting that the model has delivered Odean's estimate of the disposition effect with an analysis of a single investor.

5 The Tversky & Kahneman (1992) model and comparative statics

Tversky and Kahneman (1992) propose power functions of the form:

$$U(y) = \begin{cases} y^{\alpha_+}, & y \geqslant 0; \\ -k(-y)^{\alpha_-}, & y < 0 \end{cases} \tag{5}$$

where $0 < \alpha_\pm < 1$. The parameters $1-\alpha_+$ and $1-\alpha_-$ represent the coefficients of risk aversion and risk seeking, respectively. The parameter $k \geqslant 1$ governs loss aversion, introducing an asymmetry about the origin. Experimental results of Tversky and Kahneman (1992) give estimates of $\alpha_+ = \alpha_- = 0.88$ and $k = 2.25$. The TK parameters arise from experimental settings with small gamble sizes and we would expect higher levels of risk aversion in a financial trading setting. Wu and Gonzalez (1996) estimate $\alpha_+ = 0.5$ when they use the TK weighting parameterization. Furthermore, Ingersoll and Jin (2013) consider $\alpha_+ = 0.5, \alpha_- = 0.9$ as one of their base parameter sets. For consistency, we will also adopt $\alpha_+ = 0.5$, $\alpha_- = 0.9$ as our base case.[22] Our base loss aversion parameter level is $k = 1.25$. For all parameters we will consider a range of values when we look at comparative statics.

Tversky and Kahneman (1992) propose the probability weighting functions

$$w_\pm(p) = \frac{p^{\delta_\pm}}{(p^{\delta_\pm} + (1-p)^{\delta_\pm})^{1/\delta_\pm}} \tag{6}$$

for $0.28 < \delta_\pm \leqslant 1$. Alternative forms of $w_\pm$ proposed in the literature include Goldstein and Einhorn (1987) and Prelec (1998).

Estimates of the TK probability weighting parameters have been quite consistent across experimental and empirical studies. TK estimate the probability weighting parameters as: $\delta_+ = 0.61, \delta_- = 0.69$ (Barberis (2012)). Wu and Gonzalez (1996) find experimentally that $\delta_+ = 0.71$. Baele, Driessen, Ebert, Londono and Spalt (2017) estimate the degree of probability weighting from S&P 500 equity and option data and report a range of 0.72-0.79. Reflecting these findings, we take base parameters of $\delta_+ = \delta_- = 0.7$.

[22] Ingersoll and Jin (2013) comment that weak risk aversion on gains is a problem, even in the absence of probability weighting, and they would require unreasonable values for their discount parameter if they used the TK value for α_+.

The final ingredient of our model is a specification of the asset price process. We will assume that the price process follows geometric Brownian motion so that $P = (P_t)_{t\geq 0}$ solves

$$dP_t = P_t(\kappa dt + \sigma dB_t) \tag{7}$$

for constant expected return κ and volatility σ with $\kappa < \sigma^2/2$. The hypothesis that $\kappa < \sigma^2/2$ ensures the price does not reach arbitrarily high levels with probability one. Recall the definition $\beta := 1 - \frac{2\kappa}{\sigma^2}$ which involves the return-for-risk-per-unit-variance κ/σ^2 and thus reflects the expected performance of the asset. We assume $\kappa \geq 0$ so that in expectation P is non-decreasing and then $\beta \leq 1$. Our assumption $\kappa < \sigma^2/2$ implies that $\beta > 0$.

We require a further parameter restriction given in the following proposition. This is to avoid situations leading to infinite expected value whereby the investor simply waits indefinitely to take advantage of the favourable asset.[23]

Proposition 3. *For the problem to be well-posed we need* $\alpha_+ < \beta\delta_+$.

To simultaneously satisfy each of these restrictions whilst respecting our choices of other parameters, we take a base parameter for β of 0.9. We will also assume $R = P_0 = 1$; then losses are bounded by 1. The model we have described here (TK value and weighting functions, geometric Brownian motion, with the associated parameter values) is the model used to generate the figures and numerical results in the paper.

We first recap the form of the solution in the absence of probability weighting, when $\delta_+ = \delta_- = 1$. In this case (see Henderson (2012)), the optimal strategy is a threshold sale strategy. There will be a gain threshold level and a loss threshold level, and the optimal strategy is to stop the first time the price process leaves this interval. The corresponding prospect is a distribution on exactly two points. Typically the gain threshold is very close to the reference level. For realistic price parameters, the loss threshold is at zero, and it is never optimal to sell at a loss. (Instead, sales in the loss regime are postponed indefinitely). The convexity of the utility and loss aversion together mean that the investor prefers to continue to gamble and delay any losses.[24]

[23] Similar conditions arise in standard infinite horizon portfolio problems.

[24] When expected return κ is negative there will be a positive loss threshold, which is usually much further

5.1 The impact of probability weighting

In this section we are interested in the impact of probability weighting upon the optimal prospect. We assume that the probability weighting functions on gains and losses are identical and consider varying $\delta = \delta_+ = \delta_-$. As we introduce probability weighting by reducing the values of δ, we see the optimal prospect on gains completely changes character and switches from a point mass to a distribution with unbounded support. The tail of this distribution gets larger as probability weighting increases in strength. Why is this the case? Risk aversion alone makes small gains attractive. However now the investor overweights extreme events — in particular, extreme gains — and this encourages him to place some probability mass on these extreme wins. His distribution over gains is right-skewed in that most mass is still concentrated on lower gain levels, but probability weighting causes him to want to gamble on the best wins by placing some mass there.

Rather than present the quantile functions for each δ, in Figure 5 we summarize the distribution of the optimal prospect with three numbers: the location of the mass on losses (given by the dot-dash line), the location of the mass on gains and the location of the 99th percentile (given by the solid lines). The 99th percentile is a proxy for the upper tail of the distribution which is unbounded.

As probability weighting becomes sufficiently strong ($\delta_\pm$ below about 0.75 in Figure 5 thus including our base parameter of 0.7), we see that there will also be a strictly positive lower loss threshold at which the investor voluntarily takes losses. There are two forces driving this. First, the convexity and loss aversion are encouraging the investor to wait and avoid taking a loss. But now the investor overweights extreme events - in particular, extreme losses - which encourages him to cut-losses at some threshold. Importantly, the parameter region where a non-trivial loss threshold is present includes the levels of probability weighting commonly estimated in experimental and empirical studies.

Also clear from Figure 5 is that as probability weighting becomes stronger, the investor places more and more mass on extreme wins. However the location of the atom on gains is

from the reference level than the gain threshold. Thus, if losses are realized, they are typically much larger in size than gains. This is because the marginal utility of a gain or loss is decreasing with size, so small gains and large losses are preferable.

largely insensitive to δ.

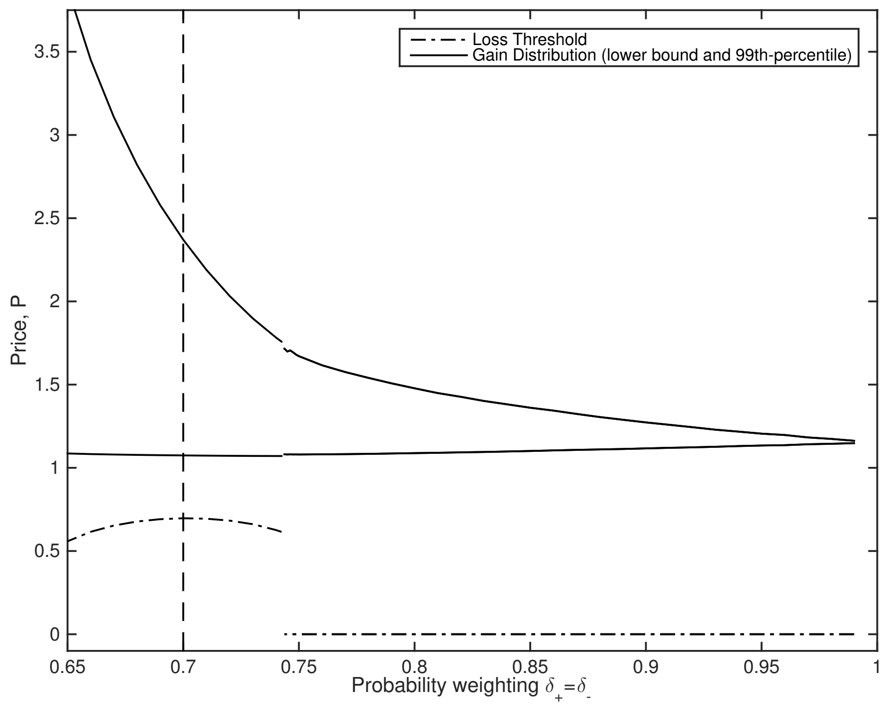
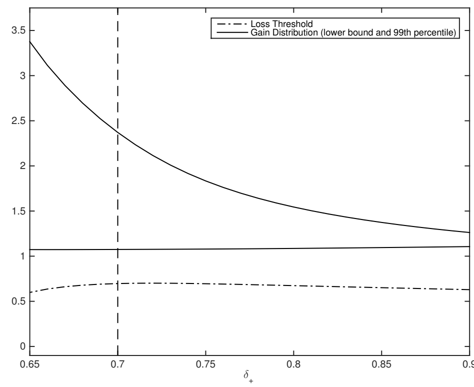


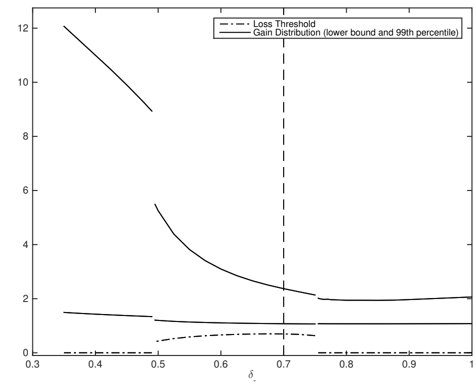
Figure 5: Summary statistics for the optimal quantile function in the model with Tversky and Kahneman value and weighting functions. The figure shows the location of the single loss threshold and the location of the mass on gains together with the upper 99th percentile of the distribution, each as a function of the probability weighting parameter $\delta_+ = \delta_- = \delta$. The vertical dashed line indicates the base parameter value of $\delta_\pm = 0.7$. Base parameters used are $\alpha_+ = 0.5$, $\alpha_- = 0.9$, loss aversion parameter $k = 1.25$, $\beta = 0.9$, reference level $R = 1$ and $P_0 = 1$. Note that a loss threshold of zero in the figures represents the situation where the investor never voluntarily realizes losses.

5.2 The impact of other model parameters

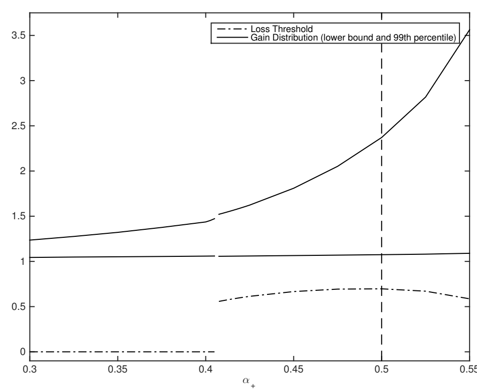
In Figure 6 we consider the impact of the individual probability weighting parameters and the loss aversion and risk aversion/risk seeking parameters. Panels (a) and (b) vary the probability weighting parameters individually whilst keeping the other fixed. As might be expected, we see that the main impact of δ_+ is on the tail of the distribution on gains. As



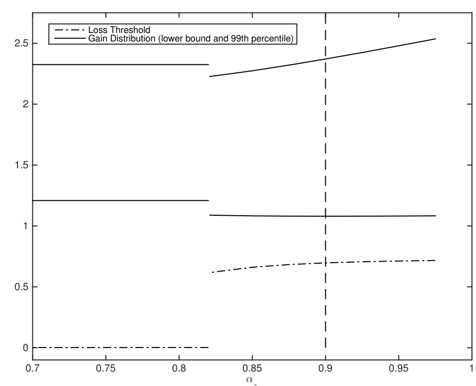
(a) probability weighting on gains, δ_+



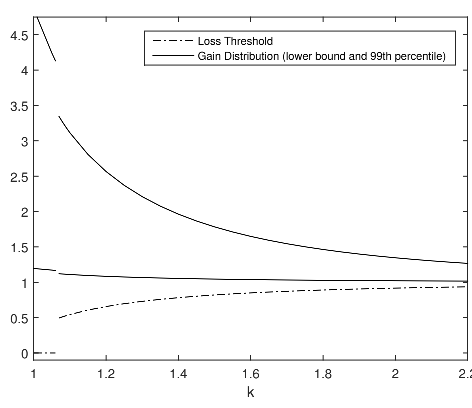
(b) probability weighting on losses, δ_-



(c) risk aversion on gains, α_+



(d) risk seeking on losses, α_-



(e) loss aversion, k

Figure 6: Comparative statics with respect to parameters with TK value and weighting functions. The optimal distribution consists of a single loss threshold together with a distribution over the gains region. We display summary statistics consisting of the loss threshold together with the lower bound and upper 99th percentile of the distribution over the gains regime. Each panel varies one parameter at a time, keeping the others fixed at base values. The vertical line marks the location of the relevant base parameter in each panel. Base parameters used are $\alpha_+ = 0.5$, $\alpha_- = 0.9$, $\delta_+ = \delta_- = 0.7$, loss aversion parameter $k = 1.25$, $\beta = 0.9$, reference level $R = 1$ and $P_0 = 1$.

probability weighting on gains increases (δ_+ decreases) the incentive to sometimes try for extreme gains increases, and the optimal prospect on gains becomes more disperse. This is clearly seen in the 99th percentile, and is also evident in the location of the atom on gains which decreases as probability weighting becomes stronger.

The dependence of the optimal prospect on δ_- is more complicated. In panel (b) and in other panels we see that the optimal prospect is not continuous in the parameters. This is because the objective function in the optimization of Proposition 1 is complicated, and it may have several local maxima. A small change in parameters may result in the global maximum being attained at a different mode. The primary impact of δ_- is to govern whether the distribution on losses contains an atom strictly above zero, or whether it contains an atom at zero. If probability weighting is too small, then the convexity of the value function on losses means that the investor aims for extreme losses, and the prospect contains an atom at zero. When probability weighting is larger the atom moves to an interior point. When probability weighting on losses is too strong, we again find that the agent never voluntarily incurs a loss.

Panels (c) and (d) vary the risk aversion and risk seeking parameters separately, whilst holding all other parameters fixed at their base values. In panel (c) we observe that higher levels of risk aversion (lower α_+) results in the distribution over gains being pulled down closer to the reference level. If risk aversion over gains is sufficiently strong, below about 0.4 in the panel, the investor no longer realizes losses. At the other extreme, we know that if risk aversion over gains is not strong enough, it violates the condition in Proposition 3 and the investor instead waits indefinitely. For the parameters in the graph, this would occur for values of $\alpha_+ > 0.63$. In particular, we see that under the original TK parameter $\alpha_+ = 0.88$ (particularly low level of risk aversion), the investor violates the condition in Proposition 3 and thus waits forever to sell, unless the expected return on the asset is unrealistically large and negative. Ingersoll and Jin (2013) encounter a similar issue (even in the absence of probability weighting) hence their choice of $\alpha_+ = 0.5$.

In panel (d) we see that when the value function on losses is close to linear (α_- near to 1) probability weighting dominates and there is a positive loss threshold. This means we are able to take the TK choice of the risk seeking parameter $\alpha_- = 0.9$. However, for α_- small enough, risk seeking on losses means that the agent prefers a prospect which places mass at zero. Further changes in α_- beyond this point make no difference as $U(-1) = -k|-1|^{\alpha_-} = -k$ and the value function does not depend on α_-. As seen in Figure 6, for α_- below about 0.82 the optimal prospect is insensitive to α_-.

In panel (e) we vary loss aversion. As loss aversion becomes stronger, the investor chooses a loss threshold which is closer to the reference level. We can understand this dependence by considering the problem in the absence of probability weighting. As loss aversion increases the investor becomes less willing to tolerate losses and wants to take losses sooner, and is even prepared to accept smaller gains. This has the impact of concentrating the optimal prospect closer to zero. The same intuition applies in the case with probability weighting.

6 Concluding remarks

Prospect theory has been very successful in explaining puzzles from economics and finance. The intuition associated with loss aversion and risk-seeking behavior on losses, and the overweighting of probabilities of extreme events, fits well with observed behavior in static situations, including many laboratory experiments. However, investors often face more complicated dynamic problems and decisions, and models based on the S shaped value function of PT have been only partially successful in explaining empirical patterns of behavior. Adding probability weighting to the analysis brings major technical challenges. However, it is important to overcome these challenges since probability weighting significantly alters the form of the PT investor's optimal strategy. In this paper we investigate the impact of probability weighting. Understanding the implications of probability weighting in a dynamic setting is crucial in understanding the behavior of PT investors.

The trading behavior of PT investors in dynamic models but in the absence of probability weighting has been studied by Ingersoll and Jin (2013), Henderson (2012) and Barberis and Xiong (2012). They find that the optimal stopping rule in this setting is a two-sided threshold strategy. This is a poor match to empirical data and laboratory findings on gains (although there is support for stop-loss strategies). Moreover, the locations of the stop-loss and stop-gain thresholds are such that a gain is much more likely than a loss, corresponding to a target law with a negative skew and an extreme disposition effect (well beyond that found in empirical data).

We show that introducing probability weighting greatly improves the predictive power of models of PT investors in all these aspects. With probability weighting set to levels estimated in the literature we find the PT investor has an optimal stopping rule which is stop-loss but not take-gain. Instead, the investor trades to achieve an optimal prospect which on gains is a long-tailed distribution chosen to reflect the overweighting of extreme gains. Overall, her target prospect can have right skew, and yet simultaneously have a model-based disposition ratio which matches the levels predicted by Odean (1998).

In this paper we provide the first stopping model for a precommitment agent to feature *all* the components of Tversky and Kahneman's prospect theory - S shape value function, loss aversion and probability weighting. We find that the PT agent with probability weighting provides a better fit to observed behavior than the PT agent without probability weighting, or a classical maximizer of expected utility. A caveat is that we have studied a partial equilibrium model and it may be difficult to extend to general-equilibrium implications.

An assumption throughout this paper is that the investor commits to an optimal strategy at time zero. However, it is well known (see eg. Machina (1989), Barberis (2012)) that probability weighting induces time inconsistency. There are several alternative ways in which we may model the reaction of agents to time-inconsistency, and each of these alternatives deserve their own study. Recent conclusions from PT stopping models for naive agents (naive agents "never stop", Ebert and Strack (2015)) and sophisticated agents without committment (sophisticates "never get started", Ebert and Strack (2017)) highlight the challenges in modeling the stopping behavior of PT agents. In the case where agents can commit, we have shown that we can obtain model-based predictions for PT investors which closely match experimental and empirical results.

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Appendix

The Appendix contains four sections. The main development of our solution to the optimal stopping problem for the prospect theory investor is presented in Section A where Theorem 1 (which provides a qualitative description of the optimal prospect) and Proposition 9 (which provides a quantitative description) are given. Proofs of some of the results used in Section A are deferred to Section B. Detailed proofs to show that the assumption on self-elasticity holds (Assumption 2 and Proposition 5) for several popular classes of value functions and probability weighting functions are given in Section C. Finally, a general version of Proposition 2 concerning the disposition ratio is deferred to Section D.

A Solution of the optimal stopping problem

Our goal in Section A is to prove Theorem 1 and Proposition 9. We define the concept of self-elasticity (see Section A.2) and give results (Propositions 4 and 5) which are required to demonstrate that Assumption 2 (Elasticity Assumption) holds. In Section A.4 we describe known key results from Xu and Zhou (2013). A.5 constructs the main result.

A.1 Asset price dynamics and optimal prospects

In general, we can model the asset price $P = (P_t)_{t \geqslant 0}$ by a time-homogeneous diffusion with state space $\mathcal{J}$, given by

$$dP_t = \kappa_P(P_t)dt + \sigma_P(P_t)dB_t. \tag{A-1}$$

Here $B = (B_t)_{t \geqslant 0}$ is a standard Brownian motion and $\kappa_P : \mathcal{J} \to \mathbb{R}$ and $\sigma_P : \mathcal{J} \to (0, \infty)$ are Borel functions. We assume $\mathcal{J}$ is an interval with endpoints $-\infty \leq a_J < b_J \leq \infty$ and that P is regular in (a_J, b_J). We will later specialize to the most popular asset price specification where P is a geometric Brownian motion (or equivalently P is lognormal). In that case $\mathcal{J} = (0, \infty)$, $\kappa_P(p) = \kappa p$ and $\sigma_P(p) = \sigma p$ for constants κ and σ.

Following Henderson (2012) it is convenient to reformulate the objective (2) by transforming the asset price into a martingale. We define $X_t := s(P_t)$ where the scale function s ensures X is a (local) martingale.[25] We are free to normalize s such that $s(R) = 0$, and hence $X_t \geqslant 0$ when $P_t \geqslant R$ and $X_t \leqslant 0$ when $P_t \leqslant R$. Then X_t represents the transformed gains and losses relative to

[25] The scale function of P can be identified as the increasing, non-degenerate solution (which is unique up

the reference level. If we take the reference level to be the initial asset price $R = P_0$, then $X_0 = 0$, but unlike in the main text, in the appendix we allow $P_0 \neq R$ and $X_0 \neq 0$.

The state space of X is an interval with endpoints $L = s(a_J)$ and $M = s(b_J)$. Then $L < 0$ represents the potential maximum loss. We assume $L > -\infty$ to ensure the problem is non-degenerate.

Proof of Proposition 1. Define $v(x) := U\left(s^{-1}(x) - R\right) = U(p - R)$. Then the investor's objective (2) can be rewritten as

$$\sup_{\tau} \left(\int_0^{v(M)=U(b_J-R)} w_+ \left(\mathbb{P}\left(v(X_\tau) > y\right)\right) dy - \int_{U(a_J-R)=v(L)}^{0} w_- \left(\mathbb{P}\left(v(X_\tau) < y\right)\right) dy \right). \quad \text{(A-2)}$$

One of the insights of Xu and Zhou (2013) is that the argument in (A-2) only depends on the law of X_τ. Hence, (A-2) can in turn be rewritten as

$$\sup_{\nu \in \mathcal{A}} \left(\int_0^{v(M)} w_+ \left(1 - F_\nu(v^{-1}(y))\right) dy - \int_{v(L)}^{0} w_- \left(F_\nu(v^{-1}(y))\right) dy \right) \quad \text{(A-3)}$$

where $\mathcal{A}$ is the set of attainable laws of X_τ and F_ν is the cumulative distribution function of the law ν. The set of attainable laws $\mathcal{A}$ can be characterized by $\mathcal{A} = \{\nu : \int y\nu(dy) = X_0\}$.[26] After a change of variables this can be written as (3). $\square$

Our investor evaluates (A-3) at the outset and commits today to achieve the desired target distribution or prospect.

to positive affine transformation) to the ordinary differential equation

$$\frac{1}{2}\sigma_P^2(p)s''(p) + \kappa_P(p)s'(p) = 0.$$

Then $X = (X_t)_{t \geq 0}$ defined by $X_t := s(P_t)$ is a (local) martingale. We assume that $\kappa_P(.)$ and $\sigma_P(.)$ are sufficiently regular that there exists a weak solution to the stochastic differential equation (A-1) and that the scale function s exists (see Revuz and Yor (1999)).

[26] At this point the fact that X is a local martingale is important since it allows us to give a simple characterization of the space of attainable laws. Since $L > -\infty$ and X is bounded below, it is a supermartingale and any attainable law ν must satisfy $\int y\nu(dy) = \mathbb{E}[X_\tau] \leq X_0 = s(P_0)$; conversely the theory of Skorokhod embeddings tells us that for every law ν with $\int y\nu(dy) \leq X_0$ there is a stopping rule τ such that $X_\tau \sim \nu$. Finally, since U is increasing, in searching for the supremum in (A-3) we may restrict attention to laws satisfying $\int y\nu(dy) = X_0$. Hence we may set $\mathcal{A} = \{\nu : \int y\nu(dy) = X_0\}$. If ν^* is the optimal law arising in (A-3), so that the optimal prospect for the process in natural scale is ν, then the optimal prospect for P has law μ^* where $F_{\mu^*}(p) = F_{\nu^*}(s(p))$. If ν^* is an optimal prospect then any stopping time τ^* constructed such that X_{τ^*} has the law as ν^* is optimal. Then we also have that P^*_τ has law μ^*.

A.2 Self-elasticity: definition

In this section we introduce a quantity we call the self-elasticity measure. This is key to our proof of Theorem 2 of which Theorem 1 is a special case. Elasticity is generally a ratio of the partial derivatives of a function with respect to two different variables. Self-elasticity is the ratio of two different representations of the derivative of a univariate function, one local and non-local. The monotonicity of the self-elasticity function, applied separately to the value and probability weighting functions, gives a simple sufficient condition for the monotonicity of the prospect value in the location of the point mass on losses.

Definition 1 (Self-elasticity)**.** For a monotonic and continuously differentiable function $f : S \to \mathbb{R}$, the self-elasticity measure (parameterized by x) relative to a reference point c is defined as

$$E_{f,c}(x) = E(x; f, c) = \frac{(x-c)f'(x)}{f(x) - f(c)} = \frac{f'(x)}{\frac{f(x)-f(c)}{x-c}}$$

where $x, c \in S$ and $x \neq c$. At $x = c$, and provided $f'(c) \neq 0$, we define $E(c; f, c) = 1$ by L'Hôpital's rule.

A useful property which we make use of is the following, the proof of which is deferred to Appendix C.

Proposition 4. *Let ι denote the identity function $\iota(x) = x$, and suppose f and g are monotonic and continuously differentiable. Let $a \neq 0$ and b be constants. Then $E(x; a\iota + b, c) = 1$ and $E(x; g \circ f, c) = E(x; f, c)E(f(x); g, f(c))$.*

Proposition 4 is key to showing the following elasticity properties hold. The first result is stated for the TK value function (see Section 5) but in Appendix C it is shown to hold for a variety of value functions. The second result holds for the common weighting functions and the details are given in Appendix C.

Proposition 5.

1. *Suppose $0 < \alpha_- < 1$ and $0 < \beta \leq 1$. If $v(x) = -k\left(R - (x + R^\beta)^{1/\beta}\right)^{\alpha_-}$ for $L \leq x \leq 0$ then $E(x; v, c)$ is increasing in x for $x \in [L, 0]$ for fixed $c \in [L, 0)$.*

2. *If the weighting function w is of the form proposed by Tversky and Kahneman (1992), Goldstein and Einhorn (1987) or Prelec (1998) and has inflexion point q then $E(p; w, r)$ is decreasing in p for $0 \leq p \leq \min\{r, q\}$ for any r in $[0, 1]$.*

We will use Proposition 5 in the next section where we will make an Elasticity Assumption on value and weighting functions which will be key to our main result.

A.3 The main result

In what follows we assume that U is S shaped, $w_\pm$ is inverse-S shaped, $L = s(a_J) > -\infty$ and $M = s(b_J) = \infty$.

Assumption 1 (S shaped Assumption on v)**.** v is concave on $[0, \infty)$ and convex on $[L, 0]$. Further, $v'(0+) = \infty$ and $\lim_{x \uparrow \infty} v'(x) = 0$.

In our base case of geometric Brownian motion and TK preferences it is simple to check by differentiation that v is concave on $[0, \infty)$ and convex on $[L, 0]$. Assumption 1 is also satisfied whenever $U'(0+) = \infty$, $\lim_{p \uparrow b_J = \infty} U'(p) = 0$ and P is a martingale whence the scale function is the identity function. More generally it depends on the interplay between the value function U and the dynamics of the price process.

Assumption 2 (Elasticity Assumption)**.** $E(x; v, L)$ is increasing in x for $x \in [L, 0]$ and $E(p; w_-, r)$ is decreasing in p for $0 \leq p \leq \min\{r, q\}$ for any r in $[0, 1]$.

By Proposition 5 both parts of the Elasticity Assumption are satisfied in the TK base case model and we show in Appendix C that the assumption is also satisfied for a range of other probability weighting and value functions.

Our main theoretical result is the following. The precise form of the optimal prospect is given in Proposition 9 in Appendix A.5.3, but here we describe the qualitative form of the solution to (3).

Theorem 2. *Suppose Assumptions 1 and 2 hold. Then the optimal prospect has a distribution which consists of a point mass in the loss regime and a point mass at some point A in the gains regime, together with a continuous distribution on the unbounded interval (A, ∞).*

It follows that the optimal strategy for a PT investor is a stop-loss combined with a strategy yielding a long-tailed distribution on gains.

A.4 Existing results

Xu and Zhou (2013) consider an optimal stopping problem with probability weighting in which asset returns are always in the gain regime. In this section we summarize the relevant results. Recall that the quantile function of a random variable Y, denoted G_Y (or G if the random variable is clear), is the (left-continuous) inverse of the cumulative distribution function.

Proposition 6 (Lemmas 3.1 and 3.2 of Xu and Zhou (2013))**.** *Suppose the scaled asset price process $X = (X_t)_{t \geqslant 0}$ is always non-negative and $X_0 > 0$. Then the probability-weighted optimal stopping problem*

$$\sup_{\tau} \int_0^{\infty} w\left(\mathbb{P}\left(v(X_\tau) > x\right)\right) dx \tag{A-4}$$

has a dual representation in terms of the quantile function $G = G_X$ of X_τ in the form

$$\sup_{G \in A_{X_0}} \int_0^1 v(G(x)) w'(1-x) dx \tag{A-5}$$

where

$$A_z = \{G | \ G : (0,1) \to [0,\infty) \text{ is a left-continuous quantile function, } \int_0^1 G(x)dx = z\}.$$

In particular, if G^* is an optimizer of (A-5) *with ν^* being the associated probability law, then there exists a stopping time τ^* such that $X_{\tau^*} \sim \nu^*$, and such τ^* is optimal for* (A-4)*.*

The optimal prospects in this one-sided problem can be identified under some particular forms of v and w. The two results below are the most relevant to our current problem.

Proposition 7 (Theorem 5.2 and Lemma 4.1 of Xu and Zhou (2013))**.** *The optimizer of* (A-5) *can be characterized under the two cases below:*

1. *Suppose the target prospect can take values on $[0,\infty)$, and that the mean is constrained to be less than or equal to z. If v is concave and w is inverse S-shaped such that it is concave on $[0,q]$ and convex on $[q,1]$, then the optimizer G is of the form*

$$G(x) = a 1_{(0,1-q]} + \left(a \vee (v')^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right) 1_{(1-q,1)} \tag{A-6}$$

   for some $a \geqslant 0$ and $\lambda \geqslant 0$, where a and λ are chosen such that they respect the constraint $\int_0^1 G(x)dx = z$. The optimal prospect is an atom at a of size at least $1-q$ combined with a continuous distribution on $(a \vee (v')^{-1}(\frac{\lambda}{w'(q)}), \infty)$.

2. *Suppose the target prospect is bounded such that it can only take values on $[0,K]$ for some $0 < K < \infty$. If v is convex and w is a general probability weighting function, the optimizer G is a step-function taking values on 0, K and some $b \in (0,K)$. The optimal prospect is a three-point distribution with masses at 0, b and K.*[27]

[27] Although Xu and Zhou (2013) do not directly consider the problem with bounded payoff, their Lemma 4.1 can be trivially extended to a set of quantile functions with bounded range on $[0,K]$.

The results in the second part of the Proposition 7 translate directly to a one-sided problem involving losses. Thus, we can deduce from the results of Xu and Zhou (2013) that in the loss regime the optimal prospect consists of up to three point masses.

Our main technical contribution is to solve for the optimal prospect for a wide class of prospect theory specifications. The results we present are not valid for all set-ups. Rather, we make some additional assumptions involving our self-elasticity condition which are satisfied under our base case, and more widely under many standard formulations of the problem. Under these additional assumptions we can prove that the optimal prospect has extra structure beyond that which can be deduced from Proposition 7. This extra structure allows us to solve for the optimal prospect in the general case with probability weighting and both gains and losses.

First, on the gain regime we show that a and λ are such that $a \geq (v')^{-1}(\frac{\lambda}{w'(q)})$. Hence the point a is simultaneously the location of a point mass in the optimal prospect and the lower limit in the continuous part of the optimal prospect on gains. Second, on the loss regime, we show that the optimal prospect is a single point mass (and not three point masses) located at some point $b \in [L, 0)$.

A.5 General construction of the optimal solution

In this subsection, we solve (A-3) assuming that the scaled value function v is concave on the gain regime $[0, \infty)$ and convex on the loss regime $[L, 0]$. The probability weighting functions $w_\pm$ are inverse-S shaped, concave on $[0, q_\pm]$ and convex on $[q_\pm, 1]$. The starting level of the scaled price process X_0 is a given fixed constant. Our base case fits into this setting. Detailed proofs are given in Appendix B.

A.5.1 The problem for gains

Suppose we are given $\phi_+ \in (0, 1]$ and $z_+ \geqslant X_0^+$ which are the probability mass allocated to gains and the mean of gains. The gain problem is to find

$$D_+(\phi_+, z_+) = \sup_{G \in A^+_{\phi_+, z_+}} \int_{1-\phi_+}^1 v(G(x)) w'_+(1-x) dx \tag{A-7}$$

where

$$A^+_{\phi,z} = \{G |\ G : (0, 1) \to [0, \infty) \text{ is a quantile function}, \int_{1-\phi}^1 G(x)dx = z,\ G(x) = 0 \text{ on } (0, 1 - \phi]\}.$$

The gain problem involves an optimization for concave v and inverse-S shaped w_+. The first part of Proposition 7 can be applied to identify the form of the optimal quantile function on $(1-\phi_+, 1)$.[28] From (A-6), with q_+ the point of inflexion of w_+ we deduce that the optimizer is of the form

$$G_+(x) = G_+(x;\phi_+, z_+; a, \lambda) = a1_{(1-\phi_+,(1-q_+)\vee(1-\phi_+)]} + \left(a \vee (v')^{-1}\left(\frac{\lambda}{w'_+(1-x)}\right)\right) 1_{((1-q_+)\vee(1-\phi_+),1)} \tag{A-8}$$

for some constants $a \geqslant 0$ and $\lambda \geqslant 0$. (Note that $G_+ = 0$ on $(0, 1-\phi_+]$.) The optimal values a^* and λ^* of a and λ are obtained by maximizing the objective function in (A-7) over $A^+_{\phi_+,z_+}$. The next result is proved in Appendix B.

Lemma 1. *Suppose v' is continuous with $v'(0+) = \infty$ and $\lim_{x\uparrow\infty} v'(x) = 0$. Then for the optimal prospect we have $a^* \geq (v')^{-1}\left(\frac{\lambda^*}{w'_+(q_+\wedge\phi_+)}\right)$.*

It follows that $a^* = (v')^{-1}\left(\frac{\lambda^*}{w'_+(\psi)}\right)$ for some $\psi \leq q_+ \wedge \phi_+$ and that the optimal quantile function is of the form

$$G_+(x) = G_+(x;\phi_+, \psi, \lambda) = (v')^{-1}\left(\frac{\lambda}{w'_+((1-x)\wedge\psi)}\right) 1_{((1-\phi_+),1)} \tag{A-9}$$

(Then G_+ is identically 0 on $(0, 1-\phi_+)$, equal to a constant on $(1-\phi_+, 1-\psi)$ and continuous on $(1-\phi_+, 1)$. The corresponding distribution has an atom at a^* and a density on (a^*, ∞)). For quantile functions of the form in (A-9) the mean on gains can be written as

$$z_+ = z_+(\phi_+, \psi, \lambda) = \int_{1-\phi_+}^{1} du (v')^{-1}\left(\frac{\lambda}{w'_+((1-u)\wedge\psi)}\right) = \int_0^{\phi_+} du (v')^{-1}\left(\frac{\lambda}{w'_+(u\wedge\psi)}\right). \tag{A-10}$$

Then instead of fixing the mass and mean on gains it is convenient to fix the mass ϕ_+ only and to consider a family of candidate optimizers of the form in (A-9), parameterized by ϕ_+, λ and $\psi \leq q_+ \wedge \phi_+$.

A.5.2 The problem for losses

Suppose now we are given ϕ_- and z_- such that $\phi_- \geqslant 0$ and $X_0^- \leqslant z_- \leqslant K\phi_-$ where $K = -L$ and where ϕ_- and z_- represent the probability mass allocated to losses and the mean of losses. The loss problem is to find

$$D_-(\phi_-, z_-) = \sup_{G\in A^-_{\phi_-,z_-}} \int_0^{\phi_-} v(G(x))w'_-(x)dx \tag{A-11}$$

[28]By considering a transformation of $\widetilde{G}(x) = G((1-\phi_+)(1-x)+x)$, one can see (A-7) can be rewritten in form of (A-5).

where

$$A^-_{\phi,z} = \{G|\ G : (0,1) \to [L,0] \text{ is a quantile function, } \int_0^\phi G(x)dx = -z,\ G(x) = 0 \text{ on } (\phi,1)\}.$$

The loss problem is a maximization problem involving a convex v and an inverse-S shaped w_- over quantile functions with fixed mean and a bounded range. Using the second part of Proposition 7, the optimal quantile function on $(0, \phi_-)$ is in the form of a step-function taking values on L, 0 and some $b \in (L, 0)$. Our main result is that under some mild extra assumptions on v and w_- the solution can be further simplified to a two-point distribution which has a mass at L or a mass at an interior point, but not both, together with an atom at 0. Later we will argue that for the problem with gains and losses there cannot be a mass at zero. Proposition 8 is proved in Appendix B.

Proposition 8. *If Assumption 2 holds, then the optimal solution to problem* (A-11) *is a two-point distribution with probability mass allocated to* 0 *and a single further point in* $[L, 0)$. *The optimal quantile function is of the form*

$$G_-(x; \phi_-, z_-) = -\frac{z_-}{\eta} 1_{(0,\eta]}$$

for some η *with* $-\frac{z_-}{L} \leqslant \eta \leqslant \phi_-$.

A.5.3 The combined problem for gains and losses

Proposition 9. *Suppose that Assumptions 1 and 2 hold. Then the optimizer for* (3) *has quantile function of the form*

$$G_X(u) = \begin{cases} -\frac{1}{1-\phi}\left[\int_0^\phi dy (v')^{-1}\left(\frac{\lambda}{w'_+(\psi\wedge y)}\right) - X_0\right] & u \leq 1-\phi \\ (v')^{-1}\left(\frac{\lambda}{w'_+(\psi)}\right) & 1-\phi < u \leq 1-\psi \\ (v')^{-1}\left(\frac{\lambda}{w'_+(1-u)}\right) & 1-\psi < u \leq 1 \end{cases} \tag{A-12}$$

for some $\lambda > 0$, $\phi \in [0, 1]$, $\psi \leq q_+ \wedge \phi$ *such that*

$$X_0^+ \leq \int_0^\phi dy (v')^{-1}\left(\frac{\lambda}{w'_+(\psi \wedge y)}\right) \leq X_0 - (1-\phi)L. \tag{A-13}$$

It follows from the proposition that the optimal prospect does not allocate any probability mass to zero (which corresponds to the reference level prior to scaling). Moreover, Assumption 2 provides a simple sufficient (and decoupled) condition on the behaviors of v and w_- leading to the feature that there is only one single atom on loss.

B Proofs in the derivation of the optimal prospect

This Appendix is devoted to the proofs of several of our key results presented in Appendix A, namely Lemma 1, Propositions 8 and 9 and finally, Proposition 3.

Proof of Lemma 1. We begin by showing that the optimizer (A-6) to the one-sided gain-problem (A-5) with v concave has its parameter $a \geq (v')^{-1}(\frac{\lambda}{w'_+(q_+)})$.

Let $L(x; G, \lambda) = v(G)w'_+(1-x) - \lambda G$. This is maximized over G by $G = y_L(x; \lambda) := (v')^{-1}\left(\frac{\lambda}{w'_+(1-x)}\right)$. If $v'(0) = \infty$ and $\lim_{x\uparrow\infty} v'(x) = 0$, then $y_L(x; \lambda)$ is well-defined and strictly positive for all $0 < x < 1$. Now suppose G_0 is an optimal solution to (A-5) which has the form of (A-6) with parameters (a_0, λ_0). Consider another quantile function G_1 which is in form of (A-6) with parameters (a_1, λ_1) where $a_1 := (v')^{-1}\left(\frac{\lambda_1}{w'_+(q_+)}\right) > a_0$ and the value of λ_1 is implied by the constraint $\int_0^1 G_1(x)dx = z$.

On $0 < x \leqslant 1 - q_+$, $G_0(x) = a_0 < a_1 = G_1(x) \leqslant y_L(x; \lambda_1)$. As $y_L(\,\cdot\,; \lambda_1)$ is the maximizer of $L(\,\cdot\,; G, \lambda_1)$ we have $L(x; G_1, \lambda_1) \geqslant L(x; G_0, \lambda_1)$. On $1 - q_+ < x < 1$, $G_1(x) = y_L(x; \lambda_1)$ and then trivially $L(x; G_1, \lambda_1) \geqslant L(x; G_0, \lambda_1)$. This shows $L(x; G_1, \lambda_1) \geqslant L(x; G_0, \lambda_1)$ for all $0 < x < 1$ (with strict inequality holding for some x). It contradicts the assumption that G_0 is an optimal solution.

By extension, the optimizer (A-8) for the sub-problem of gains (A-7) must have its parameter $a \geq (v')^{-1}\left(\frac{\lambda}{w'_+(q_+)}\right)$. Two things follow. First we can rewrite the optimal solution in (A-8) as

$$G_+(x) = (v')^{-1}\left(\frac{\lambda}{w'_+(\psi_+ \wedge (1-x))}\right) 1_{(1-\phi_+,1)}$$

where λ is chosen such that

$$\int_{1-\phi}^{1} dx (v')^{-1}\left(\frac{\lambda}{w'_+(\psi_+ \wedge (1-x))}\right) = z_+$$

Second, since $a > 0$, an optimal solution on gains must allocate all the available probability mass on gains. This also implies the value of the gain problem $D_+(\phi_+, z)$ is always strictly increasing in the available probability mass ϕ_+. □

Proof of Proposition 8. From the results of Xu and Zhou (2013) we know that on losses, the optimal prospect consists of masses at up to three points, two of which must be at 0 and L. We want to show that under Assumption 2, the optimal prospect contains a single mass on the loss regime, with potentially a second mass at the origin. Suppose that the probability that the prospect takes a positive value and the mean of gains element of the prospect are given. Then the probability of the prospect taking a value on losses, and the mean loss z may also be considered as given.

Consider prospects on losses in form[29] of $\mathcal{P} = (L, p_L; x, p_x; 0, p_0)$ with $L < x < 0$. We have the relationships $p_L + p_x + p_0 = \phi$ and $Lp_L + xp_x = -z$ for fixed $\phi \in (0,1)$ and $z \in (0, -L]$. To show that under the stated assumptions the optimal prospect on losses is a two-point distribution, it is sufficient to show the existence of some feasible two-point prospects which are at least as good as $\mathcal{P}$.

Recall that w_- is an inverse S-shaped function which is concave on $[0, q_-]$ and convex on $[q_-, 1]$. By the Elasticity Assumption, $E(p; w_-, c)$ is decreasing in $p \in [0, \min(c, q_-)]$ for any $c \in [0,1]$. The prospect value of $\mathcal{P}$ is given by

$$V = w_-(p_L)v(L) + (w_-(p_L + p_x) - w_-(p_L))v(x) = w_-(p_L)v(L) + (w_-(\phi - p_0) - w_-(p_L))v(x).$$

Since $L \leq x \leq 0$ we must have $L(\phi - p_0) \leq -z$ else there is no feasible solution. Fix p_0 and z and consider varying x, p_L and p_x. The feasible range of x is given by $L \leqslant x \leqslant -\frac{z}{\phi - p_0}$. From the mean constraint $p_L L + p_x x = -z$ and the fact that $p_L + p_x = \phi - p_0$, we have

$$\frac{dp_L}{dx} = -\frac{dp_x}{dx} = \frac{p_x}{x - L}.$$

Differentiation of the prospect value function with respect to x gives

$$\begin{aligned}
\frac{\partial V}{\partial x} &= (w_-(\phi - p_0) - w_-(p_L))v'(x) - \frac{p_x}{x - L} w'_-(p_L)(v(x) - v(L)) \\
&= \frac{(w_-(\phi - p_0) - w_-(p_L))(v(x) - v(L))}{x - L} \left(\frac{(x - L)v'(x)}{v(x) - v(L)} - \frac{p_x w'_-(p_L)}{w_-(\phi - p_0) - w_-(p_L)} \right) \\
&= \frac{(w_-(\phi - p_0) - w_-(p_L))(v(x) - v(L))}{x - L} (E(x; v, L) - E(p_L; w_-, \phi - p_0)).
\end{aligned}$$

Case 1: $p_x + p_L = \phi - p_0 \leqslant q_-$.

Then $p_L \leq q_- \wedge (\phi - p_0)$ and $E(p_L; w_-, \phi - p_0)$ is decreasing in p_L and in turn decreasing in x. Together with the fact that $E(x; v, L)$ is increasing in x, $\frac{\partial V}{\partial x}$ is either positive for all $x \in [L, -\frac{z}{\phi - p_0}]$, negative for all x over the same range, or changes sign from negative to positive as x increases. Hence, either V is monotonic, or V has a minima, and the maximal prospect value is attained at either $x = -\frac{z}{\phi - p_0}$ or $x = L$. The corresponding prospects are $(-\frac{z}{\phi - p_0}, \phi - p_0; 0, p_0)$ and $(L, \frac{z}{|L|}; 0, \phi - \frac{z}{|L|})$. Since $z \leq (\phi - p_0)|L|$ we have $\phi - \frac{z}{|L|} \geqslant p_0 \geqslant 0$ and both prospects are feasible two-point solutions with at most one mass at a non-zero location.

Case 2: $\phi - p_0 > q_-$ *and* $p_L \leq q_-$.

Then again by the fact that $E(p_L; w_-, \phi - p_0)$ is decreasing in p_L and $E(x; v, L)$ is increasing in x,

[29]Following Barberis (2012), we write a prospect $\mathcal{P}$ corresponding to a discrete random variable with n atoms at $x_1 < x_2 < ... < x_n$ of sizes $p_1, p_2, ..., p_n$ as $\mathcal{P} = (x_1, p_1; x_2, p_2; ...; x_n, p_n)$.

the maximal prospect value (as we let p_L range between 0 and q_-) is attained at either $p_L = 0$, whence $x = -\frac{z}{\phi - p_0}$, or $p_L = q_-$, whence $x = -\frac{z+Lq_-}{\phi - q_- - p_0}$.

The former corresponds to a feasible two-point prospect $(-\frac{z}{\phi - p_0}, \phi - p_0; 0, p_0)$. The latter corresponds to a prospect $(L, q_-; -\frac{z+Lq_-}{\phi - q_- - p_0}, \phi - p_0 - q_-; 0, p_0)$. We show in the next case that this is not an optimal prospect, since the prospect value can be further increased by increasing p_L above q_-.

Case 3: $\phi - p_0 > q_-$ and $p_L \geq q_-$.

We compare the prospect $(L, p_L; x, p_x; 0, p_0)$ with another feasible prospect which places all its mass at L and 0 and show that the latter has at least as large a PT value as the former. We have

$$
\begin{aligned}
V &= w_-(p_L)v(L) + (w_-(\phi - p_0) - w_-(p_L))v(x) \\
&\leqslant w_-(p_L)v(L) + (w_-(\phi - p_0) - w_-(p_L))\frac{|x|}{|L|}v(L) \\
&= v(L)\left(\frac{|x|}{|L|}w_-(\phi - p_0) + \left(1 - \frac{|x|}{|L|}\right)w_-(p_L)\right) \\
&\leqslant v(L)w_-\left(\frac{|x|}{|L|}(\phi - p_0) + \left(1 - \frac{|x|}{|L|}\right)p_L\right) \\
&= v(L)w_-\left(p_L + \frac{|x|}{|L|}(\phi - p_0 - p_L)\right)
\end{aligned}
$$

where we have used the fact that v is convex on $[L, 0]$ in the second line and the fact that w_- is convex on $[q_-, 1]$ in the fourth line (together with $v(L) < 0$). But $v(L)w\left(p_L + \frac{x}{L}(\phi - p_0 - p_L)\right)$ is the value of a feasible two-point prospect $\left(L, p_L + \frac{|x|}{|L|}(\phi - p_0 - p_L); 0, p_0 + (1 - \frac{|x|}{|L|})(\phi - p_0 - p_L)\right)$. $\square$

Proof of Proposition 9. With the optimal quantile functions from the previous sub-problems for gains and losses, the combined-problem is to find

$$\sup_{(\phi_\pm, z_\pm)\in H} (D_+(\phi_+, z_+) + D_-(\phi_-, z_-))$$

where $H = \{(\phi_\pm, z_\pm) : \phi_\pm \geqslant 0, \phi_+ + \phi_- \leqslant 1, z_+ - z_- = X_0, z_+ \geqslant X_0^+, z_- \in [X_0^-, -L], \}$.

Under the assumption that $v'(0+) = \infty$, by the remarks at the end of the proof of Lemma 1 we have $\phi_- = 1 - \phi_+$. Moreover, suppose that a candidate optimal solution of the problem includes a mass at the origin in the loss-component. We could reclassify this mass as part of the gain-distribution. But then, the prospect value could be improved by redistributing this mass to become strict gains. Hence the candidate solution cannot be optimal, and there cannot be any mass at zero in the optimal prospect.

We have that the mean on gains must be greater than or equal to X_0^+. Then using (A-12) we have the left-hand inequality in (A-13). Conversely, the location of the mass on losses, as described by the first line of (A-12), must be greater than or equal to L. After some algebra we get the right-hand inequality in (A-13). $\square$

Proof of Proposition 3. Let $\widehat{w}(p) = p^{\delta_+}$. Then obviously $\lim_{p\downarrow 0} \frac{\widehat{w}(p)}{w_+(p)} = 1$. Then for some fixed $\epsilon > 0$ there exists $p^* > 0$ such that $\widehat{w}(p) < (\epsilon+1)w_+(p)$ for $0 < p < p^*$.

Consider a two-point zero-mean prospect $(-a, \frac{b}{a+b}; b, \frac{a}{a+b})$ with b being large enough such that $\frac{a}{a+b} < p^*$. Then the prospect value is given by

$$\begin{aligned} V(a,b) &= w_+\left(\frac{a}{a+b}\right)v(b) + w_-\left(\frac{b}{a+b}\right)v(-a) \\ &\geqslant \frac{1}{\epsilon+1}\widehat{w}\left(\frac{a}{a+b}\right)v(b) + v(-a) \\ &= \frac{1}{\epsilon+1}\left(\frac{a}{a+b}\right)^{\delta_+}((b+P_0^\beta)^{\frac{1}{\beta}} - P_0)^{\alpha_+} + v(-a) \\ &\to \infty \end{aligned}$$

as $b \to \infty$ if $\delta_+ < \frac{\alpha_+}{\beta}$.

To prove the well-posedness property under $\delta_+ > \frac{\alpha_+}{\beta}$, it is sufficient to show that the gain-part value $D_+(\phi_+, \mu)$ is finite for any ϕ_+ and μ. Using the cumulative distribution function formulation, we can rewrite the gain-part value as

$$D_+(\phi_+, z) = \sup_{\overline{F}\in B_{\phi_+,z}} \int_0^\infty w_+(\overline{F}(x))v'(x)dx$$

where

$$B_{\phi,z} = \{\overline{F} |\ \overline{F} : [0,\infty) \to [0,1] \text{ is a decreasing function}, \int_0^\infty \overline{F}(x)dx = z,\ \overline{F}(0) = \phi\}.$$

Since $w_+(p) \leqslant \widehat{w}(p) = p^{\delta_+}$ for all p, it is sufficient to show that $\widehat{D}_+(\phi_+, z)$ is finite where $\widehat{D}_+(\phi_+, z) := \sup_{\overline{F}\in B_{\phi_+,z}} \int_0^\infty \widehat{w}(\overline{F}(x))v'(x)dx$. Since $\widehat{w}$ is concave, the optimizer for $\widehat{D}$ can be obtained by solving a simple Lagrangian problem where the solution is

$$\overline{F}^*(x) = \min\left(\phi_+, (\widehat{w}')^{-1}\left(\frac{\lambda}{v'(x)}\right)\right) = \min\left(\phi_+, \left(\left(\frac{\alpha_+\delta_+}{\lambda\beta}\right)\frac{1}{(x+P_0^\beta)^{\frac{\beta-1}{\beta}}((x+P_0^\beta)^{\frac{1}{\beta}} - P_0)^{1-\alpha_+}}\right)^{\frac{1}{1-\delta_+}}\right)$$

and the optimal value is in form of

$$
\begin{aligned}
\widehat{D}_+(\phi_+,\mu) &= \int_0^\infty w_+(\overline{F}^*(x))v'(x)dx \\
&= C + \int_K^\infty \left(\frac{\delta_+}{\lambda}\right)^{\frac{\delta_+}{1-\delta_+}} \left(\left(\frac{\alpha_+}{\beta}\right) \frac{1}{(x+P_0^\beta)^{\frac{\beta-1}{\beta}}((x+P_0^\beta)^{\frac{1}{\beta}}-P_0)^{1-\alpha_+}}\right)^{\frac{1}{1-\delta_+}} dx
\end{aligned}
$$

for some constants C and K. This indefinite integral is convergent if $\delta_+ > \frac{\alpha_+}{\beta}$. $\square$

C On Elasticity measures of popular utility and weighting functions

In this Appendix, we first prove Proposition 4. For several popular classes of convex value functions v and inverse-S shaped probability weighting functions w, this proposition allows us to give simple proofs to show that Assumption 2 holds and prove Proposition 5.

Proof of Proposition 4. The fact that $E(x; ax+b; c) = 1$ is immediate from the definition. Also

$$
E(x;f,c)E(f(x);g,f(c)) = \frac{(x-c)f'(x)}{f(x)-f(c)}\frac{(f(x)-f(c))g'(f(x))}{g(f(x))-g(f(c))} = \frac{(x-c)(g\circ f)'(x)}{(g\circ f)(x)-(g\circ f)(c)} = E(x;(g\circ f),c)
$$

$\square$

C.1 Value functions

C.1.1 Power function

Suppose v has the form of $v(x) = x^\alpha$ defined on $[0,\infty)$ and $\alpha > 0$. Then

$$
E(x;v,c) = \frac{\alpha(x-c)x^{\alpha-1}}{x^\alpha - c^\alpha}
$$

with $c \geqslant 0$. Differentiation gives

$$
E'(x;v,c) = \frac{\alpha x^{\alpha-2}(cx^\alpha - \alpha x c^\alpha + (\alpha-1)c^{\alpha+1})}{(x^\alpha - c^\alpha)^2}.
$$

Consider $H(x) = cx^\alpha - \alpha x c^\alpha + (\alpha-1)c^{\alpha+1}$ and note that $H(c) = 0$. We have $\partial H/\partial x = H'(x) = \alpha c(x^{\alpha-1} - c^{\alpha-1})$ and note that $H'(c) = 0$. Then for $\alpha > 1$ we have $H(x)$ is convex in x and $H \geq 0$. If $\alpha < 1$ then H is concave in x and $H \leq 0$. It follows that E is monotonic increasing in x if $\alpha > 1$ and monotonic decreasing in x if $\alpha < 1$.

C.1.2 Exponential function

For $v(x) = e^{\alpha x}$ on $[0, \infty)$ and $\alpha > 0$,

$$E(x; v, c) = \frac{\alpha(x-c)e^{\alpha x}}{e^{\alpha x} - e^{\alpha c}} = \frac{\alpha(x-c)e^{\alpha(x-c)}}{e^{\alpha(x-c)} - 1}.$$

Note that $e^{\alpha(x-c)} > 1 + \alpha(x-c)$. Then differentiation gives

$$E'(x; v, c) = \frac{\alpha e^{\alpha(x-c)}(e^{\alpha(x-c)} - \alpha(x-c) - 1)}{(e^{\alpha(x-c)} - 1)^2} \geqslant 0$$

so that $E(x; v, c)$ is monotonic increasing in x.

C.1.3 Reverse-power function

Consider $v(x) = K^\alpha - (K-x)^\alpha$ on $x \in [0, K]$ for some $K \geqslant 0$ and $0 < \alpha < 1$. Then v is a non-negative convex function with $v(0) = 0$. For $x, c \in [0, K]$, we have using Proposition 4 twice,

$$E(x; v, c) = E(x; (K-x)^\alpha, c) = E(K - x; x^\alpha, K - c).$$

Since $E(y; x^\alpha, z)$ is decreasing in y we conclude that $E(x; v, c)$ is increasing in x.

C.1.4 Scaled power function from TK value function

We now prove Proposition 5 (1). Consider $v(x) = -k(h - (x + h^\beta)^{1/\beta})^\alpha$ for $-h^\beta < x < 0$ with $\alpha, \beta > 0$. Then, for $-h^\beta \leq c < 0$,

$$\begin{aligned}
E(x; v, c) &= E(x; v(z) = (h - (z + h^\beta)^{1/\beta})^\alpha, c) \\
&= E(x + h^\beta; v(z) = (h - z^{1/\beta})^\alpha, c + h^\beta) \\
&= E((x + h^\beta)^{1/\beta}; v(z) = (h - z)^\alpha, (c + h^\beta)^{1/\beta})E(x + h^\beta; y^{1/\beta}, c + h^\beta) \\
&= E(h - (x + h^\beta)^{1/\beta}; v(z) = z^\alpha, h - (c + h^\beta)^{1/\beta})E(x + h^\beta; y^{1/\beta}, c + h^\beta).
\end{aligned}$$

Now suppose that $0 < \alpha, \beta < 1$. Then $0 < \alpha < 1 < 1/\beta$ and from the discussion of the power functions in C.1.1, C.1.3 both of $E(h - (x + h^\beta)^{1/\beta}; v(z) = z^\alpha, h - (c + h^\beta)^{1/\beta})$ and $E(x + h^\beta; y^{1/\beta}, c + h^\beta)$ are increasing in x. Hence the product is increasing in x.

C.2 General probability weighting functions

Let w be an inverse-S shaped weighting function which is concave on $[0, q]$ and convex on $[q, 1]$. We want to show that for common families of weighting function $E(x; w, c)$ is decreasing in x for

$0 \leqslant x \leqslant \min(c,q)$ for any $0 \leqslant c \leqslant 1$, and thus prove Assumption 2 holds (and prove Proposition 5 (2)).

Let $G(x,c) = \frac{\partial}{\partial x} \ln E(x;w,c) = -\frac{1}{c-x} + \frac{w''(x)}{w'(x)} + \frac{w'(x)}{w(c)-w(x)}$. It is necessary and sufficient to show that $G(x,c) \leqslant 0$ for all $0 \leqslant x \leqslant \min(c,q)$ and $0 \leqslant c \leqslant 1$. Fix some x with $0 \leqslant x \leqslant q$. Then by a repeated application of l'Hôpital's rule

$$\lim_{c \downarrow x} G(x,c) = \lim_{c \downarrow x} \left(-\frac{1}{c-x} + \frac{w''(x)}{w'(x)} + \frac{w'(x)}{w(c)-w(x)} \right) = \frac{w''(x)}{2w'(x)} \leqslant 0$$

since $w''(x) \leqslant 0$ for $x \leqslant q$. Then, if one could show that $G(x,c)$ is decreasing in $c \in [x,1]$, then $G(x,c) \leqslant G(x,x) \leqslant 0$, and the result will follow.

Hence, for our desired conclusion it is sufficient to show that $\frac{\partial}{\partial c} G(x,c) = \frac{1}{(c-x)^2} - \frac{w'(x)w'(c)}{(w(c)-w(x))^2} \leqslant 0$ for $x \leq c$, or equivalently $f(x,c) = f(x,c;w) \geq 0$ for $x \leq c$ where

$$f(x,c) = w'(c)w'(x) - \left(\frac{w(c)-w(x)}{c-x} \right)^2. \tag{A-14}$$

In the remainder of this section, we use this approach to give a proof for the Goldstein and Einhorn (1987) weighting function and some analysis for the Tversky and Kahneman (1992) function. This property can also be shown to hold for the Prelec (1998) weighting function but we omit the detailed proof.

C.2.1 The Goldstein and Einhorn (1987) weighting function

The Goldstein and Einhorn (1987) weighting function is given by

$$w_{\pm}^{GE}(p) = \frac{\gamma_{\pm} p^{d_{\pm}}}{\gamma_{\pm} p^{d_{\pm}} + (1-p)^{d_{\pm}}} \tag{A-15}$$

for parameters $0 < \gamma_{\pm}, d_{\pm} < 1$. Set $\gamma = \gamma_{\pm}, d = d_{\pm}$, and abbreviate $w_{\pm}^{GE}$ to w. Then differentiation gives

$$w'(x) = \frac{d}{\gamma x^2} \left(\frac{1}{x} - 1 \right)^{d-1} \left(1 + \frac{1}{\gamma} \left[\frac{1}{x} - 1 \right]^d \right)^{-2}$$

and in turn, with $z = \frac{1}{x} - 1$ and $y = \frac{1}{c} - 1$

$$\begin{aligned} f(x,c) &= \frac{d^2}{\gamma^2 x^2 c^2} z^{d-1} y^{d-1} \left(1 + \frac{z^d}{\gamma} \right)^{-2} \left(1 + \frac{y^d}{\gamma} \right)^{-2} - \frac{1}{(c-x)^2} \left(\frac{1}{1+\frac{z^d}{\gamma}} - \frac{1}{1+\frac{y^d}{\gamma}} \right)^2 \\ &= \frac{1}{\gamma^2 (c-x)^2} \left(1 + \frac{z^d}{\gamma} \right)^{-2} \left(1 + \frac{y^d}{\gamma} \right)^{-2} \left(d^2 \left(\frac{1}{x} - \frac{1}{c} \right)^2 z^{d-1} y^{d-1} - \left(z^d - y^d \right)^2 \right). \end{aligned}$$

Since $x \le c$ we have $y \leqslant z$. The condition for $f(x, y) \geqslant 0$ is then

$$d^2(z-y)^2 z^{d-1} y^{d-1} - (z^d - y^d)^2 \geqslant 0. \tag{A-16}$$

On writing $z = \lambda y$ with $\lambda \geqslant 1$, (A-16) is equivalent to $d(\lambda - 1)\lambda^{\frac{d}{2}-\frac{1}{2}} - (\lambda^d - 1) \geqslant 0$ and in turn $g_d(\lambda) \ge 0$ where

$$g_d(\lambda) := d(\lambda^{1/2} - \lambda^{-1/2}) - (\lambda^{d/2} - \lambda^{-d/2}) \geqslant 0. \tag{A-17}$$

Write $\lambda = e^{2\theta}$ for $\theta \ge 0$. Then $g_d(\lambda) = h(\theta)$ where $h(\theta) = 2d \sinh\theta - 2\sinh(d\theta)$. But $h(0) = 0$ and $h'(\theta) = 2d[\cosh\theta - \cosh(d\theta)] \ge 0$ since $d < 1$. Hence g_d is increasing in λ for $\lambda \ge 1$. Since $g_d(1) = 0$ the result follows.

C.2.2 Tversky and Kahneman (1992) weighting function

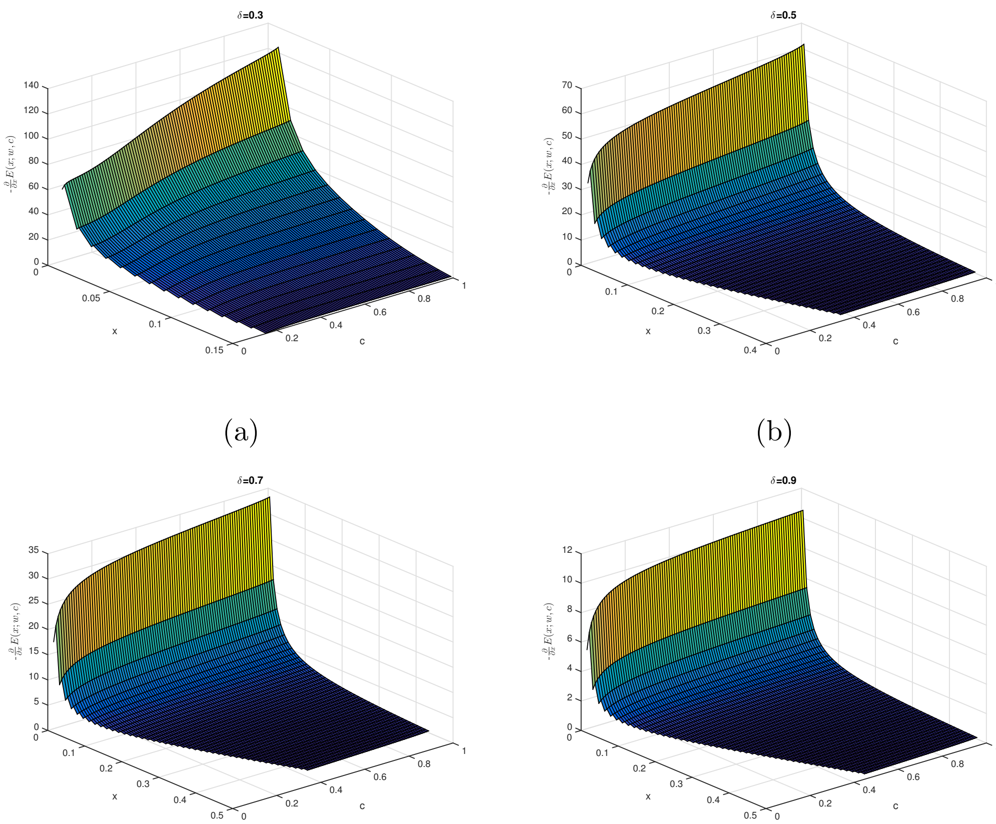
The Tversky and Kahneman (1992) probability weighting function is given in (6) as: $w(x) = \frac{x^\delta}{(x^\delta + (1-x)^\delta)^{1/\delta}}$. The decreasing elasticity property on the concave regime seems difficult to verify analytically. In Figure 7 we plot the function $-G(x, c) = -\frac{\partial}{\partial x}\ln E(x; w, c) = \frac{1}{c-x} - \frac{w''(x)}{w'(x)} - \frac{w'(x)}{w(c)-w(x)}$ for several values of δ over $0 \leqslant x \leqslant \min(c, q)$ and $0 \leqslant c \leqslant 1$ where q is the inflexion point of w. All the plots show positive values which verify the decreasing elasticity property of w on the required range.

D A proof of a general version of Proposition 2

Suppose P is a regular time-homogeneous diffusion. Let $X = s(P)$ be in natural scale and let ν be the probability law of the optimal prospect for X. Suppose X is sufficiently regular that it satisfies an SDE $dX_t = \xi(X_t)dB_t$ with initial condition $X_0 = s(P_0) = 0$. Let $L^X = (L^X_t(x))_{t\ge 0, s(a_J)\le x\le s(b_J)}$ be the local time of X at level x by time t using the standard normalization of, say, Revuz and Yor (1999). (No confusion should arise between $L = s(a_J)$ and the local time $L^X_t(x)$ since the former never has any sub- or superscripts.) By the occupation times formula (Revuz and Yor (1999) [Theorem VI.1.6]), for any Borel function Φ,

$$\int \Phi(a) L^X_t(a) da = \int_0^t \Phi(X_s) d[X]_s. \tag{A-18}$$

Proposition 2 is contained in the following result.



(a) (b)

(c) (d)

Figure 7: Plot of $-G(x,c) = -\frac{\partial}{\partial x} \ln E(x; w, c)$ with w taken to be the Tversky-Kahneman (1992) probability weighting function for several values of parameter δ. To conclude that $E(x; w, c)$ is decreasing in x for $0 \leq x \leq \min\{c, q\}$ we need $-G(x,c) \geq 0$ over the relevant range.

Proposition 10. *The disposition ratio D (recall (4)) depends on the optimal prospect, but not on the stopping rule used to generate that prospect. Further, the disposition ratio can be rewritten in the form*

$$D = \frac{\int_0^\infty \nu(dx)}{\int_0^\infty \frac{1}{\xi^2(x)} (u_\nu(x) - x)\, dx} \frac{\int_L^0 \frac{1}{\xi^2(x)} (u_\nu(x) + x)\, dx}{\int_L^0 \nu(dx)} \tag{A-19}$$

where $u_\nu(x) = \mathbb{E}^{X\sim\nu}[|X - x|]$.

Proof. Suppose ν is the probability law of the target scaled prospect, or equivalently suppose that under an optimal stopping rule $X_\tau \sim \nu$. Clearly, $\mathbb{P}(P_\tau > P_0) = \mathbb{P}(X_\tau > 0) = \int_0^\infty \nu(dx)$. Similarly $\mathbb{P}(P_\tau < P_0) = \int_L^0 \nu(dx)$. Then

$$\mathbb{E}\left(\int_0^\tau 1_{(P_u>P_0)}du\right) = \mathbb{E}\left(\int_0^\tau 1_{(X_u>0)}du\right) = \mathbb{E}\left(\int_0^\tau \frac{1_{(X_u>0)}}{\xi^2(X_u)}d[X]_u\right) = \mathbb{E}\left(\int \frac{1_{(x>0)}}{\xi^2(x)}L_\tau^X\, dx\right) = \int_0^\infty \frac{\mathbb{E}(L_\tau^x)}{\xi^2(x)}dx$$

where we use (A-18) for the penultimate equality. But, by Tanaka's formula $\mathbb{E}(L_\tau^a) = \mathbb{E}|X_\tau - a| - |X_0 - a|$. Hence, writing $u_\nu(x) := \int |z - x|\nu(dz)$

$$\mathbb{E}\left(\int_0^\tau 1_{(P_u>P_0)}du\right) = \int_0^\infty \frac{1}{\xi^2(x)} (\mathbb{E}|X_\tau - x| - |X_0 - x|)\, dx = \int_0^\infty \frac{1}{\xi^2(x)} (u_\nu(x) - x)\, dx$$

which is independent of the stopping rule used to realize ν. Similarly we can establish

$$\mathbb{E}\left(\int_0^\tau 1_{(P_u<P_0)}du\right) = \int_L^0 \frac{1}{\xi^2(x)} (u_\nu(x) + x)\, dx.$$

The representation (A-19) follows. □